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PREFACE

The following pages contain a list of the additions to the Society's library since the Cata-gue was published in May 1946 as part 2 of Number 5 of Volume 52 of the Bulletin of the merican Mathematical Society.

The use of a "+" in Section I of the Catalogue indicated that it was expected subsequent issues would be added to the library. Consequently no mention is made here of such ions.

ARNOLD DRESDEN
Librarian

anuary 1947

SECTION I

SERIAL PUBLICATIONS

- ACADEMIA brasileira de ciencias, Rio de Janeiro
  - Anais
    - 15+, 1943+
- BULLETIN des sciences mathématiques
  - s2 v63+, 1939+
- JOURNAL of physics. (Academy of sciences of the USSR) Moscow
  - 3, 7+, 1940+
- NATIONAL academy of sciences, Allahabad. Through 1935 as Academy of sciences of the United Provinces of Agra and Oudh
  - Bulletin
    - 2^4, 1933
- NATIONAL academy of sciences, Washington
  - Proceedings
    - 24, 1938
- NORSKE videnskaps-akademi, Oslo
  - Matematisk-naturvidenskapelig klasse
    - Register: 1925–39
- LYONS
  - Université
    - Annales
      - ns 3. Sciences
        - A. Science mathématiques et astronomie
          - 1+, 1936+
- SCIENCE abstracts. (Institution of electrical engineers; etc.) London
  - Sec. A. Physics
    - 48+, 1945+
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    - sA v50, 52+

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TOPOLOGICAL METHODS IN THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

MARSTON MORSE AND MAURICE HEINS

1. Introduction. Some indication of the reasons for which the authors have undertaken the study of topological methods in the theory of functions of a complex variable is appropriate.

The modern theory of meromorphic functions has distinguished itself by the fruitful use of the instruments of modern analysis and in particular by its use of the theories of integration. Its success along the latter line has perhaps diverted attention from some of the more finitary aspects of the theory which may be regarded as fundamental. In particular, the classical use of the Cauchy integral

$$\frac{1}{2\pi i}\int_{C}\frac{f''(z)}{f'(z)}\,dz$$

to find the difference between the number of zeros and poles of $f'(z)$ within C is in a sense statistical and ignores important extremal properties of the boundary values such, for example, as the extremal values of $|f(z)|$. In addition, its application requires the existence of $f''(z)$, at least almost everywhere on C, and the nonvanishing of f' on C. As we shall see there is a topological substitute for this integral under much weaker conditions on f. The needs of the classical theory usually require that the curve C be regular or rectifiable. The topological analogue makes use of Jordan curves whose images under f are "locally simple." See §3.

The study of analytic functions is of course paralleled by the study of harmonic functions. An example of a type of mathematical phenomena which the classical theory has failed to reach is given by a harmonic function $U(x, y)$ with infinitely many critical points on a Jordan region R. If $U(x, y)$ is continuous on $\bar{R}$, the closure of R, there are topological or group theoretic relations between the critical points of U on R and the relative minima of U on the boundary of R. These relations take the form of isomorphisms, termed "causal" by the authors, between an appropriately defined group of relative 1-cycles with the U-"heights"[1] of the respective "saddle points" of U, and a

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An address by Marston Morse with discussion by Maurice Heins delivered before the New York meeting of the Society on April 26, 1946, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors August 3, 1946.

[1] The U-height of a cycle is the maximum value of $U(x, y)$ on the cycle.

group of bounding 0-cycles with the U-heights of the respective relative minima of U.

To turn to another aspect of the theory, it is found that a study of harmonic functions on the basis of the topological properties of their level lines leads to a classification of the essential boundary characteristics and to basic relations between these characteristics and the critical points and logarithmic poles of U. The logarithmic poles of U are assumed finite in number. This study and the corresponding study of meromorphic functions is of such a general topological nature that the theorems and proofs apply equally well to functions of more general type. The so-called "interior transformations" $f(z)$ generalize the meromorphic functions and "pseudo-harmonic" functions generalize the harmonic functions. These extended functions will presently be defined. The term ***interior*** is used essentially in the sense of Stoilow [8].[1] In the sense of Whyburn [11], these transformations are "interior" and "light."

In the classical study of meromorphic functions $w=f(z)$ one is usually free to make a conformal transformation of the domain of definitions of f, or of the Riemann image. In the study of the zeros, poles, and branch points of interior transformations one can replace conformal transformations by arbitrary homeomorphisms of the domains of both z and w. One of the deepest theorems to be reported, Theorem 3.3, was established with the aid of such homeomorphisms after more classical methods failed.

Among the more novel topics which have been studied by the authors is the deformation or homotopy classes of meromorphic functions f with prescribed characteristic sets (zeros, poles, and branch point antecedents). The domain of definition of f has here been restricted to the open disc $|z|<1$, and the number of zeros, poles, and branch point antecedents is fixed and finite. The deformations admitted are through continuous 1-parameter families of meromorphic functions with the *same* characteristic set. A finite set of numerical invariants J_i $(i=1, \cdots, n)$ of f under the admitted deformations has been obtained. The possession of these invariants is a necessary and sufficient condition that two functions f with the same characteristic set be in the same deformation class. In general there is a countably infinite set of deformation classes. In case there is but one zero and no pole, or one pole and no zero in the characteristic set, there is but one deformation class. Up to this point no difference in deformation theory between interior transformations and meromorphic functions exists. In particular, there is a common topological definition of the invariants (J) which characterize a deformation class. An interior transformation f which possessed the same invariants (J) and the same characteristic set (α) as a meromorphic function F can be admissibly deformed through interior transformations into F.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Differences between the theory of interior transformations and meromorphic functions arise, however, if one considers "model" sequences $[f_k]$ of meromorphic functions with a given characteristic set with each f_k belonging to a different deformation class. For such a model sequence $[f_k]$ the total set of points w on the w-sphere obtained as images under the transformations of $[f_k]$ for $|z| < 1$ covers each point of the w-sphere infinitely many times, provided the given characteristic set contains both zeros and poles. No such theorem holds for model sequences of interior transformations. The preceding theorem is one of many derivable from the theory of normal families, and the properties of model sequences which we have obtained. It is obviously related to theorems of the Picard type on an essential singularity.

For a more explicit exposition three significant topics have been selected. In terms which presently will be defined these topics are: (1) pseudo-harmonic functions under boundary conditions A, (2) interior transformations with locally simple boundary images, (3) deformation classes of interior or meromorphic functions.

2. **Pseudo-harmonic functions under boundary conditions A.** Let $u(x, y)$ be a function which is harmonic and not identically constant in a neighborhood N of a point (x_0, y_0). Let the points of N be subjected to an arbitrary sense-preserving homeomorphism T leaving (x_0, y_0) fixed, carrying N into N', and carrying a point (x, y) on N into a point (x', y') of N'. Under T set

$$u(x, y) = U(x', y'). \tag{2.1}$$

The function $U(x', y')$ will be termed *pseudo-harmonic* on N'. The definition will be extended to the case in which $u(x, y)$ has a logarithmic pole at (x_0, y_0). In this case U, as defined by (2.1), will be said to have a logarithmic pole at (x_0, y_0). More generally we admit functions $U(x, y)$ which are pseudo-harmonic, except at most for logarithmic poles, in some neighborhood of each point of a region G (open). We suppose U defined and continuous (except for poles on G) at every point of $\bar{G}$ (the closure of G). We shall take G as a finite region bounded by a set of ν Jordan curves

$$(B) = (B_1, \cdots, B_\nu).$$

It follows from the known properties of a nonconstant harmonic function that the number of level arcs of U which terminate at a point (x_0, y_0) of G is even. Set $U(x_0, y_0)=c$. A point (x, y) will be said to be *above* or *below* c if $U(x, y)>c$ or $U(x, y)<c$ respectively. The level arcs of U terminating at (x_0, y_0) divide an appropriately chosen neighborhood of (x_0, y_0) into $2m$ open sectors, alternatingly above and below c. If $m=1$, the point (x_0, y_0) is *ordinary*, otherwise *critical*. When $m>1$, $m-1$ is called the *multiplicity* of the critical point (x_0, y_0). Critical points of this type will be called *saddle* points to distinguish them from critical points of U which are relative minima.

The preceding topological characterization of a saddle point of U on G can be extended to points on (B) provided appropriate conditions limit the boundary values of U. Under *boundary conditions* A the function U^i defined by U on B_i has at most a finite number of points of relative extremum. *We assume that U satisfies conditions* A.

Let $z_0=x_0+iy_0$ be a point on B_i. The point z_0 may be an extremum point of U. If this is not the case, it can be shown that there are at most a finite number of level arcs with end points at z_0, and that these level arcs divide an appropriately chosen neighborhood of z_0 (relative to $\bar{G}$) into a number of sectors, alternatingly above and below $c=U(x_0, y_0)$. Let m be the number of these sectors below c. If $m=1$, (x_0, y_0) is termed *ordinary* relative to U. If $m>1$, (x_0, y_0) is termed a critical or *saddle* point of U relative to $\bar{G}$, and $m-1$ its *multiplicity*. Under boundary conditions A the number of saddle points of U on (B) is finite.

A saddle point of U on (B) relative to $\bar{G}$ is not necessarily a saddle point of $-U$. For example, the function y^2-x^2 on the domain $|z-1|\leq 1$ is ordinary at the point $z=0$, while x^2-y^2 has a saddle point of multiplicity 1 on the same domain.

A basic theorem follows.

THEOREM 2.1. *Under boundary conditions* A *on a pseudo-harmonic function U*

$$M-S=2-\nu+s-m \tag{2.2}$$

where $M=$ the number of logarithmic poles of U on G, $S=$ the number of saddle points of U on G, $\nu=$ the number of Jordan curves bounding G, $s=$ the number of saddle points of U on (B), $m=$ the number of points of relative minimum of U on (B).

Saddle points are counted with their multiplicities.

As might be expected, the proof of this theorem eventually involves the Euler characteristic of the sets $U\leq c$ and its variation with c. However when this formal stage in the proof has been reached the major difficulties have all been passed.

In addition to the condition (2.2) one can say that $M+m>0$, $\nu>0$, $S\geqq 0$, $s\geqq 0$. These conditions and the condition (2.2) are the only conditions on the integers involved, as one can show by setting up a pseudo-harmonic function realizing any set of integers satisfying these conditions.

If $U(x, y)$ is of class C' on a neighborhood of (B), if the boundaries are regular, and U_x and U_y are never both zero on (B), then m is the number of points of relative minimum of the boundary functions U^i at which the gradient vector g of U enters G from (B), and s is the number of points of relative maximum of the functions U^i at which g enters G from (B).

Among the cases in which conditions A are not satisfied on a boundary B_i, the case in which U is constant on B_i is important. If U has a constant relative extremum on B_i, the contribution of B_i to the term $s-m$ in (2.2) is null. This happens in the case of a Green's function.

In case U is a constant c on B_i but not a relative extremum, the contribution of B_i to the term $s-m$ in (2.2) equals one-half the number of arcs at the level c which tend to B_i from G on an arbitrarily small neighborhood of B_i. The proof of the last statement is made on the assumption that U is pseudo-harmonic, except for poles, on a region which includes $\bar{G}$ in its interior, and is continuous on (B).

Theorem 2.1 is the basis of many extensions and applications. For an account the reader is referred to the papers cited. Mention will be made of the fact that one can admit points of singularity z_0 of U on G in the neighborhood of which

$$U(x, y) = k \log |z - z_0| + RF(z) \tag{2.3}$$

where k is a real constant and $F(z)$ is analytic near z_0 except for a pole of the mth order at z_0. In Theorem 2.1 such a point counts as if it were a set of $m+1$ logarithmic poles. One can also suppose that U has a representation of the type (2.3) near a boundary point z_0 and give a simple rule for the contribution of z_0 to $s-m$ in (2.2).

3. Interior transformations with locally simple boundary images. A fundamental new instrument here is the locally simple, sensed, closed curve. Such a curve is a topological generalization of the regular curve and in many problems where regular or rectifiable curves are used is more effective and less restrictive.

The curves admitted are continuous and locally 1-1 images $z=z(\theta)$, $0\leqq\theta\leqq 2\pi$, of a sensed circle with an angular parameter θ. For such a curve there exists a positive constant e such that any sub-arc of g of diameter less than e is simple; such a constant e is called a *norm of* *local simplicity* of g. A curve can have infinitely many multiple points and still be locally simple. A closed curve which traces a circle n times, $n\neq 0$, in one sense is locally simple.

The angular order p of a regular sensed closed curve g is the number of times a tangent to g rotates through 2π as g is traced in its positive sense. If g is merely locally simple, p is defined as follows. Let k be a chord of g which subtends a simple sub-arc of g. Let the end points of k trace g in its positive sense remaining distinct and always subtending a simple sub-arc of g. The number of times k rotates through 2π is independent of the choice of k subject to the above conditions and is termed the *angular order* p of g.

Let $z=a$ be a point not on g. The classical *order* $q(a)$ of g with respect to a is the number of times the vector $z(\theta)-a$ rotates through 2π as g is traced in its positive sense.

We shall *admit continuous deformations* of a locally simple closed curve through a 1-parameter family of such curves, adding the essential condition that the family possess a common norm of local simplicity. The angular order p of a locally simple sensed curve g is invariant under any admissible deformation of g. This theorem would not be true if the deformations admitted were through locally simple curves without a *common* norm of local simplicity. A necessary and sufficient condition that two locally simple sensed curves g be admissibly deformable into each other is that they have the same angular order p. In the special case in which the curves are regular this theorem was proved earlier by Whitney and Graustein. See Whitney [10].

A positively sensed circle C traced n times, where $n\neq 0$, will be denoted by C^n. Let C^0 denote a figure eight. The angular order of C^p is p. Thus any locally simple sensed closed curve g can be admissibly deformed into one and only one of the models C^p. The set of all locally simple sensed closed curves with an angular order p form a deformation class denoted by c. It is easy to define the *product* of two such classes and prove the following theorem:

THEOREM 3.1. *The deformation classes c of locally simple sensed closed curves form a group H with respect to multiplication, isomorphic with the additive group J of integers, with a class c in H corresponding to that integer p in J which equals the angular order of the curves in c.*

The unit class in H contains the figure eight.

Unless otherwise specified, the order $q(0)$ of g will be denoted by q. There exist locally simple sensed curves in which p and q are arbitrary integers. An admissible deformation in which no curve intersects the origin $z=0$ will be called an O-deformation. The pair (p, q) is invariant under any O-deformation. Conversely any two locally simple sensed closed curves which have the same invariants (p, q) admit an O-deformation into each other. Canonical curves $M_1(p, q)$ with the invariants (p, q) are readily defined. O-deformation classes and their products can then be defined. The group H_1 of O-deformation classes c with respect to multiplication is isomorphic with the additive group J_1 of pairs (p, q) of integers, with a class c in H_1 corresponding to that pair (p, q) in J which yields the orders (p, q) of the curves in c. The unit class in H_1 contains a figure eight neither loop of which encircles the origin.

Interior transformations. Locally simple curves enter naturally into the theory of interior transformations. To define such transformations one begins with a definition of an interior transformation in the neighborhood of an arbitrary point z_0 of the region G of §2. Suppose that $F(t)$ is a nonconstant, meromorphic function defined on a neighborhood N of $t=a$. One subjects N to a 1-1 continuous sense-preserving transformation $t=h(z)$ which maps a neighborhood N_1 of a onto N, holding a fast. The function

$$F[h(z)] = f(z) \tag{3.1}$$

thereby defined on N_1 is an ***interior transformation*** $w=f(z)$ from N_1 into the complex w-sphere. A transformation $w=f(z)$ from G into the w-sphere will be called interior if f is an interior transformation of some neighborhood of each point of G.

The possibility that F have a pole at $t=a$ is admitted. In that case the function f defined by (3.1) is said to have a *pole* at $z=a$. We consider interior transformations with at most a finite number of poles on G. We suppose that f is defined on $\overline{G}$ and, with respect to the metric of the w-sphere, is a continuous transformation from $\overline{G}$ to the w-sphere without exception. We do not assume that f is an interior transformation of a neighborhood of points on the boundary (B).

A point $z=a$ of G is termed a *branch point antecedent* of order m of the inverse f^{-1}, if $t=a$ is a branch point antecedent of order m of F^{-1}. The *order* of a zero or pole of f at $z=a$ is similarly defined by reference to F.

The basic theorem involves the more novel notion of a *partial branch element* E_m appearing as the Riemann image under $w=f(z)$ of a neighborhood, relative to $\overline{G}$, of a boundary point z_0. We begin with a model. The continuous Riemann image under the transformation

$$w - w_0 = re^{\theta i} \tag{3.2}$$

of the pairs (r, θ) for which

$$0 \leqq \theta \leqq (2m+1)\pi, \qquad 0 \leqq r \leqq r_0 \qquad\qquad (r_0 > 0)$$

will be represented by a Riemann surface spread over a neighborhood of w_0 in the w-plane and termed a partial branch element E_m of *order* m. The term is extended to include any Riemann element spread over a neighborhood of w_0 in the w-plane obtained by subjecting E_m to a homeomorphism in which w_0 corresponds to itself and points which cover the same point w retain this property.

The transformation f is termed *locally* 1-1 *relative* to $\bar{G}$ at a boundary point z_0 if there exists a neighborhood N of z_0 relative to $\bar{G}$ on which the transformation into the w-plane is a homeomorphism. The principal hypothesis in the following two theorems is that the w-images under f of the boundary curves B_i are locally simple,[3] by virtue of the transformation $w=f(z)$. The following theorem gives a condition under which branch point antecedents are isolated on $\bar{G}$.

THEOREM 3.2. *If the images (g) under the interior transformation $w=f(z)$ of the boundaries (B) are locally simple, then f is locally* 1-1 *relative to $\bar{G}$ at all but a finite number of points on (B), and each exceptional point on (B) has a neighborhood relative to $\bar{G}$ whose Riemann image under f is a partial branch element.*

In particular if f is meromorphic the theorem implies that the points on G at which $f'=0$ are finite in number whenever the boundary images are locally simple.

The main theorem follows:

THEOREM 3.3. *If the images (g) of the boundaries (B) under the interior transformation $w=f(z)$ are locally simple and do not intersect the point $w=a$, then*

$$2n(a) = 2 - \nu + \mu + 2q(a) - p, \tag{3.3}$$

where $n(a)$ = the number of zeros of $f(z)-a$ on G; ν = the number of boundaries B_i, $q(a)$ = the sum of the orders of the images g_i, with respect to $w=a$, p = the sum of the angular orders of the images g_i, μ = the sum of the orders of the branch elements of f^{-1} including the partial branch elements.

The preceding theorem holds even if $a=\infty$, provided one sets $q(\infty)=0$. One thus has the relation

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[3] That is, locally simple in terms of the representation acquired under the transformation $w=f(z)$.

$$2n(\infty) = 2 - \nu + \mu - p. \tag{3.4}$$

From (3.3) and (3.4) one obtains the ordinary order relation

$$n(a) - n(\infty) = q(a). \tag{3.5}$$

A generalization of a theorem of Radó is a particularly simple application of Theorem 3.3. According to Radó there exists no directly conformal $(1, n)$ map $w=f(z)$ of G onto itself when $m>1$ and $\nu>1$. In such a map $n(a)=m$ for each point a of G and $f(z)$ can be continued so as to be continuous on (B) and map each boundary curve B_i on some boundary curve B_j' covered m times. It then follows that p in the theorem is $(2-\nu)m$. Our generalization, which includes Radó's theorem, makes no assumption as to the values of $n(a)$, and the boundary images may intersect themselves and each other. The transformation f is merely interior.

THEOREM 3.4. *Under the hypotheses of the preceding theorem with* $n(\infty)=0$ *and* $\nu>2$, *it is impossible that* $p=(2-\nu)m$ *unless* $m=1$.

With $p=(2-\nu)m$, and $n(\infty)=0$ it follows from (3.3), on setting $a=\infty$, that $\mu=(\nu-2)(1-m)$. Theorem 3.4 is immediate.

4. Deformation classes of meromorphic functions or interior transformations. We consider sense-preserving interior transformations $w=f(z)$ from the open disc $S=[|z|<1]$ to the w-sphere. We shall restrict ourselves to the case in which f has a finite set of zeros

$$(a_0, a_1, \cdots, a_r) \qquad (r > 0) \tag{4.1}$$

and poles

$$(a_{r+1}, \cdots, a_n) \qquad (n > 1) \tag{4.2}$$

with branch point antecedents

$$(b_1, \cdots, b_\mu) \qquad (\mu \geq 0).$$

We shall assume that the zeros, poles, and branch points have the order 1, although the case of higher orders can be successfully treated provided the orders do not change during the deformations of f. The set

$$(\alpha) = (a_0, \cdots, a_n, b_1, \cdots, b_\mu)$$

will be called the *characteristic set* of f.

We shall admit deformations D of f of the form $w=F(z, t)$, defined for z on S and $0\leq t\leq 1$. Here t is the deformation parameter and $F(z, 0)\equiv f(z)$. We require that F map (z, t) continuously into the w-

sphere and yield an interior transformation of S for each fixed t. We shall here report on the case in which the characteristic set is independent of t. Deformations of f in which (α) is fixed are termed *restricted*. In the general theory the sets (α) are permitted to vary over circuits and return to an initial set, or return to the initial set with a permutation of the zeros, poles, and branch point antecedents among themselves. An account of this general theory is included in a paper to be published by the authors [6].

Given an interior transformation f with the characteristic set (α) we shall define a set (J) of n numbers $J_i(f, \alpha)$, $i=1, \cdots, n$, associated with the respective pairs (a_0, a_i), such that a necessary and sufficient condition that two interior transformations f_1 and f_2 defined on S with the same characteristic set (α) be in the same restricted deformation class is that f_1 and f_2 have the same set (J).

The difference order $d[k]$. The definition of the invariants (J) involves an extension of the notions of order and angular order in the form of a *difference order* $d(k)$ defined for a locally simple arc k. These arcs are continuous and locally 1-1 images

$$w(t) = u(t) + iv(t) \qquad (0 \leqq t \leqq t_0)$$

of an interval $[0, t_0]$, and shall intersect fixed end points $w(0)=a$ and $w(t_0)=b$ only when $t=0$ and t_0 respectively. As previously, a norm $e>0$ of local simplicity of k is a constant such that any sub-arc of k whose diameter is less than e is simple. We shall admit deformations D of k through a continuous family of arcs joining a to b with a common norm of local simplicity; the arcs of the family shall intersect a and b only as end points. Normal forms for such arcs under admissible deformations are desired, together with numerical invariants characterizing such normal forms.

The two cases needed for the definition of the invariants (J) are the cases in which $a=b$ (finite) and the case (referred to the w-sphere) where a is finite and $b=\infty$. In case $a=b$ and the arc $w(t)$ is regular, the difference order $d(k)$ is defined as the algebraic increment in

$$\frac{1}{2\pi} \arg w'(t) \qquad (0 \leqq t \leqq t_0) \tag{4.3}$$

minus the algebraic increment in

$$\frac{1}{2\pi} \arg [w(t) - a] \qquad (0 < t < t_0) \tag{4.4}$$

as t increases from 0 to t_0, and the argument varies continuously. The limiting directions in (4.4) as t tends to 0 or t_0 are respectively the initial direction in (4.3) and the terminal direction in (4.3) reversed in sense. It follows that $d(k)$ is a half-integer.

In the general case in which k is not regular, $d(k)$ can be readily defined on replacing the tangent vector $w'(t)$ in (4.3) by a short chord and appropriately modifying (4.4). In any case $d(k)$ is a half-integer taking on the values

$$\cdots, -3/2, -1/2, 1/2, 3/2, \cdots.$$

A positively sensed circle C through the point $a=b$ has the difference order 1/2. Reversing sense of an arc changes the sign of a difference order. To obtain a model arc K_r with a difference order $n=(2r+1)/2$ for $r>0$ one attaches a small positively sensed circle C_1 to C within C, with C_1 tangent to C at some point a_1 other than $a=b$; one then traces C until a_1 is reached, then traces C_1 r times in the positive sense, continuing to b on C. Any admissible locally simple arc joining a to itself with a difference order $d(k)$ can be admissibly deformed into the model arc with the same difference order. The difference order is invariant under admissible deformations.

The difference order for the case of a locally simple arc joining $w=a$ to $w=\infty$ is separately defined, but the values of such orders are integers. All integers are realizable as orders as before, and normal forms are readily obtained.

The invariants J_i. Let f be an interior transformation from S to the w-sphere with the characteristic set

$$(\alpha) = (a_0, a_1, \cdots, a_n, b_1, \cdots, b_n) \qquad (n > 1).$$

Without loss of generality we can suppose that a_0 is a zero. (In case there are no zeros we can replace f by its reciprocal.) Let h_i be a simple curve joining a_0 to a_i on S, and let h_i' be the image of h_i under f. The difference order $d(h_i')$ will be independent of any continuous deformation of h_i through simple arcs joining a_0 to a_i provided h_i does not at any time intersect other points of (α). To obtain a number which is independent of the choice of h_i among simple arcs joining a_0 to a_i on S we introduce the function

$$C_i(z) = \left[\frac{(z-b_1), \cdots, (z-b_n)}{(z-a_1), \cdots, (z-a_n)}\right](z-a_i).$$

Corresponding to a variation of z along h_i set

$$V(h_i) = \frac{1}{2\pi}\Big[\arg C_i(z)\Big]_{z=a_0}^{z=a_i}.$$

The difference

$$d(h_i^f) - V(h_i) = J_i(f, \alpha) \qquad (i = 1, \cdots, n)$$

is our definition of J_i. It is independent of the choice of h_i among simple arcs which join a_0 to a_i on S and do not intersect other points of (α), and is invariant under admissible deformations of f since $d(h_i^f)$ is so invariant.

It is not in general an integer. But if F is any one admissible interior transformation with the characteristic set (α) and f is any other, then

$$J_i(f, \alpha) = J_i(F, \alpha) + r_i \qquad (i = 1, \cdots, n)$$

where r_i is an integer. This follows from the fact that $V(h_i)$ is the same for f and F, and that

$$d(h_i^f) - d(h_i^F)$$

is an integer. Each integer r_i can be realized by a proper choice of an f with the characteristic set (α). One is thus led to define a countably infinite set of models f corresponding to the countably infinite ensemble of sets (J).

The fundamental theorem here is as follows:

THEOREM 4.1. *A necessary and sufficient condition that any two admissible interior transformations f_1 and f_2 of S with the same characteristic set belong to the same restricted deformation class is that they possess the same invariants (J). If f_1 and f_2 are meromorphic, the deformation can be made through meromorphic functions.*

The general proof is in two steps. One first shows that an interior transformation f can be admissibly deformed into a meromorphic function. Then by an interpolatory process the meromorphic function is admissibly deformed into a canonical model f with the given invariants (J). The Stoilow uniformization theorem enters in the first stage of the proof.

The canonical functions f. With characteristic set (α) prescribed, the topological theory discloses a countably infinite ensemble of sets J_i, $i = 1, \cdots, n$, which are invariants of the restricted deformation classes. It remains to establish the existence of a meromorphic function in each deformation class, that is, a meromorphic function with a prescribed characteristic set (α) and a topologically admissible set of invariants J_i.

To that end one sets

$$A(z) = (z - a_0)(z - a_1) \cdots (z - a_n), \qquad B(z) = (z - b_1) \cdots (z - b_p).$$

If the required meromorphic function f existed, one could set

$$\frac{f'(z)}{f(z)} = \phi(z) \frac{B(z)}{A(z)}$$

thereby defining a *residual* function $\phi(z)$ which, except for removable singularities, is analytic on S, never zero, and satisfies the relations

$$\phi(a_j) = e_j \frac{A'(a_j)}{B(a_j)} \qquad (j = 0, \cdots, n) \tag{4.5}$$

where $e_j = 1$ or -1 according as a_j is a zero or a pole. One also has the remarkable relation

$$J_i = \arg \phi(z) \Big|_{z=a_0}^{z=a_i} \qquad (i = 1, \cdots, n) \tag{4.6}$$

where any continuous branch of the argument can be used.

The necessary properties of the residual function ϕ as enumerated are characteristic. Corresponding to a prescribed characteristic set (α) and topologically admissible invariants (J) there exists a function $\phi(z)$ which is analytic on S, never zero, and satisfies (4.5) and (4.6). Such a function ϕ is the residual function of the function

$$f = K \exp \left\{ \int \phi(z) \frac{B(z)}{A(z)} dz \right\}. \qquad (K \neq 0)$$

where K is constant. The function f has the prescribed characteristic set and invariants (J).

Covering properties of sequences of meromorphic transformations of S. We consider infinite sequences $[f_k]$ of meromorphic transformations of S with the same characteristic set (α) and with no two functions f_k in the same restricted deformation class. Such a sequence will be termed a *model sequence.* We are concerned with the set W of points $w = f_k(z)$, $k = 1, 2, \cdots$, on the w-sphere given by a model sequence for z on S. When the characteristic set includes both zeros and poles the set W covers each point of the w-sphere infinitely many times. When the characteristic set includes no poles, the set W will cover each point of the w-sphere infinitely many times ($w = \infty$ excepted) provided $[f_k]$ does not converge uniformly to zero on every compact subset of S.

These results are a consequence of well known theorems on normal families (see Montel [3]) and of the following preliminary lemma. Set $(a) = (a_0, a_1, \cdots, a_n)$. A model sequence which does not converge continuously to 0 or ∞ on $S-(a)$ (in the sense of Carathéodory [2, p. 58]) is not a normal family.

The covering theorems stated above for model sequences of meromorphic functions do not hold for sequences of interior transformations. The above covering theorems can be greatly extended and refined in the terms familiar in the theory of normal families. The following theorem is a consequence of Bloch's theorem. See Bieberbach [1, p. 230].

THEOREM 4.2. *If* $[f_k]$ *is a model sequence of meromorphic transformations of* S, *no subsequence of which converges continuously to* 0 *on* S, *and if the characteristic set* (α) *includes no poles, there then corresponds to any positive constant* r, *no matter how large, a member* $f_{k(r)}$ *of the sequence and a circular disc* D_r *of radius* r *in the* w*-plane, such that* D_r *is the one-to-one image under* $f_{k(r)}$ *of some subdomain of* S.

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PROBABILITY IN FUNCTION SPACE

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Introduction. The mathematical theory of probability is now ordinarily formulated in terms of measure theory. The formalization of a succession of n trials is in terms of n-dimensional measure; that of a sequence of trials is in terms of (denumerably) infinite-dimensional measure; that of a continuous set of trials (a set dependent on a continuous time parameter say) is in terms of measure in function space. The latter case is the subject of this lecture.

There still is a residual tendency among mathematicians to deal with distribution functions rather than with the chance variables which have those distribution functions; in this way probability theorems are reduced to theorems on monotone functions, convolutions, and so on. This preference was once justified by a natural distrust of the mathematical foundations of probability theory, so that any discussion of chance variables seemed to be nonmathematical, physics, or at best mathematical physics rather than pure mathematics, and therefore not rigorous. By now this distrust of chance variables is no longer justified; in practice, although it has almost disappeared when finite or denumerably infinite sequences of chance variables are studied, its presence is still noticeable in studies of continuous families of chance variables. Thus there is now no hesitancy in evaluating the probability that a series whose terms depend on chance will converge, but there is still considerable hesitancy in discussing the probability that a function whose values depend on chance should have a limiting value at ∞, be continuous, or be Lebesgue integrable. It is the purpose of the present lecture to dispel some inhibitions by summarizing the basic knowledge on measure theory in function space, as applied to probability. In the course of doing this, some problems still to be solved will be discussed. Although the theory is mathematically rigorous, it cannot yet be considered fully satisfactory. Even mathematicians, although they can live on rigor alone, do not invariably enjoy it.

The general point of view adopted in analyzing a succession of trials, depending on the parameter t (usually representing time), is the following. A set T of values of t is preassigned, the times when events occur, or measurements are made. This set of times may be finite or infinite. At each such moment a number $x(t)$ is obtained.

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This number depends on chance; the value obtained comes from a probability distribution whose distribution function

$$P\{x(t) < k\} \tag{0.1}$$

is preassigned.[1] The process is fully specified by the complex of all possible joint distributions: if $t_1, \cdots, t_n$ is any finite subset of T, and if $-\infty \leqq a_j < b_j \leqq +\infty$, the probability

$$P\{a_j < x(t_j) < b_j, j = 1, \cdots, n\} \tag{0.2}$$

is preassigned. All probabilities are evaluated in terms of these elementary probabilities. (For alternative approaches see §4.)

This *nonmathematical set-up* is treated mathematically as follows. Let Ω be the space of all functions $x(t)$, $t \in T$, $-\infty \leqq x(t) \leqq +\infty$; Ω is the set of all possible successions of sample values, that is, the set of all possible sample functions. The set of functions of t defined by the inequalities

$$a_j < x(t_j) < b_j, \qquad j = 1, \cdots, n, \tag{0.3}$$

is given as measure (probability) the number assigned in (0.2). In other words, a set function is defined on certain sets in function space, and probability problems become measure problems. A chance variable, that is a function of sample values, becomes a function defined on function space. The change in language from probability to measure language and vice versa neither increases nor decreases the difficulty of problems, but on the one hand measure theory is the precise mathematical form of the probability calculus, and on the other hand the empirical background of probability theory provides one type of external criterion of the interest and importance of a given measure problem. In the following, a non-negative completely additive function of sets, $P\{E\}$, defined on a Borel field F of sets in a space Ω, with $P\{\Omega\}$ defined and $P\{\Omega\} = 1$, will be called a probability measure. It is usually convenient to complete the measure by defining $P\{E\}$ on any set E_1 which differs from a set E_0 in F by at most a subset of a set of measure 0 as $P\{E_1\} = P\{E_0\}$. The enlarged field will be denoted by $\bar{F}$.

The set of functions in function space defined by the inequalities (0.3) will be called a neighborhood, as will the set of functions defined by (0.3) with equality allowed on the right whenever $b_j = \infty$ and on the left whenever $a_j = -\infty$. Function space is topologized by the convention that each neighborhood is a topological neighborhood of any one of its points. With this definition, Ω becomes a bicompact Hausdorff space.

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[1] The notation $P\{\ \}$ will be used to denote the probability of the event, or measure of the point set, defined by the braces.

Let F_0 be the Borel field of sets in function space Ω generated by the neighborhoods (that is, F_0 is the smallest Borel field of Ω sets which includes the neighborhoods) and let F_1 be the Borel field generated by the open sets; F_1 is the field of Borel sets. Then $F_0 \subset F_1$ and if T is finite or denumerably infinite, $F_0 = F_1$. On the other hand if T is the entire t axis, if I is any open interval, and if k is any real number, the function space sets

$$\begin{aligned} \left\{ \underset{t \in I}{\text{L.U.B.}}\, x(t) \leq k \right\} &= \prod_{t \in I} \{x(t) \leq k\}, \\ \left\{ \underset{t \in I}{\text{G.L.B.}}\, x(t) > k \right\} &= \sum_{t \in I} \{x(t) > k\} \end{aligned} \tag{0.4}$$

are respectively closed and open sets in function space, which are in F_1 but not in F_0.[2] The two sets are respectively the intersection and sum of nondenumerably many sets in F_0.

Now suppose that T is finite. Then Ω is simply n-dimensional coordinate space, and a neighborhood is an n-dimensional open rectangle with sides parallel to the axes. It has long been known that in this case if the non-negative function of $a_1, \cdots, a_n, b_1, \cdots, b_n$ defined in (0.2) obeys simple conditions of monotoneity, and so on, then this function, considered as a function of Ω neighborhoods, can be extended to become a probability measure defined on $F_0 = F_1$, the field of all the Borel sets of n-space. In this case, then, probability problems become problems in measure theory in n-space. Next suppose that T is denumerably infinite, as required for the study of sequences of chance variables. In this case Ω becomes denumerably infinite-dimensional coordinate space. Kolmogoroff proved [7, p. 27][3] that if the non-negative set function (0.2) is defined for every finite t set $(t_1, \cdots, t_n)$, and determines a probability measure in $x(t_1), \cdots, x(t_n)$ n-dimensional coordinate space, and if the set functions so obtained are mutually compatible, that is, if whenever ν pairs (a_j, b_j) are set equal to $(-\infty, +\infty)$ in (0.2) the $(n-\nu)$-dimensional set function obtained in the other $n-\nu$ variables is exactly the preassigned set function for those variables, then just as in the finite-dimensional case, the domain of definition of the set function $P\{E\}$ in (0.2) can be extended to include all the sets of $F_0 = F_1$ (and hence of $\bar{F}_0=\bar{F}_1$), in such a way that $P\{E\}$ becomes a probability measure.

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[2] In fact it is clear that any set in F_0 is obtained by imposing conditions on $x(t)$ at finitely or denumerably many t values.

[3] Numbers in brackets refer to the bibliography at the end of the paper.

In the finite and denumerably infinite-dimensional cases, the measure so defined has proved adequate for all the needs of the calculus of probabilities. The central limit theorem, laws of large numbers, and so on, can always be expressed in terms of the properties of measurable sets and functions on the coordinate space Ω. The situation changes, however, if the parameter t has a continuous range. In the following, T will be the full real line. The changes necessary if T is a line segment or half line will be obvious. The coordinate space Ω then becomes the space of functions defined on the line. As before, (0.2) defines the measure of a certain Ω set, that is, of the set of functions satisfying the stated inequalities, and Kolmogoroff's theorem, which is independent of dimensionality, states that this measure can be extended to become a probability measure defined on all the sets of the Borel field F_0. It turns out, however, that this definition of probability measure is not entirely suited to the needs of the calculus of probabilities. In fact it was noted above that each F_0 set is defined by restrictions on $x(t)$ for at most countably many t values. On the other hand, in many investigations probabilities like that in (0.4) are required; that is, probabilities of sets of functions defined by restrictions on $x(t)$ at a continuum of t values. Such sets may not be in F_0 or even in $\bar{F}_0$. For example the sets in (0.4) are never in F_0 and in general are not in $\bar{F}_0$. More precisely, if $k<\infty$, the inner P_0 measure of the first set in (0.4) is always 0 [3, p. 465]. Hence this set must have measure 0 if it is P_0 measurable. This fact implies that if $\text{L.U.B.}_{t\in I}x(t)$ is P_0 measurable, it must be $+\infty$ almost everywhere. Similarly if $\text{G.L.B.}_{t\in I}\,x(t)$ is P_0 measurable, it must be $-\infty$ almost everywhere. These cases can actually occur. For example let $F(x)$ be any distribution function with $0<F(x)<1$ for all x. Suppose that for each value of t the chance variable $x(t)$ has the distribution function F, and that for any n values $t_1, \cdots, t_n$ the chance variables $x(t_1), \cdots, x(t_n)$ are mutually independent. In other words, as t varies the family of chance variables $x(t)$ consists of mutually independent chance variables with the common distribution function F. Then clearly if $s_1, s_2, \cdots$ is any denumerable t set,

$$
\begin{aligned}
&P_0\left\{\underset{j}{\text{L.U.B.}}\ x(s_j)\leq k\right\}=0,\\
(0.5)\qquad &P_0\left\{\underset{j}{\text{G.L.B.}}\ x(s_j)>k\right\}=0
\end{aligned}
$$

for $k\neq\pm\infty$. The sets in (0.4) are included in sets of P_0 measure 0 (take the sequence $\{s_j\}$ in I) and therefore have measure 0. Aside from such rather uninteresting cases, however, L.U.B.$_{t\in I}x(t)$ and G.L.B.$_{t\in I}x(t)$ are never P_0 measurable.

The insufficiency of P_0 measure is illustrated by the following example. Suppose that $x(t)$ is intrinsically less than or equal to k. In this simple case, any satisfactory mathematical theory must contain the statement in some form or other that the sample functions $x(t)$ are less than or equal to k with probability 1. This is trivial if T is denumerable, $T=\{t_j\}$, because then one only need conclude from the hypothesis $P_0\{x(t_j)>k\}=0$ that

$$\begin{aligned} P_0\left\{\underset{j}{\text{L.U.B.}}\ x(t_j) > k\right\} &= P_0\left\{\sum_j \{x(t_j) > k\}\right\} \\ &\leq \sum_j P_0\{x(t_j) > k\} = 0. \end{aligned} \tag{0.6}$$

This reasoning cannot be used if T is nondenumerable, and it has just been seen that the desired result is not even true in that case. The situation can be saved, however, by defining a probability measure in the space Ω_1 of functions less than or equal to k rather than in the space of all functions, and Kolmogoroff's extension theorem is applicable to Ω_1 as well as to Ω. The restriction to Ω_1 is no sacrifice, because Ω_1 is the significant space in the problem, but in more complicated problems it may not be clear what the significant space is. That is, the word "significant" must be defined in this context, and defined in such a way that P_0 measure induces a probability measure in any significant space. This will be done in the next section.

1. **Extension of Kolmogoroff measure.** The problem of the introduction is simply to extend P_0 measure by assigning probabilities in a natural way to function space sets (0.4). The fact that these sets are Borel sets in function space suggests the possibility of extending P_0 measure to F_1, the class of all Borel sets. This suggestion will be discussed below, in §2. If probabilities have been assigned to the sets (0.4), then if $\{s_j\}$ is any denumerable point set in the open interval I,

$$\begin{aligned} P\left\{\underset{t\in I}{\text{L.U.B.}}\ x(t) \leq k\right\} &\leq P\left\{\underset{j}{\text{L.U.B.}}\ x(s_j) \leq k\right\}, \\ P\left\{\underset{t\in I}{\text{G.L.B.}}\ x(t) > k\right\} &\geq P\left\{\underset{j}{\text{G.L.B.}}\ x(s_j) > k\right\} \end{aligned} \tag{1.1.1}$$

for all k. It is also clear that as more points s_j are inserted on the right the probabilities decrease or remain the same, and increase or remain the same respectively, and that an extremal sequence $\{s_j\}$ can be found such that these probabilities are left unchanged by the insertion of more points. It is then natural to *define* the probabilities on the left as these extreme values. These definitions were in fact proposed by Khintchine [6, p. 69] and it appears reasonable to require that any extension of P_0 measure for the purposes of probability analysis should agree with his definitions on the sets (0.4).

As noted in the introduction, in probability problems of the type discussed the investigator is commonly interested in probabilities defined in function space subsets of a given set, for example in subsets of the set of continuous functions, of the set of positive functions, of the set of finite-valued functions, and so on. It was therefore natural that the first systematic method of treating the problems described in the introduction was to define a probability measure in the sets of a suitable subspace Ω_1 of function space Ω, with Ω_1 chosen so that sets of functions, such as the set of continuous functions, which are not P_0 measurable become measurable relative to Ω. This is done as follows.[4] Define a (Kolmogoroff) measure $P_0\{E\}$ on the Borel field F_0 as described above. The Kolmogoroff outer measure of any set is defined by

$$P_0^*\{A\} = \underset{A\subset E\in F_0}{\text{G.L.B.}}\ P_0\{E\} = \underset{A\subset G}{\text{G.L.B.}}\ P_0\{G\}, \tag{1.1.2}$$

where G is a finite or denumerably infinite neighborhood sum. Let Ω_1 be any subset of Ω with $P_0^*\{\Omega_1\}=1$. Define $\bar{F}_0/\Omega_1$ as the field of all subsets E_1 of Ω_1 of the form

$$E_1 = E_0\Omega_1, \qquad E_0 \in \bar{F}_0, \tag{1.1.3}$$

and define a set function $P\{E_1/\Omega_1\}$ on $\bar{F}_0/\Omega_1$ by

$$P\{E_1/\Omega_1\} = P_0\{E_0\}. \tag{1.1.4}$$

In order to show that this definition is unique, we must show that E_0 is determined up to a set of P_0 measure 0 by (1.1.3). In fact, if $E_0\Omega_1=G_0\Omega_1$ $(E_0,\ G_0\in\bar{F}_0)$, it follows that $E_0\Omega_1=E_0G_0\Omega_1$, so that $(E_0G_0')\Omega_1=0$.[5] Then $P_0\{E_0G_0'\}=0$, since $P_0^*\{\Omega_1\}=1$, and similarly $P_0\{E_0'G_0\}=0$, as was to be shown. The set function is clearly a probability measure.

It is possible, using this relativization method, to drop the consideration of the total space Ω in favor of a subspace Ω_1, if such a transfer is convenient. The condition that Ω_1 have outer measure 1 is the condition that the subspace be large enough. This provides a definition of "significance" as desired in the introduction. All probabilities will look the same in the subspace as in the full space, in the sense that, for example, the probability number in (0.2) remains unchanged, but is reinterpreted as the probability that the stated inequalities are true for the functions in Ω_1. The following theorem indicates one advantage that may be gained by the shift from Ω to a properly chosen Ω_1.

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[4] Cf. [2]. The notation used in the present paper differs from that in [2], and is more adapted to a general discussion and to the methodological simplification described below.

[5] In the following, the complement of a set G will be denoted by G'.

THEOREM 1.2.[6] *There is a subset* Ω_1 *of* Ω, *with* $P_0^*\{\Omega_1\}=1$, *such that in terms of the relativized measure* $P\{E/\Omega_1\}$, $\text{L.U.B.}_{t\in I}x(t)$ *and* $\text{G.L.B.}_{t\in I}x(t)$ *are measurable functions for every interval* I, *and in fact there is a denumerable sequence* $\{t_j\}$ *such that for* $x(t)\in\Omega_1$ *and* I *any open interval*

$$\begin{aligned} &\underset{t\in I}{\text{L.U.B.}}\ x(t) = \underset{t_j\in I}{\text{L.U.B.}}\ x(t_j), \\ &\underset{t\in I}{\text{G.L.B.}}\ x(t) = \underset{t_j\in I}{\text{G.L.B.}}\ x(t_j). \end{aligned} \tag{1.2.1}$$

In other words, relative to Ω_1 the sets of functions in (0.4) can be defined using only denumerably many t values, and are therefore F_σ sets. The t set $\{t_j\}$ will in general depend on the basic Kolmogoroff measure, but in the most important cases any denumerable dense set will serve. Note that the adjunction of more points to the sequence $\{t_j\}$ cannot affect (1.2.1) and hence the distribution of the chance variables is in accordance with Khintchine's definition given above.

The proof of Theorem 1.2 will be sketched. If I is any open interval, there is a denumerable t set $\{t_j(I)\}$ in I such that for any sequence $\{s_j\}$ in I,

$$\begin{aligned} &\underset{j}{\text{L.U.B.}}\ x(s_j) \leqq \underset{j}{\text{L.U.B.}}\ x(t_j(I)), \\ &\underset{j}{\text{G.L.B.}}\ x(s_j) \geqq \underset{j}{\text{G.L.B.}}\ x(t_j(I)), \end{aligned} \tag{1.2.2}$$

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excluding possibly a set of functions $x(t)$ in Ω of P_0 measure 0. (The extremal sequences can be taken as those giving extreme values to the integrals over Ω of arctan $\text{L.U.B.}_j\, x(s_j)$ and arctan $\text{G.L.B.}_j\, x(s_j)$ for sequences $\{s_j\}$ in I.) Let $\{t_j\}$ be any sequence containing all the points $t_j(I)$ for all open intervals I with rational end points. Then for each t, almost all functions in Ω (P_0 measure) satisfy

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[6] [3, Theorem 2.4].

$$(1.2.3)\qquad \lim_{\epsilon\downarrow 0}\ \underset{|t_j-t|<\epsilon}{\text{G.L.B.}}\ x(t_j) \leqq x(t) \leqq \lim_{\epsilon\downarrow 0}\ \underset{|t_j-t|<\epsilon}{\text{L.U.B.}}\ x(t_j).$$

Define Ω_1 as the set of all functions $x(t)$ satisfying (1.2.3) simultaneously for all t. (The set Ω_1 may contain infinite-valued functions.) With this definition, Ω_1 satisfies (1.2.1). In order to prove that $P_0^*\{\Omega_1\}=1$ we show that if Γ is any finite or denumerably infinite sum of neighborhoods, and if $\Omega_1\subset\Gamma$, then $P_0\{\Gamma\}=1$. Let $\{\tau_j\}$ be the t values involved in the definition of Γ, and let Γ_0 be the Ω set of functions $x(t)$ satisfying (1.2.3) for $t=\tau_1, \tau_2, \cdots$. Then

$$(1.2.4)\qquad \Omega_1 \subset \Gamma_0 \subset \Gamma.$$

But since (1.2.3) is true for $t=\tau_n$, with P_0 measure 1, for each n, $P_0\{\Gamma_0\}=1$. Hence $P_0\{\Gamma\}=1$, as was to be proved.

Using this theorem, it is clear that many of the problems involving a continuum of conditions on the functions of Ω are reducible to problems involving denumerably many conditions, relativizing the measure so that $P_0\{E\}$ is replaced by $P\{E/\Omega_1\}$. For example the set C of continuous functions, which is not measurable relative to Ω, becomes measurable relative to Ω_1; in fact the equality

$$(1.2.5)\qquad \Omega_1 C = \Omega_1 \prod_{n=1}^{\infty}\sum_{\nu=1}^{\infty}\left\{\underset{|t_j-t_k|<1/\nu,\,|t_j|<n}{\text{L.U.B.}}\ \left|x(t_j)-x(t_k)\right| < 1/n\right\},$$

where the set $\{t_j\}$ is that described in Theorem 1.2, shows that the relative measure of C is the P_0 measure of the multiplier of Ω_1 on the right. It is Theorem 1.2 or its equivalent that justifies the avoidance of measurability problems (beyond P_0 measurability) by considering the functions in function space as if they were defined only on some appropriate everywhere dense denumerable set.

This method of attack by means of relative measure, although it has proved usable in practice, is rather inelegant. The following equivalent approach is more elegant. It is desired to have certain sets measurable which are not so in F_0. This can be accomplished by extending the domain of measurability rather than by relativizing the measure. Let Ω_1 be any Ω set of outer P_0 measure 1, and let F_1 be the Borel field generated by Ω_1 and the sets of F_0. If Ω_1 is a Borel Ω set, $F_0\subset F_1\subset F_2$. For example, if Ω_1 is chosen as described in Theorem 1.2, it is a Borel set, the intersection of denumerably many Borel sets of the form

$$(1.3.1)\qquad \underset{t\in I}{\text{L.U.B.}}\ x(t) = \underset{t_j\in I}{\text{L.U.B.}}\ x(t_j),\qquad \underset{t\in I}{\text{G.L.B.}}\ x(t) = \underset{t_j\in I}{\text{G.L.B.}}\ x(t_j).$$

Any set in F_1 can be written in the form

$$E_1 = E_0\Omega_1 + F_0\Omega_1', \qquad E_0, F_0 \in F_0 \tag{1.3.2}$$

and we define

$$P_1\{E_1\} = P_0\{E_0\}. \tag{1.3.3}$$

This definition of $P_1\{E_1\}$ is unique if E_0 is determined up to a set of P_0 measure 0 by (1.3.1); that is, if the equation $E_0\Omega_1 = G_0\Omega_1$, when $E_0, G_0 \in F_0$, implies that E_0 and G_0 differ by at most a set of P_0 measure 0. This has already been shown in the course of defining relativized measure. In particular, the equation $\Omega_1 = \Omega\Omega_1$ shows that $P_1\{\Omega_1\} = 1$, and if $E_0 \in F_0$ the equation $E_0 = E_0\Omega_1 + E_0\Omega_1'$ shows that $P_1\{E_0\} = P_0\{E_0\}$. Thus (1.3.3) defines an extension of $P_0\{E\}$, extending the domain of definition to F_1. The extension to $\overline{F}_1$ can be made in the usual way, or simply by allowing E_0 and F_0 be to in $\overline{F}_0$ in (1.3.2). If Ω_1 is chosen so that (1.2.1) is satisfied (for a proper sequence $\{t_j\}$), all functions in Ω_1, and hence almost all (P_1 measure) functions in function space, satisfy (1.2.1). The functions on the left in (1.2.1) are therefore P_1 measurable functions since those on the right are even P_0 measurable. In practice Ω_1 is always chosen to have this property. Applying $P_1\{E\}$ to (1.3.2) we have $P_1\{E_1\} = P_1\{E_0\}$; thus to any P_1 measurable set E corresponds a P_0 measurable set E_0 differing from it by at most a set of P_1 measure 0. This implies that if f_1 is any P_1 measurable function, there is a P_0 measurable function f_0 differing from f_1 on at most a set of P_1 measure 0.

2. **Discussion of solution.** The definition of a stochastic process (in the present case, with the parameter t running through the line or half line) can now be formulated. A stochastic process is the combination of function space Ω with a probability measure defined on the sets of $\overline{F}_1$. The set Ω_1 adjoined to the field F_0 to obtain F_1 has outer P_0 measure 1, and is chosen to satisfy whatever conditions compatible with this are dictated by the needs of the problem at hand.

This definition is somewhat unfortunate in that it makes the definition of probability depend essentially on the rather arbitrary choice of the set Ω_1. For example suppose in (0.2) that the probability is zero unless the a_j's are all negative and the b_j's all positive, that is we suppose that for each t, $x(t) = 0$ with probability 1. It is easy to prove that Ω_1 can be taken as the single function $x(t) \equiv 0$; this function, considered as a set in function space, has outer P_0 measure 1. If Ω_1 is so chosen, then in terms of the final probabilities (P_1 measure), $x(t) \equiv 0$ with probability 1, and this is the desired result, since P_0 measure has been defined as it would be for an experiment in which it is certain that the observed value at each moment of time is 0. On the other hand, it is easy to show that Ω_1' also has outer measure 1, and if this set instead of Ω_1 is adjoined to F_0 to get F_1, Ω_1 will have measure 0; the probability that $x(t)\equiv 0$ is now zero! A theory which allows this choice is hardly satisfactory, unless the choice is shown to be inevitable, when judicious rationalization can be used to make the choice even desirable.

What has been done in the past is to choose Ω_1 to consist of the most regular functions possible, as the set of continuous functions, or the set continuous except for jumps, and so on. This has usually been accomplished by choosing Ω_1 to satisfy the conditions of Theorem 1.2. Suppose for example that P_0 measure has the property that whenever $s_n\to s$, $x(s_n)\to x(s)$ for almost all functions of Ω. This is true in a wide class of cases. Then it is easy to see that *any* everywhere dense denumerable set $\{t_j\}$ will serve in Theorem 1.2. Suppose that for at least one such set $\{t_j\}$ it is possible to prove that the P_0 measurable Ω set

$$(2.1.1)\qquad \prod_{n=1}^{\infty}\sum_{\nu=1}^{\infty}\left\{\underset{|t_j-t_k|<1/\nu,\,|t_j|<n}{\text{L.U.B.}}\ |x(t_j)-x(t_k)|<1/n\right\}$$

has P_0 measure 1, that is, that almost all functions $x(t)$ in function space coincide on $\{t_j\}$ with everywhere continuous functions. Then if Ω_1 is chosen as in Theorem 1.2, and if this sequence $\{t_j\}$ is used, the bounds of each $x(t)$ in Ω in any open t interval become the bounds considering only the points of $\{t_j\}$ in the interval, so that $x(t)$ must be everywhere continuous. That is, in terms of the extended measure, P_1 measure, almost all functions $x(t)$ are everywhere continuous. Moreover let $\{t_j'\}$ be *any* everywhere dense denumerable t set. Then, in terms of the P_1 measure already introduced, almost all functions $x(t)$ coincide on the set $\{t_j'\}$ with everywhere continuous functions. Hence in terms of P_0 measure, almost all functions coincide on the set $\{t_j'\}$ with everywhere continuous functions. In fact, as is seen in (2.1.1), this coincidence is P_0 measurable. Hence no matter what sequence $\{t_j\}$ is used in Theorem 1.2 to define Ω_1, the end result will be, in terms of relativized measure or P_1 measure, that almost all functions $x(t)$ will be everywhere continuous. In probability language this is stated in the form "$x(t)$ is everywhere continuous, with probability 1." Thus, at least if Ω_1 is chosen to satisfy Theorem 1.2, P_1 measure has important properties independent of the set Ω_1 adjoined to F_0 to get F_1. It is important to note, however, that the set Λ of nonmeasurable functions in function space always has outer P_0 measure 1. Hence even in this example, if Λ is adjoined to F_0 to obtain F_1, in terms of this P_1 measure almost *no* function $x(t)$ is measurable, much less continuous.

The choice of Ω_1 to satisfy conditions dictated by common sense and the desire for simplicity has been satisfactory in practice, but this is not mathematically satisfactory, since mathematics and common sense are not in the same universe of discourse. Moreover more complicated examples than those given might make the common sense decision somewhat difficult.

Two ways out of the difficulty are conceivable. One possibility would be to accept the fact that any one of the possible P_1 measures corresponding to a given P_0 measure is as justified as any other, from a purely mathematical point of view, and that some new principle must be introduced if a choice is to be made. For example, from a physical point of view it may be considered desirable to accept only the P_1 measures obtained using an adjoined set Ω_1 which satisfies Theorem 1.2. The mathematician might invoke the principle of least action to choose Ω_1 so that Theorem 1.2 is satisfied and so that (t,ω) measurability, as discussed in §3, is obtained, if possible.

A second and more elegant way out would be to define a measure on a field of Ω sets large enough to fulfill the needs of all problems. For example it would be satisfactory if a single Ω_1, that is, a single field F_1, would do for all problems, and it might appear that a choice of Ω_1 as in Theorem 1.2 might well suffice. This is not true; no single denumerable set $\{t_j\}$ is large enough, although any everywhere dense set suffices for most problems. Kakutani, in conversations with the speaker before the war, proposed extending P_0 measure from F_0 to F_1, the field of Borel sets of function space. When T, the range of t, is a finite or denumerable aggregate, the Borel Ω sets have been seen above to be a sufficiently large class. The extension to F_1 can be done as follows. Let G be any open Ω set. Kakutani then defines $P_1^*\{G\}$ by

$$P_1^*\{G\} = \underset{E\subset G}{\text{L.U.B.}}\, P_0\{E\}, \qquad E\in F_0. \tag{2.2.1}$$

That is, $P_1^*\{G\}$ is the inner P_0 measure of G. If A is any Ω set, $P_1^*\{A\}$ is defined by

$$P_1^*\{A\} = \underset{G\supset A}{\text{G.L.B.}}\, P_1^*\{G\} \tag{2.2.2}$$

where G is open. It can be proved that $P_1^*\{A\}$ has the usual properties of an outer measure. The (Carathéodory) measurable sets include all Borel sets and the measure, P_1 measure, is an extension of P_0 measure. It agrees with Khintchine's definition on sets of the type (0.4).

It would then appear that Kakutani's extension of Kolmogoroff measure has all the desired properties, but this is not quite true, or at least it has not been proved. To see what is lacking, the measurability of a process will be discussed.

3. **Measurable processes.** Let $f(t, \omega)$ be a function of the two variables t, ω, where t varies on a line or half line, and ω varies in an abstract space. Then for each value of ω, $f(t, \omega)$ is a function of t. Let Ω_1 be the aggregate of functions of t obtained in this way. If an ω measure is defined, it induces a measure in function space, or more precisely, in subsets of Ω_1. Assume that ω measure is a probability measure, and that $f(t, \omega)$ is measurable in ω for each fixed value of t. Then if $(t_1, \cdots, t_n)$ is any finite t set, the ω measure of the ω set

$$(3.1.1) \qquad \{a_j < f(t_j, \omega) < b_j, j = 1, \cdots, n\}$$

can be identified with the probability, or measure, of the set of functions in function space defined by the inequalities (0.3). This approach was used to give rigorous meaning to measure in function space (or at least to measure in some of the subsets of a subset Ω_1 of function space) long before the direct attacks of the preceding section were developed. For purposes of the ergodic theorem, harmonic analysis, and similar work, where some sort of regularity in t was required, it was supposed, as a minimum condition, that f was measurable in the two variables t, ω. That is, a measure in (t, ω) space was defined in the usual multiplicative way as the direct product of Lebesgue t measure and the given ω measure, and f was supposed (t, ω) measurable. Note that the measure of a neighborhood in function space can be identified with the ω measure of the set (3.1.1) so that this approach in terms of a function of t and an auxiliary probability parameter ω actually determines a measure in function space ω. More precisely, P_0 probability measure is determined in the Ω sets of the field F_0 (Kolmogoroff's theorem). In terms of this P_0 measure it is easily shown that Ω_1 has outer measure 1 [4, Theorem 1], so that a relativized measure can be defined on the sets of $\overline{F}_0/\Omega_1$ as explained in §1. On the other hand, the given ω measure determines a measure on Ω_1 sets by the map $\omega \rightarrow x(t) = f(t, \omega)$. By definition, these measures are identical on neighborhoods (relative to Ω_1) and therefore the relativized Ω_1 measure and mapped Ω measures are identical on the sets of F_0/Ω_1. It is frequently very convenient to discuss a stochastic process using an explicit formula $x(t) = f(t, \omega)$ of this type for the sample functions. This has been frequently done in the past with f the sum of a series of functions of t with random coefficients. But it must be remembered that corresponding to the fact that a given P_0 measure in function space, the basic probability measure, may lead to many quite different extensions, so there may be many functions $f(t, \omega)$ of two variables with quite different measure properties even though there are identical probability evaluations for inequalities of the type (3.1.1). In fact it will be seen below that each P_1 measure leads to a function $f(t, \omega)$. Hence, corresponding to the different extensions of P_0 measure in an example already discussed, two functions $f_1(t, \omega)$, $f_2(t, \omega)$ can be exhibited, with $f_1(t, \omega) \equiv 0$ in t, ω, and $f_2(t, \omega) \not\equiv 0$ in t for any ω, which agree as far as the probabilities of inequalities of the type (3.1.1) are concerned. Using f_1 one would say that $x(t) \equiv 0$ with probability 1; using f_2 one would say that $x(t) \equiv 0$ with probability 0; yet the basic probabilities are identical. One must watch out for such things in using these two variable representations of stochastic processes. The representations may contain implications not inherent in the basic probabilities.

Conversely any stochastic process can be put in the form of a two variable representation. In fact, denote the variable function $x(t)$ in Ω by ω, and consider the product space of pairs (t, ω). Define the numerically valued function $x(t, \omega)$ as the function of (t, ω) which takes on the value $x_0(t_0)$ when $t = t_0$ and $\omega = x_0(s)$. Note that for each t, $x(t, \omega)$ is a measurable function of ω (P_0, P_1, or P_2 measurable, depending on which field of measurable sets is chosen); in fact $x(t_0, \omega)$ has the distribution function $P\{x(t_0) < k\}$. The function $x(t, \omega)$ thus has the role of the function $f(t, \omega)$ above. Note that given a P_1 measure, the function $x(t, \omega)$ will have measure properties which may differ markedly from those due to another P_1 measure extension of the same P_0 measure. The consequences of this have already been commented on in the preceding paragraph.

The proper definition of measurability of a process is now clear; a process is measurable if $x(t, \omega)$ is a measurable function of (t, ω), where (t, ω) measure is defined as the direct product of Lebesgue measure in t and the given ω measure. If a process is measurable, Fubini's theorem can be applied, to find that for almost all ω (that is, with probability 1) $x(t, \omega)$ is measurable in t; in other words almost all functions in function space are measurable. The functions $x(t)$, that is $x(t, \omega)$, can then be integrated (in t) if the function values are not too large. For example, if[7] $E\{|x(t)|\}$ is a bounded function of t, that is if the integral in ω of $|x(t, \omega)|$ is a bounded function of t, the integral $\int_a^b x(t)dt = \int_a^b x(t, \omega)dt$ exists in the usual Lebesgue sense for almost all functions ω in function space, and is itself a function of ω, P_0, P_1, or P_2 measurable as the case may be, according to the Ω measure used. The indirect definitions of this integral commonly found in the literature, usually as some sort of averaged limit of Riemann sums, beg the question of integrability of the individual functions and complicate the mathematics.

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[7] The expectation of a chance variable y will be denoted by $E\{y\}$.

Unfortunately three definitions of measurability of a stochastic process have been proposed in the preceding paragraph, according as the field of measurable Ω sets is taken as $\bar{F}_0$, $\bar{F}_1$, or $\bar{F}_2$. The first possibility is easily eliminated. In fact, no process is ever measurable if $\bar{F}_0$ is used, for it was just noted that almost all functions $x(t)$ are Lebesgue measurable if a process is measurable, whereas it is easily shown [2, p. 113] that the Ω set of Lebesgue nonmeasurable functions has outer P_0 measure 1. This is one more indication of the insufficiency of the field $\bar{F}_0$. Note that measurability using $\bar{F}_1$ depends on the adjoined set Ω_1 in the sense that a process may be measurable or not depending on the choice of Ω_1. For example if Ω_1 is the class of non-Lebesgue measurable functions, no process can be measurable using Ω_1, according to the argument given above for $\bar{F}_0$. We shall accept the definition that a process is measurable if $x(t, \omega)$ is (t, ω) measurable if $\bar{F}_1$ is used, for some choice of Ω_1.[8]

It was shown in the beginning of this section how a function $f(t, \omega)$ of t and an auxiliary probability parameter ω can be used to determine a stochastic process. For a full parallelism it must be shown that whenever $f(t, \omega)$ is (t, ω) measurable, there is a corresponding measurable process, that is, the following theorem must be proved.

THEOREM 3.1 [4, Theorem 6]. *Let $f(t, \omega)$ be a function of (t, ω), where t varies on a line and ω varies in any abstract space in which a probability measure is defined, $m_\omega\{\ \}$. Suppose that $f(t, \omega)$ is measurable in ω for each fixed t and also measurable in (t, ω). Define a stochastic process P_0 measure, a probability measure in function space, by setting*

$$P_0\{a_j < x(t_j) < b_j,\ j = 1, \cdots, n\} = m_\omega\{a_j < f(t_j, \omega) < b_j,\ j = 1, \cdots, n\}. \tag{3.1.2}$$

Then this P_0 measure can be extended to a P_1 measure in terms of which the stochastic process is measurable.

Note 1. This theorem is not true if the measurability of a stochastic process is defined in terms of P_0 measure in function space; an extension of P_0 measure is definitely needed, even in so regular an example as when $f(t, \omega) \equiv 0$, so that $x(t) = 0$ for each t, with probability 1. On the other hand it is not known as yet (see the discussion below) whether the extension to P_2 measure would always be sufficient.

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[8] This is equivalent to the definition in terms of relativized measure given in [2]. Necessary and sufficient conditions on the basic P_0 measure that there be a P_1 extension determining a measurable process have been given in [1] and [2].

Note 2. This theorem shows that no new types of measurable process would arise if extensions of P_1 measure were used in the definition of measurability of a stochastic process. That is to say, if P_0 measure is regular enough to make $x(t, \omega)(t, \omega)$ measurable using any Ω measure whatever which is an extension of the given P_0 measure, then $x(t, \omega)$ is P_1 measurable also, using a properly chosen extended field $\bar{F}_1$. In fact if $x(t, \omega)$ is (t, ω) measurable using as Ω measure some extension of the given P_0 measure, then, using $x(t, \omega)$ as the $f(t, \omega)$ of the theorem, it follows that there is a P_1 measure making $x(t, \omega)(t, \omega)$ measurable.

The proof of this theorem, which will be omitted, consists in finding a set Ω_1 to adjoin to F_0, such that, in terms of P_1 measure, $x(t, \omega)$ is (t, ω) measurable. This set Ω_1 can be chosen to satisfy the conditions of Theorem 1.2, so that measurability in (t, ω) and these desirable conditions are always mutually compatible.

Unfortunately it has never been demonstrated that the Ω_1 described in the preceding paragraph can be chosen to be a Borel Ω set. Thus the proof does not show that if the condition of the theorem is satisfied, $x(t, \omega)$ will be (t, ω) measurable when Ω measure is P_2 measure. In order that Kakutani's approach to these stochastic processes, using P_2 measure, be fully satisfactory, it is necessary that whenever $x(t, \omega)$ is (t, ω) measurable using P_1 measure, it should also be (t, ω) measurable using P_2 measure. It would be very desirable to have either a proof or a counterexample. At present, therefore, Kakutani's approach must be considered incomplete, although very elegant, and quite adequate in most problems. In [5] Kakutani exhibited a class of processes in which $\bar{F}_0 = \bar{F}_2$.

4. **Alternative approach.** The whole difficulty of defining a suitable probability measure in function space has been caused by the insistence on assigning measure (probability) to a neighborhood of functions, as in (0.2), first and going on from there. This automatically limits one to P_0 measure and its extensions. An approach that would avoid all this would be to suppose right from the start that there is a probability measure defined on some field F of Ω sets, and go on from there. The first and most obvious difficulty in this is that one must avoid being too general if theorems are to have any significance beyond their statement and proof. This means that F must be fairly strictly defined. No specific proposals in this direction have been made, unless Kakutani's proposal is considered in this light. In the second place P_0 measure arises in a natural way because the many stochastic processes that are known and interesting are, at present,

all defined in terms of what happens at a finite number of t values. For example the processes with independent increments are simply described by the statement that if $t_1 < \cdots < t_n$, $x(t_2) - x(t_1), \cdots, x(t_n) - x(t_{n-1})$ are mutually independent. Moreover besides this simple description, it is possible to write down explicitly the most general distribution of the chance variables $x(t_1), \cdots, x(t_n)$ for any such process, and in this way describe the measure of the neighborhoods. Thus in this and other examples, up to the present at least, the approach through neighborhood measures to P_0 measure and its extensions has been the natural one. It is conceivable, however, that new Ω measures will be developed starting from characteristics of the curves as a whole, rather than their characteristics at finite sets of values of the argument.

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UNIVERSITY OF ILLINOIS

THE NOVEMBER MEETING IN PRINCETON

The four hundred nineteenth meeting of the American Mathematical Society was held at Princeton University on Saturday, November 2, in connection with the year-long celebration of Princeton's Bicentennial. The attendance was approximately two hundred fifty, including the following two hundred nineteen members of the Society:

Milton Abramowitz, C. R. Adams, C. B. Allendoerfer, R. L. Anderson, T. W. Anderson, R. F. Arens, Emil Artin, R. P. Bailey, Joshua Barlaz, A. F. Bartholomay, P. T. Bateman, G. E. Bates, E. G. Begle, Richard Bellman, Stefan Bergman, Lipman Bers, J. H. Bigelow, Archie Blake, R. P. Boas, G. L. Bolton, T. A. Botts, A. T. Brauer, Paul Brock, A. B. Brown, R. C. Buck, G. E. Bullock, R. S. Burington, Hobart Bushey, J. H. Bushey, S. S. Cairns, W. B. Campbell, F. L. Celauro, Claude Chevalley, K. L. Chung, Alonzo Church, Randolph Church, Edmund Churchill, A. H. Clifford, R. F. Clippinger, I. S. Cohen, Richard Courant, V. F. Cowling, D. R. Crosby, H. B. Curry, J. H. Curtiss, J. F. Daly, M. D. Darkow, D. R. Davis, John DeCicco, L. S. Dederick, D. B. DeLury, C. H. Denbow, C. H. Dowker, Y. N. Dowker, T. L. Downs, Arnold Dresden, Nelson Dunford, W. D. Duthie, Jacques Dutka, Churchill Eisenhart, L. P. Eisenhart, Benjamin Epstein, Paul Erdös, Ky Fan, Herbert Federer, J. M. Feld, William Feller, N. J. Fine, W. B. Fite, Edward Fleisher, M. M. Flood, R. H. Fox, Hilda Geiringer, Abe Gelbart, J. H. Giese, B. P. Gill, H. E. Goheen, V. D. Gokhale, H. H. Goldstine, A. W. Goodman, W. H. Gottschalk, E. J. Gumbel, Theodore Hailperin, N. A. Hall, R. W. Hamming, O. G. Harrold, K. E. Hazard, L. A. Henkin, Einar Hille, A. P. Hillman, Banesh Hoffman, T. R. Hollcroft, E. M. Hull, L. C. Hutchinson, J. A. Jenkins, Bjarni Jónsson, Aida Kalish, Irving Kaplansky, Edward Kasner, M. E. Kellar, D. E. Kibbey, J. R. Kline, E. R. Kolchin, M. Z. Krzywoblocki, Jack Laderman, J. W. Lasley, Solomon Lefschetz, Joseph Lehner, Marguerite Lehr, R. A. Leibler, Howard Levi, Norman Levinson, Charles Loewner, E. R. Lorch, A. N. Lowan, N. H. McCoy, A. W. McMillan, Brockway McMillan, L. A. MacColl, Saunders MacLane, M. H. Martin, W. T. Martin, Imanuel Marx, A. E. Meder, A. N. Milgram, Don Mittleman, Deane Montgomery, Vladimir Morkovin, Richard Morris, Marston Morse, C. A. Nelson, David Nelson, E. D. Nering, O. E. Neugebauer, A. V. Newton, P. B. Norman, Nilan Norris, Lawrence Norwood, C. O. Oakley, E. R. Ott, J. C. Oxtoby, J. S. Oxtoby, L. G. Peck, A. M. Peiser, W. H. Pell, Anna Pell-Wheeler, A. J. Penico, B. J. Pettis, C. R. Phelps, R. S. Phillips, Everett Pitcher, F. V. Pohle, Harry Polachek, Walter Prenowitz, Hans Rademacher, L. L. Rauch, G. E. Raynor, Mina Rees, F. D. Rigby, John Riordan, E. K. Ritter, H. P. Robertson, M. S. Robertson, J. B. Robinson, R. M. Robinson, S. L. Robinson, P. C. Rosenbloom, Raphael Salem, Pierre Samuel, Arthur Sard, S. A. Schaaf, A. T. Schafer, R. D. Schafer, I. J. Schoenberg, Pincus Schub, Abraham Schwartz, I. E. Segal, C. E. Shannon, H. N. Shapiro, Max Shiffman, James Singer, L. L. Smail, V. E. Spencer, N. E. Steenrod, R. R. Stoll, E. G. Straus, L. M. Straus, M. M. Sullivan, C. C. Torrance, M. M. Torrey, H. I. Treiber, C. A. Truesdell, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, H. S. Vandiver, H. E. Vansant, Oswald Veblen, D. F. Votaw, G. L. Walker, A. D. Wallace, R. M. Walter, J. B. Walton, W. R. Wasow, G. C. Webber, F. J. Weyl, Hermann Weyl, A. L. Whiteman, P. M. Whitman, D. V. Widder, A. S. Wightman, S. S. Wilks, C. S. Williams, Clement Winston, Y. C. Wong, Bertram Yood, J. W. Young, Leo Zippin, Antoni Zygmund.

On Saturday morning there were two sections, one for papers in Algebra and Topology in which Professor N. E. Steenrod presided, and one for papers in Analysis, Geometry and Applied Mathematics, in which Professors Einar Hille and D. V. Widder presided.

The general session on Saturday afternoon, Professor W. B. Fite presiding, opened with a welcome by Dr. H. S. Taylor, Dean of the Graduate School, Princeton University. On behalf of the Society, Professor N. H. McCoy presented an expression of appreciation to Princeton University and the Committee on Arrangements and congratulations to Princeton University on its Bicentennial. Professor J. H. C. Whitehead of the University of Oxford gave an address on *Combinatorial homotopy theory.*

Members of the Society were guests of the Institute for Advanced Study at tea from 4–6 P.M.

Under the auspices of Princeton University, an informal discussion of the current state of mathematics throughout the world was held at 7:45 P.M., Professor Einar Hille presiding. The speakers and their topics were as follows: Professor Harald Cramér, on mathematics in the Scandinavian countries and the recent Scandinavian Mathematical Congress; Professor Claude Chevalley on mathematics in France during and since the war; Professor Solomon Lefschetz on mathematics in the U.S.S.R. and, more briefly, in Italy and China; Professor H. P. Robertson on mathematics and science in Europe, especially in Germany during and since the war; Professor Marston Morse on a Mathematical Congress, open to mathematicians of all nations, probably to be held in the U.S.A. in a few years. A discussion period followed during which Professor Arnold Dresden described a plan sponsored by the American Mathematical Society to assist in building up the mathematical libraries of Europe and Asia.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1–9 were presented in the section for Algebra and Topology, papers 10–18 in the section for Analysis, Geometry and Applied Mathematics, papers 19–49 were presented by title. Paper 15 was read by Professor Kasner, and paper 18 by Mr. Harry Polachek. Mr. Roberts was introduced by Professor C. C. Lin.

1. Nilan Norris: *An extension of an equality among averages.* (Abstract 52-11-396.)

2. E. R. Kolchin: *Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations.* (Abstract 52-11-349.

3. P. M. Whitman: *Finite groups with a cyclic group as lattice-homomorph.* (Abstract 52-11-355.)

4. A. T. Brauer: *Limits for the characteristic roots of a matrix.* II. (Abstract 52-11-347.)

5. R. M. Robinson: *Unsymmetrical approximation of irrational numbers.* (Abstract 52-11-351.)

6. Everett Pitcher: *Critical point inequalities for a nondegenerate functional.* (Abstract 52-11-405.)

7. R. H. Fox: *On a problem of S. Ulam concerning Cartesian products.* (Abstract 52-11-401.)

8. Ky Fan: *On partially ordered additive groups of continuous functions.* (Abstract 52-11-400.)

9. C. H. Dowker: *An extension of Alexandroff's mapping theorem.* (Abstract 52-11-398.)

10. R. C. Buck: *On some properties of arithmetic density.* Preliminary report. (Abstract 52-11-361.)

11. V. F. Cowling: *On the form of certain entire functions.* (Abstract 52-11-363.)

12. M. Z. Krzywoblocki: *A local maximum property of the fourth coefficient of schlicht functions.* (Abstract 52-11-369.)

13. P. C. Rosenbloom: *Entire associative functions.* (Abstract 52-11-373.)

14. Walter Prenowitz: *Characterization of the lattice of convex sets of a descriptive geometry.* (Abstract 52-11-391.)

15. Edward Kasner and John DeCicco: *Geometry of harmonic transformations.* (Abstract 52-9-322.)

16. S. A. Schaaf: *A cylinder cooling problem.* (Abstract 52-11-385.)

17. H. E. Goheen: *A bound for the error in computing the Bessel functions of the first kind by recurrence.* (Abstract 52-11-380.)

18. Harry Polachek and R. J. Seeger: *On the existence of solutions for three-shock Prandtl-Meyer configurations in the case of weak reflected shocks.* (Abstract 52-11-381.)

19. Warren Ambrose: *Direct sum theorem for Haar measures.* (Abstract 52-11-356-*t*.)

20. R. F. Arens: *Location of spectra in Banach *-algebras.* (Abstract 52-11-357-*t*.)

21. Lipman Bers: *A property of bounded analytic functions.* (Abstract 52-11-358-*t*.)

22. Lipman Bers: *On rings of analytic functions.* (Abstract 52-11-359-*t*.)

23. H. W. Brinkmann: *On the prime divisors of a polynomial with*

integral coefficients. (Abstract 52-11-348-*t.*)

24. J. P. Brown: *Cyclic characteristic curves of a system of differential equations.* (Abstract 52-9-286-*t.*)

25. R. C. Buck: *Interpolation series.* (Abstract 52-11-360-*t.*)

26. Herman Chernoff: *A note on the inversion of power series.* (Abstract 52-11-378-*t.*)

27. John DeCicco: *Constrained motion upon a surface under a generalized field of force.* (Abstract 52-11-388-*t.*)

28. C. L. Dolph: *Nonlinear integral equations of the Hammerstein type.* (Abstract 52-11-364-*t.*)

29. S. T. Hu: *On extension of homotopy.* (Abstract 52-11-402-*t.*)

30. R. C. James: *Inner products in normed linear spaces.* (Abstract 52-11-368-*t.*)

31. E. R. Kolchin: *Extensions of differential fields.* III. (Abstract 52-11-350-*t.*)

32. G. W. Mackey: *On the domains of closed linear transformations in Hilbert space.* Preliminary report. (Abstract 52-11-370-*t.*)

33. Deane Montgomery: *A theorem on locally euclidean groups.* (Abstract 52-11-403-*t.*)

34. A. P. Morse: *Perfect blankets.* (Abstract 52-11-371-*t.*)

35. S. B. Myers: *Banach spaces of continuous functions.* (Abstract 52-11-404-*t.*)

36. Everett Pitcher: *Exact homomorphism sequences for a nest of three spaces.* (Abstract 52-11-406-*t.*)

37. E. L. Post: *Recursive unsolvability of a problem of Thue.* (Abstract 52-11-394-*t.*)

38. R. C. Roberts: *On the lift of a triangular wing at supersonic speeds.* (Abstract 52-11-382-*t.*)

39. H. E. Salzer: *An alternative definition of reciprocal differences.* (Abstract 52-11-383-*t.*)

40. H. E. Salzer: *Tables for facilitating the use of Chebyshev's quadrature formula.* (Abstract 52-11-384-*t.*)

41. Alice T. Schafer: *The neighborhood of an undulation point on a space curve.* (Abstract 52-11-392-*t.*)

42. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* V. (Abstract 52-11-352-*t.*)

43. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* VI. (Abstract 52-11-353-*t.*)

44. I. E. Segal: *The group algebra of a locally compact group.* (Abstract 52-11-375-*t.*)

45. Fred Supnick: *Cooperative phenomena.* I. *Structure of the linear Ising model.* (Abstract 52-11-386-*t.*)

46. Fred Supnick: *On the problem of Tait.* Preliminary report. (Abstract 52-11-407-*t*.)

47. J. M. Thomas: *Eliminants.* (Abstract 52-9-279-*t*.)

48. J. H. M. Wedderburn: *Note on Goldbach's theorem.* (Abstract 52-9-280-*t*.)

49. J. E. Wilkins: *The converse of a theorem of Tchaplygin on differential inequalities.* (Abstract 52-11-376-*t*.)

T. R. Hollcroft,
Associate Secretary

BOOK REVIEW

A history of the conic sections and quadric surfaces. By J. L. Coolidge. Oxford University Press, 1946. 11+214 pp. $6.00.

Despite the wealth of the literature extant dealing with the conic sections, and the large amount of easily accessible material concerning the quadric surfaces, the extensive synopsis presented in this study by a distinguished geometer must be regarded as an altogether worthwhile endeavor. The connected account which the author gives of these loci, from the discovery of the conics by Menaechmus (pupil of Plato and Eudoxus) to the differential properties of quadric surfaces, makes this a book which both amateur and professional mathematicians will consult with pleasure and profit. The copious quotations from early writers which the book contains are interlarded with sprightly observations. Though the animated style employed is perhaps unusual in a serious historical study, it undoubtedly stimulates reader interest.

The first two chapters are devoted to the work of the Greeks. In the twelve hundred years from the time of Pappus to the beginning of the sixteenth century, there was apparently a complete loss of interest in the subject. The third chapter takes up the history with the re-awakening of interest evidenced by the work of Werner, Desargues, Pascal, and so on, and carries the account down to the great projective school, which is the subject of Chapter IV. Chapters V, VI, and VII deal with miscellaneous metrical theorems, systems of conics, and conics in space, respectively. They are followed by a chapter of six pages, concerning mechanical construction of conics, which completes the treatment of the conic sections.

The remaining fifty-five pages of the book are devoted to the quadric surfaces. The work is divided into four chapters, dealing with the synthetic, algebraic, higher algebraic, and differential methods.

The book is concluded with an excellent list of papers and books quoted in the text and an adequate subject index.

LEONARD M. BLUMENTHAL

NOTES

Foundations of algebraic geometry, by André Weil, volume 29 of the American Mathematical Society Colloquium Publications, has just appeared. Orders may be sent to the Society at 531 West 116th Street, New York 27, N. Y., U.S.A. The price is $5.50 with a 25% discount to members.

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have refereed papers: R. P. Agnew, A. A. Albert, Warren Ambrose, R.F. Arens, Valentine Bargmann, P. T. Bateman, E. F. Beckenbach, E. T. Bell, P. O. Bell, Garrett Birkhoff, G. A. Bliss, R. P. Boas, Salomon Bochner, L. M. Blumenthal, Richard Brauer, H. W. Brinkmann, A. B. Brown, O. E. Brown, R. C. Buck, Herbert Busemann, Alonzo Church, A. H. Clifford, I. S. Cohen, H. S. M. Coxeter, M. M. Day, J. L. Doob, Arnold Dresden, Nelson Dunford, Samuel Eilenberg, Paul Erdös, William Feller, A. D. Fialkow, R. H. Fox, R. E. Gilman, H. H. Goldstine, J. W. Green, Lois W. Griffiths, D. W. Hall, P. R. Halmos, O. G. Harrold, R. N. Haskell, G. A. Hedlund, M. H. Heins, E. D. Hellinger, T. H. Hildebrandt, Einar Hille, H. K. Hughes, Nathan Jacobson, R. L. Jeffery, Mark Kac, J. L. Kelley, B. O. Koopman, O. E. Lancaster, D. H. Lehmer, Joseph Lehner, Norman Levinson, A. N. Lowan, G. W. Mackey, Szolem Mandelbrojt, Morris Marden, Walther Mayer, W. E. Milne, Oystein Ore, Gordon Pall, B. J. Pettis, Harry Pollard, George Pólya, E. L. Post, Tibor Rado, P. V. Reichelderfer, Eric Reissner, C. E. Rickart, J. F. Ritt, M. S. Robertson, R. M. Robinson, Hans Samelson, A. C. Schaeffer, Henry Scheffé, O. F. G. Schilling, I. J. Schoenberg, I. E. Segal, C. L. Siegel, L. L. Silverman, P. A. Smith, I. S. Sokolnikoff, Alfred Tarski, W. C. Taylor, R. M. Thrall, H. S. Wall, A. D. Wallace, J. L. Walsh, W. J. Webber, Hermann Weyl, Hassler Whitney, D. V. Widder, J. E. Wilkins, Oscar Zariski, Antoni Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers accepted for publication in volumes 59 and 60: C. R. Adams, Warren Ambrose, Reinhold Baer, E. F. Beckenbach, P. O. Bell, R. P. Boas, H. F. Bohnenblust, T. A. Botts, R. H. Bruck, Herbert Busemann, S. S. Chern, W. Z. Chien, John DeCicco, Samuel Eilenberg, Harley Flanders, M. H. Heins, M. R. Hestenes, Einar Hille, Bjarni Jónsson, Irving Kaplansky, Harry Levy, R. M. Martin, W. T. Martin, C. W. Mendel, Karl Menger, A. P. Morse, Harry Pollard, Tibor Rado, J. F. Randolph, C. N.

Reynolds, C. E. Rickart, A. C. Schaeffer, Gabor Szegö, A. E. Taylor, Alexander Weinstein, G. S. Young, J. W. T. Youngs.

A notice has been received indicating that a new journal to be called Simon Stevin is to be published under joint editorship of Dutch and Belgian mathematicians. It will replace Christiaan Huygens, Mathematica B, and Wis- en natuurkundig Tijdschrift.

Brown University has established a Graduate Division of Applied Mathematics in order to bring together students in engineering, physics and mathematics and to carry forward research projects common to those fields.

Dr. J. G. Charney has been appointed National Research Fellow.

Professor W. H. Roever of Washington University has retired with the title emeritus.

Professor R. E. Root of the United States Naval Academy has retired with the title emeritus.

Professor Oswald Veblen of the Institute for Advanced Study has been elected to membership in the Royal Danish Academy of Sciences and the Polish Academy of Sciences.

Professor Hermann Weyl of the Institute for Advanced Study has been elected a corresponding member of the Zürcher Naturforschende Gesellschaft and a foreign member of the Royal Swedish Academy of Sciences.

Professor Harald Cramér, Director of the Institute of Mathematical Statistics at the University of Stockholm, has been visiting professor at Princeton University during the fall semester of the 1946–1947 academic year.

Dr. E. T. Davies of King's College, London, has been appointed to the chair of mathematics at University College, Southampton.

Professor Marcel Riesz of Lunds Universitets Matematiska Institution will be visiting lecturer at the University of Chicago for the winter and spring quarters.

Dr. O. H. Schmidt has been appointed librarian of the Danish Technical Library, Copenhagen, Denmark.

Messrs. J. S. Macpherson, W. A. Donaldson and D. D. McKinnon have been appointed assistants in mathematics at the University of Glasgow.

Associate Professor F. T. Adler of the University of Colorado has been appointed to an assistant professorship in physics at the University of Wisconsin.

Professor L. V. Ahlfors of the University of Zurich has been appointed to an associate professorship at Harvard University.

Professor A. A. Albert of the University of Chicago will be a visiting professor at the University of Brazil in 1947.

Professor W. E. Anderson of Miami University has retired.

Assistant Professor H. C. Ayres of the United States Naval Academy has been promoted to an associate professorship.

Dr. R. P. Bailey, formerly of Lafayette College, has been appointed to an assistant professorship at the United States Naval Academy.

Assistant Professor N. H. Ball of the United States Naval Academy has been promoted to an associate professorship.

Assistant Professor R. H. Bardell of the University of Wisconsin in Milwaukee has been promoted to an associate professorship.

Mr. S. U. Benscoter of the Bureau of Aeronautics, Navy Department, has accepted an appointment as structural engineer with the National Advisory Commission for Aeronautics, Langley Field, Virginia.

Dr. Stefan Bergman has been appointed a lecturer at Harvard University.

Professor Emeritus Felix Bernstein of New York University has been appointed lecturer at Triple-Cities College, Syracuse University, Endicott, New York.

Assistant Professor M. A. Biot of Columbia University has been appointed to a professorship at Brown University.

Associate Professor A. W. Boldyreff of the University of Arizona has been appointed to an associate professorship at Wittenberg College, Springfield, Ohio.

Dr. Leonard Bristow of the University of Illinois has been appointed to an assistant professorship at the University of Wyoming.

Dr. L. F. Brown has accepted a position as consulting sound engineer with Columbia Pictures, Incorporated, Hollywood, California.

Assistant Professor L. Virginia Carleton of Wesleyan College, Macon, Georgia, has been appointed to an assistant professorship at Centenary College, Shreveport, Louisiana.

Dr. L. H. Chambers, formerly of Marshall College, has been appointed to an associate professorship at the United States Naval Academy.

Associate Professor Harold Chatland of Montana State University has been appointed to an associate professorship at Wright Field Graduate Center, Ohio State University, Dayton, Ohio.

Mr. J. O. Chellevold has been appointed to an assistant professorship at Lehigh University.

Dr. W. Z. Chien of California Institute of Technology has been appointed to a professorship in mechanical engineering at National Tsing Hua University, Peiping, China.

Dr. J. B. Coleman has been appointed to an associate professorship at the University of Georgia.

Mr. G. S. Cook has been appointed to an assistant professorship at Colorado School of Mines.

Assistant Professor A. E. Currier of the United States Naval Academy has been promoted to an associate professorship.

Dr. G. B. Dantzig has been appointed principal statistician, United States Army Air Forces.

Assistant Professor H. R. C. Dieckmann of Occidental College has been appointed to an assistant professorship at San Jose State College.

Professor J. E. Dotterer of Indiana Central College has been appointed to a professorship at Manchester College, North Manchester, Indiana.

Dr. J. N. Eastham of Nazareth College has been appointed to an assistant professorship at the Cooper Union School of Engineering.

Associate Professor Churchill Eisenhart of the University of Wisconsin is on leave and has been appointed principal mathematician in charge of statistical work of the National Bureau of Standards.

Dr. Paul Erdös of the Institute for Advanced Study has been appointed to a research professorship at Syracuse University.

Dr. A. D. Fialkow has been appointed an adjunct professor at Brooklyn Polytechnic Institute.

Mr. Walter Fleming has been appointed lecturer at the University of Manitoba.

Mr. W. W. Gandy has been appointed to an associate professorship at Northwestern State College, Natchitoches, Louisiana.

Miss Mary A. Goins has been appointed to an assistant professorship at Marshall College.

Dr. C. C. Grove has been appointed a special lecturer at the University of Pennsylvania.

Mr. N. D. Griffin of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Mr. F. C. Hall has been appointed to an assistant professorship at Manhattan College.

Assistant Professor J. R. Hammond of the United States Naval Academy has been promoted to an associate professorship.

Associate Professor E. E. Haskins of Northeastern University has been appointed to an assistant professorship at the Army Air Forces Institute of Technology, Wright Field, Dayton, Ohio.

Dr. Archie Higdon has been appointed to an associate professorship in theoretical and applied mechanics at the Iowa State College of Agriculture and Mechanical Arts.

Mr. J. M. Holme, formerly of Pennsylvania State College, has been appointed to an assistant professorship at the United States Naval Academy.

Dr. A. S. Householder has accepted a position as principal physicist at the Clinton Laboratories, Monsanto Chemical Company, Knoxville, Tennessee.

Mr. J. P. Hoyt has been appointed to an assistant professorship at the United States Naval Academy.

Dr. Waclaw Kozakiewicz of the University of Saskatchewan has been promoted to an assistant professorship.

Associate Professor L. H. McFarlan of the University of Washington has been promoted to a professorship.

Dr. A. E. Marston has been appointed research mathematician at the Naval Research Laboratory, Washington, D. C.

Dr. M. A. Melvin of Columbia University has been promoted to an assistant professorship of physics.

Assistant Professor A. B. Mewborn of the University of Arizona has been appointed to an associate professorship at the Postgraduate School, United States Naval Academy.

Mr. J. F. Milos has been appointed to an assistant professorship at the United States Naval Academy.

Assistant Professor Thirza A. Mossman of Kansas State College of Agriculture and Applied Science has been promoted to an associate professorship.

Dr. K. L. Palmquist, formerly of Southern Methodist University, has been appointed to an assistant professorship at the United States Naval Academy.

Associate Professor R. S. Pate of the University of South Carolina has been appointed to a professorship at Michigan State Normal College, Ypsilanti, Michigan.

Dr. J. F. Paydon, formerly of Northwestern University, has been appointed to an assistant professorship at the United States Naval Academy.

Dr. E. R. Peck has been appointed to an assistant professorship of physics at Northwestern University.

Dr. Echo D. Pepper has been appointed to an assistant professorship at the University of Illinois.

Miss Ellen F. Rasor has been appointed to a professorship of Flora MacDonald College, Red Springs, North Carolina.

Assistant Professor Adrienne S. Rayl of the Birmingham Center of the University of Alabama has been promoted to an associate professorship.

Professor C. A. Reagan of Friends University, Wichita, Kansas, has been appointed acting president.

Mr. J. K. Reckzeh of the University of Kentucky has been appointed to an assistant professorship at State Teachers College, Jersey City, New Jersey.

Associate Professor R. F. Rinehart of Case School of Applied Science has been promoted to a professorship.

Mr. J. S. Ripandelli has accepted a position as actuarial assistant with Miles M. Dawson and Son, Incorporated, New York, New York.

Dr. V. N. Robinson has been appointed to an assistant professorship at the United States Naval Academy.

Dr. S. S. Saslaw, formerly of the University of Miami, has been appointed to an assistant professorship at the United States Naval Academy.

Dr. F. E. Satterthwaite of the Aetna Life Insurance Company, Hartford, Connecticut, has accepted a position as design engineer at the General Electric Company, Fort Wayne, Indiana.

Associate Professor George Sauté of Rollins College, Winter Park, Florida, has been promoted to a professorship.

Dr. Nathan Schwid has been appointed to an associate professorship at the University of Wyoming.

Professor L. W. Sheridan of the College of Mount St. Vincent has been appointed to an associate professorship at the College of St. Thomas.

Mr. W. H. Simons has been appointed to a lectureship at the University of British Columbia.

Associate Professor M. F. Smiley of Lehigh University has been appointed to an associate professorship at Northwestern University.

Mr. R. E. Smith at Allegheny College has been appointed to an associate professorship at the College of William and Mary.

Dr. S. R. Smith has been appointed to an assistant professorship at the University of Wyoming.

Assistant Professor S. S. Smith of the University of Utah has been promoted to an associate professorship.

Associate Professor Andrew Sobczyk of Oregon State College has been appointed Chief of the Mathematical Research Branch, Watson Laboratories, Cambridge, Massachusetts.

Dr. K. H. Stahl of State Teachers College, California, Pennsylvania, has been appointed to an assistant professorship at the University of Colorado.

Mr. Palmer Steen has been appointed to an assistant professorship at the University of Wyoming.

Dr. Ruth W. Stokes has been appointed to an assistant professorship at Syracuse University.

Dr. Mildred M. Sullivan of Queens College, Flushing, New York, has been promoted to an assistant professorship.

Dr. Ethel Sutherland of East Carolina Teachers College has been appointed to an associate professorship at the State Teachers College, Farmville, Virginia.

Associate Professor P. M. Swingle of the New Mexico College of Agriculture and Mechanical Arts has been appointed to a professorship at the University of Miami.

Associate Professor A. H. Taub of the University of Washington has been promoted to a professorship.

Dr. E. P. Virene has been appointed to an assistant professorship at the Newark College of Engineering.

Mr. R. S. Wentworth has been appointed to an assistant professorship at Lehigh University.

Professor J. H. Zant of Oklahoma Agricultural and Mechanical College has been appointed director of instruction of the Okmulgee Branch, newly established to give instruction to veterans.

The following appointments to instructorships are announced: University of Buffalo: Mrs. Joan S. Anderson, Mr. V. N. Behrns, Mrs. Jeanne J. Dinwoodie, Mrs. Lorraine W. Farber, Miss Lillian Gough, Miss June M. McArtney, Miss Mabel D. Montgomery, Mr. N. H. Sampson, Mr. F. C. Warner, Mrs. Ina W. Welmers; University of Colorado: Mr. George Barnes, Mr. D. L. Barrick, Mr. F. J. Casey, Mr. R. H. Glass, Mr. E. W. Grigs, Mr. P. F. Hultquist, Mr. V. J. Moore, Mr. L. W. Rutland, Jr., Mr. M. E. Sperline; Harvard University: Dr. R. C. James; Houston College: Mr. O. L. Lattimore; Iowa State College of Agriculture and Mechanical Arts: Mr. W. G. Dyer; Lehigh University: Mr. Leon Benson, Mr. W. Hibbard, Mr. H. A. Seebald, Mr. R. H. Spohn, Mr. K. C. Walters, Mr. R. W. Young; University of Maine: Dr. W. B. Caton, Mr. J. A. Harmon, Mr. Leo Lapidus, Mrs. A. D. Mawhinney, Mr. Sutton Monro, Miss Letitia Watson; Michigan State College: Miss Evelyn O. Bychinsky, Mr. H. A. Hanson, Mr. M. S. Miller, Mr. J. A. Straw; University of Michigan: Dr. W. M. Kincaid; Mississippi Southern College: Mr. John Jones, Jr., Mr. J. T. Lewandowski; University of New Hampshire: Mr. A. R. Harvey; Northwestern University: Dr. A. O. Lindstrum; Ohio State University: Mr. C. T. West; University of Pennsylvania: Mr. A. G. Makarov; Rutgers University: Dr. M. A. Hyman; Union College: Mr. E. F. Gillette, Mr. H. K. Holt, Mr. F. F. Ormsby; United States Naval Academy: Mr. M. V. Gibbons, Mr. E. C. Gras, Mr. T. A. Lamke, Mr. Joseph Milkman, Mr. J. W. Popow, Mr. R. W. Rector, Mr. J. A. Tierney, Mr. E. C. Watters, Mr. J. H. White; Washington University: Miss Theresa M. Renner; Wellesley College: Miss Kathleen E. Butcher; University of Wisconsin: Mr. W. N. Smith; University of Wyoming: Miss Rosamond Jones.

The death of Mr. E. Feldheim has been reported.

Professor Dunham Jackson of the University of Minnesota died November 6, 1946, at the age of fifty-eight years. He had been a member of the Society for thirty-five years.

Dr. Benjamin Rosenbaum died May 29, 1946. He had been a member of the Society for fifteen years.

Dean Emeritus C. S. Slichter of the University of Wisconsin died October 4, 1946, at the age of eighty-two years. He had been a member of the Society for fifty-six years.

Associate Professor I. D. Stewart of Whitman College, Walla Walla, Washington, died on July 21, 1946.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

1. A. A. Albert: *A structure theory for Jordan algebras.*

This paper gives the first general structure theory for Jordan algebras, that is, commutative algebras in which the identity $x(yx^2)=(xy)x^2$ holds. Using identities on right multiplications it is shown that every solvable algebra over a field of characteristic not two is nilpotent in the sense that there exists an integer k such that every product of k elements is zero. Also an algebra whose quantities are all nilpotent is necessarily solvable. For algebras over a nonmodular field a trace criterion is developed yielding a decomposition relative to an idempotent, and it is shown that if N is the maximal solvable ideal of A then $A-N$ is a direct sum of simple algebras. Finally, it is shown that every simple Jordan algebra is isomorphic to a Jordan algebra of linear transformations except for a certain class of algebras of order 27 over the center. (Received October 10, 1946.)

2. R. F. Arens: *Representation of rings in which $x^p=x$.*

McCoy and Montgomery, Duke Math. J. vol. 3 (1937), have partially generalized the representation of Boolean algebras of Stone, Trans. Amer. Math. Soc. vol. 40 (1936), by showing that a ring A in which $x^p=x$ and $px=0$ is a subdirect sum of the corresponding Galois field. In the present paper, the author follows Stone in making this representation sharper and unique, in a certain sense, by considering the locally compact 0-dimensional structure space of A. The case of rings in which $x^{p^n}=x$ is also considered. (Received November 19, 1946.)

3. Reinhold Baer: *Direct decompositions.*

In this paper a comprehensive refinement theorem for direct decompositions of operator loops is proved; and this theorem is shown to contain as special cases the theorems of W. Krull, O. Schmidt, V. Kořínek and A. Kurosh. (Received October 14, 1946.)

4. Reinhold Baer: *Endomorphism rings of operator loops.*

If L is an operator loop, and if A is a commutative and associative admissible subloop of L, then the set θ of all the endomorphisms of L which map L into A is a ring with respect to the customary operations of addition and multiplications. In this note the structure of θ is investigated, mainly under the hypothesis that all the endomorphisms in θ split L. (Received October 7, 1946.)

5. Reinhold Baer: *Splitting endomorphisms.*

If t is an endomorphism of the operator loop L, then its radical $R(t)$ consists of all the elements x in L such that $0=xt^n$ for some integer $n=n(x)$. The radical is a normal and admissible subloop. A complement of the endomorphism t is an admissible subloop C such that every coset of $L/R(t)$ contains one and only one element in C and such that $C=Ct$. The object of this note is to find criteria for the existence of complements. (Received October 7, 1946.)

6. Reinhold Baer: *The double chain condition in cyclic operator groups.*

If A is a cyclic abelian operator group, then it is possible that the descending, but not the ascending, chain condition is satisfied by the admissible subgroups of A. It is therefore desirable to obtain criteria for the validity of the double chain condition in a cyclic abelian operator group; and in the present note there are offered several criteria of this type. (Received November 16, 1946.)

7. J. D. Bankier: *Extended regular continued fractions.*

An *extended* regular continued fraction (e.r.c.f.) is defined recursively by the relations $y_n=\epsilon_n/(y_{n-1}-b_{n-1})$ $(n=1, 2, 3, \cdots)$, where $\epsilon_n=\pm 1$, $\epsilon_n(y_{n-1}-b_{n-1})>0$ and b_n is the integer further from y_n of the integers $[y_n]$, $[y_n]+1$, providing such an integer exists. If $y_n=I+1/2$, where I is an integer, $b_n=I+1$, if $y_n=2$, $b_n=1$, and, if $y_n=1$, $b_n=1$. A necessary and sufficient condition that an e.r.c.f. terminate is that y_0 be rational. A nonterminating e.r.c.f. converges to the generating value y_0. Necessary and sufficient conditions that a nonterminating continued fraction be an e.r.c.f. are that its elements be integers such that $b_n=1$ or 2, if $b_n=1$, $\epsilon_{n+1}=1$, if $b_n=2$, $\epsilon_{n+1}=-1$ $(n=1, 2, 3, \cdots)$, and that, corresponding to every positive integer n, there exists an integer m such that $n<m$ and $b_m=1$. For a continued fraction terminating with b_k, the additional condition is required that $b_{k-1}=b_k=-\epsilon_{k-1}=\epsilon_k=1$. An e.r.c.f. is periodic if and only if it converges to a quadratic irrational. A method is given for obtaining the e.r.c.f. expansion of a real number directly from the corresponding r.c.f. The convergents of an e.r.c.f. are shown to be an arrangement of the primary and secondary convergents of the corresponding r.c.f. The e.r.c.f. enables one to find solutions for a larger class of Pell equations than the r.c.f. (Received November 19, 1946.)

8. B. A. Bernstein: *Weak definitions of field.*

The author obtains from a typical addition-multiplication definition of field two weaker definitions. In one of these definitions the distributive law is replaced by a law involving only two variables. In the other definition each of the additive laws of the type definition is replaced by a law involving fewer variables, and the distributive law is entirely dispensed with. The definitions are postulational, with independence among the postulates established. (Received October 23, 1946.)

9. Garrett Birkhoff: *Notes on lattices.*

The congruence relations on any abstract algebra A with permutable congruence relations form a modular lattice; hence the Kurosch-Ore theorem applies. If A also contains a one-element subalgebra, then the unique factorization theorem of Wedderburn-Remak, and all Jordan-Holder-Schreier-Zassenhaus theorems hold. An abstract lattice L is isomorphic with the lattice of all subalgebras of a suitable abstract algebra if and only if (i) L is complete, (ii) in L, the meet operation is continuous in the order topology, (iii) every element a of L is the join of elements a_α, no one of which is the topological limit of smaller elements. Any complete metric lattice is a topological lattice. (Received October 25, 1946.)

10. Garrett Birkhoff: *Subdirect representation of certain algebras.*

Let A be any algebra with finitary operations, which has a finite number of non-isomorphic subdirectly irreducible homomorphic images $F_1, \cdots, F_n$. Suppose also that each F_i is finite. Then A can be represented as a totally disconnected topological algebra of continuous functions from a totally disconnected bicompact "space" (of homomorphisms) onto the union F of the F_i. (Received October 31, 1946.)

11. R. H. Bruck: *An extension theory for Moufang loops.*

For terminology, see the author's recent paper (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 245–354). In the present note he treats the following problem: Given a Moufang loop M and a group G, to determine all Moufang loops E homomorphic to M with kernel G contained in the associator of E. In sharp contrast with many other extension problems for loops, there emerges a theory of factor sets closely akin to that for groups. If M has finite order m and f is a central Moufang factor set, $f^n \sim 1$ where $n = 2m^2$ (*not* m, as in the group case). The powers f^m are nontrivial factor sets of a particularly simple form, and these are thoroughly investigated. When M is abelian, each f^m is a group factor set. When M is a noncommutative group the situation is more complex, but a variety of relations is obtained between central group factor sets and central Moufang factor sets. (Received November 18, 1946.)

12. R. P. Dilworth: *Decomposition of relatively complemented lattices.*

The following theorem is proved: *Every indecomposable, relatively complemented lattice of finite dimensions is simple.* (Received October 17, 1946.)

13. J. S. Frame: *Group decomposition by double coset matrices.*

A formula $n_i = g/\sum k_\alpha^{st}|\rho_{\alpha i}^{st}|^2$ is obtained for the degrees n_i of the common irreducible components of two isomorphic (but not necessarily distinct) representations $R^s(\gamma)$ and $R^t(\gamma)$ of a finite group G of order g by transitive groups of permutation matrices of degree n^s and n^t respectively. Matrices V_α^{ts} consisting of 0's and 1's, and such that $R^t(\gamma)V_\alpha^{ts} = V_\alpha^{ts}R^s(\gamma)$, correspond to the double cosets $H_\alpha^{st} = H^s\gamma_\alpha H^t/k_\alpha^{st}$. The orders of $\gamma_\alpha^{-1}H^s\gamma_\alpha$, H^t, and their intersection are respectively k^s, k^t, and k_α^{st}. The quantities $\rho_{\alpha i}^{st}$ appear as coefficients in the transforms of V_α^{ts} by the same matrix U which reduces the direct sum of R^s and R^t. In general $\rho_{\alpha 1}^{st} = (k^s k^t)^{1/2}/k_\alpha^{st}$, and it seems that for a suitable reducing transformation U all the $\rho_{\alpha i}^{st}$ will be algebraic integers. In certain important cases the $\rho_{\alpha i}^{st}$ are integral factors of k^t/k_α^{st}, or zero, and the degrees n_i are easily found. The irreducible group characters χ_λ^i for the class C_λ are $\chi_\lambda^i = n_i(\sum \gamma k_\alpha^{st}\rho_{\alpha i}^{st}k_{\alpha\lambda}^{st})/(k_1^{st}\rho_{1i}^{st}g_\lambda)$, where $k_{\alpha\lambda}^{st}$ is the number of the g_λ elements of the class C_λ which are in the double coset H_α^{st}. (Received November 25, 1946.)

14. Wallace Givens: *Parametric solution of linear homogeneous Diophantine equations.*

A system of r linearly independent linear homogeneous equations in n unknowns with rational integers as coefficients is considered. Every solution of such a system is proved to be obtainable by giving integral values to the parameters $p_{\rho j}$ ($\rho=1, \cdots, n-r-1; j=1, \cdots, n$) and k (which may be set equal to one if $n>r+1$) in the formulas $x_i=kF_i(p_{\rho j})$, where the F_i are homogeneous polynomials with integral coefficients which are given explicitly in terms of the coefficients of the given equations. With k allowed to be rational instead of integral, this theorem was recently proved by L. W. Griffiths (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 734–736). A new proof of Miss Griffiths' result is first given and then the stronger theorem is reduced to the special case $r=0$ where it is equivalent to a theorem due to Hermite. (Received November 22, 1946.)

15. L. K. Hua: *Geometries of matrices*. II. *Study of involutions in the geometry of symmetric matrices.*

In the first part of the paper, the explicit normal forms of involutions and anti-involutions and the manifold of fixed points of all sorts of involutions are determined. Secondly, the author enumerates the geometries keeping an involution or an anti-involution as absolute. Those obtained from anti-involutions are generalizations of non-Euclidean geometries, among which the geometry of the hyperbolic type is Siegel's symplectic geometry. Those obtained from involutions give several new types of geometries, one whose real analog is a generalization of the Möbius geometry of circles. The author shows that every symplectic transformation is a product of *two* involutions and *four* anti-involutions and that in the space of symmetric matrices of order $n=2^{\sigma}\tau$, τ being odd, are at most $\sigma+3$ pairs of points of which any two pairs separate each other harmonically. As a by-product, the author proved also that if $S_1, \cdots, S_s$ are n-rowed symmetric matrices satisfying $s_i^2=-I$, $s_i s_j=-s_j s_i$, then $s\leqq\sigma+1$. (Received October 23, 1946.)

16. L. K. Hua: *On the automorphisms of the symplectic group over any field*. Preliminary report.

Let Φ be any field with characteristic not equal to 2 and containing more than three elements. Let F be a $2n$-rowed nonsingular skew-symmetric matrix, say one with first row 0, I and second row $-I$, 0. Let Γ be the group of $2n$-rowed symplectic matrices, that is, the group is constituted by all those elements P satisfying $PFP'=F$, where P' denotes the transposed matrix of P. In this paper, all automorphisms of the group Γ are determined. More precisely, all the automorphisms $A(X)$ on Γ are of the form $R^{-1}P^{-1}X^{\sigma}PR$, where P runs over all symplectic matrices, σ runs over all the automorphisms of the field, R is a matrix with first row I, 0 and second row 0, aI, and a runs over a complete residue system of the factor group Φ/Φ_2. Φ is considered as a multiplicative group and Φ_2 is the multiplicative group of all the square elements of Φ. (Received November 20, 1946.)

17. Ralph Hull: *Generators of the rational integral symplectic groups.*

Elementary methods are employed to determine a finite set of generating elements of the group of $2n$ by $2n$ symplectic matrices with rational integral elements. The methods may also be applied to the case of symplectic groups over an algebraic number field in which there is a division algorithm. (Received October 24, 1946.)

18. Irving Kaplansky: *Semi-simple algebraic algebras.*

Representation theorems for a commutative semi-simple algebraic algebra A over a field F are given, analogous to Stone's topological representation of Boolean rings. If F is algebraically closed, A consists of all continuous functions from a compact totally disconnected space to F. In the general case the functions may take values in the various subfields of the algebraic closure of F. If F is complete in a valuation, and A is normed and complete, then the elements of A have bounded degrees. Here A can be any algebraic algebra whose radical is the set of all nilpotent elements. This generalizes a theorem announced by Mazur. (Received November 21, 1946.)

19. B.-E. Meserve: *Division sequences by cross-multiplication.*

A synthetic method is given for obtaining the division sequences of any two polynomials in one unknown with real coefficients. It is a generalization of the process of cross multiplication used by E. J. Routh (*Advanced rigid dynamics*, 6th ed., 1905, p. 226) to obtain conditions for the stability of a linear dynamical system. Special applications of the method give the Sturm sequences of a single polynomial in one unknown with real coefficients and a highest common factor of any two such polynomials. (Received October 28, 1946.)

20. L. J. Paige: *A note on finite abelian groups.*

Let $[X_\rho, A_\rho, B_\rho]$ be an ordered triple of ordered subsets $X_\rho=(x_1, \cdots, x_\rho)$, $A_\rho=(a_1, \cdots, a_\rho)$, $B_\rho=(b_1, \cdots, b_\rho)$ of an abelian group G of order N. Defining a triple to be admissible if and only if X_ρ, A_ρ, B_ρ each contain ρ distinct elements and $x_ia_i=b_i$ $(i=1, 2, \cdots, \rho)$, the author proves that an admissible triple $[X_N, A_N, B_N]$ exists if and only if G does not contain a unique element of order 2. For all G there exist admissible triples $[X_{N-1}, A_{N-1}, B_{N-1}]$. (Received October 11, 1946.)

21. H. J. Ryser: *Rational vector spaces.* Preliminary report.

Let V_n be a rational vector space of dimension n admitting rational inner product (α, β) (Bull. Amer. Math. Soc. Abstract 52-9-267). V_n is called an I-space provided it has basis $\delta_1, \delta_2, \cdots, \delta_n$ such that $(\delta_i, \delta_j)=\delta_{ij}$. If $\xi_1, \xi_2, \cdots, \xi_n$ are n linearly independent elements of V_n, the square free integer part d of the determinant of the matrix $[(\xi_i, \xi_j)]$ is an invariant of the space. If V_n has inner product (α, β) and invariant d and V_m inner product (γ, δ) and invariant D, then a direct product space $V_n \otimes V_m$ of dimension nm is defined in which $(\alpha, \beta)(\gamma, \delta)$ acts as inner product. If $n=2r$ and $m=2s$, a necessary and sufficient condition that $V_n \otimes V_m$ be an I-space relative to $(\alpha, \beta)(\gamma, \delta)$ is that the product of Hilbert symbols $(d, d)_p^s(D, D)_p^r(d, D)_p$ $=+1$ for $p=2$ and for each of the odd prime divisors of dD. Similar results hold for n odd, and both n and m odd. Algebraic number fields are studied as rational vector spaces, and an explicit determination is made of all cyclotomic I-fields, all cyclic I-fields of even degree, and all I-fields defined by the e periods of the pth roots of unity. (Received November 4, 1946.)

22. R. M. Thrall: *On ahdir algebras.* Preliminary report.

If an algebra A over a field k possesses indecomposable representations of arbitrarily high degree it will be called an *ahdir* algebra. Assume k algebraically closed. R. Brauer (*On the indecomposable representation of algebras*, Bull. Amer. Math. Soc. Abstract 47-9-334) has given three conditions each sufficient to ensure that A is ahdir. These conditions are stated in terms of the Cartan invariants of A, A/N, A/N^2, $\cdots$ where N is the radical of A. In the present work a fourth sufficient condition of the same general character is given and conversely it is proved that if $N^2=0$ and none of the four sufficient conditions hold, then A is not ahdir. In the case $N^3=0$ the four sufficient conditions can be stated in the following manner. Suppose that A/N has t simple ideals. Then the Cartan invariants are the elements of a t by t matrix C with integer elements. Set $D=C-I$. Associate a graph G of $2t$ vertices $P_1, \cdots, P_t \cdot Q_1, \cdots, Q_t$ with D by joining P_i and Q_j if and only if $d_{ij}\neq 0$. The four sufficient conditions are then (1) some $d_{ij}>1$; (2) G is not a tree; (3) some vertex of G is of order greater than 3; (4) some connected subgraph of G has more than one vertex of order 3. (Received November 21, 1946.)

23. Bernard Vinograde. *Radicals associated with equivalent semi-simple residue systems.*

This paper investigates rings wherein the radical is a homomorphic additive image of the semi-simple part and satisfies $f(xy)=xf(y)+f(x)y+f(x)f(y)$, where f is the homomorphism. $f(xy)$ affords a trioperational approach. This is an aspect of the distribution of residue systems in a semi-primary ring. (Received October 24, 1946.)

24. Daniel Zelinsky: *Nonassociative valuations.*

An ordered quasigroup G is a quasigroup, written additively, which is linearly ordered by a transitive, binary relation $>$, having the property that $x>y$ implies $x+z>y+z$ and $z+x>z+y$ for all x, y, z of G. A valuation, V, of a (nonassociative) ring R is a function on R to an ordered quasigroup with ∞ adjoined such that for all a, b of R, $V(a+b)\geqq \min\,[V(a), V(b)]$, $V(ab)=V(a)+V(b)$, $V(a)=\infty$ if and only if $a=0$. The principal theorem of this paper is the following: If R is an algebra of finite order over a field F, if R has a unity quantity and if $V(F)$ is an archimedean-ordered group, then $V(R)$ is an archimedean-ordered abelian group in which $V(F)$ has finite index. Examples of nonassociative ordered loops are obtained by simple loop extensions. The existence of a ring with arbitrary prescribed value loop and residue-class ring (without zero divisors) is proved. From these two facts follow examples showing that the hypothesis "$V(F)$ is archimedean-ordered" cannot be omitted in the theorem above. This is in strong contrast with the associative, noncommutative case. (See O. F. G. Schilling, *Noncommutative valuations*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 297–304.) (Received November 13, 1946.)

ANALYSIS

25. H. W. Becker: *Generalizations of the Epstein-Fourier series.*

These series are combinations of exponential and Fourier series (Leo F. Epstein, Journal of Mathematics and Physics vol. 18 (1939) p. 60, (19)). Where $K_0=1$, $x=r\cos\theta$, $y=r\sin\theta$, and "soc" means "sine or cosine," some generalizations are: (1) $\exp[X+r\ \mathrm{soc}\ (\ )\theta\cdot(KX+Z)]=[\mathrm{soc}(Zy+e^{Xx}\sin Xy)]\cdot[\exp(Zx+e^{Xx}\cos Xy)]$, the $(KX+Z)$ except for change of sign of X being the polynomials of Steffensen (*Some recent researches in the theory of statistics and actuarial science*, Cambridge Press, 1930, p. 24); (2) $\exp[e\{1+r\ \mathrm{soc}\ (\ )\theta\cdot K^{(x)}\}]=[\mathrm{soc}\{e^{e^x\cos y}\sin(e^x\sin y)\}]\cdot[\exp\{e^{e^x\cos y}\cos(e^x\sin y)\}]$, the $K^{(x)}$ being Bell numbers (Ann. of Math. vol. 39 (1938) p. 539). Under the substitutions $r\rightarrow rX$, $X\rightarrow X^{-1}$, $Z=0$, (1) becomes (3) $\exp[X^{-1}+r\ \mathrm{soc}\ (\ )\theta\cdot T]=[\mathrm{soc}(e^x\sin y)]\cdot[\exp(e^x\cos y)]$, where $T=XKX^{-1}$ is an umbral transform of Riordan and Kaplansky (*The problem of the rooks and its applications*). It is noteworthy that the right side of (3) is free of X, endowing the left side with a kind of invariance. (Received October 1, 1946.)

26. R. H. Cameron and W. T. Martin: *Fourier-Wiener transforms of functionals belonging to L_2 over the space C.*

The authors consider the class $L_2(C)$ of real or complex-valued functionals $F(x)$ which belong to L_2 over the space C: $\int_C^w |F(x)|^2 dw < \infty$. They define a Fourier-Wiener transform and show that every functional $F(x)$ belonging to $L_2(C)$ has a transform $G(y)$ which also belongs to $L_2(C)$ and which has $F(-x)$ as its transform. They show furthermore that Plancherel's relation holds: $\int_C^w |F(x)|^2 d_w x = \int_C^w |G(y)|^2 d_w y$. In an earlier paper (*Fourier-Wiener transforms of analytic functionals*, Duke Math. J. vol. 12 (1945) pp. 489–507) the authors showed that every functional belonging to a restricted class E_- has a Fourier-Wiener transform which belongs to E_- and which has the properties mentioned above. (The definition given there is modified slightly in the present paper to free Plancherel's formula from a factor $2^{1/2}$.) The functionals of class E_- are shown to be dense in $L_2(C)$; this property enables the authors to obtain the desired result. (Received November 6, 1946.)

27. K. L. Chung and Paul Erdős: *On the lower limit of sums of independent random variables.*

The following theorems form a counterpart of the law of iterated logarithms. Let $X_1, \cdots, X_n, \cdots$ be independent random variables having the same distribution function $F(x)$. Suppose that the absolutely continuous part of $F(x)$ does not vanish identically and that its first moment is zero, the second is one, and the absolute fifth is finite. Let $\psi(n)$ be nondecreasing and tend to infinity. Then (1) $\Pr(\liminf_{n\to\infty} n^{1/2}\psi(n)S_n = 0)$ is one or zero according as (2) $\sum n^{-1}\psi(n)^{-1}$ is divergent or convergent. If each X_n has the same Bernoullian distribution: $X_n = q$ with probability p and $X_n = -p$ with probability q where $p > 0$ and $p + q = 1$, then if (2) is divergent, (1) is equal to one. On the other hand, if p is a quadratic irrational, and (2) is convergent, (1) is equal to zero. The last statement holds in fact for almost all real p. The case when p is rational is known to be degenerate. (Received October 31, 1946.)

28. Paul Civin: *Mean values of periodic functions.*

Let $f(x)$ be a complex periodic function of period 2π for which the Fourier coefficients of order less than m vanish. Inequalities are obtained between the integral means of $f(x)$ and transforms of $f(x)$ by means of factor sequences. As a specialization, inequalities are obtained between the means of $f(x)$ and its αth integral. (Received October 28, 1946.)

29. Evelyn Frank: *On continued fraction expansions of analytic functions.*

(i) A continued fraction expansion $\gamma_0 + \sigma_0(z+\bar{\xi}_0)(1-|\gamma_0|^2)/\sigma_0\bar{\gamma}_0(z+\bar{\xi}_0) - (z-\xi_0)/\gamma_1 + \sigma_1(z+\bar{\xi}_1)(1-|\gamma_1|^2)/\sigma_1\bar{\gamma}_1(z+\bar{\xi}_1) - (z-\xi_1)/\gamma_2 + \cdots$, $\gamma_k = f_k(-\bar{\xi}_k)$, $f_{k+1} = \sigma_k(z-\xi_k)(\gamma_k - f_k)/(z+\bar{\xi}_k)(1-\bar{\gamma}_k f_k)$, $k = 0, 1, \cdots$, for functions $f(z)$ analytic in $R(z) > 0$, $|f(z)| \leq 1$, is found directly from Schur's theory by mapping $|w| < 1$ on $R(z) > 0$. Here $\xi_k = \xi$, $R(\xi) < 0$, $\sigma_k = 1$, $|\gamma_k| \leq 1$. This expansion is proved finite by use of the reduction theorem (cf. Frank, *The real parts of the zeros of a complex polynomial*, abstract 52-7-232) if and only if $f(z) = \epsilon P(z)/P^*(z)$, $|\epsilon| = 1$, $P^*(z) = \bar{P}(-z)$, $P^*(z)$ a Hurwitz polynomial, $P(-\bar{\xi}) = 1$. If $P^*(z)$ is any polynomial, then in the finite continued fraction $|\sigma_k| = |\bar{P}_k^*(-\xi)/P_k^*(-\bar{\xi}_k)| = 1$, the ξ_k not necessarily equal. (ii) The continued fraction $[z + (\bar{\xi} - \xi)/2]/(\bar{\xi} - \xi)/2 + (z+\bar{\xi})(z-\xi)(\delta_0 - \gamma_0)/[\delta_0(z-\xi) - \delta_1(z+\bar{\xi})] + (z+\bar{\xi})(z-\xi)(\delta_1$

$+\gamma_0)(\delta_1-\gamma_1)/[\delta_1(s-\xi)-\delta_1(s+\bar{\xi})]+(s+\bar{\xi})(s-\xi)(\delta_2+\bar{\gamma}_1)(\delta_2-\gamma_2)/[\delta_2(s-\xi)-\delta_2(s+\bar{\xi})]+\cdots$, $\delta_{k+1}=(\delta_k-\gamma_k)/(1-\delta_k\bar{\gamma}_k)$, $k=0, 1, \cdots$, converges uniformly over every bounded closed domain in $R(s)>0$ to a function, analytic in $R(s)>0$, $F(s)=[\delta_0(s-\xi)+(s+\bar{\xi})f(s)]/[\delta_0(s-\xi)-(s+\bar{\xi})f(s)]$, $|\delta_0|\leqq 1$, $R(\xi)<0$. (iii) Functions $F(s)$ and $[\lambda-f(s)]/[\lambda+f(s)]$, where $f(s)=\epsilon P(s)/P^*(s)$, have partial fraction expansions $Q(s)+c-\sum_{\nu=1}^{k} r_\nu(s+\bar{\eta}_\nu)/(s-\eta_\nu)$, $Q(s)$ zero or a polynomial of degree k, the η_ν zeros of the denominator of the function represented, and under certain conditions the c and r_ν pure imaginary. (Received October 17, 1946.)

30. Evelyn Frank: *On methods for the location of the zeros of complex polynomials.*

(I) The zeros of $P(s)=A_0s^n+A_1s^{n-1}+\cdots+A_n$, A_j complex, are located as follows: If $|A_0|>|A_n|$, form $\bar{A}_0P(s)-A_nP^*(s)=sP_1(s)$, $P^*(s)=s^n\bar{P}(1/s)$, then $P(s)$ has one more zero in $|s|=1$ than $P_1(s)$; if $|A_0|<|A_n|$, $P_1(s)=\bar{A}_nP(s)-A_0P^*(s)$ has as many zeros as $P(s)$ in $|s|=1$. Repeating this process on $P_1(s)$, $P_2(s)$, $\cdots$, after at most $(n-1)$ steps one finds the number of zeros in $|s|=1$ (Cohn, Math. Zeit. vol. 14). Form $P(rs)$ and vary r successively until, by this process, $P_{n-1}(rs)$ has a zero s' of modulus 1 (or as close to 1 as desired), then rs' is a zero of $P(s)$ of modulus r (or as close to r as desired). The exceptional cases $|A_\nu^{(k)}|=|A_{n-\nu}^{(k)}|$, $\nu=0, 1, \cdots$, in $P_{n-k}(s)$ are considered. The zeros of $P(s)$ are different and of modulus r if $A_n\bar{A}_\nu=\bar{A}_0A_{n-\nu}r^{2\nu}$, $\nu=1, 2, \cdots, n$, and the zeros of $P'(rs)$ lie in $|s|=1$. (II) An alternate method for finding the zeros of $P(s)$ is obtained by use of the quadratic forms of Hermite. (III) Schur's determinant criterion (J. Reine Angew. Math. vol. 148) for all zeros of $P(s)$ to lie in $|s|=1$ is compared with that of the author (Bull. Amer. Math. Soc. vol. 52) for all zeros to lie in $R(s)>0$. (Received November 20, 1946.)

31. A. M. Gleason: *A theorem on Banach spaces.* Preliminary report.

The sequence $\{\phi_n\}$ of points in a Banach space B is called a basis if every element $x\in B$ has a unique convergent representation $x=\sum a_n(x)\phi_n$. A basis is called an absolute basis if the convergence of $\sum a_n\phi_n$ implies the convergence of $\sum a_{n_i}\phi_{n_i}$ for any sequence $\{n_i\}$. Under this definition the ordinary basis of l_p $(1\leqq p<\infty)$ is absolute, but Schauder's basis for the set of continuous functions is not. Theorem: If B has an absolute basis and B^* (the conjugate of B) is separable, then B^* has an absolute basis. It is well known that the functionals $a_n(x)$ form a basis of their closed linear manifold; if there is a functional f not in this clm, one can construct an uncountable set of points in B^* each pair of which are separated by more than $k>0$, so that B^* is not separable. This construction is obtained by considering functionals $g(x)=\sum b_{n_i}a_{n_i}(x)$, where $f(x)=\sum b_na_n(x)$. An immediate corollary is the following: If B has an absolute basis and B^{**} is separable, then B is reflexive. (Received November 22, 1946.)

32. O. E. Glenn: *A differential equation that plays a part in a mathematical theory of biological variation.*

A sequence of plane curves k_1, k_2, $\cdots$ is considered, in the form of irregular (statistical) truncated spirals of amplitude less than four right angles. Draw k_1 in the (y, z) cartesian plane and k_2 in the (x, y) plane. The cylinders erected upon these curves as bases with elements $\|x$ and z, respectively, intersect in a twisted space-curve c_1. Move k_2 to the plane (y, z) and draw k_3 in (x, y) and repeat the construction, giving a space-curve c_2, and continue the process. The curves of the *community* (c_i) are permuted among themselves by a known transformation S in polar coordinates (Annali della R. Scuola Normale Superior di Pisa (2) vol. 2 (1933) p. 300), and a function $I(r, \theta, \phi, d\theta, d\phi)$ left invariant by S satisfies the equation (1) $(up\,\partial/\partial r+vq\partial/\partial\theta+up'(dr)\partial/\partial(dr)+vq'(d\theta)\partial/\partial(d\theta))I=0$, $p=ar^{l-1}+br^{l-2}+\cdots+k$, $q=\alpha\theta^{l-1}+\beta\theta^{l-2}+\cdots+\kappa$, u, $v=0$. Solution for I and reduction of $I=0$ gives (2) $-(dr)/p+\lambda(Q,\phi)(d\theta)/q+\mu(Q,\phi)d\phi=0$, λ, μ arbitrary, $Q=v\int(dr)/p-u\int(d\theta)/q$. Equation (2) is formally invariant under S and its integral surfaces are related, by theorems, to the community (c_i). (Received November 8, 1946.)

33. Casper Goffman: *Fubini interval functions.*

A function $F(R)$ of rectangles and a function $f(I)$ of line segments, both properly oriented, satisfy a condition of Fubini if $F(R)=\int f(I)dy$. Assuming that $F(R)$ and $f(I)$ satisfy a condition of Fubini, and if E, not necessarily measurable, is the set of points for which both $f'(x, y)$, the bilateral derivative of $f(I)$, and $F'_S(x, y)$, the strong derivative of $F(R)$, exist, then $f'(x, y)=F'_S(x, y)$ almost everywhere on E. A similar result for the unilateral derivative is proved for E measurable, but the theorem then holds for the approximate unilateral derivative. It is shown by example that these results do not hold for the ordinary derivative, $F'(x, y)$, of $F(R)$. Conditions are established under which the existence of $f'(x, y)$ implies the existence almost everywhere of $F'(x, y)$ and $F'_S(x, y)$. The existence of $F'_S(x, y)$ implies the existence of the approximate unilateral derivative and its equality almost everywhere with $F'_S(x, y)$. A similar result is obtained for the derivative of $f(I)$ only by assuming continuity. (Received November 21, 1946.)

34. M. R. Hestenes. *An indirect sufficiency proof for the problem of Bolza in nonparametric form.*

Until recently, sufficiency theorems in the calculus of variations have been established by direct methods. Indirect proofs have now been devised by McShane (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 344–379), by Myers (Duke Math. J. vol. 10 (1943) pp. 73–99) and by the author (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 93–118). Sufficient conditions for a strong relative minimum for the nonparametric case can be obtained from the results given in these papers by transforming our problem to a parametric problem. However the methods used are not applicable directly to the nonparametric case without modification. It is the purpose of the present paper to show what modifications are necessary in order to establish the corresponding sufficiency theorem for the nonparametric problem without transformation to a parametric one. (Received October 29, 1946.)

35. Dunham Jackson: *The boundedness of orthonormal polynomials on certain curves of the fourth degree.*

For the study of the boundedness of systems of orthonormal polynomials on algebraic curves, a property intimately associated with the convergence of the corresponding developments in series, various elementary devices and expedients have been found useful under one set of circumstances or another. The general problem is of such complexity that there is no reason to expect that it can be dealt with by any combination of such special devices. On the other hand, even if attention is limited to these particular procedures, any attempt at an exhaustive account of their possibilities in permutation and combination would be unmanageable. It seems worthwhile to continue at least a little further with an analysis of illustrative examples. In this spirit the writer discusses certain rational curves of the fourth degree, and also the nonrational curve $x^4+y^4=1$, previously studied from a different point of view (Bull. Amer. Math. Soc. vol. 43 (1937) pp. 388–393). While further examples could be multiplied without limit, it is the writer's impression that, in a certain qualitative sense, an enumeration of the simplest and most striking illustrations of the methods referred to is hereby concluded. (Received October 30, 1946.)

36. Herman Kober: *Approximation of non-bounded functions by integral functions of finite order.*

The introduction deals with a theorem by Carleman; it also mentions a new result concerning approximation to bounded functions. The non-bounded functions are required to be uniformly continuous of order n, that is, $\left|\sum(-1)^{j}C_{n,j}f(x+jh)\right| \leq \epsilon$ $(j=0, 1, \cdots, n)$ for $-\infty<x<\infty$, $|h| \leq \delta=\delta(\epsilon)$, or at least to be "uniformly bounded of order n." After discussing basic properties of these classes of functions, the following results are proved: If $f(z)$ is an entire function of exponential type, and if $f(x)$ is uniformly bounded of order n, then $f^{(n)}(x)$ is bounded in $(-\infty, \infty)$. A function $f(x)$, uniformly bounded of order n, is approximated by entire functions of exponential type uniformly in $-\infty<x<\infty$ if, and only if, it is uniformly continuous of order n. If $(x+i)^{-m}f(x)$ $(m=1$, or $2, \cdots)$ is uniformly continuous of some order, then there are entire functions $g_\alpha(z)$ of exponential type α such that, uniformly in $-\infty<x<\infty$, $(1+|x|^{m})\,|f(x)-g_\alpha(x)| \to 0$ as $\alpha\to\infty$. The results are applied to $f(x)=x^r$ $(0<r<\infty)$, also for approximation by entire functions of order 1/2. They are finally extended to the uniform approximation of non-bounded functions, analytic in the half-plane and satisfying certain conditions, by entire functions of exponential type. (Received October 16, 1946.)

37. C. E. Langenhop: *Analytic functions of matrices.*

A number of definitions of analytic functions of matrices have been given. In this paper square matrices are treated as numbers and the definitions of ordinary calculus with a few necessary modifications are applied to them, an approach which is essentially different from those reviewed by MacDuffee (*The theory of matrices*, Ergebnisse der Mathematik and ihrer Grenzegebiete, vol. 2, no. 5, pp. 99–101). Definitions of the terms function, derivative, and analyticity as applied to matrices are given. Second order matrices are extensively treated and necessary and sufficient conditions for one matrix to be an analytic function of another, including differential equations analogous to the Cauchy-Riemann equations, are derived. These conditions are utilized to solve the matrix differential equation $D_X f(X)=f(X)$, and the elements of the matrix function $f(X)$ are found in closed form. (Received October 17, 1946.)

38. J. D. Mancill: *Unilateral variations with variable end points.*

The author studies the properties of an arc E_{12} in order that it minimize an integral $I=\int_{t_1}^{t_2}F(x, y, x', y')dt$ in the class of all admissible curves joining a given curve D through the point 1 and a given curve C through the point 2. The entire arc E_{12} or any part of it may lie along the boundary of the region R of admissible curves. A simple derivation of the transversality condition is given which requires weaker hypotheses than the usual proof by means of the first variation and is independent of the Euler equations. The transversality condition involves an inequality at an end point which lies on the boundary of R. A set of sufficient conditions is given for each of the several cases, which permit the curves D and C to be tangent to a non-extremal arc E_{12}. The treatment given for the case when E_{12} is an extremal and the curves D and C are transversal to E_{12} applies to free variations and is much simpler than that given by Bliss (*Jacobi's criterion when both end points are variable*, Math. Ann. vol. 58 (1904) pp. 70–80). The entire results of the paper are applicable to the problem in nonparametric form. (Received October 17, 1946.)

39. A. D. Michal: *An existence and uniqueness theorem for a nonlinear differential equation in Banach spaces.*

Let B_1 and B_2 be Banach spaces, and let $T(y_1, x, y_2)$ be a trilinear function on $B_1B_2B_1$ to B_1. A global existence and uniqueness theorem is given for the differential system in Fréchet differentials $\delta y(x) = T(y(x), \delta x, y(x))$, $y(x_0) = y_0$. Under some natural restrictions on T, the solution $y(x)$ is shown to have a generalized Taylor's series expansion in successive Fréchet differentials valid for *all* $x \in B_2$. This theory is applied to "ordinary" linear differential equations in Banach spaces, to the matrizant functional, and to related topics. The situation is also studied in complex Banach spaces. (Received October 24, 1946.)

40. A. D. Michal: *Global groups of motions in some infinitely dimensional Riemannian spaces.*

The author shows, by furnishing examples, that there exist global continuous groups of motions in some infinitely dimensional Riemannian spaces with a variable (infinitely dimensional) Riemannian curvature. (Received October 24, 1946.)

41. A. D. Michal: *The solutions of systems of linear differential equations as entire analytic functionals of the coefficient functions.*

The results of a previous paper (Bull. Amer. Math. Soc. Abstract 53-1-39) are used to obtain theorems on the subject matter of the title. These investigations have important applications to the approximate solution of systems of linear differential equations with variable coefficients and stem from the generalized Taylor's series expansions for the solutions. (Received October 24, 1946.)

42. A. D. Michal, R. C. James, and Max Wyman: *Topological abelian groups with ordered norms.*

A study is made of the minimal normed linear spaces that contain such groups. A theory of differentials of functions is given. The theory is related to Fréchet differentials in normed linear spaces and to M_1 differentials in topological abelian groups. (Received October 24, 1946.)

43. W. D. Munro: *Orthogonal trigonometric sums with auxiliary conditions.*

The author considers the construction of a system of trigonometric sums, orthonormal over a period interval and subject to certain auxiliary conditions of the form $U_i(f) = 0$, the $U_i(f)$ being linear, homogeneous functionals. Conditions for convergence of a series of such sums at an arbitrary point are obtained, and an analogue of a theorem of Korous on the boundedness of orthogonal polynomials is proved. (Received October 21, 1946.)

44. M. O. Reade: *On areolar monogenic functions.*

In this paper, results similar to those derived by Haskell (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 332–337) are obtained. Some of the results are extensions of Haskell's results, and some pertain to areolar monogenic polynomials of certain form. This paper will appear in the Bulletin. (Received November 26, 1946.)

45. P. C. Rosenbloom: *Studies on the heat equation.*

New results on the uniqueness and representation of solutions of the heat equation are obtained, extending and sharpening results of Tykhonov, Widder, and Pollard. It is shown that methods which have been used in the theory of harmonic functions can be applied to the study of the heat equation. Thus, theorems analogous to the Fatou theorem and the Phragmén-Lindelöf theorem are obtained. (Received November 21, 1946.)

46. Robert Schatten: *An evaluation of the bound for certain functionals.*

The present notation is that of Bull. Math. Soc. Abstract 51-9-162. The bound on $\mathfrak{B}_1 \otimes_\gamma \mathfrak{B}_2$ of an invariant under equivalence functional $\mathfrak{F}(\sum_{i=1}^n f_i \otimes g_i)$ of expressions which satisfy the condition $|\mathfrak{F}(\sum_{i=1}^n f_i \otimes g_i + \sum_{j=1}^m h_j \otimes k_j)| \leq |\mathfrak{F}(\sum_{i=1}^n f_i \otimes g_i)| + |\mathfrak{F}(\sum_{j=1}^m h_j \otimes k_j)|$ can be expressed in terms of expressions of $f \otimes g$ of rank 1 as $\sup |\mathfrak{F}(f \otimes g)| / \|f\| \, \|g\|$ for $f \in \mathfrak{B}_1$, $g \in \mathfrak{B}_2$; $\|f\| = \|g\| = 1$. In particular if the S_i and T_i denote linear transformations on $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively, then for any crossnorm α, $\sum_{i=1}^n S_i \otimes T_i$ represents a linear transformation from $\mathfrak{B}_1 \otimes_\gamma \mathfrak{B}_2$ into $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$, with a bound equal to $\sup \alpha(\sum_{i=1}^n S_i f \otimes T_i g)/\|f\| \, \|g\|$. (Received November 20, 1946.)

47. I. J. Schoenberg: *On totally positive functions, Laplace integrals and entire functions of the Laguerre-Pólya-Schur type.*

A real-valued function $\Lambda(x)$, defined for all real x, is said to be *totally positive* if it satisfies the following three conditions: (α) $\Lambda(x)$ is measurable. (β) If $x_1 < x_2 < \cdots < x_n$ and $t_1 < t_2 < \cdots < t_n$, then the inequality $\det \|\Lambda(x_i - t_j)\| \geq 0$ should hold. This condition should be satisfied for $n = 1, 2, \cdots$. (γ) $\Lambda(x)$ should be different from zero for at least two values of x. A trivial example of such functions is $\Lambda(x) = \exp(ax+b)$. G. Pólya (*Algebraische Untersuchungen uber ganze Funktionen vom Geschlechte Null und Eins*, J. Reine Angew Math. vol. 145 (1915) pp. 224–249) has raised the question as to the possibility of representing by a bilateral Laplace integral the reciprocal $1/\Psi(s)$ of an entire function $\Psi(s)$ which is the uniform limit of a sequence of real polynomials with only real roots. It is shown here $\epsilon/\Psi(s)$, where $\epsilon = 1$ or -1 and $\Psi(s) \neq \exp(as+b)$, allows in every one of its vertical strips of regularity a representation $\int_{-\infty}^{\infty} \exp(sx)\Lambda(x)dx$, where $\Lambda(x)$ is totally positive. Conversely, a Laplace integral of this type, where $\Lambda(x) \neq \exp(ax+b)$, always converges in a vertical strip and represents there a reciprocal $1/\Psi(s)$. (Received November 18, 1946.)

48. M. H. Stone: *Pseudo-norms and partial orderings in abelian groups.*

In this paper it is shown that if an abelian group X has a pseudo-norm then it can be partially ordered and provided with a new pseudo-norm which is equivalent to the old and naturally related to the ordering introduced. The case where X is a Banach space and the case where the pseudo-norm for X is a norm are both discussed, the latter being fully characterized. (Received October 15, 1946.)

49. W. M. Stone: *On Samuelson's generalized Laplace transformation.*

P. A. Samuelson (Bull. Amer. Math. Soc. Abstract 52-3-86) pointed out that a generalized Laplace transformation, $L(s:f(t))_n=\sum_0^\infty f(n)s^{-n-1}$, yields operational methods for solution of difference equations exactly like those for differential equations. Further applications are developed in this paper. Relationships between sums and integrals which define convolution in both the generalized and ordinary sense are brought out by use of a table of function pairs such that the ordinary transform of one is equal to the generalized transform of the other. The summation of certain Taylor and Fourier series is reduced to an integration and a real inversion formula corresponding to that of Post (Trans. Amer. Math. Soc. vol. 32 (1930) pp. 723–781) is developed from the analogy with a Taylor series expansion. (Received October 21, 1946.)

50. Walter Strodt (National Research Fellow): *On the principal solution of a difference equation.* Preliminary report.

Let (1) $f[x, y(x+\omega_1), \cdots, y(x+\omega_n)]=\phi(x)$ be a difference equation, with $\phi(x)$ a given function of the complex variable x, analytic throughout a region $\mathfrak{B}$ containing the position axis, with f a given polynomial in y having coefficients analytic throughout $\mathfrak{B}$ and having no term free of y, and with $\omega_1, \cdots, \omega_n$ given complex numbers. Let $q_1, \cdots, q_n$ be complex functions of any positive number b such that $b(1-q_k)$ tends to ω_k as b becomes infinite. Let t be a non-negative integer. A *special solution* $y(x, b)$ of (2) $f[x, y(q_1(x-b)+b), \cdots, y(q_n(x-b)+b)]=(1-x/b)^t\phi(x)$ is any series of *real* powers of $x-b$ satisfying (2) in a circle with center b, cut along the right-most radius. A *principal solution* of (1) is any $y(x)$ which in some suitable subregion of $\mathfrak{B}$ is the uniform limit of an analytic continuation of $y(x, b)$, as b becomes infinite on some sequence of positive numbers. This principal solution is shown to be a broad generalization of the principal solution as defined and studied by Nörlund (*Differenzenrechnung*, 1924). It is conjectured that the principal solution becomes the totality of analytic solutions if the word "real" in the definition of special solution is replaced by "complex." (Received October 17, 1946.)

51. Otto Szász: *On closed sets of rational functions.*

Theorem 1. Let $\{z_n\}$ be a sequence of points in the half-plane $Rz>0$; a necessary and sufficient condition that either of the sequences $\{(x^2+z_n^2)^{-1}\}$, $\{x(x^2+z_n^2)^{-1}\}$ be closed in $L_2(0, \infty)$ is that $\sum Rz_n/(1+|z_n|^2)=\infty$. Theorem 2. Let $\{\zeta_n\}$ be a sequence of points in the unit circle $|z|<1$; the sequence of functions $\{(1-\zeta_n z)^{-1}\}$ is closed in H_2 if and only if $\sum(1-|z_n|)=\infty$. The method used for the proof of these theorems is elementary and essentially the same as used partly by Müntz and in a simpler and more general form by the author to solve the analogous problem for the sequence $\{x^{\lambda_n}\}$ in $L_2(0, 1)$. Closure and completeness in other spaces and for other sequences of functions are also considered. (Received November 9, 1946.)

52. Gabor Szegö: *On an inequality due to P. Turan concerning Legendre polynomials.*

P. Turan proved recently the following theorem. Let $P_n(x)$ be the nth Legendre polynomial. Then the inequality $P_{n-1}(x)P_{n+1}(x)-(P_n(x))^2\leqq 0$ holds in the interval $-1\leqq x\leqq 1$, with equality only for $x=\pm 1$. For this theorem three proofs are given based on different principles. (Received October 24, 1946.)

53. H. P. Thielman: *On continuous and discontinuous solutions of generalized Cauchy functional equations.*

One of the functional equations considered is reduced to the form $f(x+y+nxy)=f(x)+f(y)$, $n>0$, $x>-1/n$, $y>-1/n$. It is shown that every continuous solution of this equation is of the form $f(x)=k\log(1+nx)$, where k is any real number. On the basis of Zermelo's postulate that the real continuum can be well ordered, discontinuous solutions of this equation are constructed. The method used is similar to the one used by Hamel (Math. Ann. vol. 60 (1905) pp. 459–462). Some of the results obtained by Blumberg (Trans. Amer. Math. Soc. vol. 20 (1919) pp. 40–44), Sierpinski (Fund. Math. vol. 1 (1920) pp. 116–122), and Ostrowski (Comment. Math. Helv. vol. 1 (1929) pp. 157–159) for convex functions are shown to apply to the solutions of the given functional equation. In particular, every solution which is measurable is continuous. Other functional equations considered can be reduced to the following forms: $g(x+y+nxy)=g(x)\cdot g(y)$, $h(x)+h(y)+nh(x)h(y)=h(x+y)$, and $k(x)+k(y)+nk(x)k(y)=k(xy)$. (Received October 17, 1946.)

54. W. J. Thron. *Periodic iterated fractions.*

Let $w=T_n(z)$ be an arbitrary linear fractional transformation of the z-plane into the w-plane. Let $F_n(z)=T_1T_2\cdots T_n(z)$. Then $\{F_n(z)\}$ is called an "infinite iterated fraction." In this paper necessary and sufficient conditions for the convergence of periodic infinite iterated fractions ($T_{rk+m}=T_m$ for all $r\geqq 0$; $m=1, 2, \cdots, k$) are given. Further a theorem analogous to the Galois-Pringsheim theorem for periodic continued fractions is proved for iterated fractions. (Received October 25, 1946.)

55. Leonard Tornheim: *Harmonic series in two variables.*

The value of $\sum_{p,q=1}^{\infty}1/p^mq^r(p+q)^s$ in terms of $c_n=\sum_{p=1}^{\infty}1/p^n$ is found for $t=m+r+s$ odd and for t even under any of the following conditions: m, r, or s is 1; $m=r=s$; $s=0$; $r=0$ and $m=s$; or $s=r+1=m+2$. The following result is used: if $f(p)$ is a monotone function decreasing to zero, then $\sum[2f(q)/p(p+q)-f(p+q)/pq]=2f(1)$, where the sum is for pairs p, q which are positive and relatively prime. (Received November 21, 1946.)

56. W. J. Trjitzinsky: *Singular integral equations with complex-valued kernels.*

This work, to appear in the Annali di Matematica, is an extensive investigation of various types of integral equations with singular kernels, which are limits of complex-valued regular kernels, having possibly complex-valued characteristic values and real characteristic functions; use is made of a suitable spectral theory. (Received November 20, 1946.)

57. C. A. Truesdell: *On a class of differential-difference equations.*

The equation $\partial f(x, \alpha)/\partial x = G(f(x, \alpha+n), f(x, \alpha+n-1), \cdots, f(x, \alpha), x, \alpha)$ is studied. Existence and uniqueness theorems for the boundary condition $f(x_0, \alpha) = \phi(\alpha)$ are given, where $\phi(\alpha)$ is defined either at the points $\alpha = \alpha_0 + i$, $i = 0, 1, 2, \cdots$, or in a right half complex plane. It is mentioned that the earlier proof of Bruwier (Liége Memoires (3) vol. 17 (1932) pp. 1–48) is inadequate because it treats only the case $\phi = \text{const}$. Proofs are by successive approximation, generalizing the existence proof for an ordinary differential equation of first order. The existence of periodic solutions is discussed. If the function G is periodic of period 1 in α, it is shown that a solution satisfying the condition $f(x_0, \alpha) = \phi(\alpha)$ is periodic of period 1 in α if and only if $\phi(\alpha)$ is periodic of period 1. Examples show that if G is not periodic the periodicity of ϕ is not sufficient to insure the periodicity of the corresponding solution f, and if G is periodic, but ϕ is not, the corresponding solution is not necessarily periodic, so the above theorem is the strongest possible. It is shown, further, that a necessary and sufficient condition for the existence of a single periodic solution is that $G(y, y, \cdots, y, x, \alpha+1) = G(y, y, \cdots, y, x, \alpha)$. (Received November 20, 1946.)

58. Alexander Weinstein: *On the generalised Stokes-Beltrami equations.*

Let $\phi(x, y)$ and $\psi(x, y)$ be a pair of solutions of the equations (*) $y\Delta\phi + p\phi_y = 0$, $y\Delta\psi - p\psi_y = 0$, where p is a non-negative real number and ϕ and ψ can be interpreted as an axially symmetric potential and the corresponding stream function in a "space" of $n = 2 + p$ dimensions. The author establishes the obvious identity $y^{k-1}(y\Delta f + kf_y) = y\Delta g + (2-k)g_y$ where $g(x, y) = y^{k-1}f(x, y)$. This identity shows, for $k \geqq 0$, the connection between the solutions of (*) in spaces of different dimensions. This result combined with the divergence theorem for the potential ϕ yields a new approach to the theory of a class of discontinuous integrals involving Bessel functions (see A. Weinstein, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 240, 431, 823). (Received November 18, 1946.)

59. J. E. Wilkins: *The growth of the solutions of a certain linear integro-differential equation.*

The equation to be solved is of the form $d\phi/dt = -\lambda \int K(u)\phi(t-u)du$, in which the integration is extended over the interval $(0, a)$. Here a and λ are positive constants. Preliminary results on the growth of solutions $\phi(t)$ which are of finite exponential type are obtained when the kernel $K(u)$ is positive, as it usually is in the physical applications. More complete results are then obtained when $K(u)$ is subjected to hypotheses of monotonicity and convexity. (Received November 18, 1946.)

60. František Wolf: *On a decomposition of functions.*

A function $f(x)$ will be called of finite class $\mathfrak{C}$, if there exist m and n such that $\int_0^x |x-y|^{-m} f(y)dy = O(|x|^n)$. Such a function can be expressed by means of a generalized integral of the Bochner type which is (C, k) summable to $f(x)$ for large enough k's. In this sense it has a meaning to speak of functions of $\mathfrak{C}$ with a non-negative spectrum. Any function of $\mathfrak{C}$ can be expressed as the sum of two functions f_1, f_2 of $\mathfrak{C}$ of which one has a non-negative and the other a nonpositive spectrum. The decomposition is unique up to a polynomial. f_1 is the boundary function of a function of $\mathfrak{C}$ analytic in the upper half-plane, f_2 has the same property w.r.t. the lower half-plane. From here we obtain a result by T. Carleman (*L'intégrale de Fourier*, Uppsala, 1944, p. 42). (Received October 19, 1946.)

61. J. W. T. Youngs: *Remarks on the isoperimetric inequality for closed Fréchet surfaces.*

The author treats the particular isoperimetric inequality $36\pi V^2 \leq A^3$, where A is the Lebesgue area of a closed Fréchet surface S, and V is the volume enclosed by S. These terms are defined in a forthcoming paper by Radó, who establishes the inequality for every S. In the present paper, the purpose is twofold: first, to offer a variation of the definition of enclosed volume which generally increases the left side of the inequality; second, to investigate the consequences of equality, under both definitions of enclosed volume. The basic concept employed is that of the cyclic decomposition of a closed Fréchet surface S. It is shown, for example, that if $36\pi V^2 = A^3$ and $0 < A < \infty$, the cyclic decomposition of S reduces to precisely one closed surface. (Received November 20, 1946.)

APPLIED MATHEMATICS

62. N. R. Amundson: *Unsymmetrically loaded orthotropic thin plates on elastic foundations.*

The author considers orthotropic thin plates of infinite extent on elastic foundations of two different kinds. In the first case the foundation exerts a reactive pressure proportional to the deflection. The solution of the plate equation for this case, $D(u) = au_{xxxx} + 2bu_{xxyy} + cu_{yyyy} = q(x, y) - ku(x, y)$ with suitable conditions at infinity, is $u(x, y) = (4\pi^2)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\alpha, \beta)d\alpha d\beta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}q(s, t)\exp(-i[\alpha(x-s)+\beta(y-t)])dsdt$ where $q(x, y)$ is the loading function and $F(\alpha, \beta) = (a\alpha^4 + 2b\alpha^2\beta^2 + c\beta^4 + k)^{-1}$. In the second case the foundation is the homogeneous istropic semi-infinite medium of classical elasticity theory (Timoshenko, *Theory of elasticity*, p. 332). The solution of the plate equation for this case, $D(u) = q(x, y) - p_s(x, y)$, where $p_s(x, y)$ is the reactive pressure of the foundation, is obtained from an equivalent integral equation, $u = k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[q(s, t) - D(u)]rdsdt$, and is $u(x, y) = (4\pi^2)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\alpha, \beta)d\alpha d\beta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}q(s, t)\exp(-i[\alpha(x - s) + \beta(y - t)])dsdt$ where $r = [(x - s)^2 + (y - t)^2]^{-1/2}$ and $G(\alpha, \beta) = [(\alpha^2 + \beta^2)^{1/2}(2\pi k)^{-1} + a\alpha^4 + 2b\alpha^2\beta^2 + c\beta^4]^{-1}$. These solutions are obtained by the use of the double Fourier transform. The solution of the first problem can also be obtained from the finite plate as a limiting case. Various special cases are considered. This work is connected with that of Holl, Hogg, Woinowski-Krieger, Happel, Lewe, et al. (Received October 25, 1946.)

63. J. W. Beach: *Flow of slow viscous fluid between rotating cylinders.*

A solution is obtained for the biharmonic equation in the region between eccentric cylinders when the stream function and its normal derivative are given on both cylinders. The solution is set up in terms of two functions of a complex variable and a transformation of co-ordinates is made so that the bounding cylinders become concentric. The form of the two functions is determined with the single-valued parts given as infinite series. Unknown coefficients are determined and the solution obtained. The limit of this solution as the cylinders become concentric is obtained and agrees with the known solution for concentric cylinders. (Received October 23, 1946.)

64. F. E. Bortle: *Boundary problem in partially clamped rectangular plates.*

The problem of a rectangular plate with two adjacent edges clamped and the other edges simply supported is solved by the method of superposition. In order to satisfy the edge conditions, one is lead to linear infinite systems of equations in infinitely many unknowns. It is shown that a solution of the infinite systems exists. A numerical example is carried out. (Received October 24, 1946.)

65. D. G. Bourgin. *Representations of compositions.*

A representation theory is developed in terms of transforms, analogous to the Laplace transform, for the composition of certain kernels not of the closed cycle type. (Received October 26, 1946.)

66. G. W. Brown. *The stability of feedback solutions of simultaneous linear equations.*

The feedback solution of n simultaneous equations, represented by the equation $y=Ax$, where A is a square matrix of rank n with $-1\leqq A_{ij}\leqq 1$, involves n identical high gain amplifiers, whose vector input is $y-Ax$, where x is the n-vector output of the system of amplifiers. If the amplifier gain is sufficiently high at the signal frequency the steady state vector x is a good approximation to the solution of the system $y=Ax$, essentially independent of the actual gain. The stability of such a simultaneous system is equivalent to the simultaneous stability of the n orthogonal loops, each consisting of a single amplifier identical to those mentioned above, with a feedback network of characteristic λ_i, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the characteristic roots of the matrix A. Sufficient conditions are imposed on the amplifier characteristics to insure stable solution of simultaneous systems whose characteristic roots all have positive real parts. Applications to least squares solutions and universal stability are discussed. In particular, the least squares solution for m equations of rank $r\leqq m$ can be obtained with universal stability, using $m+r$ amplifiers with realizable amplifier characteristics. (Received October 21, 1946.)

67. R. E. Carr: *Enantiomorphism in mathematical models of organic molecules.* Preliminary report.

Organic molecules can be described mathematically as skeletal frameworks with n available positions $x_1, x_2, \cdots, x_n$, to each of which is attached a chemical radical (figure). The available radicals (figure supply) can be described as $\Phi^{(\lambda)}$ $(\lambda=1, 2, 3, \cdots)$ where $\Phi^{(\lambda)}$ contains α_λ atoms of type α, β_λ atoms of type β, and so on. Molecules with the same sets of atoms which differ in space arrangement are called stereoisomers. Those stereoisomers which are not superimposable on their own mirror images are called enantiomorphs. A method of enumerating stereoisomers has been published by G. Pólya (Acta Math. vol. 68 (1937) pp. 145–253). Basic in his method is definition of a permissible group of interchanges of radicals between the positions on the skeletal framework. Since reflection through a mirror sometimes changes the space arrangement of the radicals as well as their positions on the skeletal framework, the method of Pólya is not directly applicable to enumeration of enantiomorphs. In this paper new skeletal frameworks and new figure supplies are abstracted from those used in enumeration of stereoisomers, making it possible for the author to apply the method of Pólya to enumeration of the enantiomorphous alcohols, alkenes, and paraffines. (Received October 8, 1946.)

68. Nathaniel Coburn: *Application of the Kármán-Tsien relation to the two-dimensional supersonic flow of fluids in jets.*

The application of the Kármán-Tsien relation to the two-dimensional, supersonic, steady, irrotational flow of a compressible fluid in a symmetric jet issuing from two straight walls inclined at an angle θ to the x-axis and bounded by two free stream lines is considered. By mapping the flow into the hodograph plane, a boundary value problem for the stream function as a solution of the wave equation is obtained. Thus, the problem reduces to the determination of the proper regions over which the wave equation may be integrated. By use of some mapping theorems which relate the hodograph plane to the plane determined by the stream function and the velocity potential, these regions are determined and the proper solutions of the wave equation are found. These solutions are then mapped into the physical plane by use of some general results previously obtained by the author (Quarterly of Applied Mathematics vol. 3 (1945). (Received October 30, 1946.)

69. H. S. M. Coxeter: *The product of three reflections.*

The product of reflections in the sides of a spherical (or plane) triangle is reduced to a rotatory reflection (or glide-reflection) consisting of the reflection in a side of the pedal triangle combined with a rotation (or translation) along that side, of an amount equal to the perimeter of the pedal triangle. In the spherical case this is essentially the dual of a result of Synge (Quarterly of Applied Mathematics vol. 4 (1946) pp. 166–176). In the plane case it is a corollary of a result of Schwarz (Ges. Math. Abhandlungen vol. 2 (1890) pp. 344–345). The analogous problem for n reflections in n dimensions is mentioned briefly. (Received October 17, 1946.)

70. Bernard Epstein: *A method for the approximate solution of the Dirichlet problem for certain types of domains.* Preliminary report.

Let a plane domain S be divisible into two nonoverlapping domains for each of which the Green's function is known, the division being performed by adding arcs to the boundary of S. The solution to the Dirichlet problem for the domain S is expressed in each of the two sub-domains in terms of the prescribed boundary values, the (unknown) values of the solution along the added arcs, and the Green's functions. By expressing the mean-value property along the added arcs there is then obtained an integral equation of the second kind with singular kernel for the values of the solution along the added arcs. This equation can be solved by a method of successive approximations (not requiring the iteration of the kernel). The solution inside the two sub-domains may then be obtained by the usual formula involving the boundary values and the Green's function. The same underlying idea may be applied when the original domain is divided into three or more sub-domains. In particular the method appears promising in the case where S consists of the entire plane cut by a finite number of collinear slits. Several numerical examples are presented for this type of domain. (Received November 18, 1946.)

71. A. E. Heins: *The scattering and transmission properties of a pair of semi-infinite parallel plates.*

The excitation of a pair of semi-infinite parallel metallic plates of perfect conductivity and zero thickness by an electromagnetic plane wave may be formulated as a pair of simultaneous Wiener-Hopf integral equations for the surface current density. It is shown how these equations may be solved by two different methods. One method takes advantage of the symmetry of the structure and the one uses a method of matrix decomposition. Both techniques employ the Fourier transform in the complex domain (Paley and Wiener, *The Fourier transform in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 19, chap. 4). From the Fourier transforms of the surface current densities, it is possible to calculate the physically interesting parameters. (Received November 19, 1946.)

72. Herbert Jehle: *Hydrodynamical self-consistent fields in stellar dynamics.*

In order to avoid mathematically too involved problems, stellar dynamics was usually confined to investigations of steady solutions of Boltzmann-Gibbs' equation $0=\partial f/\partial t=\sum_1^3\{\partial f/\partial t+v_r\partial f/\partial x_r-(\nabla_r\overline{U})\partial f/\partial v_r\}$. Of particular interest are two extensions of this problem, (1) to include time-dependent solutions with given $\overline{U}$, and (2) to investigate self-consistency, that is, a combination of the above hydrodynamical equation with the Poisson equation between $\overline{U}$ and the smoothed-out density $\rho=\iiint f dv_1 dv_2 dv_3$, a nonlinear problem. These problems, however, still admit reversibility in time. It is only when the statistical shaking effect of force fluctuations is introduced (causing a nonvanishing $\partial f/\partial t$ and thereby a slow diffusion superimposed on the smooth streaming in phase space) that we can expect to get the f distribution tending to develop in one direction. The case $\overline{U}=0$ is exemplified in the kinetic theory of gases by the approach towards a Maxwell-Boltzmann distribution, whereas the case $\overline{U}\neq 0$ concerns the Brownian movement in a field of force. The time-dependent and the self-consistent transient case provides some particularly interesting problems as it establishes a trend of evolution for spatial distributions $\rho(x_1, x_2, x_3, t)$. (Received November 21, 1946.)

73. Cornelius Lanczos and G. E. Forsythe: *Approximation of the eigen-vectors of a nonsymmetric matrix.* Preliminary report.

The eigen-values λ_k, the eigen-vectors u_k of a given square matrix $A=\|a_i^j\|$ $(i, j=1, 2, \cdots, n$; a_i^j complex numbers), and the eigen-vectors u_k' of the transposed matrix $A'=\|a_j^i\|$ are defined by the relations $Au_k=\lambda_k u_k$, $A'u_k'=\lambda_k u_k'$ $(k=1, 2, \cdots, n)$. Suppose $|\lambda_1|>|\lambda_2|>\cdots>|\lambda_n|$. In a certain type of flutter analysis it is necessary to approximate λ_1, λ_2, λ_3 and u_1, u_2, u_3. In the usual method (see Frazer, Duncan and Collar, *Elementary matrices*, Cambridge, 1938, pp. 140–145) λ_1, u_1, u_1' are approximated by a pair of iterative processes of form $A^m x$, $(A')^m x$. A is then reduced to a new matrix of order $n-1$, and the entire procedure is repeated for each new λ_k, u_k $(k=2, 3)$. The present investigation shows how to approximate λ_2, u_2, λ_3, u_3 from the same iterations which yield λ_1, u_1, u_1'. The process is largely self-checking, and trials indicate it to be more accurate in several respects than the usual procedure. For $5\leq n\leq 8$ the elimination of matrix reductions and iterations results in a saving of from 45 to 70 per cent of the usual lengthy computation time. A separate procedure, based on the moments $\sum_k\lambda_k^m$, is useful when only λ_1, λ_2, λ_3 are desired. The latter procedure, when used to approximate the complex roots of an algebraic equation, has advantages over the customary method of Graeffe. (Received October 18, 1946.)

74. Kenneth May: *Technological change as a functional variation.*

Innovations, or more generally any qualitative changes in the productive process, may be represented as modifications in the production functions relating output to input of factors. Innovation is introduced in this manner into a static macroeconomic model and results derived in terms of the character of the functional increment. The dynamic implications of technological change are studied by introducing into Keynes' equation of effective demand a production function varying with time. The resulting differential equation provides a simplified model of the trade cycle in terms of employment, profits, wages, capital accumulation and innovation. (Received October 19, 1946.)

75. Kenneth May: *The aggregate effect of technological changes in a two-industry model.*

It has been shown that a general economic equilibrium system of one degree of freedom determines a one-industry model, $U=\psi(N)$ and $p=\psi'(N)$, where N, U and p are suitable aggregates of employment, output, and the real wage rate (Econometrica vol. 14 (1946) pp. 285–298). The function ψ depends on the functions of the general equilibrium. If the latter is Evans' two-industry system, which involves the production functions $\phi(U_1^{(1)}, N_1)$ and $\theta(U_1^{(2)}, N_2)$, $\psi=\theta(U_1^{(2)}, N_2)$ where $U_1^{(2)}$ and N_2 are replaced by their values in terms of N found by solving the equations of the two-industry model. It is shown that if technological changes occur in the two industries, so that ϕ and θ are subjected to small functional changes $\delta\phi$ and $\delta\theta$, then the resulting aggregate technological change is given by $\delta\psi(N)=\delta\theta[U_1^{(2)}(N), N_2(N)]+\theta_1[U_1^{(2)}(N), N_2(N)]\delta\phi[U_1^{(1)}(N), N_1(N)]$. The method is applicable to studying the effect upon any macro-economic model of changes in the functions involved in a micro-economic model from which it can be derived. (Received November 18, 1946.)

76. W. B. Stiles. *Bending of clamped plates.*

A method of obtaining approximate solutions for small deflections of thin elastic plates with clamped edges, based on the minimization of energy, is presented in this paper. This solution was proposed by Weinstein and applied to a square plate with a uniform load by Weinstein and Rock. The method is extended here to include rectangular plates with either uniform or point loads and also to plates with mixed boundary conditions in which part of the boundary is clamped and the remainder is pinned. The approximation functions are obtained from the characteristic functions of static and vibrating membranes. Analytical results are compared with experimentally determined deflections and stresses. (Received October 21, 1946.)

77. J. L. Synge: *The general problem of antenna radiation.*

An antenna is regarded as a closed surface, some of this surface being a perfect conductor ($E_{\text{tang}}=0$) and the rest being a "gap" (E_{tang} assigned). A Maxwellian field with time factor exp $(-ikct)$ is assumed to exist outside the antenna, and to satisfy at infinity the usual conditions of outward radiation. Application of the method of Helmholtz gives an integro-differential equation for H_{tang} on the surface of the antenna. In the case of an antenna of revolution, considerable simplification takes place, and a comparatively simple *exact* integro-differential equation for the current in the antenna is obtained. In seeking approximate solutions, two parameters are involved,

η/a (the ratio of the length of the gap to the diameter of the antenna) and η/l (the ratio of the length of the gap to the length of the antenna). The classical sinusoidal current distribution is obtained in the limiting case where η/a is large and η/l is small. A general method of successive approximations is set up, but no proof of convergence is given. (Received October 31, 1946.)

78. Gabor Szegö: *The capacity of a circular plate-condenser.*

Consider two thin circular discs of radius a with a common axis and at a distance d, $d/a = q$, charged to constant and opposite potentials, $V = \pm V_1$. If the charges are $\pm Q$, respectively, the constant $C = Q/V_1$ is called the capacity of the condenser. G. Kirchoff (*Gesammelte Abhandlungen*, p. 112) gave the following approximate formula for this important quantity: $a^{-1}C = (4q)^{-1} + (4\pi)^{-1} \log (1/q) + \alpha(q)$, $\limsup \alpha(q) \leq (4\pi)^{-1} \cdot (\log (16\pi) - 1) = K$ as $q \to 0$. Recently (Acad. des Sciences l'URSS, 1932) Ignatowsky gave the following sharper result: $\lim \alpha(q) = (4\pi)^{-1}(\log 8 - 1/2) = I$. The proofs are in both cases somewhat incomplete. In the present paper Kirchoff's proof is revised by using Dirchlet's principle. Moreover by means of the so-called Thomson principle a very simple proof is given for $\liminf \alpha(q) \geq I$. Finally the case of thick plates is discussed. (Received November 23, 1946.)

79. H. L. Turrittin: *Stokes multipliers for asymptotic solutions of a certain differential equation.*

If ν is a positive integer, the differential equation $d^ny/dx^n - x^\nu y = 0$, $n \geq 2$, has n independent solutions $y_j = x^j(1 + a_{1j}x^p + a_{2j}x^{2p} + \cdots + a_{mj}x^{mp} + \cdots)$, $p = \nu + n$, convergent for all x. If the complex x-plane, $x = re^{i\theta}$, is divided into $2p$ sectors by the radial lines $\theta = k\pi/p$, $k = 0, 1, \cdots$, Trjitzinsky (Acta Math. (1934) pp. 167–226) has shown that to each sector there corresponds n independent solutions $\tilde{y}_k \sim \xi_k^{(1-n)/2p} \exp \xi_k\{1 + b_1/\xi_k + b_2/\xi_k^2 + \cdots\}$ where $\xi_k = (n/p)x^{p/n}e^{2\pi ki/n}$. These asymptotic representations are valid *uniformly* throughout the sector (edges included). Therefore there exists a nonsingular linear relationship $y_j = \sum_{k=0}^{n-1} c_{jk}\tilde{y}_k$, $j = 0, 1, \cdots, n-1$. These constants c_{jk}, which change from sector to sector, are the *Stokes multipliers* that have been computed. To do so the author borrowed heavily from the Ford-Newsom-Hughes theory of asymptotic expansion (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 456–461). However this theory does not yield directly the desired uniform asymptotic representation in all cases, nor even the desired form when the real part of ξ_k is negative. The F–N–H theory is extended to supply the requisite information. Scheffé (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 127–154) computed two of the n multipliers corresponding to each j. (Received October 7, 1946.)

GEOMETRY

80. L. M. Blumenthal: *Superposability in elliptic space.* II.

Let f denote a one-to-one correspondence between the points of two subsets P, Q of the elliptic space $E_{n,r}$. Two corresponding subsets A_P, B_Q of P, Q, respectively, are called f-superposable provided there exists a congruence Γ of $E_{n,r}$ with itself which gives the same correspondence between A_P and B_Q as f does. The writer defines a space to have superposability order σ provided any two subsets of the space are superposable whenever a one-to-one correspondence f between the points of the subsets exists such that each two corresponding σ-tuples are f-superposable. A principal result of this paper is that $E_{n,r}$ has minimum superposability order $n+1$. Two subsets of $E_{n,n}$ in a one-to-one correspondence f, which are not f-superposable though each two corresponding n-tuples are f-superposable are called pseudo f-superposable sets of degree n. A metric characterization of such sets of $n+1$ points is given. Whether pseudo f-superposable sets of degree n necessarily consist of exactly $n+1$ points has not yet been determined for $n>2$. (Received October 24, 1946.)

81. S. C. Chang: *Contributions to projective theory of singular points of space curves.*

In this paper a covariant coordinate system for a space curve at an inflection point is described in terms of configurations depending on neighborhoods of order no greater than eight. The principal configurations used are the quadric surface having eighth order contact with the curve at the inflection point and the family of uniplanar cubic surfaces having second order contact with the quadric and eighth order contact with the curve. It is shown that the uniplanar points of all the cubic surfaces lie on the quadric. (Received November 29, 1946.)

82. P. B. Johnson: *A contribution to parallel displacement theory in Riemannian space.*

The equations of Levi-Civita parallel displacement are integrated by means of the matrizant functional. The results in A. D. Michal, *An existence and uniqueness theorem for a nonlinear differential equation in Banach spaces* (Bull. Amer. Math. Soc. Abstract 53-1-39) are applied. The displaced vector is given as a series of terms linear in the undisplaced vector and of increasing order in the components of the directrix. The directrix is the curve along which the vector is displaced with components $x^i = x^i(t)$. The results are applied to the special case of the closed directrix. The change in the displaced vector caused by shifting the directrix is discussed. It is shown that the 1923 result of Paul Dienes giving under certain conditions the first approximation to the change in the displaced vector can be written in matrix form $\delta' V_1 = \int_0^1 \Omega_t^1 R_{ab}(dx^a/dt)\zeta^b \Omega_0^t dt V_0$ (where R_{ab} is the Riemann tensor, Ω_t^s is the matrizant functional, ζ^b is the change in the directrix) and is indeed the first Fréchet differential of V with increment ζ^b. The higher order Fréchet differentials with increments ζ^b are discussed. The results are specialized for closed directrices. (Received October 24, 1946.)

83. Edward Kasner and John DeCicco: *A partial differential equation of fourth order connected with rational functions of a complex variable.*

A certain partial differential equation of fourth order is obtained of which any polynomial solution $\phi(x, y)$ represents the numerator of the real part (when expressed as the quotient of two relatively prime polynomials) of a rational function of a complex variable. This equation states that $\log[\phi(\phi_{xx}+\phi_{yy})-(\phi_x+\phi_y)]$ is a harmonic function of (x, y). This new class of rational fractional potential polynomials contains the class of harmonic polynomials as a proper subset. The totality of analytic solutions of this partial differential equation of fourth order consists of the real parts of the special class of polygenic functions defined as the product of an analytic function of $z = x+iy$ by another independent analytic function of the conjugate complex variable $\bar{z} = x-iy$. For this class of polygenic functions, the analogues of the Cauchy-Riemann equations are discussed. (Received October 23, 1946.)

84. Edward Kasner and John DeCicco: *Rational harmonic curves.*

A rational harmonic curve is defined by setting the real part of a rational function of $x+iy$, of degree r, equal to zero. Any such curve is given by $P(x, y)=0$, where P is a polynomial of (x, y) of degree $2r-k$, where $0 \leqq k \leqq r$. The curve has k real asymptotes, all of which pass through a fixed point and which make equal angles with each other. The angle between consecutive asymptotes is π/k. The remaining asymptotes are minimal and $2(r-k)$ in number. The theorem generalizes to rational harmonic curves a theorem of Briot and Bouquet concerning the asymptotes of polynomial harmonic curves, which are defined by setting the real part of any rational integral function of a complex variable equal to zero. Other properties are given by means of the foci of systems of confocal curves. Additional results are found about satellites. (Received October 4, 1946.)

85. C. E. Springer: *Union torsion of a curve on a surface.*

The geodesic torsion at a point of a curve on a surface is the torsion of the geodesic which is tangent to the curve at the point. In this paper the union torsion at a point of a curve C on a surface is defined as the torsion of the union curve in the direction of the curve C, the union curve being defined relative to a given rectilinear congruence. The union torsion is given by a formula which reduces to the expression for geodesic torsion in case the congruence is normal to the surface. It is shown that a union curve is a plane curve if, and only if, it is tangent to a curve of intersection of a developable of the congruence with the surface. (Received October 3, 1946.)

86. A. E. Taylor: *A geometric theorem and its application to biorthogonal systems.*

Let S be a bounded and closed point set in E_n (Euclidean space of n dimensions). Let O be a point such that O and S together are not contained in any subspace of $n-1$ dimensions (such a subspace is hereafter called a plane). Then there exist n linearly independent vectors $x_1, \cdots, x_n$ emanating from O, with terminal points $P_1, \cdots, P_n$ in S, and n planes $p_1, \cdots, p_n$ satisfying the following conditions: (a) p_i contains P_i; (b) p_i is parallel to the plane determined by $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$; (c) S and O both lie in the same one of the two closed half-spaces into which p_i divides E_n. This theorem is proved, and then applied to demonstrate the following: If Y_n is an n-dimensional subspace of a normed linear space X, there exist n elements $x_1, \cdots, x_n$ of unit norm in Y_n and n linear functionals $f_1, \cdots, f_n$ of unit norm defined on X, such that $f_i(x_j)=\delta_{ij}$. (Received October 25, 1946.)

STATISTICS AND PROBABILITY

87. G. E. Forsythe: *On Nörlund summability of random variables to zero.*

The present paper is an incomplete extension to regular Nörlund summability methods of some previous results (G. E. Forsythe, Duke Math. J. vol. 10 (1943) pp. 397–428, §5) on Cesàro summability in probability of random variables to zero. Corresponding to a sequence $p_0\ (=1), p_1, p_2, \cdots$ of non-negative constants, the Nörlund method N_p is defined by the triangular Toeplitz matrix $\|a_{nk}\|$, where $a_{nk} = p_{n-k}(\sum_{k=0}^{n} p_k)^{-1}$. It is conjectured that if $N_p \subset N_q$ with respect to the summability of sequences of real numbers, then $N_p \subset N_q$ (meaning that $N_p \subset N_q$ with respect to the summability in probability to zero of normal families $\{x_k\}$ of independent, real-valued, symmetric random variables). In the present paper the conjecture is proved for only two special cases: (i) when N_p is the Cesàro method C_1; (ii) when N_q is C_1 and when the $\{p_k\}$ defining N_p are nonincreasing. If in case (i) the $\{q_k\}$ defining N_q are nondecreasing it is proved that $C_1 = N_q$ (meaning that $C_1 \subset N_q$ and $N_q \subset C_1$). Sufficient conditions are given for $N_p \subset N_q$ and for $N_p = N_q$. A regular Nörlund method N_p is exhibited for which $N_p \subset C_1$ (and $C_1 \not\subset N_p$) with respect to sequences of real numbers, but for which $N_p = C_1$. Hence the converse of the above conjecture is false. (Received November 23, 1946.)

88. E. J. Gumbel: *The asymptotic distribution of the range.*

Consider a large sample taken from an initial symmetrical unlimited distribution $\phi(x)$ for which all moments exist. Then the joint distribution of the largest and of the smallest values x_n and x_1 splits into the product of the asymptotic distributions of the largest value $f_n(x_n)$, and of the smallest value $f_1(x_1)$. Let u_n be the most probable largest value, and put $\alpha_n = n\phi(u_n)$. Then the asymptotic distribution $\Psi'(R)$ of the reduced range $R = \alpha_n(x_n - x_1 - 2u_n)$ obtained from the convolution of the asymptotic distributions of the two extremes is $\Psi'(R) = e^{-R}\int_{-\infty}^{+\infty} \exp\left[-e^{y} - e^{-y-R}\right]dy$. The numerical values of the cumulative probability $\Psi(R) = \int_{-\infty}^{+\infty} \exp\left[y - e^{y} - e^{-y-R}\right]dy$ and of the distribution $\Psi'(R)$ may also be calculated from the differential equation $\Psi'' + \Psi' - e^{-R}\Psi = 0$. The distribution $g(w)$ of the range $w = x_n - x_1$ itself is obtained from $\Psi'(R)$ by the usual linear transformation. The asymptotic probabilities and the asymptotic distributions of the mth range and of the range for assymetrical distributions are obtained by the same method and lead to similar integrals which may also be evaluated by numerical methods. (Received November 8, 1946.)

89. A. M. Mood: *On Hotelling's weighing problem.*

The paper contains some solutions of the weighing problem proposed by Hotelling (Ann. Math. Statist. vol. 15 (1944) pp. 297–306). The experimental designs found are essentially two-level multifactorial designs for determining main effects. They are particularly applicable to the problem of measuring several objects. The chemical balance problem (in which objects may be placed in either of the two pans of the balance) is almost completely solved by means of designs constructed from Hadamard matrices. Designs are provided both for a balance which has a bias and for one which has no bias. The spring balance problem (in which objects may be placed in only one pan) is completely solved when the balance is biased. For an unbiased spring balance, designs are given for small numbers of objects and weighing operations. Also the most efficient designs are found for the unbiased spring balance, but it is shown that in some cases these cannot be used unless the number of weighings is as large as the binomial coefficient $C_{p,p/2}$ or $C_{p,(p+1)/2}$ where p is the number of objects. It is found that when p objects are weighed in $N \geq p$ weighings, the variances of the estimates of the weights are of the order of σ^2/N in the chemical balance case (σ^2 is the variance of a single weighing), and of the order of $4\sigma^2/N$ in the spring balance case. (Received October 23, 1946.)

90. Isaac Opatowski: *Simple Markoff chains with reverse transitions: the time moments.*

Consider a chain of transitions $(i \rightarrow i+1)$, $(i+1 \rightarrow i)$ $(i = 0, 1, \cdots, n-1)$ between the states $0, 1, \cdots, n$. Let the usual conditional probabilities of these transitions within any time Δt be respectively $k_{i+1}\Delta t+o(\Delta t)$ and $g_i\Delta t+o(\Delta t)$. Let the probability of any other transition during Δt be $o(\Delta t)$. Let k_i's and g_i's be constant. Let $P_i(t)$ be the probability of the existence of the state i at the time t if the state 0 existed at the time $t=0$. The paper gives $\int_0^\infty t^m P_i(t)dt$ or $\int_0^\infty t^m P_i'(t)dt$ as algebraic expressions of the k_i's and g_i's. The result is obtained by using the complete homogeneous symmetric functions of the poles of the Laplace transform of $P_i(t)$ or $P_i'(t)$. (Received November 20, 1946.)

91. Gerhard Tintner: *The statistical estimation of the dimensionality of a given set of observations.*

There is a set of N observations of p variables $X_{it}=M_{it}+y_{it}$, where M_{it} is the systematic part and y_{it} the random element $(i=1, 2, \cdots, p; t=1, 2, \cdots, N)$. The y_{it} are normally and independently distributed with means zero. There are in the population R linear independent relationships of the form: $k_{s0}+\sum_i k_{si}M_{it}=0$ $(s=1, 2 \cdots R)$. A method based upon results of R. A. Fisher (Annals of Eugenics, 1938) and P. L. Hsu (ibid. 1941) assumes that we have a large sample estimate of the covariance matrix of the y_{it}: $\|V_{ij}\|$. The determinantal equation is $|a_{ij}-\lambda V_{ij}|=0$, where the a_{ij} are the sample covariances of the X_{it}. λ_1 is the smallest root of the equation, λ_2 the next smallest, and so on. $\Lambda_r=(N-1)\sum_1^r\lambda_i$. These sums of squares are distributed like χ^2 with $r(N-p-1+r)$ degrees of freedom and may be used to estimate the number R of linear relations between the M_{it}. By inserting the R smallest roots into the determinantal equation matrices are found for the computation of the coefficients k_{si} (G. Tintner, Ann. of Math. Statist., 1945). (Received October 15, 1946.)

92. Jacob Wolfowitz: *On the efficiency of unbiased sequential estimates.*

Let $f(x, \theta)$ be the distribution function (density function or probability function) of a chance variable X, which depends upon the parameter $\theta=\theta_1, \cdots, \theta_l$. Let n successive independent observations be made on X, where n is itself a chance variable, and the decision to terminate the drawing of observations depends upon the observations already obtained. Let $\theta_1^*(x_1, \cdots, x_n), \cdots, \theta_l^*(x_1, \cdots, x_n)$ be joint unbiased estimates of $\theta_1, \cdots, \theta_l$. Let $\|\lambda_{ij}\|$ be the nonsingular matrix of their covariances, and $\|\lambda^{ij}\|$ its inverse. Under certain regularity conditions it is proved that the concentration ellipsoid $\sum\lambda^{ij}(k_i-\theta_i)(k_j-\theta_j)=l+2$ always contains within itself the ellipsoid $\sum\mu_{ij}(k_i-\theta_i)(k_j-\theta_j)=l+2$, where $\mu_{ij}=EnE((\partial \log f/\partial\theta_i)(\partial \log f/\partial\theta_j))$. (Received October 21, 1946.)

TOPOLOGY

93. R. F. Arens: *Pseudo-normed algebras.*

A pseudo-norm defined on a linear algebra A over, for example, the reals, is a real-valued function having the formal properties of a norm in a normed ring except that the pseudo-norm of some nonzero elements may vanish. Consequently, a pseudo-normed algebra is required by definition to have a complete system of pseudo-norms. The continuity of inversion is considered; and the singular elements related to the closed divisorless proper ideals. The space of the latter, and conditions for its compactness, are considered. As a by-product a characterization of rings of all continuous complex-valued functions on a topological space is obtained. Special attention is given to the question of completeness of quotient rings. (Received November 19, 1946.)

94. R. H. Bing: *Solution of a problem proposed by R. L. Wilder.*

It is shown that there exists a locally connected set which is not a simple closed curve but which is the sum of two sets M_1 and M_2 such that each is irreducibly connected from the point A to the point B and such that the common part of M_1 and M_2 is A and B. This answers a question posed by R. L. Wilder (*Concerning simple continuous curves and related point sets*, Amer. J. Math. vol. 53 (1931) pp. 30–55.) (Received October 17, 1946.)

95. R. H. Bing: *Some characterizations of a simple closed curve.*

We say that R cuts A from B in M if R contains neither A nor B, M contains a continuum intersecting both A and B, and each such continuum contains a point of R. If R cuts two points from each other in M, it cuts M. It is shown that a locally peripherally compact nondegenerate continuum is a simple closed curve if it is cut by no one but by each pair of its points. Furthermore, a locally peripherally compact nondegenerate continuum is a simple closed curve provided it is neither cut by any point nor separated by any of its subcontinua. (Received October 1, 1946.)

96. Claude Chevalley and Samuel Eilenberg: *Cohomology theory of Lie groups and Lie algebras.*

Given a Lie algebra L and a representation P of L with representation space V, cohomology groups $H^q(L, P)$ are defined. If L is the Lie algebra of a compact connected Lie group G, V is the field of real numbers and P is the trivial representation, these cohomology groups give the topological cohomology groups of the manifold of G. For semi-simple Lie algebras nothing is lost by limiting the definition of trivial representations. The second cohomology group is related to extensions to Lie algebras; this study yields Levi's theorem. Relative cohomology groups are constructed whenever a subalgebra L' of L is given; with the aid of this definition the main result is extended to homogenous spaces. (Received November 20, 1946.)

97. Samuel Eilenberg: *Simplicial products of simplicial complexes.*

Given two geometric simplicial complexes K_1 and K_2 the simplicial product $K_1\Delta K_2$ is a simplicial complex defined as follows. The vertices of $K_1\Delta K_2$ are pairs (v, w) where v is any vertex of K_1 and w any vertex of K_2. The distinct vertices $(v_0, w_0), \cdots, (v_n, w_n)$ of $K_1\Delta K_2$ determine an n-simplex of $K_1\Delta K_2$ provided $v_0, \cdots, v_n$ are in a simplex of K_1 and $w_0, \cdots, w_n$ are in a simplex of K_2. The simplicial product has the same invariants as the cartesian product $K_1\times K_2$ (which is not a simplicial complex) and therefore can be used to simplify various formalisms in homology theory. (Received November 20, 1946.)

98. C. J. Everett: *Representation of a sequence of finite sets.*

A sequence of finite subsets T_i of a set S has a set of distinct representatives $t_i \in T_i$, if and only if, for every $n=1, 2, \cdots$, the union of every n of the T_i contains at least n distinct elements of S. This modification of Theorem 1 of P. Hall (*On representation of subsets*, J. London Math. Soc. vol. 10 (1935) pp. 26–30) is obtained by use of the Cantor theorem on closed compact sets in a suitably metrized space of sequences $\{t_i\}$, $t_i \in T_i$. It is then shown that Hall's subsequent theorems extend to the infinite case. The existence of a common representation of left and right cosets of a group modulo a finite subgroup of countable index is obtained as a corollary. (Received November 1, 1946.)

99. Edwin Hewitt: *Rings of unbounded continuous functions.*

The set $C(X, R)$ of all continuous functions from a completely regular space X to the real number system R forms a linear space which is also a commutative ring with unit, under the usual definitions of addition and multiplication. For $f \in C(X, R)$, let $Z(f)$ be the set of points where f vanishes. Let $\mathfrak{Z}(X)$, for any completely regular X, be the family of all sets $Z(f)$, for $f \in C(X, R)$. A subfamily $\mathcal{A}$ of $\mathfrak{Z}(X)$ not containing 0, containing finite intersections of its elements, and containing arbitrary supersets (in $\mathfrak{Z}(X)$) of its elements is said to be maximal if no proper superfamily (in $\mathfrak{Z}(X)$) enjoys these properties. A completely regular space X is said to be a Q-space if every maximal family in $\mathfrak{Z}(X)$ has total intersection nonvoid or contains a countable subfamily with void intersection. It is proved that for every completely regular space X, there exists a Q-space αX such that $C(\alpha X, R) = C(X, R)$, $X^- = \alpha X$. Furthermore, if X and Y are Q-spaces such that $C(X, R) = C(Y, R)$, then X is homeomorphic to Y. The ring $C(X, R)$ determines the space X if X is a Q-space satisfying the first axiom of countability. In particular, $C(X, R)$ determines X if X is any separable metric space. (Received November 13, 1946.)

100. F. B. Jones: *Concerning non-aposyndetic continua.*

If a continuum M contains a point P such that $M-P$ is not strongly (continuumwise) connected, P is said to be a (weak) cut point of M. It is shown that every compact metric continuum, which is non-aposyndetic at each of its points, contains a (weak) cut point. Indecomposable continua and related types are characterized by their non-aposyndetic properties. A scheme of classification of continua (which are not locally connected) is introduced and partially investigated. (Received October 25, 1946.)

101. E. E. Moise: *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua.*

Using the existence of the continuum described, the author solves a problem of M. Mazurkiewicz, proposed in Fund. Math. vol. 2 (1921) p. 285. (Received November 18, 1946.)

102. W. T. Puckett: *On a problem in connected finite closure algebras.*

In answer to a question left open by J. C. C. McKinsey and Alfred Tarski (Ann. of Math. vol. 45 (1944) p. 160) it is shown that the closure algebra over Euclidean space is a universal algebra for the class of all connected finite closure algebras. The problem is reduced to demonstrating the existence of certain interior transformations by the following results: (1) A connected finite closure algebra is isomorphic with the closure algebra over a connected finite topological space. (2) In order that the closure algebra over a topological space S_1 be isomorphic with a subalgebra of the algebra over a space S, it is sufficient that there exist an interior transformation $f(S) = S_1$. (Received October 23, 1946.)

103. M. H. Stone: *Remarks on metrizability.*

It is proved that a continuous family (that is, an upper semi-continuous collection) of mutually disjoint nonvoid compact subsets of a metric space S is, under its weak topology, *metrizable* (being in addition *separable* whenever S is). From this result and from theorems contained in a paper of the author (Trans. Amer. Math. Soc. vol. 41

(1937) pp. 375–481), it follows immediately that every separable regular space is metrizable (as is, of course, well known). (Received October 12, 1946.)

104. Sister Petronia Van Straten: *Toroidal and non-toroidal graphs.*

While the 10 point-30 arc Desargues configuration is irreducibly non-toroidal (for the definition of the Desargues and Pappus configurations, see Bull. Amer. Math. Soc. Abstract 52-9-345), the 9 point-27 arc Pappus configuration is saturated toroidal. By this we mean that while the configuration can be topologically embedded into the torus, it loses this property if we add any arc joining two points which are not joined in the Pappus configuration. This result is a consequence of the following lemma. Any subset of the torus which is homeomorphic to a Pappus configuration decomposes the torus into triangular regions. (Received November 12, 1946.)

105. G. S. Young: *On compact fiberings of the plane.*

Montgomery and Samelson have conjectured (Duke Math. J. vol. 13 (1946) pp. 49–57) that there exists no compact fibering of Euclidean n-space. This note shows that the conjecture is true for two dimensions by proving the more general theorem: If an interior transformation of the plane, $f(P)=A$, is such that the inverses of each two points of A are homeomorphic, and such that each component of the inverse of a point of A is compact, then f is light. If the entire inverse of a point of A is compact, then f is a homeomorphism. (Received November 20, 1946.)

106. G. S. Young: *On converging sequences of 2-manifolds.*

A new characterization of a 2-manifold (with or without boundary) is given: Let M be a locally compact, locally connected metric space with no local cut points, and such that there is some positive number ϵ such that every simple closed curve of diameter less than ϵ separates M. Then M is a 2-manifold. This result is used to prove that if $\{M_n\}$ is a sequence of compact 2-manifolds converging 1-regularly to a compact set M and either for each n the boundary of M_n is empty or the boundaries of the sets M_n converge 0-regularly, then M is homeomorphic to almost all M_n. The manifold characterization is based on Bing's characterization of a sphere (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 644–653). The result on convergence generalizes, and is based on, results of Begle (Duke Math. J. vol. 11 (1944) pp. 441–450) and of Whyburn (Fund. Math. vol. 25 (1935) pp. 408–426). (Received November 20, 1946.)

IRREDUCIBLE REPRESENTATIONS OF OPERATOR ALGEBRAS

I. E. SEGAL

1. **Introduction.** In this paper we investigate the representation on (not necessarily separable) Hilbert spaces of a suitably closed algebra of operators on a Hilbert space, with regard to the existence of irreducible representations and the connection between representations and states of the algebra (we use the term "state" to mean a complex-valued linear function on the algebra which is real and non-negative on positive semi-definite operators, and which satisfies a certain normalizing condition—more commonly, this is called an expectation value in a state). Our results are, first, that there is a mutual correspondence between "normal" representations (those for which there is an element of the Hilbert space whose transforms under the representation operators span the space) and states of the algebra, in which irreducible representations correspond to pure states (a state is pure if it is not a linear combination with positive coefficients of two other states), and second, that there exists a complete set of irreducible representations for a given algebra (or alternatively, a complete set of pure states).

The operator algebras which we consider are assumed to be: (1) closed in the uniform topology (in which the distance between two operators is defined as the bound of their difference), (2) self-adjoint (that is, if an operator is in the algebra then so is its adjoint); we call such an algebra a C^*-algebra. By a representation of a C^*-algebra $\mathcal{A}$ we mean a function ϕ on $\mathcal{A}$ to the collection of bounded operators on a Hilbert space which is an algebraic homomorphism, continuous, and such that for $U \in \mathcal{A}$, $\phi(U^*) = (\phi(U))^*$, the asterisks denoting adjoints. The representation is called irreducible if the Hilbert space on which $\mathcal{A}$ is represented contains no nontrivial closed invariant subspace. A collection of representations of $\mathcal{A}$ is called complete if the only element of $\mathcal{A}$ which is mapped into zero by every representation is zero.

Our results make possible a broader and more rigorous treatment of certain parts of quantum mechanics, particularly of the principle that the spectral values of an observable are determined by the behavior of the observable in the irreducible representations of the operator algebra describing the physical system in question. The original mathematical model for the observables in quantum theory was the

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class of all self-adjoint elements of the algebra $\mathcal{C}$ of all bounded operators on a Hilbert space (frequently unbounded operators were included, but these could be treated in terms of bounded operators, and did not affect the states or representations of the system of observables). For a variety of reasons, however, it seems desirable to consider the collection of self-adjoint elements in less restricted algebras of operators as a possible model for the observables, the most notable reason being: (a) the lack of a compelling physical reason why every self-adjoint operator should correspond to an observable, (b) the serious mathematical difficulties in quantum electrodynamics, (c) in a postulational treatment of quantum mechanics which we hope to present elsewhere, the collection of self-adjoint elements of a C^*-algebra satisfies our axioms for the system of observables. The concept of pure state was introduced by von Neumann and Weyl (independently) in a quite general fashion and in the case of the algebra $\mathcal{C}$ just mentioned it was shown that an element of the Hilbert space of norm unity (a wave function) gave rise to a pure state. In more general systems the question of the existence of pure states was left open, but we should mention that our proof of this existence is based on a theorem of Krein, Milman, and Shmulyan [4][1] which was proved fairly recently.

Our interest in operator algebras grew out of our investigations of group algebras of locally compact groups, a group algebra being isomorphic with a certain self-adjoint operator algebra. Applying our theorem in the case of the latter algebra, we obtain as a corollary a recent result of Gelfand and Rykov, to the effect that an arbitrary locally compact group has a complete set of strongly continuous irreducible unitary representations (which we show can be assumed to be derived from representations of the C^*-algebra which is the uniform closure of that algebra). Our procedure here is somewhat similar to that much used in the theory of finite groups and also in [6], in which it is first shown that an algebra built on the group is, in some sense, semi-simple, and therefore has a complete set of irreducible representations, from which the existence of the desired group representations may be concluded.

Our proofs make use of devices previously employed by Gelfand and Neumark [2], Gelfand and Rykov [3], and by ourselves [6]. Like Gelfand and Rykov we utilize the theorem of Krein, Milman, and Shmulyan mentioned above; unlike them we make no use of positive definite functions, in deriving the group-theoretical corollary mentioned in the preceding paragraph.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

We are indebted to W. Ambrose for suggestions which resulted in a significant extensions of our theory and in important simplifications in the proofs.

2. **Definitions.** Let $\mathcal{A}$ be a C^*-algebra, by which we mean a uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space. In other words, if U_n $(n=1, 2, \cdots)$, V, and W are elements of $\mathcal{A}$, and if $\lim_n \|U_n - U\| = 0$, where $\|T\| = \text{L.U.B.}_{(x,x)=1}(Tx, Tx)^{1/2}$, and (x, y) is the Hilbert space inner product, then the following operators are all in $\mathcal{A}$: U, VW, $V+W$, V^*, and αV, for an arbitrary complex number α. We use the term Hilbert space for a space over the complex field which satisfies the usual axioms, except that the axiom of separability is not necessarily satisfied. The complex conjugate of an expression is denoted by enclosing it in parentheses and using the right superscript c.

DEFINITION 1. *A state of* $\mathcal{A}$ is a complex-valued linear function ω on $\mathcal{A}$ such that $\omega(U^*)$ is the complex conjugate of $\omega(U)$, $\omega(U) \geqq 0$, if U is positive semi-definite and such that $\text{L.U.B.}_{\|U\| \leqq 1}\omega(UU^*) = 1$. A *pure state* is one which is not a linear combination with positive coefficients of two other states.

For example, if x is an element of the Hilbert space of norm unity, setting $\omega(U) = (Ux, x)$ defines a state. In the special case that $\mathcal{A} = C$, this state is pure, but in general the state is not pure. We observe that if $\mathcal{A}$ contains the identity operator I, then the condition $\text{L.U.B.}_{\|U\| \leqq 1}\omega(UU^*) = 1$ is equivalent to the simpler condition $\omega(I) = 1$, because if $\|U\| \leqq 1$, then $I - UU^*$ is positive semi-definite.

DEFINITION 2. *A representation of* $\mathcal{A}$ *on a Hilbert space* $\mathcal{H}$ is a mapping ϕ on $\mathcal{A}$ to the collection of bounded linear operators on $\mathcal{H}$ such that for general U and V in $\mathcal{A}$ and an arbitrary complex number α: $\phi(U+V) = \phi(U) + \phi(V)$, $\phi(UV) = \phi(U)\phi(V)$, $\phi(U^*) = (\phi(U))^*$, $\phi(\alpha U) = \alpha\phi(U)$. A representation is called ***normal*** if there exists an element x in $\mathcal{H}$ such that the span of[1] $[\phi(U)x \mid U \in \mathcal{A}]$ is $\mathcal{H}$; if x is of norm unity, the function ω defined by $\omega(U) = (\phi(U)x, x)$ is called a ***normalizing function***. A representation is called ***irreducible*** if there exists no proper closed invariant subspace of $\mathcal{H}$ other than the null-space. Two representations are called ***equivalent*** if there is a unitary transformation on one Hilbert space to the other which maps one representation into the other.

DEFINITION 3. A collection of states of $\mathcal{A}$ is called ***complete*** if there is no element of $\mathcal{A}$ except zero which vanishes in every state of the collection. A collection of representations of $\mathcal{A}$ is called *complete* if no element of $\mathcal{A}$ except zero is mapped into zero by every representation in the collection.

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[1] The notation $[f(a) \mid a \in S]$ is used to denote the range of values of $f(a)$ as a varies over S.

3. The relation between states and normal representations. In the present section we show that there is a natural mutual correspondence between states and normal representations of a C^*-algebra $\mathcal{A}$, in which pure states and irreducible representations correspond. This is interesting from the standpoint of quantum theory because it shows the general connection between two concepts of importance in that theory which have been independently introduced.

THEOREM 1. *A normalizing function of a normal representation of a C^*-algebra on a Hilbert space is a state of the algebra. Conversely, a state of the algebra is the normalizing function of some normal representation, and this representation is uniquely determined by the state, within equivalence. The state is pure if and only if the corresponding representation is irreducible.*

The proofs are somewhat simpler in the case that the C^*-algebra contains the identity operator, but it is essential for applications to group representations to consider the more general case in which the identity is not necessarily contained in the algebra. In order to discuss this more general case we prove a lemma to the effect that an arbitrary C^*-algebra contains a kind of approximate identity. This approximate identity plays roughly the same part in our proofs as the identity would, in case the treatment were limited to algebras containing the identity.

DEFINITION 4. An approximate identity of a C^*-algebra $\mathcal{A}$ is a Moore-Smith directed system V_μ of elements of $\mathcal{A}$ such that $\|V_\mu\| \leq 1$ and $\lim_\mu V_\mu U = U$ for every U in $\mathcal{A}$.

LEMMA 1.1. *A C^*-algebra contains an approximate identity.*

Let μ be a general finite subset of the C^*-algebra $\mathcal{A}$. It is plain that the μ's form a directed system relative to set inclusion. For $\mu = [U_1, \cdots, U_n]$, put $W = \sum_{i=1}^{n} U_i U_i^*$, and define $V_\mu = W(\epsilon I + W)^{-1}$, where $\epsilon = n^{-1}$. Then $V_\mu \in \mathcal{A}$ and $(V_\mu - I) W (V_\mu - I)^* = \epsilon^2 W(\epsilon I + W)^{-2}$. If we write $X \leq Y$ for SA (self-adjoint) operators X and Y if $Y - X$ is positive semi-definite, it is clear that $\epsilon^2 W(\epsilon I + W)^{-2} \leq \epsilon I/2$. Hence

$$\sum_{i=1}^{n} [(V_\mu - I) U_i][(V_\mu - I) U_i]^* \leq \epsilon I/2,$$

and it follows that each term in the sum is less than or equal to $\epsilon I/2$.

This implies that $\|(V_n-I)U\| \leq 1/2n$, and therefore $\lim_n V_n U = U$ for arbitrary $U\in\mathcal{A}$, completing the proof of the lemma.

Now we show that a normalizing function of a normal representation ϕ of an algebra $\mathcal{A}$ on a Hilbert space $\mathfrak{H}$ is a state of $\mathcal{A}$. Except for the condition

$$\underset{|U|\leq 1}{\text{L.U.B.}}\ \omega(UU^*) = 1, \tag{1}$$

where $\omega(U) = (\phi(U)z, z)$, and z is an element of $\mathfrak{H}$ such that $[\phi(U)z \mid U\in\mathcal{A}]$ spans $\mathfrak{H}$, it is trivial to verify that the conditions of Definition 1 are satisfied. We begin the verification of (1) by showing that the bound of ϕ is unity.[3] Let $\mathcal{A}'$ be the algebra generated by $\mathcal{A}$ and the identity operator I. It is readily seen that $\mathcal{A}'$ is uniformly closed and self-adjoint. We extend the representation ϕ to $\mathcal{A}'$ by defining: $\phi'(\alpha I+U) = \alpha I+\phi(U)$, where U is an arbitrary element of $\mathcal{A}$ and I is the identity operator on $\mathfrak{H}$. Clearly, ϕ' is a representation of $\mathcal{A}'$ on $\mathfrak{H}$. Now if $\|U\|\leq 1$, then $I-UU^*$ is positive semi-definite, and therefore has the form VV^* for some $V\in\mathcal{A}$. Since $\phi(I-UU^*) = \phi(V)(\phi(V))^*$, it follows that $\phi(UU^*)\leq I$, and $\|\phi(UU^*)\|\leq 1$. Since $\|\phi(UU^*)\| = \|\phi(U)\|^2$, this proves $\|\phi(U)\|\leq 1$. Putting $\text{L.U.B.}_{|U|\leq 1}\|\phi(U)\| = M$, it follows easily that $\text{L.U.B.}_{|U|\leq 1}\|\phi([UU^*]^{1/4})\| = M^{1/2}$; and since $(UU^*)^{1/4}\in\mathcal{A}$ when $U\in\mathcal{A}$, it results that $M^{1/2}\leq M$. It is trivial to conclude that $\text{L.U.B}_{|U|\leq 1}\|\phi(U)\| = 1$. Clearly,

$$\underset{|U|\leq 1}{\text{L.U.B.}}\ \omega(U^*U) = \underset{|U|\leq 1}{\text{L.U.B.}}\ (\phi(U)z, \phi(U)z) \leq 1.$$

To complete the proof, let V_n be an approximate identity of $\mathcal{A}$. It is plain that $\lim_n(\phi(V_n)z, \phi(U)z) = (z, \phi(U)z)$ for $U\in\mathcal{A}$. Now $\|\phi(V_n)z\|\leq 1$ because $\|V_n\|\leq 1$, and $[\phi(U)z \mid U\in\mathcal{A}]$ is dense in $\mathfrak{H}$. It follows easily that $\lim_n(\phi(V_n)z, x) = (z, x)$ for arbitrary x in $\mathfrak{H}$. Substituting $x = z$ and applying Schwarz' inequality to $(\phi(V_n)z, z)$, it follows that

$$(z, z) \leq \limsup_n\, (\phi(V_n)z, \phi(V_n)z)^{1/2},$$

which implies $\text{L.U.B.}_n(\phi(V_n)z, \phi(V_n)z)\geq 1$, and therefore $\text{L.U.B.}_{|U|\leq 1}\,\omega(U^*U)\geq 1$. Combining the two inequalities completes the proof that ω is a state.

Now we show that a state ω of $\mathcal{A}$ is a normalizing function for some normal representation of $\mathcal{A}$, using a procedure due in part to Gelfand and Neumark [2]. Set $\mathcal{K}$ for the class of elements U of $\mathcal{A}$ such that

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[3] We are indebted to G. W. Mackey for the observation that a representation of a C^*-algebra is necessarily continuous.

$\omega(VU)=0$ for every $V\in\mathcal{A}$. It is easy to verify that $\mathcal{K}$ is a left ideal. We denote the residue class of an operator modulo $\mathcal{K}$ by superposing a bar, and define $(\bar{U},\bar{V})=\omega(V^*U)$, for arbitrary elements of $\mathcal{A}$, U and V. To see that this is a unique definition, observe that $\omega((V+W_1)^*(U+W_2))=\omega(V^*U)+\omega(W_1^*U)+\omega(V^*W_2)+\omega(W_1^*W_2)$; and if W_1 and W_2 are in $\mathcal{K}$, then $\omega(W_1^*U)=(\omega(U^*W_1))^*=0$, $\omega(V^*W_2)=0=\omega(W_1^*W_2)$. That is, $\omega(V^*U)$ is actually a function of $\bar{U}$ and $\bar{V}$.

The function $(\bar{U},\bar{V})$, with $\bar{U}$ and $\bar{V}$ ranging over the quotient linear space $\mathcal{A}/\mathcal{K}$ of $\mathcal{A}$ modulo $\mathcal{K}$, has the usual properties of an inner product. Specifically, it is linear in $\bar{U}$, semi-linear in $\bar{V}$, is taken into its complex conjugate by interchange of $\bar{U}$ and $\bar{V}$, and is positive definite; these facts are quite evident except for the positive definiteness, which we now prove. Since the usual proof of Schwarz' inequality in a Hilbert space makes no use of completeness,

$$|(\bar{U},\bar{V})|\leqq(\bar{U},\bar{U})^{1/2}(\bar{V},\bar{V})^{1/2}.$$

Hence, if $(\bar{U},\bar{U})=0$, $\omega(V^*U)=0$ for all $V\in\mathcal{A}$, which shows that $\bar{U}=0$. Now we complete $\mathcal{A}/\mathcal{K}$ to a Hilbert space with respect to the given inner product. First, $\mathcal{A}/\mathcal{K}$ may be regarded as a linear metric space with the norm $\|\bar{U}\|=(\bar{U},\bar{U})^{1/2}$ (the usual proof of the triangle inequality for Hilbert space makes no use of completeness). Second, $(\bar{U},\bar{V})$ is plainly a uniformly continuous function on $(\mathcal{A}/\mathcal{K})\times(\mathcal{A}/\mathcal{K})$, and therefore its domain of definition can be extended to be $Z\times Z$, where Z is the completion of $\mathcal{A}/\mathcal{K}$ in such a way that it remains uniformly continuous. This extension is unique. It is easy to verify that Z is a Hilbert space relative to this extended inner product.

For U in $\mathcal{A}$ we set $\phi_0(U)$ for the operator defined on $\mathcal{A}/\mathcal{K}$ by $\phi_0(U)\bar{V}=\eta(UV)$;[4] this definition is unique, $\mathcal{K}$ being a left ideal. Now we show that $\phi_0(U)$ is a bounded operator on $\mathcal{A}/\mathcal{K}$, and may therefore be extended in a continuous fashion to Z. Since $\|U\|^2I-U^*U\geqq 0$, $V^*(\|U\|^2I-U^*U)V\geqq 0$, where U and V are arbitrary elements of $\mathcal{A}$. Hence $\omega(V^*U^*UV)\leqq\|U\|^2\omega(V^*V)$, from which it is plain that $(\phi_0(U)\bar{V},\phi_0(U)\bar{V})\leqq\|U\|^2(\bar{V},\bar{V})$ showing that: (a) $\|\phi(U)\|\leqq\|U\|$, (b) ϕ is a representation of $\mathcal{A}$ on Z, (c) this representation is adjoint-preserving, because

$$(\phi(U)\bar{V},\bar{W})=\omega(W^*UV)=(\bar{V},\phi(U^*)\bar{W}),$$

W being arbitrary in $\mathcal{A}$.

To see that the representation is normal, let $\{V_n\}$ be an approximate identity for $\mathcal{A}$ and observe that $\lim_{n}(\overline{W}, V_{n}^{*}) = \omega(W)$ for $W \in \mathcal{A}$. It follows easily that V_{n}^{*} is weakly convergent, and Hilbert space being complete in the weak neighborhood topology, there exists an element z in Z such that $(\overline{W}, z) = \omega(W)$, $W \in \mathcal{A}$. For arbitrary $X \in \mathcal{A}$,

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[4] We introduce a function η defined by $\eta(U)=\bar{U}$.

$$(\overline{X}, \phi(W)z) = (\overline{W^{*}X}, z) = \omega(W^{*}X) = (\overline{X}, \overline{W}),$$

showing that $\phi(W)z = \overline{W}$, and that the representation is normal. A normalizing functional is plainly $(\phi(U)z, z) = \omega(U)$, and the proof that a state of $\mathcal{A}$ is a normalizing function for some normal representation of $\mathcal{A}$ is complete.

Our next step is to show the essential uniqueness of the representation for which a given state is a normalizing function: Let ϕ' be a normal representation of $\mathcal{A}$ on a Hilbert space Z' with normalizing function $\omega(U) = (\phi(U)z', z')$, z' being an element of Z' such that $[\phi(U)z' \mid U \in \mathcal{A}]$ is dense in Z'. We shall show that the preceding ϕ and ϕ' are equivalent. To do this we must exhibit a unitary transformation on Z to Z' which takes ϕ into ϕ'. This transformation θ is defined as follows. For $\overline{U} \in Z$, $\theta(\overline{U}) = \phi'(U)z'$—this is a unique definition because if $\overline{U} = \overline{V}$, $\omega(W^{*}U) = \omega(W^{*}V)$ for $W \in \mathcal{A}$, which implies $(\phi'(U)z', \phi'(W)z') = (\phi'(V)z, \phi'(W)z')$; hence $(\phi'(U)z', x') = (\phi'(V)z', x')$ for $x' \in Z'$, and $\phi'(U)z' = \phi'(V)z'$. Moreover, θ is isometric, because

$$(\overline{U}, \overline{U}) = \omega(U^{*}U) = (\phi(U)z', \phi(U)z') = (\theta(\overline{U}), \theta(\overline{U})).$$

On the other hand, the range of θ (as $\overline{U}$ varies over $\mathcal{A}/\mathcal{K}$) is clearly dense in Z'. It follows that there is a unique isometric extension of the definition of θ from the domain $\mathcal{A}/\mathcal{K}$ to the domain Z, and that this extension has the range Z', and hence is unitary. Now ϕ and ϕ' are equivalent provided $\theta\phi(U)\theta^{-1} = \phi'(U)$, $U \in \mathcal{A}$. Since $\theta\phi(U)\theta^{-1}$ and $\phi'(U)$ are bounded operators, it is sufficient to check this equality on a dense domain, such as $[\phi(U)z' \mid U \in \mathcal{A}]$. This check proceeds as follows:

$$\theta\phi(U)\theta^{-1}\phi'(V)z' = \theta\phi(U)\overline{V} = \theta\eta(UV) = \phi'(UV)z' = \phi'(U)\phi'(V)z'.$$

We prove next that the representation corresponding to a pure state is irreducible, employing here and in the following paragraph a method similar to that used by Gelfand and Rykov [3]. As the hypothesis of an indirect proof, assume that ω is pure and that the proper nonzero closed linear subspace $\mathcal{S}$ of Z is invariant. If we put $\mathcal{S}\perp$ for the orthogonal complement of $\mathcal{S}$, it is plain that if $x \in \mathcal{S}$ and $y \in \mathcal{S}\perp$,

$$(x, \phi(U)y) = (\phi(U^{*})x, y) = 0,$$

showing that $\mathcal{S}\perp$ is invariant. It follows readily that the projection P on $\mathcal{S}$ commutes with $\phi(U)$ for every $U\in\mathcal{A}$. If we define $\sigma(U)=(\phi(U)Pz, Pz)$ and $\tau(U)=(\phi(U)(I-P)z, (I-P)z)$, it is easy to verify that $\omega=\sigma+\tau$, and that σ and τ are scalar multiples of states. Since ω is pure, $\sigma=\alpha\omega$ for some real number α. From $\sigma(V^*V)=\alpha\omega(V^*V)$ it follows directly that $(\bar{V}, P\bar{V})=\alpha(\bar{V}, \bar{V})$. Since $\mathcal{A}/\mathcal{K}$ is dense in $\mathcal{Z}$, it results that for any $x\in\mathcal{Z}$, $(x, Px)=\alpha(x, x)$. If x is set equal, successively, to elements of $\mathcal{S}$ and $\mathcal{S}\perp$, a contradiction is obtained with the assumed proper nonzero character of $\mathcal{S}$.

Now we show, conversely, that a normalizing function of an irreducible representation is pure. Assume that the state ω is a normalizing function for the irreducible representation ϕ of $\mathcal{A}$ on $\mathcal{Z}$, and that $\omega=\alpha\sigma+\beta\tau$, where σ and τ are states of $\mathcal{A}$. Define a new inner product on $\mathcal{A}/\mathcal{K}$ as follows: $[\bar{U}, \bar{V}]=\sigma(V^*U)$. Clearly $[\bar{U}, \bar{V}]$ can be extended, by uniform continuity as above, to an inner product on $\mathcal{Z}$ (which is positive semi-definite but not necessarily positive definite). By the Riesz representation theorem for linear functionals, $[\bar{U}, \bar{V}]$ has the form $(B\bar{U}, \bar{V})$, where B is a self-adjoint bounded linear operator on $\mathcal{Z}$. Now we show that B commutes with $\phi(W)$ for every W in $\mathcal{A}$,

$$\begin{aligned}(B\phi(W)\bar{U}, \bar{V}) &= \sigma(V^*WU) = \sigma((W^*V)^*U) = (B\bar{U}, \phi(W^*)\bar{V})\\ &= (B\bar{U}, \phi(W^*)\bar{V}) = (\phi(W)B\bar{U}, \bar{V}).\end{aligned}$$

It is easy to verify that the range of any projection in the spectral resolution of B is invariant under $\phi(W)$, $W\in\mathcal{A}$, and by the irreducibility of ϕ, $B=\beta I$. This shows that $\sigma=\beta\omega$—that is, ω is pure—and thereby completes the proof.

4. Existence of irreducible representations.

Next we show that a C^*-algebra has a complete set of irreducible representations, or alternatively, a complete collection of pure states. This result follows from Theorem 1 combined with a theorem of Krein, Milman, and Shmulyan [4] which asserts that a convex set, in the conjugate space of a Banach space, which is compact in the weak (neighborhood) topology, is the weak compactification of the convex core of its extreme points, an extreme point being one which is not an inner point of a segment contained in the set. A pure state is clearly precisely an extreme point of the set of all states, and the proof is largely concerned with showing that this set satisfies the hypothesis of the theorem just quoted.

THEOREM 2. *A C^*-algebra has a complete set of irreducible representations.*

First we observe that it is sufficient to give the proof in the case of an algebra which contains the identity operator. For if $\mathcal{A}$ does not contain I, the algebra $\mathcal{A}'$ generated (algebraically) by I and $\mathcal{A}$ is readily seen to be a C^*-algebra; it is clear that an irreducible representation of $\mathcal{A}'$ induces an irreducible representation of $\mathcal{A}$. Now set Ω for the collection of all states of the C^*-algebra $\mathcal{A}$, which is assumed to contain I. If ω_ν is a directed system of elements of Ω which converges weakly to a linear function σ on $\mathcal{A}$, it is clear that $\sigma(U^*)$ is the complex conjugate of $\sigma(U)$, that $\sigma(U^*U)\geqq 0$, and that $\sigma(I)=1$. Hence σ is a state of $\mathcal{A}$, and therefore Ω is complete. Also Ω is totally bounded because for $\omega\in\Omega$, $|\omega(U)|^2\leqq\omega(U^*U)\leqq\|U\|^2$, showing that Ω is contained in the (compact) unit sphere of the conjugate space of $\mathcal{A}$. It follows that Ω is compact.

Next we show that $\mathcal{A}$ has a complete set of pure states. The collection Π of pure states of $\mathcal{A}$ is clearly the same as the collection of extreme points of Ω. If the collection were not complete, there would exist a nonzero element U of $\mathcal{A}$ such that $\omega(U)=0$ for $\omega\in\Pi$. It would then follow from the theorem of Krein, Milman, and Shmulyan, that $\omega(U)=0$ for all $\omega\in\Omega$. In particular, $(Ux, x)=0$ for every x in the original Hilbert space, which implies $U=0$, a contradiction.

It is trivial to conclude, utilizing Theorem 1, that $\mathcal{A}$ has a complete set of irreducible representations.

5. Unitary representations of groups.

In the present section the theorem of Gelfand and Rykov, quoted in the introduction, is proved by applying Theorem 2 to an operator algebra $\mathcal{A}$ which is defined in terms of Haar measure on the group. From representations of $\mathcal{A}$ are derived strongly continuous unitary representations of the group, such a representation being defined in Definition 5.

DEFINITION 5. A *unitary representation* of a topological group G is a mapping Φ on the group to the collection of unitary operators on a Hilbert space $\mathcal{K}$, such that $\Phi(ab)=\Phi(a)\Phi(b)$ for general elements of the group a and b, and $\Phi(e)=I$, where e is the group identity. Such a representation is called *strongly continuous* in case $\Phi(a)x$ is a continuous function on G to $\mathcal{K}$ for every fixed $x\in\mathcal{K}$. It is irreducible if $\mathcal{K}$ contains no proper closed invariant subspace other than the null-space.

We mention that the weakly closed operator algebra generated by the faithful unitary representation of G obtained by Haar is not suitable for use as $\mathcal{A}$. While a complete set of irreducible representations of G can be obtained from that algebra, these representations may fail (except when G is compact) to be continuous—such failure is in evidence when G is the additive group of the reals. Actually $\mathcal{A}$ is taken to be a certain weakly dense subalgebra of the former algebra, and is the uniform closure of an operator algebra isomorphic to the group algebra.

COROLLARY 2.1 (*Gelfand and Rykov*). *A locally compact group has a complete set of irreducible, strongly continuous, unitary representations on Hilbert spaces.*

Let L_2 be the Hilbert space of complex-valued functions on the group G which are square-integrable relative to (left) Haar measure, the inner product of the function p with the function q being defined as $\int_G p(a)(q(a))^* da$, da referring to Haar measure. Putting L_1 for the space of complex-valued functions on G which are integrable relative to Haar measure, we define U_f for $f \in L_1$ as the following operator on L_2:

$$(U_f p)(b) = \int_G f(b)p(b^{-1}a)db,$$

for $p \in L_2$. U_f is a bounded operator of bound not greater than $\int_G |f(a)| da$, and it is straightforward to verify that (a) $(U_f)^* = U_{f^*}$, where $f^*(a) = \delta(a)(f(a^{-1}))^*$, $\delta(a)$ being defined by the equation $da^{-1} = \delta(a)da$; (b) $U_{f+g} = U_f + U_g$, (c) $U_{\alpha f} = \alpha U_f$ if α is a complex number, (d) $U_{fg} = U_f U_g$ where fg is defined by:

$$(fg)(a) = \int_G f(b)g(b^{-1}a)db.$$

(Compare [7]. We shall have no occasion to consider the binary operation of ordinary, pointwise, multiplication of functions on G, and therefore do not need to introduce any special symbols for the operation just defined.) Calling the collection of operators of the form U_f (with $f \in L_1$) $\mathcal{A}_0$, it follows that the uniform closure $\mathcal{A}$ of $\mathcal{A}_0$ is a C^*-algebra.

The proof shows incidentally that for any normal representation ϕ of $\mathcal{A}$, there is a strongly continuous representation $a \rightarrow B_a$ of G such that for $f \in L_1$,

$$\phi(U_f) = \int_G f(a)B_a da,$$

where the integral is appropriately defined. Actually the main burden of the proof is to show, utilizing an extension of a method in [7], that this is the case. Corollary 2.1 then follows directly from a known result in the theory of vector-valued integrals.

Now let ω be a state of $\mathcal{A}$ and let ϕ, $\mathcal{Z}$, and so on, be as in the proof of Theorem 1. For $a\in G$, R_a is defined to be the operator on L_2 which takes p into p_a, where $p_a(b)=p(a^{-1}b)$ for $p\in L_2$, and A_a is defined to be the operator on $\mathcal{A}/\mathcal{K}$ given by $A_a\overline{W}=\eta(R_aW)$, $W\in\mathcal{A}$. To show that this definition is unique, we first observe that $R_a\mathcal{A}_0\subset\mathcal{A}_0$, because $R_aU_f=U_g$, where $g(b)=f(a^{-1}b)$. Since R_a is unitary it follows that $R_a\mathcal{A}\subset\mathcal{A}$; thus, R_aW is actually an element of $\mathcal{A}$. Secondly, if $W_1\equiv W_2$ modulo $\mathcal{K}$, then $R_aW_1\equiv R_aW_2$ modulo $\mathcal{K}$. For, setting $X=W_1-W_2$, we have $X\in\mathcal{K}$, that is, $\omega(VX)=0$ for all $V\in\mathcal{A}$. Clearly $\omega(V(R_aX))=\omega((VR_a)X)$, and it is easy to verify that $VR_a\in\mathcal{A}$, whence $\omega(V(R_aX))=0$, for $V\in\mathcal{A}$. It follows that $R_aX\in\mathcal{K}$ and $R_aW_1\equiv R_aW_2$ modulo $\mathcal{K}$.

It is plain that $(A_a\overline{W}, A_a\overline{W})=\omega(W^*W)=(\overline{W}, \overline{W})$, that is, A_a is isometric, and therefore uniformly continuous, on $\mathcal{A}/\mathcal{K}$. It follows that A_a can be (uniquely) extended to an isometric operator B_a defined on $\mathcal{Z}$. It is plain that $A_aA_{a^{-1}}$ is the identity on $\mathcal{A}/\mathcal{K}$. Hence $B_aB_{a^{-1}}$ is the identity on $\mathcal{Z}$, implying that the range of B_a is $\mathcal{Z}$, and therefore B_a is unitary.

Now B_a is evidently a representation of G. Next we show that this representation is strongly continuous. For this purpose we note that (see [8]), if $f\in L_1$, the mapping $a\rightarrow f_a$, where $f_a(b)=f(a^{-1}b)$, is continuous on G to L_1. Hence the mapping $a\rightarrow R_aW$ is continuous on G to $\mathcal{A}$, if $W\in\mathcal{A}_0$. If W is any element of $\mathcal{A}$ we choose $W_n\in\mathcal{A}$ such that $\|W_n-W\|\rightarrow 0$; then $\|R_aW_n-R_aW\|\rightarrow 0$ uniformly in a, showing that the mapping is continuous for $W\in\mathcal{A}$. Now

$$
\begin{aligned}
((B_a-B_b)\overline{W}, (B_a-B_b)\overline{W}) &= (\eta(R_aW)-\eta(R_bW), \eta(R_aW)-\eta(R_bW))\\
&= (\eta((R_a-R_b)W), \eta((R_a-R_b)W))\\
&\leqq \|(R_a-R_b)W\|^2.
\end{aligned}
$$

Hence $B_a\overline{X}$ is continuous on G to $\mathcal{Z}$; and $\mathcal{A}/\mathcal{K}$ being dense in $\mathcal{Z}$ and B_a being unitary, it follows, by another application of the principle that a uniform limit of continuous functions is continuous, that B_ax is a continuous function on G to $\mathcal{Z}$ for every $x\in\mathcal{Z}$.

To show that the group representation is irreducible when the same is true of the representation of $\mathcal{A}$ we observe that the following integral, whose integrand has its values in a Banach space Υ, exists in the sense of Bochner [1] (as extended by Dunford and Pettis from the case of functions on a Euclidean space relative to Lebesgue measure to the case of functions on an arbitrary decomposable measure space):

$$\int_H y(a)f(a)da,$$

where $f \in L_1$, $y(a)$ is a bounded continuous function on G to Υ, and H is a countable sum of compact subsets of G such that $\int_{G-H}|f(a)|\,da = 0$. For such an integral exists provided: (a) the norm of the integrand is majorized by an integrable real-valued function—which is plainly the case here; (b) there exists a sequence of finitely-valued functions (that is, functions whose range of values is finite and whose sets of constancy are measurable) on G to Υ which converge almost everywhere (a.e.) to the integrand—which is now shown to be valid in the present case. Let $f_n(a)$, $n = 1, 2, \cdots$, be a sequence of finitely-valued functions on G to the complex numbers which converge a.e. to $f(a)$. Let C_n, $n = 1, 2, \cdots$, be a monotone increasing sequence of compact subsets of G whose union is H. Now cover C_n by the compact sets $D_{n1}, \cdots, D_{nm}$, on each of which the oscillation of $y(a)$ is less than n^{-1} (each set D_{nj} being contained in C_n). Let a_{nj} be any point of D_{nj}, and define $y_n(a)$ as follows: (a) for $a \in D_{n1}$, $y_n(a) = y(a_{n1})$, (b) for $a \in D_{n(j+1)} - U_{k \leq j} D_{nk}$, $y_n(a) = y(a_{n(j+1)})$; (c) $y_n(a) = 0$ if $a \in G - C_n$. It is clear that $\lim y_n(a)f_n(a) = y(a)f(a)$ a.e., and that $y_n(a)f_n(a)$ is finitely-valued.

In particular, the integral exists in case either $y(a) = R_a W$ or $y(a) = R_a p$, with $p \in L_2$ (see [7] for a proof that the mapping $a \rightarrow p_a$, where $p_a(b) = p(a^{-1}b)$, is continuous on G to L_2). Now $\int_H R_a pf(a)da = U_f p$, because for any $q \in L_2$

$$\int_G (U_f p)(b)(q(b))^* db = \int_G \left(\int_H R_a pf(a)da(b) \right) \eta(q(b))db,$$

by the Fubini theorem and the fact that continuous linear operations commute with integration. If we set $p = Wr$, and observe that $\int_H R_a Wrf(a)da = (\int_H R_a Wf(a)da)r$, it follows that $U_f W = \int_H R_a Wf(a)da$. Since the operation $X \rightarrow \overline{X}$ is a continuous linear operation on $\mathcal{A}$ to $\mathcal{A}/\mathcal{K}$, $\phi(U_f)\overline{W} = \int_H B_a \overline{W} f(a)da$; and it is easily shown, from the fact that $\int_H B_a xf(a)da$ exists for $x \in Z$ and is a continuous function of x, that $\phi(U_f)x = \int_H B_a xf(a)da$ for arbitrary $x \in Z$. Now $\int_H B_a xf(a)da$, like any Bochner integral, is in the span of the range of values of the integrand, that is, in the span of $[B_a x \mid a \in G]$. Hence a closed linear manifold which is invariant under the group representation is invariant under $\phi(U_f)$ for every $f \in L_1$, and therefore invariant under the representation of $\mathcal{A}$.

6. The completeness of irreducible representations in quantum

theory. In this section we prove a theorem which implies that for the computational purposes of quantum theory it is sufficient to consider only those representations of a C^*-algebra (which in quantum theory is the algebra generated by the observables) which are irreducible. A special case of the theorem asserts that the spectrum of a self-adjoint operator in a C^*-algebra is determined by the collection of spectra of the operator in the irreducible representations of the algebra. In practice only the irreducible representations are used, in general. Our result shows that this limitation to irreducible representations is valid for an arbitrary physical system which is describable by an algebra of operators on a Hilbert space (in the case of the Schrödinger-Heisenberg non-relativistic theory this fact had been established by special considerations by von Neumann [**5**]).

In order to treat the simultaneous analysis of a commutative collection of SA operators, as well as the analysis of a single operator, we introduce the concept of observation of an algebra.

Definition 6. An observation of a C^*-algebra is a state ω such that $\omega(UV)=\omega(U)\omega(V)$ for all U and V in the algebra.

To give an interesting example of an observation, let $\mathcal{A}$ be the C^*-algebra generated by a bounded SA operator T, and let α be an arbitrary point in the spectrum of T. It is plain from the spectral theorem that every element of $\mathcal{A}$ has the form $f(T)$, where f is a complex-valued function which is defined and continuous on the spectrum of T, and conversely, if f is any such function, and if $f(0)=0$, then $f(T)\in\mathcal{A}$. Now define a linear functional ω on $\mathcal{A}$ as follows: $\omega(f(T))=f(\alpha)$. It is easy to verify that ω is an observation. On the other hand, every observation of $\mathcal{A}$ is of the preceding form. This follows from a known result about an algebra of all continuous functions on a compact Hausdorff space, when it is observed that $\mathcal{A}$ is isomorphic with the algebra of all continuous functions on the spectrum of T, except that only functions vanishing at 0 are included if 0 is in the spectrum of T (it is trivial to modify the known result so as to apply to this case also).

This example shows that the following result is a special case of Corollary 2.2. If α is in the spectrum of a self-adjoint element U of a C^*-algebra $\mathcal{A}$, then there exists an irreducible representation ϕ of $\mathcal{A}$ and a nonzero element x of the space on which $\mathcal{A}$ is represented such that

$$\phi(U)x=\alpha x.$$

Thus there is in a sense always a wave function for a point in the spectrum of U, whether it is in the point or continuous spectrum; this wave function, naturally, will usually not be in the space on which U operates. Corollary 2.2 has the obvious partial converse, that if ϕ is a normal representation of a C^*-algebra $\mathcal{A}$, then every observation ω' on the C^*-algebra $\phi(\mathfrak{B})$, $\mathfrak{B}$ being a commutative C^*-subalgebra of $\mathcal{A}$, defines an observation ω on $\mathfrak{B}$, by the equation $\omega(U)=\omega'(\phi(U))$.

COROLLARY 2.2. *If τ is an observation of a C^*-subalgebra $\mathfrak{B}$ of a C^*-algebra $\mathcal{A}$, then there exists an irreducible representation ϕ of $\mathcal{A}$ and a nonzero element x of the space on which $\mathcal{A}$ is represented such that for $U\in\mathfrak{B}$,*

$$\phi(U)x = \tau(U)x.$$

First we prove a lemma of some general interest.

LEMMA. *An observation of a C^*-subalgebra of a C^*-algebra is pure on the subalgebra and coincides on the subalgebra with a pure state of the algebra.*

Let τ be an observation of the C^*-subalgebra B of the C^*-algebra $\mathfrak{B}$. If the notation of the proof of Theorem 1 is utilized, with $\mathcal{A}$ replaced by $\mathfrak{B}$, it is plain that for arbitrary U and V in $\mathfrak{B}$,

$$\tau(UV) = (\phi(UV)z, z) = (\phi(V)z, \phi(U^*)z).$$

On the other hand, $\tau(UV) = \tau(U)\tau(V) = (\phi(U)z, z)(\phi(V)z, z)$. Now by Schwarz' inequality,

$$\begin{aligned}(\phi(V)z, \phi(U^*)z) &\leqq (\phi(V)z, \phi(V)z)^{1/2}(\phi(U^*)z, \phi(U^*)z)^{1/2}\\ &= (\phi(V^*V)z, z)^{1/2}(\phi(UU^*)z, z)^{1/2}\\ &= |\tau(U)\tau(V)|.\end{aligned}$$

Since the equality holds for Schwarz' inequality only when the vectors involved are proportional, it follows that $\phi(V)z$ and $\phi(U^*)z$ are proportional. In other words, the representation space Z is one-dimensional. This implies, clearly, that ϕ is irreducible. Hence, applying Theorem 1, the normalizing function τ of ϕ is pure.

To prove the remainder of the lemma we show that it is sufficient to consider the case in which $I\in\mathfrak{B}$. Because if $\mathfrak{B}$ does not contain I, the observation τ of $\mathfrak{B}$ may be extended to an observation τ' of the C^*-algebra $\mathfrak{B}'$ generated by $\mathfrak{B}$ and I as follows: for $U\in\mathfrak{B}$, $\tau'(\alpha I+U) = \alpha+\tau(U)$. (As noted above, the algebra generated algebraically by $\mathfrak{B}$ and I is a C^*-algebra.) It is clear that $\tau'(VW) = \tau'(V)\tau'(W)$ for V and W in $\mathfrak{B}'$. In order to complete the verification that τ' is a state it is plainly sufficient to show that $\tau'(\alpha I+U)\geqq 0$ if $\alpha I+U$ is SA and not less than 0. Now $\alpha I+U\geqq 0$ implies that $\alpha I+U = V^2$ with V SA and in $\mathcal{B}'$ and hence $\tau'(\alpha I+U)=(\tau'(V))^2 \geqq 0$. Assuming that the lemma had been proved for the case $I \in \mathcal{B}$, it would follow that τ' coincides on $\mathcal{B}'$ with a pure state ω' of $\mathcal{A}'$, where $\mathcal{A}'$ is the C^*-algebra generated by $\mathcal{A}$ and I. Putting ψ' for a normal representation of $\mathcal{A}'$ with normalizing function $\omega'(U)=(\psi'(U)x', x')$, it follows that ψ' is irreducible, and moreover the representation ψ obtained by restricting ψ' to $\mathcal{A}$ is irreducible and has the normalizing function ω, where ω is obtained by restricting ω' to $\mathcal{A}$. Hence ω is pure. Thus τ coincides on $\mathcal{B}$ with the pure state ω of $\mathcal{A}$.

Now it is assumed that $I \in \mathcal{B}$. Put $\mathcal{A}_0$ and $\mathcal{B}_0$ for the collection of SA elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. We show next, by the method of proof of the Hahn-Banach theorem, that τ coincides on $\mathcal{B}_0$ with a real-valued linear function χ_0 on $\mathcal{A}_0$ which is non-negative on positive semi-definite elements of $\mathcal{A}_0$. Let U be an arbitrary element of $\mathcal{A}_0$ which is not in $\mathcal{B}_0$. Put $\mathcal{P}=[V \mid V \in \mathcal{A}_0, V \geqq U]$ and $\mathfrak{Q}=[V \mid V \in \mathcal{A}_0, V \leqq U]$. Since $-\|U\| I \leqq U \leqq \|U\| I$, neither $\mathcal{P}$ nor $\mathfrak{Q}$ is empty. Defining $\gamma_- = \text{L.U.B.}_{V \in \mathfrak{Q}} \tau(V)$ and $\gamma_+ = \text{G.L.B.}_{V \in \mathcal{P}} \tau(V)$, it is clear that $\gamma_- \leqq \gamma_+$. Let γ be a number such that $\gamma_- \leqq \gamma \leqq \gamma_+$. Now we define χ_0 on the space $\mathcal{B}_0^+$ spanned by $\mathcal{B}_0$ and U as follows: for $V \in \mathcal{B}_0$, $\chi_0(\alpha U+V)=\alpha\gamma+\tau(U)$. It is obvious that χ_0 is linear on $\mathcal{B}_0^+$. To see that χ_0 is non-negative on positive semi-definite elements of $\mathcal{B}_0^+$, note that if $\alpha U+V \geqq 0$, and if $\alpha>0$, then $U \geqq -\alpha^{-1}V$, $-\alpha^{-1}V \in \mathfrak{Q}$, $\chi_0(-\alpha^{-1}V)=\tau(-\alpha^{-1}V) \leqq \gamma_- \leqq \gamma = \chi_0(U)$, and finally, $\chi_0(\alpha U+V) \geqq 0$; and a similar proof is valid if $\alpha<0$. It is plain that by transfinite induction we may obtain a function χ_0 on $\mathcal{A}_0$ with the stated property. It is easy to verify that $\mathcal{A}$ is the (real) linear direct sum of $\mathcal{A}_0$ and $i\mathcal{A}_0$. Hence the definition

$$\chi(U+iV)=\chi_0(U)+i\chi_0(V), \qquad U \text{ and } V \text{ in } \mathcal{A}_0,$$

extends χ_0 to $\mathcal{A}$. It is plain that χ is a state of $\mathcal{A}$, and that it coincides on $\mathcal{B}$ with τ. Thus the class Π of states of $\mathcal{A}$ which coincide on $\mathcal{B}$ with τ is nonvacuous.

It is obvious that Π is convex. To see that it is compact in the weak neighborhood topology, observe that $\pi(U)$ is a continuous function of π, for fixed U. It was shown in the proof of Theorem 2 that Ω is compact. Now Π is an intersection of closed subsets of Ω and therefore is itself closed, and hence compact. It follows, if we apply again the theorem of Krein, Milman, and Shmulyan, that Π contains an extreme point ω.

Next we show that ω is pure. Suppose that $\omega=\alpha\rho+\beta\sigma$ where ρ and σ are states of $\mathcal{A}$ and α and β are non-negative numbers. Then $\omega(I)=\alpha\rho(I)+\beta\sigma(I)$, which implies $\alpha+\beta=1$. Now we adopt the following notation for a state of $\mathcal{B}$ which is obtained from a given state of $\mathcal{A}$ by restricting its domain of definition to $\mathcal{B}$: for $\xi \in \Omega$, ξ_B denotes the state of $\mathcal{B}$ which coincides with ξ on $\mathcal{B}$. It is plain that $\omega_B = \alpha\rho_B + \beta\sigma_B$. But $\omega_B = \tau$ and is therefore pure. Hence $\rho_B = \sigma_B = \tau$. It follows that ρ and σ are elements of Π. On the other hand, ω is an extreme point of Π, and hence the equation $\omega = \alpha\rho + \beta\sigma$ implies that $\rho = \sigma = \omega$—that is, ω is pure. The proof of the lemma is now complete.

To conclude the proof of Corollary 2.2, let ω be a pure state of $\mathcal{A}$ which coincides with τ on $\mathcal{B}$, and let ϕ be a normal representation of $\mathcal{A}$ with normalizing function ω, and adopt the rest of the notation of the proof of Theorem 1. It is clear that for U and V in $\mathcal{B}$,

$$\omega((UV - \omega(U)V)(UV - \omega(U)V)^*) = 0.$$

Hence $UV - \omega(U)V$ maps into zero under the natural mapping on $\mathcal{A}$ to $\mathcal{A}/\mathcal{K}$, or

$$\phi(U)\mathbf{V} = \omega(U)\mathbf{V}.$$

The proof is completed by observing that $\mathbf{V} \neq 0$ for some V in $\mathcal{B}$, because otherwise $\omega(V^* V) = 0$ for all V in $\mathcal{B}$, which is evidently not the case.

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INSTITUTE FOR ADVANCED STUDY

ON FUNCTIONS HAVING SUBHARMONIC LOGARITHMS

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Saks has shown that if $f(x, y)$ is subharmonic in a domain G and if $\mu(e)$ is the completely additive, non-negative function of Borel sets associated with $f(x, y)$, then

$$\lim_{\rho\to 0} \frac{8}{\rho^2}\left[\frac{1}{\pi\rho^2}\iint f(x+\xi, y+\eta)\,d\xi d\eta - f(x, y)\right] = D_s\mu(x, y),$$

and

$$\lim_{\rho\to 0} \frac{4}{\rho^2}\left[\frac{1}{2\pi\rho}\int f(x+\xi, y+\eta)\,ds - f(x, y)\right] = D_s\mu(x, y)$$

hold almost everywhere in G [7].[1] Here the first integral is extended over all (ξ, η) such that $\xi^2+\eta^2<\rho^2$, the second integral is extended over all (ξ, η) such that $\xi^2+\eta^2=\rho^2$, and $D_s\mu(x, y)$ is the symmetric derivative of $\mu(e)$ at (x, y).

The main result of this paper is an analogue of Saks' result for continuous functions having subharmonic logarithms. For such functions $f(x, y)$, it is shown that if $\sigma(e)$ is the completely additive, non-negative function of Borel sets associated with $\log f(x, y)$, then

$$\lim_{\rho\to 0} \frac{4}{\rho^2}\left\{\left(\frac{1}{2\pi\rho}\int f(x+\xi, y+\eta)\,ds\right)^2 - \frac{1}{\pi\rho^2}\iint f^2(x+\xi, y+\eta)\,d\xi d\eta\right\} = f^2(x, y)D_s\sigma(x, y)$$

holds almost everywhere in G.

Let G denote a domain (non-null connected open set) in the x, y-plane, $D(x, y; \rho)$ the open circular disc with center at (x, y) and radius ρ, and $C(x, y; \rho)$ the boundary of $D(x, y; \rho)$. If $f(x, y)$ is continuous in G, then $f(x, y)$ is said to be ***subharmonic*** in G if and only if

$$(1)\qquad f(x, y) \leqq A(f; x, y; \rho) = \frac{1}{\pi\rho^2}\iint_{D(x,y;\rho)} f(\xi, \eta)\,d\xi d\eta$$

holds for each $D(x, y; \rho)$ in G [4]. It is well known that (1) can be replaced by either [4]

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$f(x, y) \leqq L(f; x, y; \rho) = \frac{1}{2\pi\rho}\int_{C(x,y;\rho)} f(\xi, \eta)ds,$$

or

$$A(f; x, y; \rho) \leqq L(f; x, y; \rho).$$

An important subclass of the class of functions subharmonic in G consists of those functions having subharmonic logarithms. These functions, studied by Beckenbach and Radó [1], are defined as follows. A function is said to be of class PL in G if and only if (i) $f(x, y) \geqq 0$, (ii) $f(x, y) \not\equiv 0$, (iii) $\log f(x, y)$ is subharmonic in G. It is fundamental in the theory of functions of class PL in G that $f(x, y)$ is of class PL in G if and only if

$$A(f^2; x, y; \rho) \leqq [L(f; x, y; \rho)]^2$$

holds for each $D(x, y; \rho)$ in G [1].

If $f(x, y)$ has continuous partial derivatives of the second order in G, then $f(x, y)$ is subharmonic in G if and only if $\Delta f(x, y) \geqq 0$ in G, and $f(x, y)$ is of class PL in G if and only if

$$f^2\Delta \log f = f\Delta f - \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 \geqq 0 \tag{2}$$

in G [4]. Here Δ is the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\cdot$$

If $f(x, y)$ is subharmonic in G, then Riesz [6] has shown that there exists a unique, completely additive, non-negative function $\mu(e)$ of Borel sets e (for which the closure $\bar{e} \subset G$) with the following property. If D is a subdomain of G, such that $\bar{D} \subset G$, then $f(x, y)$ has the representation

$$f(x, y) = f(P) = -\frac{1}{2\pi}\iint_D \log\frac{1}{PQ}\, d\mu(e_Q) + H(P), \qquad P \in D, \tag{3}$$

where $PQ = ((x-\xi)^2+(y-\eta)^2)^{1/2}$, $H(P)$ is harmonic in D, and where the integral is a Stieltjes-Radon integral [8].

Since the *density* of $\mu(e)$ at (x, y) is defined by [8]

$$D_s\mu(x, y) = \lim_{\rho\to 0} \frac{\mu[D(x, y; \rho)]}{\pi\rho^2} \tag{4}$$

(which is known to exist almost everywhere [8]), then Saks' result may be stated as follows. If $f(x, y)$ is subharmonic in G, and if $\mu(e)$ is the set function used in (3), then

$$\begin{aligned}\lim_{\rho\to 0}\frac{8}{\rho^2}\,[A(f;\,x,\,y;\,\rho)-f(x,\,y)]\qquad\qquad&\\ =\lim_{\rho\to 0}\frac{4}{\rho^2}\,[L(f;\,x,\,y;\,\rho)-f(x,\,y)]&=D_s\mu(x,\,y)\end{aligned}\tag{5}$$

holds almost everywhere in G. Saks' proof of (5) depends upon the representation (3) for $f(x, y)$.

If $f(x, y)\geqq 0$ is continuous and subharmonic in G, then $f^2(x, y)$ is continuous and subharmonic in G [4]. Hence by the "representation theorem" of Riesz, noted above, there exist unique, completely additive, non-negative set functions $\mu(e)$ and $\nu(e)$, for $\bar{e}\subset G$, associated with $f(x, y)$ and $f^2(x, y)$, respectively. Then the following lemmas hold.

LEMMA 1.

$$\begin{aligned}\lim_{\rho\to 0}\frac{1}{\rho^2}\,[L^2(f;\,x,\,y;\,\rho)-A(f^2;\,x,\,y;\,\rho)]\qquad\qquad&\\ =\frac{f(x,\,y)D_s\mu(x,\,y)}{2}-\frac{D_s\nu(x,\,y)}{8}&\end{aligned}\tag{6}$$

holds almost everywhere in G.

PROOF. It is well known [4] that $L(f;\,x,\,y;\,\rho)\to f(x,\,y)$ and $A(f^2;\,x,\,y;\,\rho)\to f^2(x,\,y)$ on $\bar{e}\subset D$, as $\rho\to 0$. The relation (6) now follows from (5) and the identity

$$\begin{aligned}\frac{L^2(f;\,x,\,y;\,\rho)-A(f^2;\,x,\,y;\,\rho)}{\rho^2}&\\ =[L(f;\,x,\,y;\,\rho)+f(x,\,y)]&\left[\frac{L(f;\,x,\,y;\,\rho)-f(x,\,y)}{\rho^2}\right]\\ &-\frac{A(f^2;\,x,\,y;\,\rho)-f^2(x,\,y)}{\rho^2}.\end{aligned}$$

LEMMA 2. *If* e *is a Borel set,* $\bar{e}\subset G$, *then*

$$\nu(e)=2\iint_e f(P)d\mu(e_P)+2\iint_e\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]dxdy.\tag{7}$$

PROOF. Let D be a subdomain of G, such that $\bar{D}\subset G$. It follows from the proofs of the representation (3) for subharmonic functions, given by Evans [2] and Riesz [6], that $\mu(e)$ and $\nu(e)$ may be obtained as follows. If iterated averages of $f(x, y)$ are defined as

$$A_2(f;\, x,\, y;\, \rho) = A(A;\, x,\, y;\, \rho), \quad A_3(f;\, x,\, y;\, \rho) = A(A_2;\, x,\, y;\, \rho),$$

then there exists a sequence $\{\rho_n\} \searrow 0$, as $n \to \infty$, such that the set functions

$$\mu_n(e) = \iint_e \Delta A_3(f;\, x,\, y;\, \rho_n)\,dx\,dy, \qquad \bar{e} \subset D, \tag{8}$$

$$\nu_n(e) = \iint_e \Delta[A_3(f;\, x,\, y;\, \rho_n)]^2\,dx\,dy, \qquad \bar{e} \subset D, \tag{9}$$

converge to $\mu(e)$ and $\nu(e)$, respectively; that is,

$$\lim_{n\to\infty} \mu_n(e) = \mu(e),$$

if e is open and μ-regular (that is, $\mu(\bar{e}-e) = 0$) and

$$\lim_{n\to\infty} \nu_n(e) = \nu(e),$$

if e is open and ν-regular (that is, $\nu(\bar{e}-e) = 0$).

Now if R is an *oriented* rectangle in D, and if R is both μ- and ν-regular, and if the substitution

$$A_3(f;\, x,\, y;\, \rho_n) = \mathfrak{A}_n(x,\, y)$$

is made, then it follows from (8) and (9) that

$$\begin{aligned} \nu(R) = {} & \lim_{n\to\infty} 2 \iint_R \mathfrak{A}_n(P)\,d\mu_n(e_P) \\ & + \lim_{n\to\infty} 2 \iint_R \left[\left(\frac{\partial \mathfrak{A}_n}{\partial x}\right)^2 + \left(\frac{\partial \mathfrak{A}_n}{\partial y}\right)^2\right] dx\,dy \end{aligned} \tag{10}$$

holds. However, Frostman has shown [3] that if R is μ-regular, then

$$\lim_{n\to\infty} \iint_R \mathfrak{A}_n(P)\,d\mu_n(e_P) = \iint_R f(P)\,d\mu(e_P),$$

and Evans has shown [2] that

$$\lim_{n\to\infty} \iint_R \left[\left(\frac{\partial \mathfrak{A}_n}{\partial x}\right)^2 + \left(\frac{\partial \mathfrak{A}_n}{\partial y}\right)^2\right] dx\,dy = \iint_R \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dx\,dy;$$

so that (10) may be written

$$\nu(R) = 2 \iint_R f(P)\,d\mu(e_P) + 2 \iint_R \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dx\,dy. \tag{11}$$

Now each open oriented rectangle R in D is the point set limit of a monotone increasing sequence $\{R_n\}$ of open μ- and ν-regular rectangles, such that [5]

$$\lim_{n\to\infty} \mu(R_n) = \mu(R), \tag{12}$$

$$\lim_{n\to\infty} \nu(R_n) = \nu(R). \tag{13}$$

Hence it follows from (11), (12) and (13) that (11) must hold for *all* open oriented rectangles R in D. By a familiar argument used in the theory of set functions [5, 7], it follows that (7) holds for all Borel sets in D.

Since D was an arbitrary subdomain of G, the lemma now follows.

COROLLARY.

$$D_s\nu(x, y) = 2\left[f(x, y)D_s\mu(x, y) + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \tag{14}$$

almost everywhere in G.

PROOF. The relation (14) follows from (4), (7) and the classic theorem due to Lebesgue on the derivation of integrals [8].

THEOREM 1.

$$\begin{aligned}\lim_{\rho\to 0} \frac{4}{\rho^2} [L^2(f; x, y; \rho) - A(f^2; x, y; \rho)]\\ = f(x, y)D_s\mu(x, y) - \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2\end{aligned} \tag{15}$$

almost everywhere in G.

PROOF. (15) follows from (6) and (14).

In the following, it is assumed that $f(x, y)$ is also of class PL in G. Hence $f(x, y) \equiv \exp u(x, y)$, where $u(x, y)$ is continuous and subharmonic in G, with associated set function called $\sigma(e)$.

LEMMA 3. *If e is a Borel set, $\bar{e}\subset G$, then*

$$\begin{aligned}\mu(e) = \iint_e \exp \mu(P) d\sigma(e_P)\\ + \iint_e \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \exp u(x, y) dx dy.\end{aligned} \tag{16}$$

PROOF. It is inherent in the proof of (3) given by Evans [2] and Riesz [6], that $\mu(e)$ and $\sigma(e)$ may be found as follows. If the definition

$$A_3(u;\ x,\ y;\ \rho_n) \equiv a_n(x,\ y)$$

is made, then there exists a sequence $\{\rho_n\} \searrow 0$, as $n \to \infty$, such that

$$\mu_n^*(e) \equiv \iint_e \Delta[\exp a_n(x,\ y)]dxdy \tag{17}$$

and

$$\sigma_n(e) \equiv \iint_e \Delta a_n(x,\ y)dxdy \tag{18}$$

converge to $\mu(e)$ and $\sigma(e)$, respectively; that is

$$\lim_{n\to\infty} \mu_n^*(e) = \mu(e) \tag{19}$$

for each open μ-regular set e, and

$$\lim_{n\to\infty} \sigma_n(e) = \sigma(e) \tag{20}$$

for each open σ-regular set e.

Now an argument similar to that used in the proof of Lemma 2 shows that (16) follows from (17)–(20).

COROLLARY.

$$\begin{aligned} D_s\mu(x,\ y) &= \exp u(x,\ y)\cdot D_s\sigma(x,\ y) \\ &\quad + \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \exp u(x,\ y) \end{aligned} \tag{21}$$

almost everywhere in G.

PROOF. The corollary follows at once from (4), (16) and the theorem of Lebesgue on the derivation of integrals.

THEOREM 2.

$$\lim_{\rho\to 0} \frac{4}{\rho^2}\left[L^2(f;\ x,\ y;\ \rho) - A(f^2;\ x,\ y;\ \rho)\right] = f^2(x,\ y)D_s\sigma(x,\ y) \tag{22}$$

almost everywhere in G.

PROOF. The theorem, which is an analogue of Saks' result (5), follows at once from (15) and (21).

The relations (5), (6), (15) and (22) are examples of "generalized Laplacians" [7, 9]. For example, if $f(x, y)$ is sufficiently smooth in G, then (22) yields

$$\lim_{\rho\to 0} \frac{4}{\rho^2} \left[L^2(f;\, x,\, y;\, \rho) - A(f^2;\, x,\, y;\, \rho) \right] = f^2(x, y) \Delta \log f(x, y),$$

which bears an important relation to the defining inequality (2) for smooth functions of class PL.

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PURDUE UNIVERSITY

A PARTICULAR GENERALIZED LAPLACIAN

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In a preceding paper, the author developed a generalized Laplacian for functions having subharmonic logarithms [2].[1] The purpose of this note is to indicate how generalized Laplacians may be used to weaken differentiability requirements; in particular, the generalized Laplacian

$$\Delta^* f(x, y) = \limsup_{\rho \to 0} \frac{4}{\rho^2} \left[L(f;\, x, y;\, \rho) - f(x, y) \right] \tag{1}$$

is used to weaken differentiability requirements in certain theorems due to Kierst and Saks [4] and the author [3].

The definitions and notation used in [2] will be used here. In addition, use will be made of the following known result.

THEOREM A [1]. *If $f(x, y)$ is continuous in a domain G, then a necessary and sufficient condition that $f(x, y)$ be subharmonic in G is that*

$$\Delta^* f(x, y) \geqq 0 \tag{2}$$

hold throughout G.

A slightly more general version of a theorem due to Kierst and Saks is the following one.

THEOREM 1. *Let $F(t)$ have a continuous second derivative, with $F'(t) > 0$, for $-\infty < t < \infty$. If $f(x, y)$ has continuous partial derivatives of the first order in a domain G, and if $F[\alpha x + \beta y + f(x, y)]$ is subharmonic in G for every choice of the real constants α, β, then $f(x, y)$ is subharmonic in G.*

PROOF. Let (x_0, y_0) be a fixed, arbitrary point of G. Then after expanding $F(t)$ and $f(x, y)$ in Taylor series about $t_0 = \alpha x_0 + \beta y_0 + v(x_0, y_0)$ and (x_0, y_0), respectively, one obtains

$$\begin{aligned} L(\phi_{\alpha,\beta};\, x_0, y_0;\, \rho) &- \phi_{\alpha,\beta}(x_0, y_0) \\ &= F'(t_0)\left[L(f;\, x_0, y_0;\, \rho) - f(x_0, y_0)\right] \\ &\quad + \frac{\rho^2 F''(t_0)}{4}\left[(\alpha + f_x)^2 + (\beta + f_y)^2\right] + o(\rho^2), \end{aligned} \tag{3}$$

where

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$\phi_{\alpha,\beta}(x, y) = F[\alpha x + \beta y + f(x, y)],$$

$$f_x = \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)}, \qquad f_y = \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)},$$

and $o(\rho^2)$ is a quantity (not always the same quantity) such that

$$\lim_{\rho\to 0} \frac{o(\rho^2)}{\rho^2} = 0. \tag{4}$$

Now set $\alpha = -f_x$, $\beta = -f_y$. Since $\phi_{\alpha,\beta}(x, y)$ is subharmonic in G for all choices of the real constants α, β, it follows from (1), (3), (4) and Theorem A that (2) holds at (x_0, y_0). But (x_0, y_0) was an arbitrary point of G, so that (2) holds throughout G; therefore, by Theorem A, $f(x, y)$ is subharmonic in G. This completes the proof.

In a similar manner one may prove the following more general version of a theorem due to the author [3].

THEOREM 2. *Let $F(t)$ and $f(x, y)$ have the properties noted in Theorem 1. If the function $F\{\log[(x-\alpha)^2+(y-\beta)^2]+f(x, y)\}$ is subharmonic in G for every choice of the real constants α, β, then $f(x, y)$ is subharmonic in G.*

The same technique may be applied to other results [1, 3] to obtain slightly more general theorems. However, it would be desirable to remove all conditions of differentiability (on $f(x, y)$)—which the usual averaging process does not appear to do.

Other generalized Laplacians may be used to obtain results similar to those above; for example, either

$$\limsup_{\rho\to 0} \frac{8}{\rho^2} [A(f; x, y; \rho) - f(x, y)],$$

or

$$\limsup_{\rho\to 0} \frac{8}{\rho^2} [L(f; x, y; \rho) - A(f; x, y; \rho)]$$

may be used.

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PURDUE UNIVERSITY

ON AREOLAR MONOGENIC FUNCTIONS

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Let $f(z)=u(x, y)+iv(x, y)$, $z=x+iy$, be a complex-valued function defined in the unit circle D: $|z|<1$. $f(z)$ is said to be *areolar monogenic* in D if and only if $u(x, y)$ and $v(x, y)$ (and hence $f(z)$) have continuous partial derivatives of the second order such that

$$(1) \qquad u_{xy} = -2^{-1}(v_{xx} - v_{yy}), \qquad v_{xy} = 2^{-1}(u_{xx} - u_{yy})$$

hold in D [3].[1] It is known that an areolar monogenic function has partial derivatives of all orders [3].

Whereas (1) is a differential characterization of areolar monogenic functions, it is the integral characterization contained in the following theorem that forms the basis for this note.

THEOREM A [3]. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that there exist a function $g(z)$, analytic in D, such that*

$$(2) \qquad g(z) = \frac{1}{\pi r^2 i}\int_{C(z,r)} f(\zeta)d\zeta, \qquad \zeta = \xi + i\eta,$$

holds for all circles $C(z; r)$, with center z and radius r, in D.

It should be noted that a symbol once introduced holds its meaning throughout the paper.

If $f(z)$ is continuous in D, then the right-hand member of (2) is a function of z and r, defined for z in the circle D_r: $|z|<1-r$, and for all r such that $0<r<1$. Now if the definition

$$(3) \qquad G_r(z) = \frac{1}{\pi r^2 i}\int_{C(zr)} f(\zeta)d\zeta, \qquad 0 < r < 1, \ |z| < 1 - r,$$

is made, then the following is an extension of Theorem A.

THEOREM 1. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that $G_r(z)$ be analytic in D_r, for all r, $0<r<1$.*

Necessity. This is precisely the necessity part of Theorem A.

Sufficiency. Suppose that in addition $f(z)$ has continuous partial

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

derivatives of the first order in D. Then from (3) and Green's lemma it follows that

$$G_r(z) = \frac{1}{\pi r^2}\iint_{D(z;r)} G(\zeta)\,d\xi d\eta, \tag{4}$$

where

$$G(z) = U(x, y) + iV(x, y) \equiv (u_x - v_y) + i(u_y + v_x), \tag{5}$$

and where $D(z; r)$ is the closed circular disc with center at z and radius r. Since $f(z)$ has continuous partial derivatives of the first order in D, it follows from (4) and (5) that $G_r(z) \rightarrow G(z)$, as $r \rightarrow 0$, on each closed subset of D. Hence, since $G_r(z)$ is analytic in D_r, for all r, it follows that $G(z)$ is analytic in D. Therefore $U(x, y)$ and $V(x, y)$ are conjugate harmonic functions such that (4) and the Gauss mean-value theorem for harmonic functions yield

$$G_r(z) = \frac{1}{\pi r^2}\iint_{D(z;r)} G(\zeta)\,d\xi d\eta = G(z)$$

for all z in D_r, $0 < r < 1$.

It now follows from Theorem A that $f(z)$ is areolar monogenic in D.

Now if $f(z)$ is merely continuous in D, then for sufficiently small ρ, the mean-value function

$$A(f; z; \rho) = \frac{1}{\pi \rho^2}\iint_{D(z;\rho)} f(\zeta)\,d\xi d\eta \tag{6}$$

satisfies the hypotheses of this lemma in the circle D_ρ, for $0 < \rho < 1$; moreover, $A(f; z; \rho)$ has continuous partial derivatives of the first order in D_ρ [1]. Therefore by the preceding part of this proof, it follows that for all sufficiently small ρ, the function

$$\frac{1}{\pi r^2 i}\int_{C(z;r)} A(f; \zeta; \rho)\,d\zeta, \qquad 0 < r + \rho < 1,$$

is analytic in $D_{r+\rho}$ and independent of r. But $A(f; z; \rho) \rightarrow f(z)$ as $\rho \rightarrow 0$, on each closed subset of D [1]. Hence it follows that for $0 < r < 1$, the right-hand member of (2) is analytic and independent of r in D_r. Hence, by Theorem A, $f(z)$ is areolar monogenic in D.

COROLLARY 1. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that $A(f; z; \rho)$ be areolar monogenic in D_ρ, for all ρ, $0 < \rho < 1$.*

COROLLARY 2. *If $f(z)$ is areolar monogenic in D, then $f(z)$ has the following mean-value property:*

$$\int_{C(z;r)} f(\zeta)d\zeta = \int_{C(z;r)} A(f;\zeta;\rho)d\zeta$$

for each $C(z;r)$ *in* D_ρ.

Proofs of the corollaries are contained in the proof of Theorem 1.

The equation (2) and recent results on the polygonal mean-values of harmonic polynomials [2] suggest the analogue of (2) wherein $C(z;r)$ is replaced by a regular n-gon $p_n(z;r;\phi)$, $n\geqq 3$. Let $P_n(z;r;\phi)$ denote the closed, finite region bounded by the regular n-gon $p_n(z;r;\phi)$ whose center is at z and whose inscribed circle has radius r; ϕ denotes the angle from R to N, $-\pi/n\leqq\phi<\pi/n$, where R is the ray extending horizontally to the right from z and N is the exterior normal at the point where R emerges from the polygon.

Here n denotes a fixed, positive integer, $n\geqq 3$, and ϕ always denotes an (arbitrary) angle, $-\pi/n\leqq\phi<\pi/n$. For brevity, $p_n(z;r;\phi)$ will be denoted by $p(z;r)$ and $P_n(z;r;\phi)$ will be denoted by $P(z;r)$; $|P|$ will denote the area of $P(z;r)$.

The analogue of (2) referred to above is

$$F_{r,\phi}(z) = \frac{1}{|P|\,i}\int_{p(z;r)} f(\zeta)d\zeta, \tag{7}$$

which is defined on an open subset $D_{r,\phi}$ of D, for sufficiently small r.

The following result is comparable to the preceding theorem.

THEOREM 2. *If* $f(z)$ *is continuous in* D, *then a necessary and sufficient condition that* $f(z)$ *be areolar monogenic in* D *is that* $F_{r,\phi}(z)$ *be analytic in* $D_{r,\phi}$ *for each pair* r, ϕ.

Necessity. If $f(z)$ is areolar monogenic in D, then $f(z)$ has continuous partial derivatives (of all orders) in D [3]. Hence (7) and Green's lemma yield the following representation for $F_{r,\phi}(z)$:

$$F_{r,\phi}(z) = \frac{1}{|P|}\iint_{P(z;r)} G(\zeta)d\xi d\eta, \tag{8}$$

where, by (1) and (5), $G(z)$ is analytic in D. Now (8) shows that $F_{r,\phi}(z)$ is an integral mean of $G(z)$, so that $F_{r,\phi}(z)$ is analytic wherever defined, that is, in $D_{r,\phi}$.

Sufficiency. First suppose that $f(z)$ has continuous partial derivatives of the first order in D. Then it follows from (7) and Green's lemma that $F_{r,\phi}(z)$ can be written in the form (8). It now follows from (8) that $F_{r,\phi}(z)\to G(z)$, as $r\to 0$, on each closed subset of D. Since $F_{r,\phi}(z)$ is analytic in $D_{r,\phi}$, for all sufficiently small r, it follows that

$G(z)$ is analytic in D. Hence, as in the proof of Theorem 1, it follows that $f(z)$ is areolar monogenic in D.

The requirement that $f(z)$ have continuous partial derivatives of the first order in D may be removed as in the proof of Theorem 1. This completes the proof.

It should be noted that if $f(z)$ is an arbitrary areolar monogenic function (hence, if $G(z)$ is an arbitrary analytic function) in D, then $F_{r,\phi}(z)$ is analytic in $D_{r,\phi}$, though not necessarily independent of r. Indeed, if $F_{r,\phi}(z)$ is to be both analytic and independent of r, in $D_{r,\phi}$, then the following result holds.

THEOREM 3. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $F_{r,\phi}(z)$ be both analytic and independent of r in $D_{r,\phi}$, for fixed ϕ, is that $f(z)$ be areolar monogenic in D, with the representation*

$$f(z) = 2^{-1}\sum_{0}^{n-1} c_k(\bar{z}z^k - z^{k-1}) + \Psi_y + i\Psi_x, \tag{9}$$

where $\bar{z}=x-iy$, where the c_k are arbitrary complex constants, and where $\Psi(x, y)$ is an arbitrary function harmonic in D.

To prove Theorem 3, the following lemma is needed.

LEMMA. *If $f(z)$ is areolar monogenic in D, then a necessary and sufficient condition that $G(z)$ be a polynomial in z of degree at most $(n-1)$ is that $f(z)$ have the representation* (9).

Necessity. Let $G(z)$ have the representation

$$G(z) = \sum_{0}^{n-1} c_k z^k. \tag{10}$$

Now Haskell has shown [3] that for areolar monogenic $f(z)$ there exist real functions $\mu(x, y)$, $\nu(x, y)$, $\Psi(x, y)$, with $\Psi(x, y)$ harmonic in D, such that

$$f(z) = (\mu_x + \nu_y) + i(\nu_x - \mu_y) + \Psi_y + i\Psi_x, \tag{11}$$

where

$$\mu + i\nu = -\frac{1}{2\pi}\int\int_D \log \frac{1}{|z-\zeta|} G(\zeta)d\xi d\eta. \tag{12}$$

If the substitutions $\zeta = re^{i\theta}$, $z=\rho e^{i\psi}$ are made in (10) and (12), then (12) yields

$$
\begin{aligned}
\mu + i\nu = & -\frac{1}{2\pi}\int_0^{\rho}\int_0^{2\pi}\left[\log\frac{1}{\rho} + \sum_1^{\infty}\frac{r^k}{k\rho^k}\cos k(\theta-\psi)\right] \\
& \cdot\left[\sum_0^{n-1} c_k r^k e^{ik\theta}\right] r\,dr\,d\theta \\
& -\frac{1}{2\pi}\int_{\rho}^{1}\int_0^{2\pi}\left[\log\frac{1}{r} + \sum_1^{\infty}\frac{\rho^k}{kr^k}\cos k(\theta-\psi)\right] \\
& \cdot\left[\sum_0^{n-1} c_k r^k e^{ik\theta}\right] r\,dr\,d\theta \\
= & \frac{1}{4}\left[\sum_0^{n-1}\frac{c_k}{k+1}\bar{z}z^{k+1} - \sum_1^{n-1}\frac{c_k}{k}z^k - c_0\right].
\end{aligned}
\tag{13}
$$

The representation (9) for $f(z)$ now follows from (11) and (13).

Sufficiency. If $f(z)$ is given by (9), then a computation shows that $G(z)$, given by (5), has the form (10).

PROOF OF THEOREM 3. If $f(z)$ is areolar monogenic in D, with representation (9), then it follows from the lemma that $G(z)$ has the form (10), such that $U(x, y)$ and $V(x, y)$ are harmonic polynomials of degree at most $(n-1)$. It is known that such harmonic polynomials satisfy

$$
\begin{aligned}
U(x, y) &= U(z) = \frac{1}{|P|}\iint_{P(z;r)} U(\zeta)\,d\xi\,d\eta, \\
V(x, y) &= V(z) = \frac{1}{|P|}\iint_{P(z;r)} V(\zeta)\,d\xi\,d\eta,
\end{aligned}
\tag{14}
$$

for each $P(z; r)$ in D [2]. From (5), (8) and (14), it follows that $F_{r,\phi}(z)$ is independent of r, ϕ. This proves the necessity part of the theorem.

On the other hand, if $F_{r,\phi}(z)$ is analytic and independent of r in $D_{r,\phi}$, then by the lemma, $f(z)$ is areolar monogenic in D. Moreover, it follows that the real and imaginary parts of $G(z)$ satisfy (14) and hence $U(x, y)$ and $V(x, y)$ have the representations implied by

$$
G(z) = U + iV = \sum_0^{n-1} c_k z^k + c_n I(z_k\alpha), \qquad \alpha = e^{i\psi}, \tag{15}
$$

where the symbol "I" means "the imaginary part of" [2]. However, since $U(x, y)$ and $V(x, y)$ are conjugate harmonic functions, it follows that $c_n = 0$ in (15). Hence $G(z)$ is a polynomial of degree at most $(n-1)$; therefore, by the lemma, $f(z)$ is areolar monogenic in D with representation (9). The proof is now complete.

The author is indebted to the referee for the observation that Theorem 3 above is true for variable ϕ.

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PURDUE UNIVERSITY

A CANTOR FUNCTION CONSTRUCTED BY CONTINUED FRACTIONS

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1. **Introduction.** The well known Cantor function was treated in a paper by Hille and Tamarkin [5][1], who collected together a few properties of that function, which is often used as an example in the theory of functions of a real variable. In the definition of this function, the representation of the numbers of the closed interval [0, 1] in the arithmetic scales of radices 3 and 2 (ternary and binary scales of notation) was used. Gilman [3] generalized the above function by replacing the radices 3 and 2 by α and β, respectively, where α and β are integers such that $\alpha>\beta>1$ and $\alpha-1$ is divisible by $\beta-1$. He thus obtained a whole class of functions of the Cantor type.

The present paper gives an example of a function of the same type, based on the representation of numbers as simple continued fractions. While in the main our function has the same properties as those treated in [5] and [3], there are also some interesting differences between them, to some of which we shall call attention as the occasion arises. Since some of the results of [5] are a special case of those of [3] (with $\alpha=3$, $\beta=2$), it will in most cases suffice to compare our results with those of [3].

2. **The set *E*.** Let $0<x<1$. If the positive integers $a_1, a_2, a_3, \cdots$ are successively the denominators (partial quotients) appearing in the expansion of x in a simple continued fraction, we shall write

$$x=\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots=\{a_1, a_2, a_3, \cdots\}.$$

In the case of a rational x, $0<x<1$, that is, of a finite simple continued fraction, we have two such expansions, namely, $x=\{a_1, \cdots, a_{n-1}, a_n\}=\{a_1, \cdots, a_{n-1}, a_n-1, 1\}$, where $a_n\geqq 2$. Both of these expansions will be admitted in this paper, also the expansion $1=\{1\}$.

Throughout the paper we shall use the letter c (with or without subscript) to denote positive integers.

We now define the set E as *the set consisting of* $x=0$ *and of all* x, $0<x<1$, *which can be expanded in a simple continued fraction none of whose denominators equals unity*. In other words, E contains $x=0$ and all finite and infinite simple continued fractions of the form

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$x=\{c_1+1,\ c_2+1,\ \cdots\}$. The above wording is to imply that $x=\{c_1+1,\ \cdots,\ c_n+1\}$ does belong to E although it can also be written as $x=\{c_1+1,\ \cdots,\ c_n,\ 1\}$.

The set complementary to E with respect to the interval $[0, 1]$ will be denoted by D. Thus D consists of $x=1$ and of all open intervals with end points ξ and ξ', where[2]

$$\begin{aligned} \xi &= \{c_1+1, c_2+1, \cdots, c_{n-1}+1, 1\} \\ (1) \qquad &= \{c_1+1, c_2+1, \cdots, c_{n-1}+2\}, \\ \xi' &= \{c_1+1, c_2+1, \cdots, c_{n-1}+1, 2\}. \end{aligned}$$

Such an interval will be called a D_n-*interval* in the following. (For examples of D_n-intervals, see the table at the end of the paper.) There is one D_1-interval, $(1/2, 1)$, but infinitely many D_n-intervals for each $n\geqq 2$. It is easily seen that any two D_n-intervals (with equal or unequal values of n) do not overlap and do not even have an end point in common.

The points of E (except for $x=0$) can be divided into two classes, E_R and E_I. E_R is to contain the rational $x\in E$, other than zero, and E_I the irrational $x\in E$. It is clear from the above that $x\in E_R$ if and only if x is the end point of a D_n-interval (except $x=1$, which does not belong to E at all). $x=0$ will be considered as belonging neither to E_R nor to E_I.

We shall now derive a few of the main properties of the set E.

(2.1) *E is a closed set.*

Consider D as the union of the interval $1/2<x\leqq 1$ and all open D_n-intervals, $n=2, 3, \cdots$. All these intervals are open sets with respect to $[0, 1]$. Thus D is an open set and E is a closed one.

(2.2) *E is a perfect set.*

The following three relations, referring to the cases $x=0$, $x\in E_R$, $x\in E_I$, respectively, show that E is perfect. (In each case $m\to+\infty$ through integral values.)

$$0 = \lim_m \{m\},$$

$$\{c_1+1, c_2+1, \cdots, c_n+1\} = \lim_m \{c_1+1, c_2+1, \cdots, c_n+1, m\},$$

$$\{c_1+1, c_2+1, \cdots\} = \lim_m \{c_1+1, c_2+1, \cdots, c_m+1\}.$$

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[2] If $n=1$ the numbers in (1) are to be interpreted as meaning $\xi=\{1\}=1$ and $\xi'=\{2\}=1/2$. We also remark that $\xi<\xi'$ ($\xi>\xi'$) when n is even (odd). See [8, p. 41, Theorem 8].

(2.3) *E has the power of the continuum.*

E contains as a subset the set of all infinite simple continued fractions $\{c_1+1, c_2+1, \cdots\}$ with each $c_i=1$ or 2.

(2.4) *E is of Lebesgue measure zero.*

This is a result due to F. Bernstein [1, p. 429, Remark]. (See also [4, p. 334, corollary].)

(2.5) *E is nowhere dense in* $[0, 1]$.

The complementary set D is by (2.4) of measure one, hence everywhere dense in $[0, 1]$. It is also open in that interval by (2.1). Hence (see [2, p. 64, definition]) E is nowhere dense in $[0, 1]$.

Comparing our set E with the corresponding sets $P_{\alpha\beta}=P$ of [3], we see that they agree in the five properties, derived above. The subsets E_R and E_I of E correspond, respectively, to the subsets P^-+P^+ and P^0 of P. (See [3, p. 434].) However, while P is symmetric about $x=1/2$, E has no such symmetry. A more essential difference is that the intervals δ_{pk} of [5, p. 255] for fixed p are finite in number and of equal length, while in our case we have infinitely many D_n-intervals for fixed $n\geqq 2$ and their lengths are not equal.

3. **Definition and main properties of $\phi(x)$.** We shall first define our function for $x\in E$ by the following three equations, which apply to the cases $x=0$, $x\in E_R$, $x\in E_I$, respectively.

$$\phi(0) = 0. \tag{2a}$$

$$x = \{c_1+1, c_2+1, \cdots, c_n+1\} : \phi(x) = \{c_1, c_2, \cdots, c_n\}. \tag{2b}$$

$$x = \{c_1+1, c_2+1, \cdots\} : \phi(x) = \{c_1, c_2, \cdots\}. \tag{2c}$$

Before extending this definition to values of $x\in D$, we first define $\phi(1)=1$ and show that the values of $\phi(x)$ at both end points of each D_n-interval are equal to one another. This is true for the D_1-interval $(1/2, 1)$, since $\phi(1/2)=\phi(\{2\})=\{1\}=1=\phi(1)$, and for a D_n-interval with $n\geqq 2$ we see from (2b) that, when ξ and ξ' are given by (1), we have

$$\phi(\xi) = \{c_1, c_2, \cdots, c_{n-1}+1\} = \{c_1, c_2, \cdots, c_{n-1}, 1\} = \phi(\xi').$$

We are now able to define $\phi(x)$ for any interior point of a D_n-interval as the common value of $\phi(x)$ at the end points of that interval. (For examples, see the table at the end of the paper.) The definition of $\phi(x)$ in $[0, 1]$ is thus complete.

We shall now derive some of the more important properties of $\phi(x)$.

(3.1) *$\phi(x)$ is constant in every (closed) D_n-interval. The union of the intervals of constancy of $\phi(x)$ is a set of measure one.*

This follows at once from the definition of $\phi(x)$ for $x \in D$ and from (2.4).

(3.2) *$\phi(x)$ is a nondecreasing function of x in $[0, 1]$; moreover, when $0 \leqq x < x' \leqq 1$ then $\phi(x) \leqq \phi(x')$, the equality holding if and only if x and x' lie in the same (closed) D_n-interval.*

On account of (3.1), it suffices to consider the case in which x and x' belong to E. Since by (2a) the case $x = 0$ is trivial we may assume furthermore that $0 < x < x' \leqq 1/2$. Now let x and x' be written in the form (2b) or (2c) and assume that they agree in the first $m-1$ denominators, but no further ($m \geqq 1$). Then when m is even (odd) either the mth denominator of x' is larger (smaller) than that of x, or x' (x) has only $m-1$ denominators. (See [8, pp. 40–41, Theorems 7 and 8].) The definitions (2b) and (2c) of $\phi(x)$ are such that these same relations will hold true for the simple continued fractions for $\phi(x)$ and $\phi(x')$, so that in general $\phi(x) < \phi(x')$. And it is easily seen that the only exceptional case, leading to $\phi(x) = \phi(x')$, is that in which x and x' are the smaller and larger, respectively, of the numbers ξ and ξ' (with $n \geqq 2$) of (1). This completes the proof.

(3.3) *The function $y = \phi(x)$ assumes every value of y of the interval $0 \leqq y \leqq 1$. Moreover, if y is rational and $0 < y \leqq 1$ then y is assumed by $\phi(x)$ in one whole D_n-interval, while $y = 0$ and every irrational y, $0 < y < 1$, is assumed at one point x.*

This follows at once from the definitions (2a), (2b) and (2c) of the function $\phi(x)$ and its definition for $x \in D$.

(3.4) *$\phi(x)$ is continuous in $[0, 1]$.*

A function $f(x)$ which is defined and monotone in a closed interval $[a, b]$ is continuous in that interval if and only if it assumes every value between $f(a)$ and $f(b)$. Thus (3.4) follows from (3.2) and (3.3).

(3.5) *For every positive λ, the λ-variation* (see [2, p. 511, Definition]) *of $\phi(x)$ in $[0, 1]$ is equal to unity; $\phi(x)$ is not absolutely continuous.*

We shall show the first part of (3.5) by proving the more general theorem below, whose hypotheses are satisfied for $\phi(x)$ by (3.2) and (3.1). The second part of (3.5) follows from the first, but could be directly concluded from the fact that $\int_0^1 \phi'(x)dx = 0$, but $\phi(1) - \phi(0) = 1$.

THEOREM 1. *If $f(x)$ is monotone*[3] *in the closed interval $[a, b]$ and if the intervals of constancy of $f(x)$ cover $[a, b]$ except for a set of measure zero, then the λ-variation of $f(x)$ in $[a, b]$ equals $|f(b) - f(a)|$ for every positive λ.*

Let $f(x)$ be nondecreasing. (If $f(x)$ is nonincreasing consider the function $-f(x)$.) Let $\lambda>0$ be given. We denote the intervals of constancy of $f(x)$, of which there are at most denumerably many, in some order by I_j ($j=1, 2, \cdots$) and their lengths by $|I_j|$. Since $\sum_j |I_j| = b-a$ we can in any case choose k such that $\sum_{j=1}^{k} |I_j| > b-a-\lambda$. We now replace these k intervals I_j by intervals I_j^* such that the end points of I_j^* are inner points of I_j ($j=1, 2, \cdots, k$) and such that we still have $\sum_{j=1}^{k} |I_j^*| > b-a-\lambda$. Let these k intervals I_j^* be denoted in their natural order (from left to right) by $(x_1, x_2), (x_3, x_4), \cdots, (x_{2k-1}, x_{2k})$. We know then that $f(x_{2i-1}) = f(x_{2i})$ for $i=1, 2, \cdots, k$. The complementary intervals $(a, x_1), (x_2, x_3), \cdots, (x_{2k-2}, x_{2k-1}), (x_{2k}, b)$ have a combined length less than λ, while (with $x_0=a, x_{2k+1}=b$)

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[3] Continuity of $f(x)$ is not assumed.

$$\sum_{i=0}^{k} [f(x_{2i+1}) - f(x_{2i})] = f(b) - f(a).$$

This completes the proof of Theorem 1.

(3.6) *The curve* $y=\phi(x)$ *is of arc length*[4] 2.

This will be proved by the following theorem. Its hypotheses are satisfied by $\phi(x)$ on account of (3.2) and (3.5).

THEOREM 2. *If* $f(x)$ *is monotone*[3] *in the closed interval* $[a, b]$ *and if its* λ*-variation in* $[a, b]$ *equals* $|f(b)-f(a)|$ *for every positive* λ, *then the arc length*[4] *of the curve* $y=f(x)$ *is* $b-a+|f(b)-f(a)|$.

Let $f(x)$ be nondecreasing. That the curve $y=f(x)$ is rectifiable follows from the fact that $f(x)$ is of bounded variation. Since the length of any inscribed polygon can obviously not exceed $b-a+f(b)-f(a)$ it suffices to show the existence of an inscribed polygon of length greater than $b-a+f(b)-f(a)-\epsilon$ for every given positive ϵ. By the hypothesis, we can select a set of intervals $(x_1, x_2), (x_3, x_4), \cdots, (x_{2k-1}, x_{2k})$ with $x_j < x_{j+1}$ ($j=1, 2, \cdots, 2k-1$) such that

$$\sum_{j=1}^{k} (x_{2j} - x_{2j-1}) < \frac{\epsilon}{2}, \tag{3}$$

$$\sum_{j=1}^{k} [f(x_{2j}) - f(x_{2j-1})] > f(b) - f(a) - \frac{\epsilon}{2}\cdot \tag{4}$$

Let P_i be the point $(x_i, f(x_i))$, $i=1, 2, \cdots, 2k$, and form the polygon

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[4] If $y=f(x)$ is of bounded variation in $[a, b]$ then the two definitions of arc length by Peano and by Jordan and Scheeffer (see [6, p. 338]) are equivalent for functions $f(x)$ which have the additional property that for each x, $a<x<b$, either $f(x-) \leqq f(x) \leqq f(x+)$ or $f(x-) \geqq f(x) \geqq f(x+)$. (See [7, pp. 300–302].) Monotone functions obviously have this property. The definition used in the proof of Theorem 2 is Peano's definition.

$P_0P_1P_2 \cdots P_{2k}P_{2k+1}$, where[5] $P_0=(a,f(a))$ and $P_{2k+1}=(b,f(b))$. By (3) the projections of the segments P_0P_1, P_2P_3, $\cdots$, $P_{2k-2}P_{2k-1}$, $P_{2k}P_{2k+1}$ on the x-axis have a combined length greater than $b-a-\epsilon/2$; and by (4) the projections of the segments P_1P_2, P_3P_4, $\cdots$, $P_{2k-1}P_{2k}$ on the y-axis have a combined length greater than $f(b)-f(a)-\epsilon/2$. This completes the proof of Theorem 2.

The function $\phi(x)$ has the properties, which were derived in this section, in common with the functions $\omega_{\alpha\beta}(x) \equiv \omega(x)$ of [3]. In fact, properties (3.5) and (3.6) were proved simultaneously for all of these functions by Theorems 1 and 2, respectively. It is to be observed, however, that according to (3.3) our curve $y=\phi(x)$ has all rational numbers y, $0<y\leqq 1$, as ordinates of its horizontal segments, while the functions $\omega_{\alpha\beta}(x)$ of [3] have as such only certain rational numbers y, $0<y<1$, namely, those of the form $y=m/\beta^r$, r and m being positive integers and β being the radix, used for the values of y.

4. **The derivatives of $\phi(x)$.** The problem of determining the derivatives of the functions $\omega_{\alpha\beta}(x)$ of [3] was dealt with by Gilman [3, pp. 438–442], who obtained conditions for the existence of the derivative on the left and that on the right at the points of the set $P_{\alpha\beta}$. The various possibilities are listed on p. 442 of [3]. The corresponding problem for our function $\phi(x)$ seems to be more difficult. Except for one case (see footnote 7 below), we shall obtain the values of the four derivatives of $\phi(x)$ at all points x of $[0, 1]$.

We introduce first the following notations. In this section $x_0 = \{c_1+1, c_2+1, \cdots\}$ will always denote a fixed number such that $x_0 \in E$, $x_0 \neq 0$. If the simple continued fraction for x_0 has at least n denominators we shall denote the nth complete quotient of x_0 by c_n+1+s_n, where $s_n=0$ if x_0 has only n denominators and otherwise

$$s_n = \{c_{n+1} + 1, c_{n+2} + 1, \cdots\}. \tag{5}$$

In either case we have $s_n \in E$, so that $0 \leqq s_n \leqq 1/2$, and we conclude from [8, p. 27, Rule A] that

$$x_0 = \{c_1 + 1, c_2 + 1, \cdots, c_{n-1} + 1, c_n + 1 + s_n\}. \tag{6}$$

It follows readily from (2a), (2b) and (2c) in conjunction with (5) that

$$\phi(x_0) = \{c_1, c_2, \cdots, c_{n-1}, c_n + \phi(s_n)\}. \tag{7}$$

We denote the mth convergent of the simple continued fraction for x_0 by P_m/Q_m and that for $\phi(x_0)$ by[6] p_m/q_m ($m=1, 2, \cdots$) and obtain

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[5] If $x_1=a$ the polygon is to start with P_1 and if $x_{2k}=b$ it is to end with P_{2k}.

[6] As usual we put $P_{-1}=p_{-1}=Q_0=q_0=1$ and $P_0=p_0=Q_{-1}=q_{-1}=0$.

$$x_0 = \frac{(c_n + 1 + s_n)P_{n-1} + P_{n-2}}{(c_n + 1 + s_n)Q_{n-1} + Q_{n-2}};\tag{8}$$
$$\phi(x_0) = \frac{[c_n + \phi(s_n)]p_{n-1} + p_{n-2}}{[c_n + \phi(s_n)]q_{n-1} + q_{n-2}}$$

from (6) and (7). If now x is any number such that $x \in E$, $x \neq 0$, $x \neq x_0$, and (at least) the first $n-1$ denominators of the simple continued fraction of x agree with those of x_0 we may write in analogy to (6)

$$x = \{c_1 + 1, c_2 + 1, \cdots, c_{n-1} + 1, c + 1 + s\},\tag{9}$$

where, as in (6), c is a positive integer and $s \in E$, so that $0 \leqq s \leqq 1/2$. Formulas analogous to (7) and (8) hold also for x. By the use of the formulas (8) for x_0 and x and the relation $P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1} = p_{n-1}q_{n-2} - p_{n-2}q_{n-1} = (-1)^n$ we then obtain the following formula:

$$\begin{aligned}\Delta(x, x_0) &= [\phi(x) - \phi(x_0)]/(x - x_0)\\ &= \frac{c - c_n + \phi(s) - \phi(s_n)}{c - c_n + s - s_n} \cdot \frac{(c_n + 1 + s_n)Q_{n-1} + Q_{n-2}}{[c_n + \phi(s_n)]q_{n-1} + q_{n-2}}\\ &\quad \cdot \frac{(c + 1 + s)Q_{n-1} + Q_{n-2}}{[c + \phi(s)]q_{n-1} + q_{n-2}}.\end{aligned}\tag{10}$$

We shall also need the following inequalities. If $x \in E$, $x \neq 0$, we can write x in the form (9) with $n = 1$, that is, $x = \{c+1+s\}$ with positive integral c, and $s \in E$; by (7), $\phi(x) = \{c+\phi(s)\}$ and hence

$$\phi(x)/x = (c + 1 + s)/[c + \phi(s)],\tag{11}$$

so that $\phi(x)/x > 1$. Therefore, $\phi(x)/x \leqq (c+1+s)/(c+s) \leqq (c+1)/c \leqq 2$. In view of $0 \leqq s \leqq 1/2$ we conclude now that $\phi(x)/x \geqq (c+1+s)/(c+2s) \geqq (c+3/2)/(c+1) \geqq 1+1/4c$. We thus obtained for $x \in E$, $x \neq 0$,

$$1 < 1 + 1/4c \leqq \phi(x)/x \leqq 1 + 1/c \leqq 2,\tag{12}$$

where $c+1$ is the first denominator of the simple continued fraction for x. We now write the second fraction on the right of (10) in the form

$$\frac{Q_{n-1}}{q_{n-1}} \, \frac{c_n + 1 + s_n + t_{n-1}}{c_n + \phi(s_n) + \phi(t_{n-1})},\tag{13}$$

where

$$\begin{aligned} t_{n-1} &= \{c_{n-1} + 1, \cdots, c_2 + 1, c_1 + 1\} = Q_{n-2}/Q_{n-1},\\ \phi(t_{n-1}) &= \{c_{n-1}, \cdots, c_2, c_1\} = q_{n-2}/q_{n-1},\end{aligned}\tag{14}$$

when $n>1$, and $t_0=\phi(t_0)=0$. Obviously, $t_{n-1}\in E$ for $n\geq 1$. Applying the "outer" estimates of (12) to $\phi(s_n)$ and $\phi(t_{n-1})$ and using $0\leq s_n+t_{n-1}\leq 1$, we see that the second fraction in (13) is always bounded between 1 and 2. Treating the third fraction on the right of (10) similarly, we obtain from (10)

$$G_n(Q_{n-1}/q_{n-1})^2 \leq \Delta(x, x_0) \leq 4G_n(Q_{n-1}/q_{n-1})^2, \tag{15}$$

where

$$G_n = (c - c_n + \phi(s) - \phi(s_n))/(c - c_n + s - s_n). \tag{16}$$

We shall now state our results, concerning the derivatives of $\phi(x)$. We shall use the notations D_-, D^-, D_+, D^+ for the lower and upper left-hand, lower and upper right-hand derivatives, respectively. If all four are equal we shall use the notation $\phi'(x)$. We remark that, in investigating the behavior of the difference quotient $\Delta(x, x_0)$ as $x\to x_0$, where $x_0\in E$, we may restrict also x to values of E, since for $x\in D$, $\Delta(x, x_0)$ lies between $\Delta(\xi, x_0)$ and $\Delta(\xi', x_0)$, where ξ and ξ' are the end points of the D_n-interval to which x belongs.

(4.1) *When* $x\in D$ *then* $\phi'(x)=0$.

This follows at once from (3.1).

(4.2) $\phi'(0)=1$.

By (2a) we have to show that $\Delta(x, 0)=\phi(x)/x\to 1$, as $x\to 0+$. We write x in the form $x=\{c+1+s\}$ (see above). When $x\to 0+$, $c\to+\infty$, while $0\leq s\leq 1/2$; thus our statement follows from (11).

(4.3) *When* $x_0\in E_R$ *and* x_0 *is the right-hand end point of a* D_n*-interval* $(n\geq 2)$ *we have* $D_-\phi(x_0)=D^-\phi(x_0)=0$, *while* $D_+\phi(x_0)=D^+\phi(x_0)=(Q/q)^2$, *where* Q *and* q *are the denominators of the reduced fractions for* x_0 *and* $\phi(x_0)$, *respectively. When* x_0 *is the left-hand end point of a* D_n*-interval* $(n\geq 1)$ *the above relations hold with the indices* $-$ *and* $+$ *interchanged.* (For examples, see the table at the end of the paper.)

Let $x_0=\{c_1+1, c_2+1, \cdots, c_n+1\}$ and assume that x_0 is the right-hand end point of an interval of constancy of $\phi(x)$, that is, that n is even. The first statement of (4.3) follows at once from (3.1). To investigate $\lim \Delta(x, x_0)$ as $x\to x_0+$, we may restrict x to numbers of E in the interval $x_0<x<\{c_1+1, c_2+1, \cdots, c_{n-1}+1, c_n+2\}$. Such an x can be written in the form (9) with $c=c_n$ and $s\in E$, $s\neq 0$. By (6) we have in this case $s_n=0$. Substituting these values of c and s_n in (10) and using the reduction formulas for the Q_m and q_m (see [8, p. 28, (6) and (7)]), we obtain

$$\Delta(x, x_0) = \frac{\phi(s)}{s}\,\frac{Q_n}{q_n}\,\frac{(c_n+1+s)Q_{n-1}+Q_{n-2}}{[c_n+\phi(s)]q_{n-1}+q_{n-2}}.$$

As $x \to x_0+$, we have $s \to 0+$ and hence, by (4.2), $\Delta(x, x_0) \to (Q_n/q_n)^2 = (Q/q)^2$, since $x_0 = P_n/Q_n$ and $\phi(x_0) = p_n/q_n$. The proof for the left-hand end point of an interval of constancy of $\phi(x)$ requires only the changing of a few symbols in the above proof.

The result of (4.3) represents an interesting difference between our function $\phi(x)$ and the functions $\omega_{\alpha\beta}(x)$ of [3]. For the latter the derivative at the points of the corresponding sets P^- and P^+ is zero on one side, but $+\infty$ on the other side. See [3, p. 438, Theorem 3].

Before considering the case $x_0 \in E_I$, we shall prove the following lemma, concerning the fractions Q_n/q_n, which in this case are defined for all positive integral n.

LEMMA. *When $x_0 \in E_I$ the fractions Q_n/q_n form an increasing sequence which approaches a finite limit $L > 1$ or $+\infty$, according to whether the infinite series*

$$\sum_{j=1}^{\infty} 1/c_j \tag{17}$$

converges or diverges.

We have $Q_1/q_1 = 1 + 1/c_1$. Let $j > 1$. Then by (14), $(Q_j/q_j)/(Q_{j-1}/q_{j-1}) = (q_{j-1}/q_j)/(Q_{j-1}/Q_j) = \phi(t_j)/t_j$, and hence, by the "inner" estimates in (12),

$$1 + 1/4c_j \leqq (Q_j/q_j)/(Q_{j-1}/q_{j-1}) \leqq 1 + 1/c_j.$$

Multiplying over j from $j = 2$ to $j = n$, we obtain for $n = 1, 2, \cdots$ $\prod_{j=1}^{n}(1+1/4c_j) < Q_n/q_n \leqq \prod_{j=1}^{n}(1+1/c_j)$. From these inequalities the lemma follows easily.

(4.4) *Let $x_0 = \{c_1+1, c_2+1, \cdots\} \in E_I$.*

(i) *If the c_j are bounded then $\phi'(x_0) = +\infty$.*

(ii) *If the c_j are unbounded but the series* (17) *diverges, then*[7] $D^-\phi(x_0) = D^+\phi(x_0) = +\infty$.

(iii) *If the series* (17) *converges then $D^-\phi(x_0)$ and $D^+\phi(x_0)$ are finite numbers greater than unity, while $D_-\phi(x_0) = D_+\phi(x_0) = 0$.*

Let $x \in E$, $x \neq 0$, $x \neq x_0$, and let the simple continued fractions for x and x_0 agree in the first $n-1$ denominators but no further. We can then[8] write x in the form (9) with $c \neq c_n$ and make use of (15). We observe that $x \to x_0$ means that $n \to \infty$, while c in (9) may assume any positive integral value, other than c_n, and s in (9) any value of E. Finally, if $c > c_n$ we have $x > x_0$ ($x < x_0$) when n is even (odd), while if $c < c_n$ the situation is reversed.

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[7] No statement is being made about $D_-\phi(x_0)$ and $D_+\phi(x_0)$ in this case. In fact, this is the only problem, concerning the derivatives of $\phi(x)$, not answered in this section.

[8] There is one exceptional case: the convergents of x_0, that is, the numbers $x = \{c_1+1, c_2+1, \cdots, c_{n-1}+1\}$, $n = 2, 3, \cdots$; cannot be written in the form (9) with $c \neq c_n$ and $s \in E$. A moment's consideration shows that their omission (as well as that of $x = 0$) from the values of x in $\Delta(x, x_0)$ has no influence on the behavior of $\Delta(x, x_0)$ as $x \to x_0-$ or $x \to x_0+$.

In the case (i), the series (17) diverges, hence, by the lemma, $Q_n/q_n \to +\infty$. Therefore, by (15) it will suffice to show that G_n is greater than a positive constant for all values of n, c and s (see above). Since the c_j are bounded and in view of (5), there exists a δ, $0 < \delta < 1/4$, such that $\delta < s_n < 1/2 - \delta$ for all n, and hence by (3.2) an ϵ, $0 < \epsilon < 1/2$, such that $\epsilon < \phi(s_n) < 1 - \epsilon$ for all n. Thus the denominator of G_n (see (16)) is numerically less than $|c - c_n| + 1/2$, while its numerator is numerically greater than $|c - c_n| - (1 - \epsilon)$. In view of $|c - c_n| \geqq 1$, this shows that $G_n > 2\epsilon/3$.

In the case (ii) we still have $Q_n/q_n \to +\infty$. Hence in view of (15), we have only to show the existence of two sequences of numbers in E, approaching x_0, one from the left and one from the right, such that G_n for all members of these sequences is greater than a positive constant. These sequences are obtained by putting $c = c_n + 1$ and $s = s_n$ for each n, in which case (16) gives $G_n = 1$. And for even (odd) n we have approach from the right (left).

In the case (iii), Q_n/q_n approaches a finite limit $L > 1$ (see the lemma). In order to show that $D^-\phi(x_0)$ and $D^+\phi(x_0)$ are finite, it suffices by (15) to show that G_n is bounded above by a constant for all n, c and s. But it is easily seen that the denominator of G_n in (16) is numerically greater than or equal to $|c - c_n| - 1/2$, its numerator is numerically less than or equal to $|c - c_n| + 1$ and hence $G_n \leqq 4$. In order to show that $D^-\phi(x_0)$ and $D^+\phi(x_0)$ are greater than unity, we use the method used above in case (ii). In fact, the same sequences, defined by $c = c_n + 1$, $s = s_n$, hence $G_n = 1$, will suffice here.[9] Finally, in order to show that $D_-\phi(x_0) = D_+\phi(x_0) = 0$, we remark that by (3.2) these derivatives could not be less than zero and hence the method of case (ii) can again be used, only that here (see (15)) we have to choose c and s in such a way that $G_n \to 0$. Since, in case (iii), $c_n \to \infty$ and hence, by (5), $s_n \to 0+$, we may put (for sufficiently large n) $c = c_n - 1$ and $s = 1/2$, so that, by (16), $G_n = \phi(s_n)/(1/2 + s_n) \to 0$, as $n \to \infty$. This completes the proof of (4.4).

We remark that the cases (i), (ii) and (iii) of (4.4) divide the set E_I into three subsets, each of which has the power of the continuum.

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[9] By the use of more careful estimates it is not difficult to show that $D^-\phi(x_0) = D^+\phi(x_0) = 2L^2$, where $L = \lim_n (Q_n/q_n)$. The details of the proof of this fact are not given here.

We have thus exhibited a continuum of points at which[10] $\phi'(x)=+\infty$ (case (i)) and also one at which neither the left-hand nor the right-hand derivative exists (case (iii)). These two subsets correspond to the types I and IV, respectively, of [3, p. 442].

The methods used in this section can also be applied to the problem of whether the function $\phi(x)$ satisfies a Lipschitz condition: $\phi(x')-\phi(x)\leq M(x'-x)^{\nu}$ for $0\leq x<x'\leq 1$. (Compare [3, p. 437, Theorem 2].) We mention without proof that $\phi(x)$ satisfies a Lipschitz condition of order $\nu=\log b/\log a=.546$, where $a=\{2, 2, \cdots\}=2^{1/2}-1$ and $b=\phi(a)=\{1, 1, \cdots\}=(5^{1/2}-1)/2$, but that $\phi(x)$ does not satisfy a Lipschitz condition of any higher order. The "best" value for the constant M, if $\nu=\log b/\log a$, is $10^{\nu}/2=1.758$.

5. **Table.** The following table, which we give for the convenience of the reader, contains a few examples of D_n-intervals, which we denote here by (x_1, x_2); it gives the value of $\phi(x)$ for each of these intervals and the left-hand (right-hand) derivative at its left (right) end point. The values of x and $\phi(x)$ are written as simple continued as well as ordinary fractions.

TABLE

n	D_n-interval $=(x_1, x_2)$		$\phi(x)$ for $x_1\leq x\leq x_2$	$D_-\phi(x_1)$ $D^-\phi(x_1)$	$D_+\phi(x_2)$ $D^+\phi(x_2)$
	x_1	x_2			
1	$\{2\}=1/2$	$\{1\}=1$	$\{1\}=1$	4	—
2	$\{3\}=1/3$	$\{2,2\}=2/5$	$\{2\}=1/2$	9/4	25/4
2	$\{4\}=1/4$	$\{3,2\}=2/7$	$\{3\}=1/3$	16/9	49/9
2	$\{5\}=1/5$	$\{4,2\}=2/9$	$\{4\}=1/4$	25/16	81/16
3	$\{2,2,2\}=5/12$	$\{2,3\}=3/7$	$\{1,2\}=2/3$	16	49/9
3	$\{2,3,2\}=7/16$	$\{2,4\}=4/9$	$\{1,3\}=3/4$	16	81/16
3	$\{3,2,2\}=5/17$	$\{3,3\}=3/10$	$\{2,2\}=2/5$	289/25	4
4	$\{2,2,3\}=7/17$	$\{2,2,2,2\}=12/29$	$\{1,1,2\}=3/5$	289/25	841/25

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[10] In this connection, see [5, p. 258, footnote 2]. We also note that all quadratic irrationalities between 0 and 1 fall under case (i).

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NOTE ON THE DEGREE OF CONVERGENCE OF SEQUENCES OF POLYNOMIALS

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The object of this note is to establish the following result:

THEOREM. *Let the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

have the radius of convergence ρ (>1), *and let* $p_n(z)$ *denote the polynomial of degree* n *of best approximation to* $f(z)$ *in the closed region* $|z| \leq 1$ *in the sense of Tchebycheff. A necessary and sufficient condition for*

$$\lim_{n\to\infty} [\max |f(z) - p_n(z)|, \text{ for } |z| \leq 1]^{1/n} = 1/\rho \tag{2}$$

is that $f(z)$ *not be of lacunary structure.*

It is well known[1] that the equation

$$\limsup_{n\to\infty} [\max |f(z) - p_n(z)|, \text{ for } |z| \leq 1]^{1/n} = 1/\rho \tag{3}$$

is valid for every $f(z)$ defined by a power series as in (1) with radius of convergence ρ. The significance of the theorem is that the stronger relation (2) is valid except for functions of lacunary structure, as defined by Bourion.[2]

If and only if $f(z)$ is of lacunary structure, the partial sums $s_n(z) = \sum_0^n a_k z^k$ are polynomials of degree n of which a suitably chosen subsequence $s_{n_k}(z)$ has the property (Bourion, loc. cit.)

$$\limsup_{n_k\to\infty} [\max |f(z) - s_{n_k}(z)|, \text{ for } |z| \leq r]^{1/n_k} < r/\rho, \quad 0 < r < \rho. \tag{4}$$

If (4) holds for a single r, $0 < r < \rho$, it holds for every such r.

If $f(z)$ is of lacunary structure, then for the extremal polynomials $p_n(z)$ of best approximation we have

$$[\max |f(z) - p_{n_k}(z)|, \text{ for } |z| \leq 1] \\ \leq [\max |f(z) - s_{n_k}(z)|, \text{ for } |z| \leq 1],$$

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[1] Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, 1935, chap. 4.

[2] *L'ultraconvergence dans les séries de Taylor*, Actualités Scientifiques et Industrielles, no. 472, 1937, pp. 9 ff.

whence from (4)

$$\limsup_{n_k\to\infty}\left[\max\left|f(z)-p_{n_k}(z)\right|,\ \text{for } |z|\leq 1\right]^{1/n_k} < 1/\rho, \tag{5}$$

and (2) is not satisfied.

Conversely, if (2) is not satisfied, then for a suitably chosen sequence n_k the first member of (5), which we denote by $1/\rho'$, is less than $1/\rho$. For values of z interior to the unit circle C we have the relations

$$f(z)-s_n(z)=\frac{1}{2\pi i}\int_C\frac{z^{n+1}f(t)\,dt}{t^{n+1}(t-z)},$$

$$0=\frac{1}{2\pi i}\int_C\frac{z^{n+1}p_n(t)\,dt}{t^{n+1}(t-z)},$$

$$f(z)-s_n(z)=\frac{1}{2\pi i}\int_C\frac{z^{n+1}\left[f(t)-p_n(t)\right]dt}{t^{n+1}(t-z)},$$

$$\limsup_{n_k\to\infty}\left[\max\left|f(z)-s_{n_k}(z)\right|,\ \text{for } |z|\leq r<1\right]^{1/n_k}\leq r/\rho' < r/\rho,$$

whence $f(z)$ is of lacunary structure. The theorem is established.

The generalization of equation (2), where we now consider approximation on a suitably chosen more general point set than $|z|\leq 1$, furnishes a generalization of the concept of functions of lacunary structure.

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SUBSERIES OF SERIES WHICH ARE NOT ABSOLUTELY CONVERGENT

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1. **Introduction.** It is the object of this note to give two theorems on series, of real or complex terms, which fail to converge absolutely. The second is a corollary of the first. They say, roughly, that each such series becomes or remains divergent after "nearly all" of its terms, suitably selected, are discarded or replaced by zeros.

THEOREM 1. *If* $a(1)+a(2)+a(3)+\cdots$ *is a series of real or complex terms which fails to converge absolutely, then there is an increasing sequence*

$$1 \leqq n_1 < n_2 < n_3 < \cdots \tag{1}$$

of integers such that $n_{k+1}-n_k\to\infty$ *as* $n\to\infty$ *and the series*

$$a(n_1) + a(n_2) + a(n_3) + \cdots \tag{2}$$

is divergent.

THEOREM 2. *If* $a(1)+a(2)+a(3)+\cdots$ *is a series of real or complex terms which fails to converge absolutely, then there is a sequence* $x_1, x_2, x_3, \cdots$ *of which each element is either* 0 *or* 1, *such that*

$$\lim_{n\to\infty} (x_1 + x_2 + \cdots + x_n)/n = 0 \tag{3}$$

and the series

$$a(1)x_1 + a(2)x_2 + a(3)x_3 + \cdots \tag{4}$$

is divergent.

2. **Series of non-negative terms.** In this section, we obtain the conclusions of Theorems 1 and 2 for the case in which $a(1)+a(2)+\cdots$ is a series of real non-negative terms which fails to converge absolutely and accordingly $a(1)+a(2)+\cdots$ is a divergent series of real non-negative terms. Choose integers

$$1 = \alpha_2 < \beta_2 < \alpha_3 < \beta_3 < \alpha_4 < \beta_4 < \cdots \tag{5}$$

such that, for each $p=2, 3, 4, \cdots$,

$$\sum_{k=\alpha_p}^{\beta_p-1} a(k) > p, \tag{6}$$

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$$\alpha_{p+1} - \beta_p > p, \tag{7}$$

and

$$\beta_p - \alpha_p = p\gamma_p \tag{8}$$

where $\gamma_2, \gamma_3, \gamma_4, \cdots$ are integers. For each index p,

$$\sum_{k=0}^{p-1} \sum_{j=0}^{\gamma_p-1} a(\alpha_p + k + jp) = \sum_{k=\alpha_p}^{\beta_p-1} a(k) > p \tag{9}$$

and hence there must be an index k_p such that $0 \leqq k_p < p$ and

$$\sum_{j=0}^{\gamma_p-1} a(\alpha_p + k_p + jp) > 1. \tag{10}$$

Let $x_n = 0$ when $\alpha_p \leqq n < \alpha_{p+1}$ except that $x_n = 1$ when

$$n = \alpha_p + k_p + jp, \qquad j = 0, 1, \cdots, \gamma_p - 1. \tag{11}$$

The series (4) is divergent because

$$\sum_{n=\alpha_p}^{\beta_p-1} a(n)x_n = \sum_{j=0}^{\gamma_p-1} a(\alpha_p + k_p + jp) > 1. \tag{12}$$

Let $n_1, n_2, n_3, \cdots$ denote in increasing order the values of n for which $x_n = 1$. The series (2) then becomes the series obtained by omitting zero terms from (4) and hence (2) is divergent. Moreover the sequence $n_1, n_2, n_3, \cdots$ is so constructed that $n_{k+1} - n_k \to \infty$ as $n \to \infty$, and therefore (3) holds. This gives the conclusions of the theorems for the case in which the series have real non-negative terms.

3. **Series of complex terms.** Let $a(1) + a(2) + \cdots$ be a series of complex terms for which $\sum |a(n)| = \infty$. Let $a(n) = b(n) + ic(n)$ where $b(n)$ and $c(n)$ are real. Then at least one of

$$\sum_{k=1}^{\infty} |b(k)| = \infty, \qquad \sum_{k=1}^{\infty} |c(k)| = \infty \tag{13}$$

holds. If the first holds, let $d(k) = b(k)$; otherwise, let $d_k(t) = c_k(t)$. Let

$$d^+(k) = 2^{-1}[|d(k)| + d(k)], \quad d^-(k) = 2^{-1}[|d(k)| - d(k)] \tag{14}$$

so that $d^+(k) \geqq 0$, $d^-(k) \geqq 0$ and $d(k) = d^+(k) - d^-(k)$. Then at least one of

$$\sum_{k=1}^{\infty} d^+(k) = \infty, \qquad \sum_{k=1}^{\infty} d^-(k) = \infty \tag{15}$$

holds. If the first holds, let $e(k)=d^+(k)$; otherwise, let $e(k)=d^-(k)$. Then $\sum e(k)$ is a divergent series of real non-negative terms. Therefore, as was proved in §2, there is an increasing sequence n_k such that $n_{p+1}-n_p\to\infty$ and the series

$$e(n_1)+e(n_2)+e(n_3)+\cdots \tag{16}$$

is divergent. Let

$$e(n_1')+e(n_2')+e(n_3')+\cdots \tag{17}$$

be the subseries of (16) obtained from (16) by omitting all zero terms. Then $n_{p+1}'-n_p'\to\infty$ and (17) is divergent. Since $d^+(k)=0$ when $d^-(k)\neq 0$, and $d^-(k)=0$ when $d^+(k)\neq 0$, it follows that the series

$$d(n_1')+d(n_2')+d(n_3')+\cdots$$

is divergent. Hence at least one of $\sum b(n_k')$ and $\sum c(n_k')$ is divergent and therefore $\sum a(n_k')$ is divergent. This completes the proof of Theorem 1 and hence also that of Theorem 2.

CORNELL UNIVERSITY

ABSOLUTE AND UNCONDITIONAL CONVERGENCE

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A series $\sum_{n=1}^{\infty} x_n$ whose terms are elements of a Banach space B is said to converge *absolutely* if $\sum |x_n|$ converges, *unconditionally* if every rearrangement converges. If a series converges absolutely it converges unconditionally; absolute convergence and unconditional convergence are equivalent in any finite-dimensional Euclidean space, whereas in L^2, for instance, there are simple examples of series which converge unconditionally but not absolutely. The question in what spaces the two definitions are equivalent is left open by Banach [1, p. 240].[1] It may be conjectured that unconditional convergence does not imply absolute convergence in any infinite-dimensional space (cf. [2, p. 30]), but the problem does not seem to have been explicitly treated. The purpose of this note is to obtain a criterion for the equivalence of the two notions in a given Banach space. As examples of its use we shall show that unconditional convergence does not imply absolute convergence in the spaces L and l. The result for L is already known (see for example [3, p. 45]), but the result for l has not, to the author's knowledge, been stated elsewhere.

We shall need the following definitions. Let S be any (finite) sequence of elements $\xi_1, \xi_2, \cdots, \xi_p$ in B. We do not require that the ξ_i be all distinct, and we shall understand by *addition*, S_1+S_2, the mere adjoining of the two sequences. Thus we might write $(1, 2, 2, 3)+(3, 2, 1)=(1, 2, 2, 3, 3, 2, 1)$. By cS (c real) we understand the sequence $c\xi_1, c\xi_2, \cdots, c\xi_p$. We shall use two norms,

$$|S| = \sum |\xi_i|, \qquad |S|^* = \sup_{\sigma} \left| \sum_{\sigma} \xi_i \right|,$$

where σ is any subset of $1, 2, \cdots, p$. It is easily verified that $|S_1+S_2| = |S_1| + |S_2|$, $|S_1+S_2|^* \leq |S_1|^* + |S_2|^*$, $|cS| = |c|\,|S|$, and $|cS|^* = |c|\,|S|^*$. Define further, when $|S| \neq 0$,

$$G(S) = |S|^*/|S|.$$

Note that $0 < G(S) \leq 1$. Let $g = \inf G(S)$, taken over all sequences $S \subset B$.

THEOREM.[2] *If $g=0$, unconditional convergence does not imply absolute convergence in B. If $g>0$, the two are equivalent.*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Suggested to the author by Garrett Birkhoff, in conversation.

PROOF. (i) If $g=0$, we can find a sequence $\{S_n\}$ such that $G(S_n)<4^{-n}$. Put $c_n=2^n/|S_n|$, and $S_n'=c_nS_n$. Then $|S_n'|=c_n|S_n|=2^n$, and $|S_n'|^*<4^{-n}|S_n'|=2^{-n}$. A well known necessary and sufficient condition for unconditional convergence of a series $\sum x_n$ is that given $\epsilon>0$, we can find N such that $|\sum_s x_n|<\epsilon$, where s is any finite set of integers, all greater than N. From this it is easily seen that the series formed by summing the elements of $S_1'+S_2'+S_3'+\cdots$ converges unconditionally but not absolutely.

(ii) If $g>0$, we may write $|S|\leqq|S|^*/g$, and it is now easily seen that unconditional convergence implies absolute convergence, and so the two are equivalent.

COROLLARY 1. *Unconditional convergence does not imply absolute convergence in L.*

PROOF. Let $r_n(t)$ denote the Rademacher orthogonal functions on $0\leqq t\leqq 1$, defined by $r_n(t)=2e_n(t)-1$, where $e_n(t)$ is the nth digit in the binary expansion of t, the value 0 being assigned at points that have two such expansions. Thus $r_n(t)$ takes the values -1 and $+1$ alternately on intervals of length $1/2^n$.

We shall show that if $S_n=(r_1, r_2, \cdots, r_n)$, we have

$$|S_n|^*/|S_n|\rightarrow 0, \tag{1}$$

from which the desired result follows by the theorem. Evidently $|S_n|=n$. Moreover, when σ is a subset containing p of the numbers $1, 2, \cdots, n$, use of the Schwarz inequality and the orthogonality of the functions $r_k(t)$ gives

$$\int_0^1\left|\sum_\sigma r_k(t)\right|dt\leqq\left\{\int_0^1\left|\sum_\sigma r_k(t)\right|^2 dt\right\}^{1/2}=p^{1/2}.$$

Hence, since $p\leqq n$, $|S_n|^*\leqq n^{1/2}$. Therefore (1) holds, and the proof is complete.

COROLLARY 2. *Unconditional convergence does not imply absolute convergence in l.*

PROOF. Since there is no necessity for the sets S_n to have common elements, we may write

$$\begin{aligned}
S_1 &= \{(-1,\ \ 1,\ \ 0,\ \ 0,\ \ 0,\cdots)\},\\
S_2 &= \{(-1,\ -1,\ \ 1,\ \ 1,\ \ 0,\ \ 0,\ \ 0,\cdots),\\
&\qquad (-1,\ \ 1,\ -1,\ \ 1,\ \ 0,\ \ 0,\ \ 0,\cdots)\},
\end{aligned}$$

$$\begin{aligned} S_3 = \{&(-1,-1,-1,-1,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\cdots),\\ &(-1,-1,\ 1,\ 1,-1,-1,\ 1,\ 1,\ 0,\ 0,\ 0,\cdots),\\ &(-1,\ 1,-1,\ 1,-1,\ 1,-1,\ 1,\ 0,\ 0,\ 0,\cdots)\},\end{aligned}$$

and so on. Then $|S_n| = 2^n n$. Denote the sequences which make up S_n, taken in the order indicated, by $R_1^n, R_2^n, \cdots, R_n^n$, and denote the mth term of R_k^n by $R_k^n(m)$. Then

$$\sum_{m=1}^{\infty}\left|\sum_{k\in\sigma} R_k^n(m)\right| = 2^n\int_0^1\left|\sum_{k\in\sigma} r_k(t)\right| dt \leqq 2^n n^{1/2},$$

as in Corollary 1. We then have $|S_n|^*/|S_n| \to 0$, and the result follows as before.

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2. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw and Lwów, 1935.

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ACADIA UNIVERSITY

A PROPERTY OF DERIVATIVES

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If a real function defined over a closed interval $[a, b]$ is differentiable at each point of the interval, it is well known that its derivative possesses the Darboux property: if $f'(c)<\xi<f'(d)$, then there is a point e between c and d with $f'(e)=\xi$.

Now let α, β, with $\alpha<\beta$, be any two fixed reals, and consider the set $E(\alpha, \beta)=E\{x/\alpha<f'(x)<\beta\}$. It is easily seen, as a consequence of the Darboux property, that any such set $E(\alpha, \beta)$ must contain a continuum of points, unless it is empty. The question of the measure of $E(\alpha, \beta)$ does not seem to be covered in the literature, except in the case in which the given interval is either $(-\infty, \beta)$ or $(\alpha, +\infty)$. We prove that any such set $E(\alpha, \beta)$ is either empty or of positive measure.

We remark that this result cannot be deduced from the Darboux property alone; Lebesgue exhibited a function[1] which possesses that property without satisfying the measure condition. Another example is the following. Let C be the Cantor closed nondense set of measure zero and power c in the unit interval, and let $\{T_n\}$ $(n=1, 2, 3, \cdots)$ be a sequence of linear transformations such that the sets $T_n(C)$ are disjoint, and such that any sub-interval of $[0, 1]$ contains some $T_n(C)$. We take T_1 to be the identity. Let the function $g(x)$ be defined on C in such a way as to assume all values from zero to one inclusive; on $T_n(C)$ let $g(x)=g(T_n^{-1}(x))$. On all remaining points of the unit interval set $g(x)=0$. It is clear that this function g possesses the Darboux property, but that the set $E\{x/1/2<g(x)<1\}$ will be nonvoid and of measure zero.

THEOREM. *If $f(x)$ is real and everywhere differentiable in the closed interval $[a, b]$, then for any two reals α, β $(\alpha<\beta)$, the set*

$$E(\alpha, \beta) = E\{x/\alpha < f'(x) < \beta\}$$

is empty or of positive measure.

PROOF. We start with the following known result:[2] if a continuous function $f(x)$ is differentiable in the interval $[a, b]$, with the possible exception of a denumerable set of points x, and if $f'(x)$ is non-negative almost everywhere, then $f(x)$ is nondecreasing. It follows that if $f'(x)$ exists for all x in $[a, b]$, and $f'(x)\geq\lambda$ [or $f'(x)\leq\mu$] for almost all x,

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[1] Lebesgue, *Leçons sur l'intégration*, 2d ed., Paris, 1928, p. 97.

[2] Saks, *Théorie de l'intégrale*, Warsaw, 1933, p. 141.

then $f'(x) \geqq \lambda$ $[f'(x) \leqq \mu]$ for all x in $[a, b]$ (by considering the functions $f-\lambda x$, $f-\mu x$). In other words, if a function is differentiable everywhere, then the least upper and greatest lower bounds of its derivative are not changed if sets of measure zero are neglected in computing them: this is the theorem which results from our formulation if either $\alpha = -\infty$ or $\beta = +\infty$. We shall make use of this remark in the following argument.

Consider α, β arbitrary and finite: write E for $E(\alpha, \beta)$. Suppose E not void, and $mE = 0$; we shall show that a contradiction results. We write $E_\alpha = E\{x/f'(x) \leqq \alpha\}$, $E_\beta = E\{x/f'(x) \geqq \beta\}$, so that the interval $[a, b] = E + E_\alpha + E_\beta$, the three sets being disjoint.

First, $E \subset E'_\alpha \cdot E'_\beta$. To prove this relation, assume the contrary and let x_0 belong to E but not to E'_α; then x_0 is in the interior of some interval U, $U \cdot E_\alpha$ void. But then in the interval U the set of points where $f' < \beta$ is not empty, since it contains x_0, but is of measure zero, since it is a subset of E. This contradicts the special case of our theorem which was referred to in our preliminary remark. Thus $E \subset E'_\alpha$, and by a similar argument, $E \subset E'_\beta$ also.

The derivative $f'(x)$ belongs to Baire's first class, and hence, by a known theorem,[3] if A is any subset of $[a, b]$, and f' be considered for the moment on the domain A alone, its points of discontinuity must form a set of the first category relative to A. Take for the subset A the closure of our set E: this set, $\bar{E}$, being closed, is of second category relative to itself, and hence we shall have a contradiction if we show that $f'(x)$ is everywhere discontinuous considered on the domain $\bar{E}$. This may be seen as follows: let x_0 be a point, first, of E itself. Since $E \subset E'_\alpha \cdot E'_\beta$, we have for any interval I which contains x_0

$$\sup_{x \in I} f'(x) \geqq \beta, \qquad \inf_{x \in I} f'(x) \leqq \alpha.$$

Because f' possesses the Darboux property, we infer

$$\sup_{x \in I \cdot E} f'(x) = \beta, \qquad \inf_{x \in I \cdot E} f'(x) = \alpha,$$

so that we have now shown that, considered on domain E, and so a fortiori on domain $\bar{E}$, the function f' is discontinuous at each point of E. On domain $\bar{E}$ the saltus of f' at each point of E has been shown to be at least $\beta - \alpha$; the same, then, will be true at each point of $\bar{E}$. The function f' has now been shown to be everywhere discontinuous considered on domain $\bar{E}$; this completes the proof.

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[3] Kuratowsky, *Topologie*, Warsaw, 1933, p. 189.

THE CONVERSE OF A THEOREM OF TCHAPLYGIN ON DIFFERENTIAL INEQUALITIES

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1. **Introduction.** Suppose that $y(x)$ is a solution of the linear differential equation

$$y'' - p_1y' - p_2y - q = 0, \qquad x \geqq x_0, \tag{1.1}$$

where $p_1(x)$, $p_2(x)$ and $q(x)$ are continuous when $x \geqq x_0$, and that

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'. \tag{1.2}$$

Then if $v(x)$ satisfies the differential inequality

$$v'' - p_1v' - p_2v - q > 0, \qquad x \geqq x_0, \tag{1.3}$$

and the same boundary conditions as $y(x)$ at x_0, it is clear that the inequality

$$v(x) > y(x) \tag{1.4}$$

holds in some right-hand neighborhood of x_0. Tchaplygin[1] has proved that the inequality (1.4) holds when $x_0 < x \leqq x_1$ provided that there exists a solution $\lambda(x)$ of the Riccati equation

$$\lambda' + \lambda^2 + p_1\lambda + (p_1' - p_2) = 0 \tag{1.5}$$

which is continuous when $x_0 < x < x_1$. Let $X(x_0)$ be the least upper bound of values x_1 for which the Riccati equation admits a continuous solution when $x_0 < x < x_1$. Then the inequality (1.4) holds when $x_0 < x \leqq X(x_0)$, and Petrov [2] has shown that if p_1 and p_2 are constants no better result is true. That is, if p_1 and p_2 are constants and $X(x_0)$ is finite, then there exists a function $v(x)$ satisfying (1.3) and (1.2) for which $v(x) = y(x)$ at a point arbitrarily close to but greater than $X(x_0)$. It is the purpose of this paper to show that this last result is true without the restriction that p_1 and p_2 are constants. We prefer to state our results in terms of and make our proofs depend

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[1] The author knows this result only by virtue of a reference to it contained in the paper of Petrov [2], and there it is not made clear whether or not Tchaplygin took the interval from x_0 to x_1 to be open, as we have written it, or closed or half-open. It follows, however, from the results obtained in §2 that this statement is true for the open interval and hence is a fortiori true for the closed and half-open intervals. Numbers in brackets refer to the bibliography at the end of the paper.

upon properties of solutions of the homogeneous linear differential equation

$$u'' - p_1u' - p_2u = 0, \tag{1.6}$$

rather than solutions of the Riccati equation (1.5). Apart from the trivial advantage that we do not need to assume that $p_1(x)$ is differentiable, it is possible to give simpler and more natural proofs using (1.6) than (1.5).

2. **The main theorem.** We are going to prove the following theorem.

THEOREM. *If* $y(x)$ *satisfies* (1.1) *and* (1.2), *and if* $v(x)$ *satisfies* (1.3) *and* (1.2), *then the inequality* (1.4) *holds when* $x_0<x\leqq x_1$ *provided that there exists a solution* $u(x)$ *of* (1.6) *which does not vanish when* $x_0<x<x_1$. *Let* $u_0(x)$ *be a solution of* (1.6) *such that* $u_0(x_0)=0$, $u_0'(x_0)=1$, *and let* $X(x_0)$ *be the first zero of* $u_0(x)$ *to the right of* x_0 *if any such zero exists; otherwise, let* $X(x_0)=+\infty$. *Then the interval where* $x_0<x\leqq X(x_0)$ *is the largest interval in which the inequality* (1.4) *can be asserted to hold. In other words, either* $X(x_0)=+\infty$ *or else there exists a function* $v(x, k)$ *for each sufficiently small positive* k *which satisfies* (1.3) *and* (1.2) *and for which* $v[X(x_0)+k, k]<y[X(x_0)+k]$.

Let us define $z(x)=v(x)-y(x)$. Then

$$z'' - p_1z' - p_2z = \phi(x) > 0, \qquad z(x_0) = z'(x_0) = 0. \tag{2.1}$$

If $u(x)$ is a solution of (1.6) and one defines the Wronskian of $u(x)$ and $z(x)$ to be

$$W(x) = u(x)z'(x) - u'(x)z(x),$$

then it is easy to see that

$$W'(x) - p_1(x)W(x) = u(x)\phi(x), \qquad W(x_0) = 0, \tag{2.2}$$

whence we have that

$$W(x) = \int_{x_0}^{x} u(s)\phi(s)P(x, s)ds, \tag{2.3}$$

where $P(x, s)$ is defined as

$$P(x, s) = \exp \int_{s}^{x} p_1(t)dt. \tag{2.4}$$

Suppose now that $z(x)$ vanishes at some point to the right of x_0, and that x^* is the first such point. Then when $x=x^*$, equation (2.3) reduces to

$$u(x^*)z'(x^*) = \int_{x_0}^{x^*} u(s)\phi(s)P(x^*, s)ds. \tag{2.5}$$

Since $z(x) > 0$ to the left of x^* we have that $z'(x^*) \leqq 0$. It now follows from equation (2.5) that $u(s)$ must change sign when $x_0 < s < x^*$, for otherwise the two sides of (2.5) could not have the same sign. This remark is equivalent to the first sentence of the theorem.

If $X(x_0) = +\infty$, then $z(x)$ can never vanish to the right of x_0. In this case the inequality (1.4) holds whenever $x > x_0$. Let us now suppose that $X(x_0)$ is finite. Let x_2 be any point to the right of $X(x_0)$ such that $u_0(x)$ has no zeros between $X(x_0)$ and x_2. Let $u(x)$ be the solution of (1.6) such that $u(x_2) = 0$, $u'(x_2) = -1$. It follows from the separation theorem for the zeros of solutions of second order linear homogeneous differential equations [1, p. 224] that there is a unique point x_1 between x_0 and $X(x_0)$ such that $u(x_1) = 0$. For this function $u(x)$ we get from equation (2.3) that

$$z(x_2) = \int_{x_0}^{x_2} u(s)\phi(s)P(x_2, s)ds.$$

To complete the proof of the theorem it is sufficient to show that for each x_2 subject to the restrictions already imposed on it a positive function $\phi(s, x_2)$ can be found such that

$$\int_{x_0}^{x_2} u(s)\phi(s, x_2)P(x_2, s)ds < 0. \tag{2.6}$$

If $b > 0$, the function $\phi(s, x_2)$ defined as

$$\begin{aligned} \phi(s, x_2) &= [1 - bu(s)]/P(x_2, s), & x_0 < s < x_1, \\ \phi(s, x_2) &= 1/P(x_2, s), & x_1 < s < x_2, \end{aligned}$$

is surely positive. For this function the integral (2.6) has the value

$$\int_{x_0}^{x_2} u(s)ds - b\int_{x_0}^{x_1} u^2(s)ds$$

and will certainly be negative if b is sufficiently large.

To round out the results of this paper we shall now show that the number $X(x_0)$ defined in the theorem coincides with the number $X(x_0)$ defined in the introduction by means of the Riccati equation (1.5). We prefer to replace (1.5) by

$$\mu' + \mu^2 - p_1\mu - p_2 = 0, \tag{2.7}$$

where $\mu = \lambda + p_1$, and define $X_1(x_0)$ as the least upper bound of numbers x_1 such that the equation (2.7) admits a solution continuous when $x_0 < x < x_1$. Since p_1 is continuous this number is the same as that defined for the equation (1.5) in case p_1 is differentiable, but $X_1(x_0)$ is defined even though p_1 is not differentiable. A function $\mu(x)$ satisfies (2.7) if and only if $\mu(x) \doteq u'(x)/u(x)$, where $u(x)$ is a nontrivial solution of (1.6). Hence solutions $\mu(x)$ of (2.7) which are continuous when $x_0 < x < x_1$ correspond in a one-to-one fashion with families of functions $cu(x)$, where c is an arbitrary nonzero constant, which satisfy (1.6) and which do not vanish when $x_0 < x < x_1$. It follows that $X_1(x_0)$ is the least upper bound of values x_1 for which the equation (1.6) admits a solution not vanishing when $x_0 < x < x_1$. Since every solution of (1.6) must vanish when $x_0 < x \leqq X(x_0)$ we have that $X_1(x_0) \leqq X(x_0)$, and since $u_0(x)$ does not vanish when $x_0 < x < X(x_0)$ we have that $X(x_0) \leqq X_1(x_0)$. Therefore, $X(x_0) = X_1(x_0)$.

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AMERICAN OPTICAL COMPANY

THE BEHAVIOR OF MEASURE AND MEASURABILITY UNDER CHANGE OF SCALE IN WIENER SPACE

R. H. CAMERON AND W. T. MARTIN

1. Introduction. Denoting by C the space of all real-valued continuous functions $x(t)$ on $0 \leqq t \leqq 1$ which vanish at $t=0$, we consider the magnification, or change of scale,

$$y(\cdot) = \lambda x(\cdot) \tag{1.1}$$

for λ a nonvanishing real number. An investigation of this transformation leads to some very surprising results. For example, we find that there exists a subset C_1 of C with measure equal to the measure of the total space,

$$m_w(C_1) = m_w(C), \tag{1.2}$$

which is transformed into a set of measure zero by every transformation of the form (1.1) except when $\lambda = \pm 1$. Thus,

$$m_w(\lambda C_1) = 0 \qquad \text{for } -\infty < \lambda < \infty, \lambda \neq \pm 1, \tag{1.3}$$

where λC_1 denotes the set of all functions $y = \lambda x$ for which $x \in C_1$.

This result is based upon the following theorem which we prove in §2:

THEOREM 1. *Denote by* $\sigma_n(x)$ *the sum*

$$\sigma_n(x) = \sum_{j=1}^{2^n} \left\{ x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right) \right\}^2. \tag{1.4}$$

Then for almost all x *the limit*

$$\lim_{n\to\infty} \sigma_n(x) = \int_0^1 |dx(t)|^2$$

exists, and

$$\int_0^1 |dx(t)|^2 = 1/2 \qquad \textit{for almost all } x. \tag{1.5}$$

Using Theorem 1 and certain lemmas to be proved in §§3 and 4, we also obtain the following theorem, which we number as Theorem 3.

THEOREM 3. *Let* $f(\lambda)$ *be a given function of* λ *defined for all positive* λ *and satisfying* $0 \leqq f(\lambda) \leqq 1$. *Then by an explicit construction (without the*

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use of Zermelo's axiom) we can construct a specific set E such that E is measurable under all magnifications and $m_w(\lambda E)=f(\lambda)$ for all $\lambda>0$.

We shall prove this theorem in §4.

Throughout this note we shall use the measure on C defined by Wiener.[1]

2. **Proof of Theorem** 1. Let m be any positive integer, and let

$$0=t_0<t_1<\cdots<t_m<1. \tag{2.1}$$

Later we shall take m to be of the form 2^n, and $t_j=j/2^n$; for the present no such specialization need be made. Denote by $s_m(x)$ the sum

$$s_m(x)=\sum_{j=1}^{m}\left\{x(t_j)-x(t_{j-1})\right\}^2. \tag{2.2}$$

We shall now evaluate the integral

$$\int_C^w\{s_m(x)-1/2\}^2d_wx=\int_C^w s_m^2(x)d_wx-\int_C^w s_m(x)d_wx+1/4. \tag{2.3}$$

Each of the two integrals on the right can be expressed as an m-fold Lebesgue integral which in turn can be easily evaluated. Thus

$$\begin{aligned}\int_C^w s_m(x)d_wx &= \pi^{-m/2}[t_1(t_2-t_1)\cdots(t_m-t_{m-1})]^{-1/2}\\ &\quad\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{j=1}^{m}(\xi_j-\xi_{j-1})^2\\ &\quad\cdot\exp\left\{-\sum_{j=1}^{m}\frac{(\xi_j-\xi_{j-1})^2}{t_j-t_{j-1}}\right\}d\xi_1\cdots d\xi_m\\ &= \pi^{-m/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{j=1}^{m}\eta_j^2(t_j-t_{j-1})\\ &\quad\cdot\exp\left\{-\sum_{j=1}^{m}\eta_j^2\right\}d\eta_1\cdots d\eta_m\\ &= \frac{1}{2}\sum_{j=1}^{m}(t_j-t_{j-1})=\frac{1}{2}.\end{aligned} \tag{2.4}$$

---

[1] *Generalized harmonic analysis*, Acta Math. vol. 55 (1930) pp. 117–258, especially pp. 214–234. See also other references to Wiener's work given in this paper. (For further properties of Wiener measure and the Wiener integral, see *Transformations of Wiener integrals under a general class of linear transformations* by R. H. Cameron and W. T. Martin, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 184–219; see also other references there.)

Similarly

$$
(2.5)\quad
\begin{aligned}
\int_C^w s_m^2(x)\,d_wx &= \pi^{-m/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big[\sum_{j=1}^m \eta_j^2(t_j-t_{j-1})\Big]^2 \\
&\qquad\cdot\exp\Big\{-\sum_{j=1}^m \eta_j^2\Big\}d\eta_1\cdots d\eta_m \\
&= \pi^{-m/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big[\sum_{j=1}^m \eta_j^4(t_j-t_{j-1})^2 \\
&\qquad+2\sum_{j=1}^{m-1}\sum_{p=j+1}^{m}\eta_j^2\eta_p^2(t_j-t_{j-1})(t_p-t_{p-1})\Big] \\
&\qquad\cdot\exp\Big\{-\sum_{j=1}^m \eta_j^2\Big\}d\eta_1\cdots d\eta_m \\
&= \frac{3}{4}\sum_{j=1}^m (t_j-t_{j-1})^2 \\
&\qquad+\frac{1}{2}\sum_{j=1}^{m-1}\sum_{p=j+1}^{m}(t_j-t_{j-1})(t_p-t_{p-1}).
\end{aligned}
$$

On specializing the t_j to be

$$
(2.6)\qquad t_j = j/m, \qquad j = 0, 1, \cdots, m,
$$

and inserting (2.4) and (2.5) into (2.3) we obtain

$$
(2.7)\quad
\begin{aligned}
&\int_C^w \Big\{s_m(x)-\frac{1}{2}\Big\}^2 d_wx \\
&\qquad= \frac{3}{4m}+\frac{1}{2m^2}\cdot\frac{m(m-1)}{2}-\frac{1}{2}+\frac{1}{4}=\frac{1}{2m}\cdot
\end{aligned}
$$

This relation enables us to prove Theorem 1; letting m range over the subsequence 2^n, $n=1, 2, 3, \cdots$, we see that (2.7) yields

$$
(2.8)\qquad \int_C^w \Big[\sigma_n(x)-\frac{1}{2}\Big]^2 d_wx = 2^{-n-1}.
$$

A familiar argument yields the convergence almost everywhere of $\sigma_n(x)$ to the value 1/2. In fact, denote by E_n the set of x's for which

$$
(2.9)\qquad |\sigma_n(x)-1/2| \geqq 2^{-n/3}.
$$

Then by (2.8)

$$
(2.10)\qquad m(E_n) \leqq 2^{-n/3}.
$$

Hence if $F_n = E_n + E_{n+1} + E_{n+2} + \cdots$,

$$m(F_n) \leqq 2^{-n/3} + 2^{-(n+1)/3} + \cdots < 6 \cdot 2^{-n/3}. \tag{2.11}$$

But on $C - F_n$

$$\left| \sigma_j(x) - 1/2 \right| < 2^{-j/3} \qquad \text{for } j = n, n+1, \cdots . \tag{2.12}$$

Hence for each positive integer n, we have

$$\lim_{j \to \infty} \sigma_j(x) = 1/2 \qquad \text{for } x \text{ on } C - F_n. \tag{2.13}$$

Thus

$$\lim_{j \to \infty} \sigma_j(x) = 1/2 \tag{2.14}$$

except possibly in the set $F_1 F_2 F_3 \cdots$. But

$$m_w(F_1 F_2 F_3 \cdots) \leqq m_w(F_n) < 6 \cdot 2^{-n/3}, \qquad n = 1, 2, 3, \cdots . \tag{2.15}$$

Thus $m_w(F_1 F_2 F_3 \cdots) = 0$ and (2.14) holds for almost all x. This concludes the proof of Theorem 1.

3. Invariant and non-invariant null sets.

We now introduce the notion of an invariant null set.

DEFINITION. A set E is called an *invariant null set* if for all $\lambda > 0$

$$m_w(\lambda E) = 0 \tag{3.1}$$

where λE is the set consisting of all functions $y(\cdot) = \lambda x(\cdot)$ for which $x(\cdot) \in E$.

Notation. The set of $x(t)$ in C such that $\int_0^1 \left| dx(t) \right|^2$ does not exist will be denoted by D.

For each $\lambda \geqq 0$, the set of $x(t)$ in C such that

$$\int_0^1 \left| dx(t) \right|^2 = \lambda^2/2$$

will be denoted by C_λ.

LEMMA 1. *The sets C_0 and D are invariant null sets.*

PROOF. Clearly $\lambda C_0 = C_0$ and $\lambda D = D$ while by Theorem 1

$$m_w(C_0) = m_w(D) = 0.$$

LEMMA 2. *The measures of the sets C_λ are as follows:*

$$m_w(C_1) = 1, \qquad m_w(C_\lambda) = 0 \quad \text{if} \quad \lambda \neq 1, \qquad m_w\Big(\sum_{0 < \lambda \neq 1} C_\lambda\Big) = 0.$$

PROOF. The first statement follows immediately from Theorem 1. Then the third statement follows from the first (since $\sum_{0<\lambda\neq 1} C_\lambda$ is contained in the complement of C_1 which is a null set) and the second follows from the third.

LEMMA 3. *The sets* C_λ (*for* $\lambda \geqq 0$) *and* D *are all disjoint, and their sum is* C.

This follows immediately from the definitions of C_λ and D.

LEMMA 4. *For all non-negative* λ *and positive* μ

$$\mu C_\lambda = C_{\mu\lambda}.$$

PROOF. Obviously we have $\sigma_n(\mu x) = \mu^2 \sigma_n(x)$, and hence if $x \in C_\lambda$,

$$\lim_{n\to\infty} \sigma_n(\mu x) = \mu^2 \lim_{n\to\infty} \sigma_n(x)$$

so that $\int_0^1 |d(\mu x(t))|^2 = \mu^2\lambda^2/2$, and $\mu x \in C_{\mu\lambda}$.

LEMMA 5.

$$m_w\left(\frac{1}{\lambda} C_\lambda\right) = 1 \qquad \textit{for all } \lambda > 0,$$

$$m_w\left(\frac{1}{\mu} C_\lambda\right) = 0 \qquad \textit{if } \mu \neq \lambda;\ \mu > 0,\ \lambda \geqq 0;$$

$$m_w\left(\sum_{0<\mu\neq\lambda} \frac{1}{\mu} C_\lambda\right) = 0 \qquad \textit{for each fixed } \lambda \geqq 0.$$

These three statements are immediate consequence of the three corresponding statements in Lemma 2 and the transformation property of Lemma 4.

COROLLARY 1. *In the decomposition of* C *into disjoint sets:*

$$C = D + C_0 + \sum_{0<\lambda\neq 1} C_\lambda + C_1$$

there are only two invariant null sets, namely D *and* C_0. *The set* C_1 *is of measure* 1, *while the other sets* C_λ $(0<\lambda\neq 1)$ *are non-invariant null sets.*

4. **Construction of sets having prescribed measures after change of scale.** In this section we prove Theorem 3 and some related results.

LEMMA 6. *Given two real numbers* λ *and* μ *satisfying the inequalities* $0 \leqq \lambda \leqq \mu \leqq 1$, *there exists a subset* E *of* C *such that*

$$m_{iw}(E) = \lambda \quad \text{and} \quad m_{ew}(E) = \mu.$$

We know that there exists a subset F of the real numbers s on the interval I: $(0\leqq s\leqq 1)$ such that $m_i(F)=\lambda$ and $m_e(F)=\mu$. Now apart from null sets, the Wiener mapping[3] T takes I into C in an essentially 1-1 measure-preserving way, and since inner measure can be defined in terms of the sup of the measure of suitable (measurable) sets and outer measure can similarly be defined in terms of the inf of the measure of suitable (measurable) sets, it follows that the set $E=TF$ in the space C has the desired property.

THEOREM 2. *Let* $f(\lambda)$ *and* $g(\lambda)$ *be two functions defined on the set of all positive* λ *and completely arbitrary except that*

$$0 \leqq f(\lambda) \leqq g(\lambda) \leqq 1 \qquad \text{on } 0 < \lambda.$$

Then there exists a set E *such that*

$$m_{iw}(\lambda E) = f(\lambda) \quad \text{and} \quad m_{ew}(\lambda E) = g(\lambda) \qquad \text{for all } \lambda > 0. \tag{4.1}$$

PROOF. By Lemma 6 and Zermelo's axiom there exists a one-parameter set of sets E_λ such that $m_{iw}(E_\lambda)=f(\lambda)$ and $m_{ew}(E_\lambda)=g(\lambda)$. Define $\tilde{E}_\lambda=C_1E_\lambda$. Then

$$E_\lambda - \tilde{E}_\lambda = E_\lambda(C - C_1),$$

and since $C-C_1$ is a null set (cf. Lemma 2), so is $E_\lambda-\tilde{E}_\lambda$. Thus $\tilde{E}_\lambda$ differs from E_λ by a null set and hence has the same inner and outer measure as E_λ:

$$m_{iw}(\tilde{E}_\lambda) = f(\lambda), \qquad m_{ew}(\tilde{E}_\lambda) = g(\lambda), \qquad 0 < \lambda. \tag{4.2}$$

Now define

$$E = \sum_{0<\mu}\left(\frac{1}{\mu}\,\tilde{E}_\mu\right). \tag{4.3}$$

We shall show that E has the required properties, (4.1). Consider

$$\lambda E = \sum_{0<\mu}\left(\frac{\lambda}{\mu}\,\tilde{E}_\mu\right) = \tilde{E}_\lambda + \sum_{0<\mu\neq\lambda}\left(\frac{\lambda}{\mu}\,\tilde{E}_\mu\right).$$

Now since $\tilde{E}_\mu\subset C_1$, we have by Lemma 4

$$\sum_{0<\mu\neq\lambda}\left(\frac{\lambda}{\mu}\,\tilde{E}_\mu\right) \subset \sum_{0<\mu\neq\lambda}\left(\frac{\lambda}{\mu}\,C_1\right) = \sum_{0<\mu\neq\lambda}\left(\frac{1}{\mu}\,C_\lambda\right),$$

and thus by Lemma 5

$$\sum_{0<\mu\neq\lambda}\left(\frac{\lambda}{\mu}\,\tilde{E}_\mu\right)$$

---

[3] Loc. cit.

is a null set. Thus λE and $\tilde{E}_\lambda$ differ by a null set and hence have the same inner and outer measures. Hence, by (4.2), (4.1) holds with E defined as in (4.3).

COROLLARY 2. *Let $f(\lambda)$ be an arbitrary function satisfying $0 \leqq f(\lambda) \leqq 1$ on $0<\lambda$. Then there is a set E which is measurable under every magnification and such that $m_w(\lambda E) = f(\lambda)$, $0<\lambda$.*

We now strengthen Corollary 2 by showing that in this case the set E can be constructed explicitly (without Zermelo's axiom). This is the content of Theorem 3 as stated in the introduction.

To prove Theorem 3, consider the quasi-interval Q_ξ with only one subdivision $t_1 = 1$ and with the corresponding lower and upper bounds ξ and ∞ respectively. Thus

$$Q_\xi: \qquad \xi < x(1).$$

Now

$$m_w(Q_\xi) = \frac{1}{\pi^{1/2}} \int_\xi^\infty e^{-v^2} dv = \pi^{-1/2} \operatorname{Erfc}(\xi).$$

It is clear that $\operatorname{Erfc}(\xi)$ is a decreasing function of ξ all the way from $-\infty$ to ∞, decreasing from $\pi^{1/2}$ at $-\infty$ to 0 at ∞. Thus $\operatorname{Erfc}^{-1}(w)$ is a single-valued decreasing function on $0 \leqq w \leqq \pi^{1/2}$, decreasing from $\operatorname{Erfc}^{-1}(0) = \infty$ to $\operatorname{Erfc}^{-1}(\pi^{1/2}) = -\infty$. Thus if v is any given real number on $0 \leqq v \leqq 1$, there is an explicitly constructed number ξ, namely

$$\xi = \operatorname{Erfc}^{-1}(\pi^{1/2} v)$$

such that the (explicitly constructed) set Q_ξ has the measure

$$m_w(Q_\xi) = v.$$

From here the proof goes as in Theorem 2: Let $E_\lambda = Q_{\xi_\lambda}$, where

$$\xi_\lambda = \operatorname{Erfc}^{-1}(\pi^{1/2} f(\lambda)).$$

Then let $\tilde{E}_\lambda = E_\lambda C_1$ and

$$E = \sum_{0<\mu} \left(\frac{1}{\mu} \tilde{E}_\mu \right).$$

It is clear that E has been explicitly constructed. Also

$$m_w(\tilde{E}_\lambda) = m_w(E_\lambda) = f(\lambda),$$

and as before

$$\lambda E = \tilde{E}_\lambda + \sum_{0<\mu\neq\lambda} \left(\frac{\lambda}{\mu} \tilde{E}_\mu \right)$$

and

$$m_w\left(\sum_{0<\mu\neq\lambda}\frac{\lambda}{\mu}\tilde{E}_\mu\right)=0,$$

so $m_w(\lambda E)=m_w(\tilde{E}_\lambda)=f(\lambda)$, and the theorem is proved.

5. **The final theorem.** Our final theorem is the following paradoxical sounding theorem.

THEOREM 4. *Let $F_\lambda(x)$ be a given one-parameter family of functionals, there being one functional for each positive value of λ and each functional being defined on C.*

Then there exists an explicitly constructed functional $F(x)$ such that for each positive λ

$$F(\lambda x)=F_\lambda(x) \qquad \textit{almost everywhere on } C. \tag{5.1}$$

For the proof we merely let

$$F(x)=F_\lambda\left(\frac{x}{\lambda}\right) \qquad \text{on } C_\lambda \text{ for all } \lambda>0 \tag{5.2}$$

and take

$$F(x)=0 \text{ on } C_0 \text{ and on } D.$$

By Lemma 3, these sets are all disjoint and hence there is no x for which $F(x)$ has been defined twice (that is, the definition is self-consistent), and again by Lemma 3 the sum of these sets is C, and hence $F(x)$ is defined for every x in C. Moreover the definition is entirely explicit (and hence Zermelo's axiom has not been used explicitly or implicitly).

Now for all positive λ and μ we have by (5.2)

$$F(\mu x)=F_\lambda\left(\frac{\mu x}{\lambda}\right) \tag{5.3}$$

for $\mu x\in C_\lambda$, that is, for x on $(1/\mu)C_\lambda=(C_{\lambda/\mu})$. Taking $\mu=\lambda$, we have

$$F(\lambda x)=F_\lambda(x) \qquad \text{on } C_1 \text{ for each } \lambda>0. \tag{5.4}$$

But $m_w(C_1)=1$, and hence for almost all x in C we have

$$F(\lambda x)=F_\lambda(x) \qquad \text{for each } \lambda>0.$$

Thus Theorem 4 is proved.

THE UNIVERSITY OF MINNESOTA AND
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ON THE SET OF VALUES OF A FINITE MEASURE

PAUL R. HALMOS

Let X be any set, let S be a σ-field of subsets of X (that is, S is a class of sets containing X and closed under complementation and the formation of countable unions), and suppose that μ is a finite, non-negative, and countably additive measure defined for the sets of S. Concerning such measures Kai Rander Buch has recently proved the following two statements.[1]

THEOREM 1. *The set of values of μ is closed.*

THEOREM 2. *If μ and ν are two finite measures defined on the same σ-field S of sets, then the set of all points of the form $(\mu(E), \nu(E))$, where $E \in S$, is a closed subset of the plane.*

Buch's proofs are long and complicated and make use of an intricate construction involving the Cantor set in order to map the measure space X on an interval.[2] It is the purpose of this note to give direct and simple proofs of Theorems 1 and 2. It is worth remarking that (a) Theorem 1 is a trivial corollary of Theorem 2 (set $\nu(E)$ identically zero), (b) there does not seem to be a completely trivial proof of Theorem 2 from Theorem 1 based on elementary properties of product spaces, and (c) possibly both theorems can be made to appear as special cases of a theorem on measures whose values are suitably general entities (such as, say, elements of an ordered abelian group).

An *atom* of a measure space X is a measurable set E of positive measure such that for every measurable subset $F \subset E$ either $\mu(F) = 0$ or $\mu(E-F) = 0$. If E is an atom of X we may replace X by the space whose points are the points of the complement of E together with a single point of measure $\mu(E)$. Since the set of values of μ is not altered by this replacement we may and do assume that all atoms contain exactly one point. Since $\mu(X) < \infty$, X can contain at most countably many distinct atoms. Let Y be the union of the atoms of X and write $Z = X - Y$.

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[1] *Some investigations of the set of values of measures in abstract space*, Matematisk-Fysiske Meddelelser vol. 21 (1945).

[2] Such constructions were previously employed for the same purpose by John von Neumann, *Zur Operatorenmethode in der klassischen Mechanik*, Ann. of Math. vol. 33 (1932) p. 602, and J. L. Doob, *Stochastic processes with an integral-valued parameter*, Trans. Amer. Math. Soc. vol. 44 (1938) p. 91.

LEMMA 1. *Every measurable set $E \subset Z$ of positive measure contains measurable subsets of arbitrarily small positive measure.*

PROOF. Since E is not an atom there exists a measurable subset $F \subset E$ for which $0 < \mu(F) < \mu(E)$. Write E_1 for that one of the two sets F and $E-F$ whose measure is not greater than $\mu(E)/2$. Similarly we may construct a set $E_2 \subset E_1$ such that $0 < \mu(E_2) \leqq \mu(E_1)/2$, and proceed so on by induction.

LEMMA 2. *The set of values of μ on measurable subsets of Z is the closed interval $0 \leqq x \leqq \mu(Z)$.*

PROOF. If $\mu(Z) = 0$ there is nothing to prove. If $0 < \alpha < \mu(Z)$, we may apply Lemma 1 to find a measurable set $E_1 \subset Z$ such that $0 < \mu(E_1) \leqq \alpha$. If the equality holds we are finished; if not we may apply Lemma 1 to find a measurable set $E_2 \subset Z - E_1$ such that $0 < \mu(E_2) \leqq \alpha - \mu(E_1)$. Proceeding in this way, by transfinite induction if necessary, we obtain a countable sequence of pairwise disjoint measurable sets the union of which has measure α.[3]

LEMMA 3. *The set of values of μ on measurable subsets of Y is closed.*

PROOF. Let $y_1, y_2, \cdots$ be the points of Y. Let Γ be the set of all sequences $\gamma = \{\epsilon_1, \epsilon_2, \cdots\}$ where $\epsilon_i = 0$ or 1. In the customary topology of Cartesian product spaces Γ is a compact topological space and each of the functions $\epsilon_i = \epsilon_i(\gamma)$ is a continuous function.[4] It follows from the finiteness of $\mu(Y)$ and the Weierstrass M-test that the function $\phi(\gamma)$ defined by the series

$$\phi(\gamma) = \sum_{i=1}^{\infty} \epsilon_i \mu(y_i)$$

is also a continuous function on Γ. Since a continuous image of a compact space is compact and therefore closed[5] and since the image $\phi(\Gamma)$ is exactly the set of all values of μ on subsets of Y, the proof of Lemma 3 is complete.

It is not difficult to put together Lemmas 2 and 3 in order to prove Theorem 1. It is however a little more convenient not to do that directly but, with the proof of Theorem 2 in mind, to establish first two easy but mildly interesting topological lemmas.

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[3] The device used in the proof of Lemma 2 finds frequent application in measure theory; it is called the method of *exhaustion*.

[4] See Solomon Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, New York, 1942, p. 19.

[5] See Lefschetz, op. cit. p. 18.

LEMMA 4. *Let S be an arbitrary set and let f be a function defined on S and taking values in a topological space R. A necessary and sufficient condition that there exist in S a topology with respect to which S is compact and f is continuous is that the image $f(S)$ be a compact subset of R.*

PROOF. The necessity of the condition asserts merely that a continuous image of a compact space is compact. To prove sufficiency suppose that $f(S)$ is compact, and consider all those subsets of S which are of the form $f^{-1}(U)$ where U is an open set in R. Defining each such set to be open in S makes S into a topological space (without any separation axioms in general) on which f is continuous. Since any open covering of S is obviously induced by an open covering of $f(S)$, S is compact with respect to the topology described.

LEMMA 5. *Suppose that S is a set which is a compact space with respect to each of two topologies T_1 and T_2. Let T be the weakest topology on S (that is, the one with fewest open sets) whose open sets include all open sets of both T_1 and T_2. Then S is compact with respect to T.*

PROOF. The class of all sets of the form UV, where U is open with respect to T_1 and V is open with respect to T_2, is a base of the open sets of T. If S is covered by sets of this form then (because of the compactness hypotheses) it is covered by a finite number of the U's that occur and also by a finite number of the V's that occur. It follows therefore that S is covered by the finite class of sets obtained by intersecting each one of the finite number of U's with each one of the finite number of V's.

LEMMA 6. *If ν_1 and ν_2 are two measures defined on the same σ-field $\mathbf{S}$ of sets and if the set of values of each of them is a closed and bounded set on the line, then the set of all points of the form $(\nu_1(E), \nu_2(E))$, where $E \in \mathbf{S}$, is a closed and bounded subset of the plane.*

PROOF. By Lemma 4 we may introduce a topology T_i into the space $\mathbf{S}$ of measurable sets so that $\mathbf{S}$ is compact and ν_i is continuous, $i = 1, 2$. By Lemma 6, $\mathbf{S}$ is compact with respect to the weakest topology T which is stronger than both T_1 and T_2, and it is clear that the introduction of additional open sets does not affect the continuity of ν_1 and ν_2. It follows that the point $(\nu_1(E), \nu_2(E))$ depends continuously on E (with respect to the compact topology T) and that consequently the set of all such points is compact. This completes the proof of Lemma 6.

We return now to the notation of Lemmas 2 and 3. If for every $E \in \mathbf{S}$ we write $\nu_1(E) = \mu(EY)$ and $\nu_2(E) = \mu(EZ)$ then Lemmas 2 and 3 assert that the conditions of Lemma 6 are satisfied by ν_1 and ν_2. It follows that the set of all points of the form $(\mu(EY), \mu(EZ))$ is a compact set. Since the function $f(s, t) = s+t$ is continuous, it follows that the set of all numbers of the form

$$\mu(E) = \mu(EY) + \mu(EZ)$$

is also compact, and this proves Theorem 1. Theorem 2 is an immediate corollary of Theorem 1 and Lemma 6. The method of proof shows, incidentally, that the obvious generalization of Theorem 2 from two to n dimensions is also true.

UNIVERSITY OF CHICAGO

LIMIT DISTRIBUTION OF THE MAXIMUM AND MINIMUM OF SUCCESSIVE CUMULATIVE SUMS OF RANDOM VARIABLES

ABRAHAM WALD

1. **Introduction.** For any positive integral value N, let $X_{N1}, X_{N2}, \cdots, X_{NN}$ be independent and identically distributed random variables each having standard deviation 1. Let μ_N denote the mean value of X_{Ni}, and let

$$S_{Nk} = X_{N1} + X_{N2} + \cdots + X_{Nk}. \tag{1.1}$$

Two cases will be considered: (1) the sequence $\{N^{1/2}\mu_N\}$ converges to a finite value as $N \to \infty$; (2) $\lim_{N\to\infty} N^{1/2}\mu_N = \infty$. In case (1) we shall obtain for any positive constants a and b the limit values of

$$P_N(a) = \text{prob } \{\max (S_{N1}, \cdots, S_{NN}) < aN^{1/2}\} \tag{1.2}$$

and

$$\begin{aligned} P_N^*(a, b) = \text{prob } \{ - bN^{1/2} < \min & (S_{N1}, \cdots, S_{NN}) \\ & \leq \max (S_{N1}, \cdots, S_{NN}) < aN^{1/2}\} \end{aligned} \tag{1.3}$$

as $N \to \infty$. In case (2), we shall obtain for any real value c the limit of

$$Q_N(c) = \text{prob } \{\max (S_{N1}, \cdots, S_{NN}) < N\mu_N + cN^{1/2}\} \tag{1.4}$$

as $N \to \infty$.

In the particular case when $\mu_N = 0$ and $a = b$, the limit values of (1.2) and (1.3) were recently obtained by Erdős and Kac [1].[1] The case when $\mu_N \neq 0$, especially when $\mu_N N^{1/2}$ converges to a finite value, is of particular importance in the theory of sequential tests of statistical hypotheses. It will be seen in §3 that the limit distribution of the number of observations required by a sequential probability ratio test can immediately be obtained from the limit values of (1.2) and (1.3), and vice versa.

2. **Proof that the limit values of** (1.2) **and** (1.3) **do not depend on the distribution of the** X**'s.** It will be assumed in this section that $\mu_N N^{1/2}$ converges to a finite value as $N \to \infty$. The independence of the limit values of (1.2) and (1.3) of the distribution of the X's was proved by Erdős and Kac [1] in the special case when $\mu_N = 0$ and $a = b$. To deal with the more general case considered here, we shall

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[1] Numbers in brackets refer to the references cited at the end of the paper.

follow essentially their method of proof. Let k be a positive integer, and let

$$N_j = \left[j \frac{N}{k} \right] \qquad (j = 1, 2, \cdots, k), \tag{2.1}$$

$$P_{N,k}(a) = \text{prob}\left\{\max\,(S_{NN_1}, \cdots, S_{NN_k}) < aN^{1/2}\right\}, \tag{2.2}$$

and

$$\begin{aligned} E_{Nr} = \text{prob}\, \{S_{Nr} \geqq aN^{1/2}, S_{N1} \\ < aN^{1/2}, \cdots, S_{N,r-1} < aN^{1/2}\}\,(r = 1, \cdots, N). \end{aligned} \tag{2.3}$$

Let, furthermore, ϵ be a positive number. Following Erdős and Kac, for $N_i < r \leqq N_{i+1}$ we write

$$\begin{aligned} E_{Nr} = \text{prob}\, \{S_{Nr} \geqq aN^{1/2}, S_{N1} < aN^{1/2}, \cdots, S_{N,r-1} \\ < aN^{1/2}, \left| S_{NN_{i+1}} - S_{Nr} \right| \geqq \epsilon N^{1/2}\} \\ + \text{prob}\, \{S_{Nr} \geqq aN^{1/2}, S_{N1} < aN^{1/2}, \cdots, S_{N,r-1} \\ < aN^{1/2}, \left| S_{NN_{i+1}} - S_{Nr} \right| < \epsilon N^{1/2}\}. \end{aligned} \tag{2.4}$$

Clearly, the first of these probabilities is equal to E_{Nr} prob $\{ | S_{NN_{i+1}} - S_{Nr} | \geqq \epsilon N^{1/2} \}$. Since

$$E(S_{NN_{i+1}} - S_{Nr})^2 \leqq (N_{i+1} - N_i) + (N_{i+1} - N_i)^2 \mu_N^2, \tag{2.5}$$

by Tchebychef's inequality we have

$$\begin{aligned} \text{prob}\, \{ \left| S_{NN_{i+1}} - S_{Nr} \right| \geqq \epsilon N^{1/2} \} \\ \leqq \frac{(N_{i+1} - N_i) + (N_{i+1} - N_i)^2 \mu_N^2}{\epsilon^2 N}. \end{aligned} \tag{2.6}$$

Since

$$\begin{aligned} \frac{N_{i+1} - N_i}{N} = \frac{1}{N}\left\{\left[(i+1)\frac{N}{k}\right]\right\} - \frac{1}{N}\left\{\left[i\frac{N}{k}\right]\right\} \\ \leqq \frac{1}{k} + \frac{1}{N} \leqq \frac{2}{k}, \end{aligned}$$

we obtain from (2.6)

$$\text{prob}\, \{ \left| S_{NN_{i+1}} - S_{Nr} \right| \geqq \epsilon N^{1/2} \} \leqq \frac{2}{\epsilon^2 k}\left(1 + \frac{2A}{k}\right) \tag{2.7}$$

where A is an upper bound of the sequence $\{N\mu_N^2\}$. From (2.4), (2.7) and the equation

$$\sum_{r=1}^{N} E_{Nr} = 1 - P_N(a) \leqq 1, \tag{2.8}$$

we easily obtain

$$\begin{aligned} 1 - P_N(a) = \sum_{r=1}^{N} E_{Nr} \leqq & \frac{2}{\epsilon^2 k}\left(1 + \frac{2A}{k}\right) \\ & + \sum_{i} \sum_{N_i < r \leqq N_{i+1}} \text{prob}\, \{S_{Nr} \geqq aN^{1/2}, \\ & S_{N1} < aN^{1/2}, \cdots, S_{N,r-1} < aN^{1/2}, \\ & \left| S_{NN_{i+1}} - S_{Nr} \right| < \epsilon N^{1/2}\}. \end{aligned} \tag{2.9}$$

Clearly, the double sum is less than the probability that at least one of the sums $S_{NN_1}, \cdots, S_{NN_k}$ exceeds $(a-\epsilon)N^{1/2}$. Hence

$$1 - P_N(a) \leqq \frac{2}{\epsilon^2 k}\left(1 + \frac{2A}{k}\right) + 1 - P_{N,k}(a - \epsilon). \tag{2.10}$$

This inequality can be written as

$$P_{N,k}(a - \epsilon) - \frac{2}{\epsilon^2 k}\left(1 + \frac{2A}{k}\right) \leqq P_N(a). \tag{2.11}$$

Let $G_{k1}, \cdots, G_{kk}$ be normally and independently distributed random variables with mean $\mu/k^{1/2}$ and variance 1, where $\mu = \lim_{N\to\infty} N^{1/2}\mu_N$. Let, furthermore,

$$R_{ki} = G_{k1} + \cdots + G_{ki} \qquad (i = 1, \cdots, k).$$

It follows from the central limit theorem that

$$\lim_{N\to\infty} P_{N,k}(a) = \text{prob}\, \{\max\, (R_{k1}, \cdots, R_{kk}) < ak^{1/2}\}. \tag{2.12}$$

From (2.11), (2.12) and the relation $P_N(a) < P_{N,k}(a)$ we obtain[2]

$$\begin{aligned} \text{prob}\, \{\max\, (R_{k1}, \cdots, R_{kk}) < (a - \epsilon)k^{1/2}\} - \frac{2}{\epsilon^2 k}\left(1 + \frac{2A}{k}\right) \\ \leqq \liminf_{N\to\infty} P_N(a) \leqq \limsup_{N\to\infty} P_N(a) \\ \leqq \text{prob}\, \{\max\, (R_{k1}, \cdots, R_{kk}) < ak^{1/2}\}. \end{aligned} \tag{2.13}$$

If the distribution of the X's is such that $|X_{Ni}|$ is constant, $\lim_{N\to\infty} P_N(a)$ exists and is a continuous function of a, as will be seen in §3. Let $P(a)$ denote this limit. Using the arguments given by Erdős and Kac [1, pp. 295–296], one can show that inequality (2.13) implies that for any arbitrary distribution of the X's we have

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[2] This inequality corresponds to inequality (1) in [1].

$$\lim_{N\to\infty} P_N(a) = P(a). \tag{2.14}$$

It will be seen in §3 that also $\lim_{N\to\infty} P_N^*(a, b) = P^*(a, b)$ exists and is a continuous function of a and b when the X's are distributed such that $|X_{Ni}|$ is constant. The proof that $\lim_{N\to\infty} P_N^*(a, b) = P^*(a, b)$ for any arbitrary distribution of the X's can be carried out in exactly the same manner as that of (2.14).

3. Determination of the limit values of (1.2) and (1.3) when the distribution of the X's is such that $|X_{Ni}|$ is constant.[3]

In this section it will be assumed that $|X_{Ni}|$ is constant and that $N^{1/2}\mu_N$ converges to a finite value as $N\to\infty$. The limits of (1.2) and (1.3) as $N\to\infty$ can easily be obtained from some results in the theory of sequential tests of statistical hypotheses (see [2] and [3]).

The sequential probability ratio test for testing a statistical hypothesis H_0 against an alternative hypothesis H_1 is defined as follows: Let H_i be the hypothesis that the elementary probability law of the random variable X under consideration is equal to $f_i(x)$ $(i = 0, 1)$. Let

$$z = \log \frac{f_1(x)}{f_0(x)}$$

and

$$z_k = \log \frac{f_1(x_k)}{f_0(x_k)}$$

where x_k denotes the kth observation on x. Thus, $z_1, z_2, \cdots$, ad inf.

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[3] This problem is intimately connected with a discrete model for the Brownian motion of a particle moving in a field of constant force (that is, gravity). Let a particle starting from the origin move along the x-axis in such a way that in each step it can move Δx to the right or Δx to the left with respective probabilities p and $q = 1 - p$. Let the duration of each step be Δt. This random walk becomes a model of the Brownian motion in a field of constant force in the limit when $\Delta x \to 0$, $\Delta t \to 0$, $2p - 1 \to 0$ in such a way that $(\Delta x)^2/2\Delta t = D$, $(2p-1)/2\Delta x = c/4D$ where c and D are physical constants. The problem of finding the probability that the particle should remain in an interval around the origin during a time interval $(0, t)$ is equivalent to the problem of finding the limit value of (1.3) when $|X_{Ni}|$ is constant.

The limit values of (1.2) and (1.3) are obtained here without difficulty from some previous results of the author [2]. In a subsequent publication [4], M. Kac treated the special case of a free particle ($c = 0$) using an interesting and entirely different method of attack. His method could be extended to treat also the case when $c \neq 0$.

are independent and identically distributed random variables. The test procedure is carried out as follows: two positive constants a and b are chosen. At each stage of the experiment, at the ith trial for each integral value i, the cumulative sum

$$Z_i = z_1 + z_2 + \cdots + z_i \qquad (i = 1, 2, \cdots, \text{ad inf.})$$

is computed. Additional observations are taken as long as $-b < Z_i < a$. The first time that this inequality does not hold, the test procedure is terminated. Let n denote the smallest integral value of i for which Z_i does not lie in the open interval $(-b, a)$. H_1 is accepted if $Z_n \geqq a$, and H_0 is accepted if $Z_n \leqq -b$.

It has been shown in [2] that for all points t in the complex plane for which the absolute value of $\phi(t) = E(e^{zt})$ is not less than 1, the following identity holds:

$$E[e^{Z_n t}\phi(t)^{-n}] = 1. \tag{3.1}$$

Assume now that z can take only two values, g and $-g$ $(g > 0)$. Let p denote the probability that $z = g$. Then the expected value of z is equal to

$$\mu = g(2p - 1) \tag{3.2}$$

and the variance of z is given by

$$\sigma^2 = g^2[1 - (2p - 1)^2]. \tag{3.3}$$

Let

$$m = \frac{\mu^2}{2\sigma^2} n, \tag{3.4}$$

$$t_1(\tau) = \frac{1}{g} \log \frac{e^{-(\mu^2/2\sigma^2)\tau} - (e^{-(\mu^2/\sigma^2)\tau} - 4p(1 - p))^{1/2}}{2p} \tag{3.5}$$

and

$$t_2(\tau) = \frac{1}{g} \log \frac{e^{-(\mu^2/2\sigma^2)\tau} + (e^{-(\mu^2/\sigma^2)\tau} - 4p(1 - p))^{1/2}}{2p} \tag{3.6}$$

where τ is a purely imaginary variable. Since the absolute value of

$$\phi[t_i(\tau)] = e^{-(\mu^2/2\sigma^2)\tau}$$

is equal to 1, we may substitute $t_i(\tau)$ for t in (3.1). We then obtain

$$E(e^{Z_n t_i(\tau)} e^{m\tau}) = 1 \qquad (i = 1, 2). \tag{3.7}$$

For any random variable u and any relation R let $E(u \mid R)$ denote the conditional expected value of u when R holds. For any value r, let $[r]$ denote the smallest integer not less than r. Since Z_n can take only the values $g[a/g]$ and $-g[b/g]$, equation (3.7) can be written as follows

$$\begin{aligned}&\text{prob}\left\{Z_n = -g\left[\frac{b}{g}\right]\right\}e^{-g[b/g]h_i(\tau)}\,E\left(e^{m\tau}\,\middle|\,Z_n = -g\left[\frac{b}{g}\right]\right)\\ (3.8)\qquad &+\text{prob}\left\{Z_n = g\left[\frac{a}{g}\right]\right\}e^{g[a/g]h_i(\tau)}\,E\left(e^{m\tau}\,\middle|\,Z_n = g\left[\frac{a}{g}\right]\right) = 1\ (i = 1, 2).\end{aligned}$$

Solving the two linear equations (3.8) in the unknowns $\psi_1(\tau) = \text{prob}\,\{Z_n = -g[b/g]\}E(e^{m\tau}|Z_n = -g[b/g])$ and $\psi_2(\tau) = \text{prob}\{Z_n = g[a/g]\}E(e^{m\tau}|Z_n = g[a/g])$, we obtain

$$\begin{aligned}&E(e^{m\tau}) = \psi_1(\tau) + \psi_2(\tau)\\ (3.9)\qquad &= \frac{e^{g[a/g]h_1(\tau)} + e^{-g[b/g]h_2(\tau)} - e^{g[a/g]h_2(\tau)} - e^{-g[b/g]h_1(\tau)}}{e^{g[a/g]h_1(\tau)-g[b/g]h_2(\tau)} - e^{g[a/g]h_2(\tau)-g[b/g]h_1(\tau)}}.\end{aligned}$$

We shall be interested in the limiting case when μ and σ take a sequence of values such that

$$(3.10)\qquad \lim \mu = 0, \qquad \lim \sigma = 0 \quad\text{and}\quad \lim \frac{\mu}{\sigma^2} = d,$$

where d is a finite value not equal to 0. It follows from (3.2) and (3.3) that (3.10) is equivalent with

$$(3.11)\qquad \lim\,(2p - 1) = 0, \qquad \lim g = 0 \quad\text{and}\quad \lim \frac{2p - 1}{g} = d.$$

It can easily be verified that

$$\frac{1}{g}\log\frac{e^{-(\mu^2/2\sigma^2)\tau}(1 \pm (1 - 4p(1 - p)e^{(\mu^2/\sigma^2)\tau})^{1/2})}{2p}$$

converges to

$$(3.12)\qquad -d(1 \pm (1 - \tau)^{1/2})$$

uniformly over any finite τ-interval as μ, σ and μ/σ^2 approach the limit values given in (3.10). Hence the characteristic function of m given in (3.9) converges to

$$(3.13)\quad \psi(\tau)=\frac{e^{-ad(1-(1-\tau)^{1/2})}+e^{bd(1+(1-\tau)^{1/2})}-e^{-ad(1+(1-\tau)^{1/2})}-e^{bd(1-(1-\tau)^{1/2})}}{e^{-ad(1-(1-\tau)^{1/2})+bd(1+(1-\tau)^{1/2})}-e^{-ad(1+(1-\tau)^{1/2})+bd(1-(1-\tau)^{1/2})}}$$

uniformly over any finite τ-interval as μ, σ and μ/σ^2 approach the limit values given in (3.10).

The characteristic function $\psi(\tau)$ has been inverted in [2] yielding the limit distribution of m. Denote this limit distribution by $F(u)$, that is, $F(u)=\text{prob}\{m<u\}$ where m is a random variable whose characteristic function is equal to $\psi(\tau)$. The value of $F(u)$ depends on the constants ad and bd, since these constants are involved in the characteristic function $\psi(\tau)$. To put this dependence in evidence, we shall also use the symbol $F(u \mid ad, bd)$.

We shall now express the limit value of $P_N^*(a, b)$ in terms of $F(u \mid ad, bd)$. It is assumed that X_{Ni} can take only two values, g_N and $-g_N$ $(g_N>0)$. The values g_N and $\text{prob}\{X_{Ni}=g_N\}$ are chosen so that the standard deviation of X_{Ni} is equal to 1 and the mean value of X_{Ni} has the prescribed value μ_N. We consider the case when $\lim_{N\to\infty}\mu_N N^{1/2}=d\neq 0$. Let the distribution of z be equal to that of $X_{Ni}/N^{1/2}$. The mean and standard deviation of z are then equal to

$$(3.14)\qquad \mu=\mu_N/N^{1/2} \quad\text{and}\quad \sigma=1/N^{1/2}$$

respectively. Hence, the limit values of μ, σ and μ/σ^2 as $N\to\infty$ are equal to those given in (3.10). Clearly,

$$(3.15)\qquad \begin{aligned} P_N^*(a, b) &= \text{prob}\,\{n > N\} \\ &= \text{prob}\left\{m > \frac{\mu^2}{2\sigma^2}N\right\} = \text{prob}\left\{m > \frac{\mu_N^2 N}{2}\right\}. \end{aligned}$$

Since $\lim_{N\to\infty}\mu_N N^{1/2}=d$ and since the limit distribution $F(u)$ of m is a continuous function of u, we obtain from (3.15)

$$(3.16)\qquad \lim_{N\to\infty} P_N^*(a, b) = 1 - F\left(\frac{d^2}{2}\,\middle|\, ad, bd\right) \qquad (d\neq 0).$$

The above formula is valid for $d\neq 0$ .We shall now determine the limit of $P_N^*(a, b)$ when $d=0$, that is, when $\lim_{N\to\infty}\mu_N N^{1/2}=0$. Since the value of $P_N^*(a, b)$ depends on the value of $d_N=\mu_N N^{1/2}$, we shall put this in evidence by writing $P_N^*(a, b \mid d_N)$.

Clearly, for any $d\neq 0$, we have

$$(3.17)\qquad \begin{aligned} P_N^*(a+2\mid d\mid, b+2\mid d\mid \mid 0) &> P_N^*(a, b\mid d) \\ &> P_N^*(a-2\mid d\mid, b-2\mid d\mid \mid 0). \end{aligned}$$

It follows from (3.17) that

$$
\begin{aligned}
(3.18)\qquad \liminf_{N\to\infty} P_N^*(a+2|d|, b+2|d| \mid 0) &\geqq 1 - F\left(\frac{d^2}{2} \,\middle|\, ad, bd\right) \\
&\geqq \limsup_{N\to\infty} P_N^*(a-2|d|, b-2|d| \mid 0).
\end{aligned}
$$

If $\mu_N = 0$, it follows from the first limit theorem of Erdős and Kac [1] that $\lim_{N\to\infty}$ prob $\{\max(S_{N1}, \cdots, S_{NN}) < aN^{1/2}\}$ and $\lim_{N\to\infty}$ prob $\{\min(S_{N1}, \cdots, S_{NN}) > -bN^{1/2}\}$ are continuous functions of a and b. This implies that $\limsup_{N\to\infty} P_N^*(a, b\mid 0)$ and $\liminf_{N\to\infty} P_N^*(a, b\mid 0)$ are continuous functions of a and b, and that

$$
\begin{aligned}
(3.19)\qquad \lim_{d=0} \limsup_{N\to\infty} [P_N^*(a+2|d|, b+2|d| \mid 0) \\
- P_N^*(a-2|d|, b-2|d| \mid 0)] = 0.
\end{aligned}
$$

It follows from (3.18) that

$$
\begin{aligned}
&\limsup_{N\to\infty} P_N^*(a+2|d|, b+2|d| \mid 0) \\
&\quad - \liminf_{N\to\infty} P_N^*(a+2|d|, b+2|d| \mid 0) \\
&\qquad \leqq \limsup_{N\to\infty} [P_N^*(a+2|d|, b+2|d| \mid 0) \\
&\qquad\quad - P_N^*(a-2|d|, b-2|d| \mid 0)].
\end{aligned}
$$

Hence, because of (3.19), we have

$$
\begin{aligned}
\lim_{d=0} [\limsup_{N\to\infty} P_N^*(a+2|d|, b+2|d| \mid 0) \\
- \liminf_{N\to\infty} P_N^*(a+2|d|, b+2|d| \mid 0)] = 0.
\end{aligned}
$$

The existence and continuity of $P^*(a, b\mid 0) = \lim_{N\to\infty} P_N^*(a, b\mid 0)$ follows from the above equation and the continuity of $\limsup_{N\to\infty} P_N^*(a, b\mid 0)$ and $\liminf_{N\to\infty} P_N^*(a, b\mid 0)$. Hence, because of (3.18), we have

$$
(3.20)\qquad P^*(a, b \mid 0) = \lim_{d=0}\left[1 - F\left(\frac{d^2}{2} \,\middle|\, ad, bd\right)\right].
$$

Let $\{d_N\}$ be a sequence of values such that $\lim_{N\to\infty} d_N = 0$. Substituting d_N for d in (3.17) and letting $N\to\infty$, we obtain

$$
(3.21)\qquad \lim_{N\to\infty} P_N^*(a, b \mid d_N) = P^*(a, b \mid 0) = \lim_{d=0}\left[1 - F\left(\frac{d^2}{2} \,\middle|\, ad, bd\right)\right].
$$

We shall now show that

$$\lim_{N=\infty} P_N(a) = \lim_{b=\infty} \lim_{N=\infty} P_N^*(a, b). \tag{3.22}$$

To prove (3.22), it is sufficient to show that

$$\lim_{b=\infty} \liminf_{N=\infty} \text{prob} \left\{\min (S_{N1}, \cdots, S_{NN}) > -bN^{1/2}\right\} = 1. \tag{3.23}$$

Let r be an upper bound of $|N^{1/2}\mu_N|$. Clearly,

$$\begin{aligned} &\text{prob} \left\{\min (\bar{S}_{N1}, \cdots, \bar{S}_{NN}) > (-b + r)N^{1/2}\right\} \\ &\qquad \leqq \text{prob} \left\{\min (S_{N1}, \cdots, S_{NN}) > -bN^{1/2}\right\} \end{aligned} \tag{3.24}$$

where $\bar{S}_{Ni} = S_{Ni} - i\mu_N$. Since according to the first limit theorem of Erdős and Kac [1] we have

$$\begin{aligned} &\lim_{b=\infty} \lim_{N=\infty} \text{prob} \left\{\min (\bar{S}_{N1}, \cdots, \bar{S}_{NN}) > (-b + r)N^{1/2}\right\} \\ &\qquad = \lim_{b=\infty} \left(\frac{2}{\pi}\right)^{1/2} \int_0^{b-r} e^{-t^2/2}dt = 1, \end{aligned}$$

(3.23) follows from (3.24). Hence (3.22) is proved.

If $\lim (\mu/\sigma^2) = d > 0$ and $b \to \infty$, the characteristic function $\psi(\tau)$ of m given in (3.13) converges to

$$\psi^*(\tau) = e^{ad(1-(1-\tau)^{1/2})}. \tag{3.25}$$

This characteristic function has been inverted in [2] and the corresponding distribution function of m is given by

$$H(m)dm = \frac{ad}{2\Gamma(1/2)m^{3/2}} e^{-(a^2d^2/4m)-m+ad}dm \qquad (0 \leqq m < \infty).$$

Hence

$$\lim_{N=\infty} P_N(a) = \text{prob} \left\{m > \frac{d^2}{2}\right\} = \int_{d^2/2}^{\infty} H(m)dm. \tag{3.26}$$

We can summarize the results of this section in the following theorem.

THEOREM. *If* $\lim_{N=\infty} \mu_N N^{1/2} = d \neq 0$, *the limit value of the probability* (1.3) *is given by*

$$P^*(a, b \mid d) = \text{prob} \left\{m > \frac{d^2}{2}\right\}$$

where m is a random variable whose characteristic function is given in (3.13). *If $d=0$, the limit value of* (1.3) *is equal to* $\lim_{a\to 0} P^*(a, b \mid d)$. *For any finite value d, the limit value of* (1.2) *is equal to* $\lim_{b\to\infty} P^*(a, b \mid d)$. *If $d>0$, the limit value of* (1.2) *is given explicitly in* (3.26).

4. **Derivation of the limit value of** (1.4) **when** $\lim \mu_N N^{1/2} = \infty$. In this section we shall determine the limit value of the probability $Q_N(c)$ defined in (1.4) assuming that

$$\lim \mu_N N^{1/2} = \infty. \tag{4.1}$$

We can assume without loss of generality that $\mu_N > 0$ for all N. Let r be a positive number and let $\lambda(N)$ be a positive integral-valued function of N such that

$$\lambda(N) < N, \qquad \lim_{N\to\infty} \frac{N-\lambda(N)}{N} = 0, \tag{4.2}$$

$$\lim_{N\to\infty} \frac{N-\lambda(N)}{N^{1/2}} \mu_N = r' \tag{4.3}$$

where $\infty \geqq r' > r$. It follows from (4.1) that such a function $\lambda(N)$ exists. Because of (4.2) and (4.3) we have for sufficiently large N

$$\begin{aligned} &\text{prob } \{\max (S_{N1}, \cdots, S_{N\lambda(N)}) < N\mu_N + cN^{1/2}\} \\ &\quad > \text{prob } \{\max (S_{N1}, \cdots, S_{N\lambda(N)}) < \lambda(N)\mu_N + (c+r)N^{1/2}\} \\ &\quad > \text{prob } \{\max (\bar{S}_{N1}, \cdots, \bar{S}_{N\lambda(N)}) < (c+r)N^{1/2}\} \end{aligned} \tag{4.4}$$

where

$$\bar{S}_{Ni} = S_{Ni} - i\mu_N. \tag{4.5}$$

Let $\epsilon > 0$. Since for $\mu_N = 0$ we have $\lim_{c\to\infty} \lim_{N\to\infty} P_N(c) = 1$, there exists a fixed value r_0 (independent of N) such that

$$\lim_{N\to\infty} \text{prob } \{\max (\bar{S}_{N1}, \cdots, \bar{S}_{N\lambda(N)}) < (c+r_0)N^{1/2}\} = 1 - \frac{\epsilon}{2}. \tag{4.6}$$

Putting $r = r_0$, we obtain from (4.4) and (4.6)

$$\text{prob } \{\max (S_{N1}, \cdots, S_{N\lambda(N)}) < N\mu_N + cN^{1/2}\} \geqq 1 - \epsilon \tag{4.7}$$

for sufficiently large N. Hence

$$\begin{aligned} &\text{prob } \{\max (S_{N1}, \cdots, S_{NN}) < N\mu_N + cN^{1/2}\} \\ &\quad \leqq \text{prob } \{\max (S_{N,\lambda(N)+1}, \cdots, S_N) < N\mu_N + cN^{1/2}\} \\ &\quad \leqq \text{prob } \{\max (S_{N1}, \cdots, S_{NN}) < N\mu_N + cN^{1/2}\} + \epsilon \end{aligned} \tag{4.8}$$

for sufficiently large N. From (4.2) and the first limit theorem of Erdös and Kac [1, p. 292] it follows easily that

$$\lim_{N\to\infty} \text{prob} \left\{\max \left(-\bar{S}_{NN} + \bar{S}_{N,N-1}, \cdots, -\bar{S}_{NN} + \bar{S}_{N,\lambda(N)+1}\right) < \delta N^{1/2}\right\} = 1 \tag{4.9}$$

for any positive δ. Since the inequality

$$\max \left(-\bar{S}_{NN} + \bar{S}_{N,N-1}, \cdots, -\bar{S}_{NN} + \bar{S}_{N,\lambda(N)+1}\right) < \delta N^{1/2}$$

implies the validity of $\max (S_{N,\lambda(N)+1}, \cdots, S_{NN}) \leqq S_{NN}+\delta N^{1/2}$, we obtain from (4.9)

$$\lim_{N\to\infty} \text{prob} \left\{S_{NN} \leqq \max (S_{N,\lambda(N)+1}, \cdots, S_{NN}) \leqq S_{NN}+\delta N^{1/2}\right\} = 1. \tag{4.10}$$

Since

$$\lim_{N\to\infty} \text{prob} \left\{S_{NN} < N\mu_N + cN^{1/2}\right\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{c} e^{-t^2/2}dt, \tag{4.11}$$

we obtain from (4.10)

$$\begin{aligned} \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{c} e^{-t^2/2}dt & \\ &\geqq \limsup_{N\to\infty} \text{prob} \left\{\max (S_{N,\lambda(N)+1}, \cdots, S_{NN}) < N\mu_N + cN^{1/2}\right\} \\ &\geqq \liminf_{N\to\infty} \text{prob} \left\{\max (S_{N,\lambda(N)+1}, \cdots, S_{NN}) < N\mu_N + cN^{1/2}\right\} \\ &\geqq \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{c-\delta} e^{-t^2/2}dt. \end{aligned} \tag{4.12}$$

Since δ can be chosen arbitrarily small, it follows from (4.12) that

$$\lim_{N\to\infty} \text{prob} \left\{\max (S_{N,\lambda(N)+1}, \cdots, S_{NN}) < N\mu_N + cN^{1/2}\right\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{c} e^{-t^2/2}dt. \tag{4.13}$$

Finally, it follows from (4.8) and (4.13), since ϵ can be chosen arbitrarily small, that

$$\lim_{N\to\infty} \text{prob} \left\{\max (S_{N1}, \cdots, S_{NN}) < N\mu_N + cN^{1/2}\right\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{c} e^{-t^2/2}dt. \tag{4.14}$$

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NOTE ON THE ZEROS OF $P_n^m(\cos\theta)$ AND $dP_n^m(\cos\theta)/d\theta$ CONSIDERED AS FUNCTIONS OF n

C. W. HORTON

In many physical problems in which the boundary conditions are specified over the surface of a cone, it is necessary to know the roots of the equations

$$(1)\qquad P_n^m(\cos\theta)=0$$

and

$$(2)\qquad dP_n^m(\cos\theta)/d\theta=0$$

considered as functions of n. This problem has been solved by Bholanath Pal.[1] In these papers he develops infinite series for the roots n which converge rapidly and are very suitable for numerical computation. In deriving his solution Pal introduced a parameter k which takes on successive integer values and thereby yields successive roots of the equations.

It is the purpose of this note to point out that the value $k=1$ with which Pal commenced the series does not always give the first root of the equation, and sometimes it gives a number which is not a root of the equation. For example, in treating the equation $P_n^2(\cos\theta)=0$, Pal gives three roots: $n=4.77$, 2.26, 1.52, corresponding to values of θ equal to 15°, 30°, 45°, respectively. That these values are not roots

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[1] Bull. Calcutta Math. Soc. vol. 9 (1917–1918) p. 85; vol. 10 (1918–1919) p. 187.

of the equation can be established by examining the *Tables of associated legendre functions.*[2]

In numerous instances the value $k=1$ does not give the first root of the equation. In several of these cases the series for n will converge when k is set equal to zero, although in one or two instances the series apparently does not converge.

The tables given by Pal for the roots of $dP_n^1(\cos\theta)/d\theta=0$ start with the second root, and apparently he was not aware of the fact that a smaller root existed-which could be obtained from his formulae by setting $k=0$. This occurs also in his table of roots for the equation $dP_n^2(\cos\theta)/d\theta=0$ for $\theta=15°$ and $30°$, although in the case of $\theta=45°$, $k=1$ does give the first root.

Thus, although Pal's formulae give satisfactory methods of computing the roots of equations (1) and (2), they must be used with discretion, since it is difficult to ascertain if the formulae yield more or less than the correct number of roots. Amended lists of the roots of equations (1) and (2) are given in Tables I and II respectively.

TABLE I

Roots of $P_n^m(\cos\theta)=0$ treated as a function of n

$\theta=15°$		$\theta=30°$		$\theta=45°$	
m	n	m	n	m	n
0	8.45[3]	0	4.13[3]	0	2.59[3]
	20.60		10.03		6.52
	32.55		16.02		10.51
	44.53		22.01		14.51
	56.53		28.01		18.50
	68.52		34.01		22.50
±1	14.14	±1	6.83	±1	4.40
	26.30		12.91		8.44
	38.36		18.93		12.46
	50.39		24.95		16.47
	62.41		30.96		20.47
±2	19.21[4]	±2	9.39[4]	±2	6.15[4]
	31.67		15.62		10.28
	43.90		21.72		14.34
	56.03		27.78		18.37

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[2] A. N. Lowan, Columbia University Press, 1945.

[3] Values overlooked by Pal.

[4] In these sequences Pal gives a smaller value that does not exist.

TABLE II

Roots of $dP_n^m(\cos\theta)/d\theta=0$ treated as a function of n

$\theta=15°$		$\theta=30°$		$\theta=45°$	
m	n	m	n	m	n
0	14.12	0	6.83	0	4.42[3]
	26.29		12.89		8.33
	38.35		18.93		12.34
	50.39		24.95		16.34
	62.41		30.96		20.35
					24.35
±1	6.80[3]	±1	3.19[3]	±1	2.00[3]
	19.88		9.71		6.32
	32.11		15.82		10.33
	44.22		21.87		14.34
	56.28		27.89		18.34
	68.32		33.91		22.34
±2	11.80[3]	±2	5.75[3]	±2	3.73
	25.06		12.38		8.15
	37.58		18.58		12.24
	49.83		24.68		16.26
	61.96		30.74		20.28
	74.03		36.78		

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FROM AMONG n CONJUGATE ALGEBRAIC INTEGERS, $n-1$ CAN BE APPROXIMATELY GIVEN

TH. MOTZKIN

1. **Summary.** Given n arbitrary complex numbers $z_1, \cdots, z_n$, it is easy to construct (see 6.2) an irreducible algebraic equation

$$\zeta^n + \alpha_1\zeta^{n-1} + \cdots + \alpha_n = 0$$

with complex rational coefficients α_ν, such that its n roots $\zeta_1, \cdots, \zeta_n$ lie within assigned neighbourhoods of $z_1, \cdots, z_n$. If the numbers $z_1, \cdots, z_n$ are symmetric to the real axis, then there exists an equation with real rational coefficients whose roots are near $z_1, \cdots, z_n$. If, however, the coefficients α_ν are required to be integers, then in sufficiently small neighbourhoods there will be no system $\zeta_1, \cdots, \zeta_n$, except possibly the system $z_1, \cdots, z_n$ itself, as follows immediately from the continuity of the coefficients as functions of the roots.

In this note we prove (Theorem 3.2) that for every $n-1$ given numbers $z_1, \cdots, z_{n-1}$, and every $\epsilon>0$, there exists an irreducible equation with complex integral coefficients $\alpha_1, \cdots, \alpha_n$ and with roots ζ_ν such that $|\zeta_\nu - z_\nu| < \epsilon$ for $\nu = 1, \cdots, n-1$. The same is true (§5) for real integral coefficients provided that the numbers $z_1, \cdots, z_{n-1}$ are symmetric to the real axis. Some remarks on the possible location of the free root ζ_n are added (§4).

The proof of the main theorems employs the well known facts of the solubility of a system of linear Diophantine inequalities under certain conditions, and of the uniform continuity of the roots of an algebraic equation

$$b_0y^n + \cdots + b_n = 0$$

with general complex coefficients b_ν, as functions of the ratios of these coefficients. A topological proof of the latter theorem is prefixed (§2); uniform continuity, and incidentally ordinary continuity, of the roots of an algebraic equation is exhibited as a consequence of the classical proposition of set theory that asserts the uniform continuity of every topological mapping of a compact space.[1]

2. **The theorem on uniform continuity.** 2.1. Every polynomial

$$z^n + a_1z^{n-1} + \cdots + a_n$$

corresponds to a point $a = (a_1, \cdots, a_n)$ of the complex affine n-space

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[1] For example Sierpiński, *General topology*, p. 99.

A_n. The system $(z_1, \cdots, z_n)$ of the roots of the polynomial can be considered as a point of the complex symmetric-affine n-space Z_n that is obtained from the ordinary affine n-space by identifying points $(z_1, \cdots, z_n)$ that have the same coordinates z_ν in different order. Neither A_n nor Z_n is compact.

The closure of $A_1 = Z_1$ formed by adding the point ∞ is the complex projective one-space B_1. By allowing z_ν to take the value ∞, the space Z_n is enlarged and becomes the symmetric-multiprojective space Y_n that contains as points all the systems $(y_1, \cdots, y_n)$ of roots of equations $b_0 y^n + \cdots + b_n = 0$. The coefficients $b_0, \cdots, b_n$ are the coordinates of a point of the complex projective n-space B_n; to every point of B_n there correspond infinitely many points of A_{n+1} with normalized coordinates, that is, such that $\sum |b_\nu|^2 = 1$. Any two normalized representations of the same point of B_n differ by a factor λ with $|\lambda| = 1$.

The spaces Y_n and B_n are compact, and since the mapping of $y = (y_1, \cdots, y_n)$ on $b = (b_0, \cdots, b_n)$ is one-to-one, continuous, and defined for every y, this mapping is topological.

2.2. An appropriate metric of B_n is obtained by defining the distance $\rho(b, b')$ between two points of B_n as that between the two infinite subsets of A_{n+1} corresponding to the two given points, that is, as the least ρ for which there exist normalized representations of b and b' with $(\sum |b_\nu - b'_\nu|^2)^{1/2} = \rho$, or alternatively, $\max_\nu |b_\nu - b'_\nu| = \rho$.[2] The normalized representation of one of the points, say b, can be arbitrarily chosen when determining the distance; there follows at once $\rho(b, b') + \rho(b, b'') \geqq \rho(b', b'')$.

In particular the distance $\rho(y, y')$ between two points y and y' of $B_1 = Y_1$ is defined as the least ρ with $(|z_1 - z'_1|^2 + |z_2 - z'_2|^2)^{1/2} = \rho$ for a suitable representation $y = z_1/z_2$, $y' = z'_1/z'_2$ with $|z_1|^2 + |z_2|^2 = |z'_1|^2 + |z'_2|^2 = 1$, or equivalently by use of the spherical distance.

In the same way a metric of Y_n can be introduced by defining the distance $\rho(y, y')$ between two points of Y_n as that of the sets of identified points, that is, as the least ρ for which there exists a permutation $1', \cdots, n'$ of $1, \cdots, n$ such that all $\rho(y_\nu, y'_{\nu'}) \leqq \rho$.

With these definitions of distances in B_n and Y_n, the theorem on the uniform continuity of every topological mapping of a compact space entails:

If the distance between the coefficients of two polynomials of the same degree is less than ϵ, the distance between their systems of roots is less than

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[2] Or by any other natural extension of a metric from one to more dimensions, in the sense of Th. Motzkin, *Sur le produit des espaces métriques*, Comptes Rendus du Congrès International des Mathématiciens Oslo 1936, vol. 2, p. 137.

ϵ', and $\epsilon' \to 0$ for $\epsilon \to 0$.[3]

The example of the polynomials $\epsilon z^n - 1$ and 1 shows that uniform convergence independently of the degree n does not hold for the above distance; perhaps it holds for a similar definition in regard to $\sum (b_\nu z)^\nu = 0$.

3. The theorem on approximation by conjugate integers. 3.1. *For any given complex number z and every $\epsilon > 0$ there exists a root ζ of an irreducible algebraic equation $\zeta^2 + \alpha_1 \zeta + \alpha_2 = 0$ with complex integer coefficients α_1 and α_2 such that $|\zeta - z| < \epsilon$.*

Proof. The function $\alpha^{1/2}$ maps the parallels to the axes through complex integers into two orthogonal systems of equilateral hyperbolas. For integral $\alpha \to \infty$ their consecutive intersections become arbitrarily near to each other. The non-integral numbers $\zeta = \alpha^{1/2} - \alpha'$ with complex integral α and α' are therefore dense on the whole plane.

Alternatively, this theorem is a particular case of 3.2 and can be proved in the same way, every step of the proof becoming much simpler.

3.2. *For any $n-1$ given complex numbers*

$$z_1, \cdots, z_{n-1}$$

and every $\epsilon > 0$ there exists a system

$$\zeta_1, \cdots, \zeta_{n-1}$$

of roots of an irreducible algebraic equation $\zeta^n + \alpha_1 \zeta^{n-1} + \cdots + \alpha_n = 0$ with complex integral coefficients α_ν such that

$$|\zeta_\nu - z_\nu| < \epsilon, \qquad \nu = 1, \cdots, n-1.$$

3.3. Proof. The numbers z_ν are roots of an equation $z^{n-1} + a_1 z^{n-2} + \cdots + a_{n-1} = 0$ with general complex coefficients $a_\nu = c_{2\nu-1} + i c_{2\nu}$.

We assume the real numbers

$$1, c_1, \cdots, c_{2n-2}$$

to be rationally independent, that is, not to fulfill any homogeneous linear equation with coefficients that are rational and not all 0. If necessary we can achieve this independence by slightly changing the numbers $c_1, \cdots, c_{2n-2}$ one after the other, always avoiding the enumerable set of values with a rational relation between them and the preceding numbers. Because of the continuity of the roots of an algebraic equation, the new roots z_ν are near to the old ones, so that numbers ζ_ν near enough to the new roots z_ν are also near to the given ones.

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[3] With the same proof, the theorem on uniform continuity may be enunciated for polynomials with n real roots, even before introducing complex numbers. Indeed, the theorem on uniform continuity and that on the existence of the roots of an algebraic equation (the "fundamental theorem of algebra") are independent of each other. For a topological proof of the "fundamental theorem," see W. L. Chow, Math. Ann. vol. 116 (1939) p. 463; for an elementary proof see Th. Motzkin and A. Ostrowski, Preuss. Akad. Wiss. Sitzungsber. 1933, p. 255.

3.4. We also assume the numbers z_ν, and every subset of them, not to be a system of roots of an algebraic equation with complex integral coefficients. Otherwise replace the z_ν by $z_\nu\lambda$ with a suitable real rational λ near 1. This does not affect the above property of rational independence, since the a_ν are only multiplied by real rational factors.

Then the existence of a system $\zeta_1, \cdots, \zeta_n$ of roots of an algebraic equation with complex integral coefficients entails its irreducibility for sufficiently small ϵ. For if there existed a sequence of reducible systems with $\epsilon\rightarrow 0$, then infinitely many of them would have a subsystem belonging to the same indices $\nu<n$. Passing to the limit, the corresponding z_ν would form a system of roots of an algebraic equation with complex integral coefficients.

3.5. Now if there exist complex integers $\alpha_1, \cdots, \alpha_n$ such that

$$\left|(\alpha_{\nu+1} - a_\nu\alpha_1) - (a_{\nu+1} - a_\nu a_1)\right| < \epsilon_1, \qquad \nu = 1, \cdots, n-1,$$

where $a_n=0$, put $a_1-\alpha_1=z_n$. Then the coefficients $\alpha_{\nu+1}$ of $\zeta^n+\alpha_1\zeta^{n-1} + \cdots +\alpha_n$ differ from those

$$a'_{\nu+1} = a_{\nu+1} - a_\nu z_n = a_{\nu+1} - a_\nu a_1 + a_\nu\alpha_1$$

of $(z-z_n)(z^{n-1}+a_1z^{n-2}+ \cdots +a_{n-1})$ by less than ϵ_1, and the coefficient of z^{n-1} is $a_1' =\alpha_1$ in both polynomials. Hence the distance between the points $(1, \alpha_1, \cdots, \alpha_n)$ and $(1, a_1', \cdots, a_n')$ in complex projective n-space is less than ϵ_2, and $\epsilon_2\rightarrow 0$ for $\epsilon_1\rightarrow 0$. By the theorem on the uniform continuity of the roots of algebraic equations, the roots $\zeta_1, \cdots, \zeta_n$ and $z_1, \cdots, z_n$ may be so arranged that the spherical distance (or distance as defined in 2.2) between corresponding roots is less than ϵ_3, and $\epsilon_3\rightarrow 0$ for $\epsilon_2\rightarrow 0$. Hence $|\zeta_\nu-z_\nu|<\epsilon$, $\nu=1, \cdots, n-1$, and $\epsilon\rightarrow 0$ for $\epsilon_3\rightarrow 0$.

Since $a_1' =\alpha_1$, we have $\zeta_1+ \cdots +\zeta_n=z_1+ \cdots +z_n$, whence $|\zeta_n-z_n|<(n-1)\epsilon$.

3.6. Now a theorem of Kronecker[4] states that, for rationally independent real $1, c_1, c_2, \cdots$, arbitrary real $d_1, d_2, \cdots$, and $\epsilon_1>0$, there are real integers $\alpha_1>0, \gamma_1, \gamma_2, \cdots$ such that $|\alpha_1c_\mu-\gamma_\mu-d_\mu|<\epsilon_1$, $\mu=1, 2, \cdots$. Taking $d_1, d_2, \cdots$ as the real and imaginary parts of $a_1a_\nu-a_{\nu+1}$, we see that the Diophantine inequalities at the beginning of 3.5 can be solved for complex integers $\alpha_1>0$, $\alpha_2=\gamma_1+i\gamma_2, \cdots$. This completes the proof of 3.2.

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[4] For example Hardy-Wright, *Theory of numbers*, p. 370, Theorem 442.

For a given α_1 there is at most one solution, if $\epsilon_1 \leqq 1/2$.

4. **Situation of the remaining root ζ_n.** 4.1. *For every* $N>0$ *there is an* $\epsilon(N)>0$ *such that for* $\epsilon<\epsilon(N)$ *either* $|\zeta_n|>N$ *or* $\zeta_\nu=z_\nu$, $\nu=1, \cdots, n-1$.

Proof. There exists either no value ζ with $|\zeta|\leqq N$ such that $z_1, \cdots, z_{n-1}, \zeta$ are the roots of an algebraic equation of degree n with complex integral coefficients, or one value, or (for the roots $z_1, \cdots, z_{n-1}$ of an algebraic equation of degree $n-1$ with complex integral coefficients) every complex integer ζ with $|\zeta|\leqq N$ is appropriate; at any rate the number of the possible ζ is finite. For a given ζ, a system ζ_ν close enough to the numbers $z_1, \cdots, z_{n-1}, \zeta$ must coincide with them; let ϵ_0 be a distance ensuring coincidence for every ζ. Now if there existed for every $\epsilon'=1, 1/2, 1/3, \cdots$ values ζ_n with $|\zeta_n|\leqq N$ and $|\zeta_n-\zeta|\geqq\epsilon'$ for every possible ζ, then a subsequence of these ζ_n would converge to a new ζ. Hence no such ζ_n exists for a certain ϵ'; then we may put $\epsilon(N)=\min(\epsilon', \epsilon_0)$.

4.2. There exist, however, values ζ_n near any given direction. More precisely:

The numbers $\zeta_1, \cdots, \zeta_n$ *may be required to have their sum in a given half strip that contains a complex integer; or, what is less,* ζ_n *may be required to be within a given angle with arbitrary vertex and arbitrarily small aperture.*[5]

Proof. The second part of the theorem follows immediately from the first part. Indeed, since every angle contains a half strip with infinitely many complex integers, the sum $-\alpha_1$ may be required to be within a given angle. Hence the same is true for $z_n=a_1-\alpha_1$ and, by the inequality $|\zeta_n-z_n|<(n-1)\epsilon$ at the end of 3.5, for ζ_n.

To prove the first contention, we remember that by 3.5, it is enough to show that the complex integer α_1 can be determined such that the numbers $a_\nu\alpha_1$ are, mod integers, near given numbers. If the given half strip contains an integer ray $\gamma+\delta x$, $x>0$, we put $\alpha_1=\gamma+\delta\alpha'$, $\alpha'=1, 2, \cdots$. The Diophantine inequalities for α', α_2, $\alpha_3, \cdots$ are soluble, as in 3.6, if 1 and the real and imaginary parts of the numbers $a_\nu\delta$ are rationally independent. Now every rational relation between them might be written as an equation $\sum\delta_\nu a_\nu\delta+\delta'=c$ with complex rational δ_ν and δ' and real c; putting $\delta_\nu\delta=\delta_\nu'$, this would become a similar relation for the numbers a_ν. But the real and imaginary part of these numbers, and 1, have been supposed to be rationally independent.

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[5] A strip between, but not including, two parallel straight lines is divided by a non-parallel line into two half strips. A strip or half strip contains infinitely many complex integers either if it contains an integral ray (half straight line through two complex integers) or if its direction is irrational; otherwise it contains no complex integer.

On the other hand, if the given half strip has an irrational direction, that is, if it contains a ray $x+(cx+c')i$ with variable real $x>N$ (or $x<N$) and given real c' and real irrational c, then the imaginary part γ' of $\alpha_1=\gamma+i\gamma'$ is required to be near $c\gamma+c'$. Hence the numbers $c\gamma$, $a_\nu(1+ic)\gamma$ have to be, mod integers, near given numbers, for a $\gamma>N$ (or $\gamma<N$). Such numbers exist provided that 1, c, and the real and imaginary parts of $a_1(1+ic), \cdots, a_{n-1}(1+ic)$ are rationally independent, which can be attained, as in 3.3, by small changes of the numbers a_ν.

5. The approximation theorem for real coefficients. 5.1. *For any given numbers $z_1, \cdots, z_{n-1}$ that are symmetric to the real axis and every $\epsilon>0$ there exists a system $\zeta_1, \cdots, \zeta_n$ of roots of an algebraic equation $\zeta^n+\alpha_1\zeta^{n-1}+\cdots+\alpha_n=0$ with real integral coefficients α_ν such that $|\zeta_\nu-z_\nu|<\epsilon$ for all $\nu\leq n-1$.*

Moreover, ζ_ν may be required to be real if the corresponding z_ν is real, and ζ_n may be required to be on a given real ray.

The proof is as in the general case. The real numbers z_ν can be changed a little so as to be different from each other; then the symmetry of the corresponding roots ζ_ν will compel these roots to be real. This change should be effectuated before the displacement ensuring that the numbers $1, a_1, \cdots, a_{n-1}$ are rationally independent and that the z_ν have no subset which is the system of roots of an algebraic equation with real integral coefficients. The latter condition entails irreducibility, the former the solubility of the Diophantine inequalities for real integral α_ν and α_1 on the given ray, whence the theorem.

5.2. Rational independence as required for this proof can also be attained by a simultaneous translation, according to the lemma:

For every finite set $z_1, \cdots, z_{n-1}$ that is symmetric to the real axis and every $\epsilon>0$ there exists an ϵ' with $0<\epsilon'<\epsilon$ such that the coefficients $1, a_1', \cdots, a_{n-1}'$ of the equation with the roots $z_1+\epsilon', \cdots, z_{n-1}+\epsilon'$ are rationally independent.

PROOF. Otherwise, for every ϵ', there would hold a rational relation with coefficients not all 0. The set of rational relations being enumerable, and the set of ϵ' between 0 and ϵ not, there would exist a relation holding for infinitely many ϵ'. Since the a_ν' are polynomials in ϵ', this relation belongs to every ϵ'. But such a relation between the coefficients $a(\epsilon), a'(\epsilon), \cdots, a^{(n-1)}(\epsilon)/(n-1)!$ of $1, z, \cdots, z^{n-1}$ in a polynomial $a(z+\epsilon)$ cannot exist, since $a(\epsilon), \cdots, a^{(n-1)}(\epsilon)$ are of degree $n-1, \cdots, 0$.

6. **Miscellaneous remarks.** 6.1. Let $\beta(\eta)=\beta_0\eta^n+\beta_1\eta^{n-1}+\cdots+\beta_n$ have complex integral coefficients β_ν, and let q be a complex integer.

The polynomial $p\beta(\eta)+q$, *where* p *is a complex prime not dividing* $q\beta_0$, *is irreducible.*[6]

Proof. Otherwise $p\beta(\eta)+q=\beta'(\eta)\beta''(\eta)$. The product of the highest powers in $\beta'(\eta)$ and $\beta''(\eta)$ with coefficients not divisible by p is $q \pmod{p}$, hence these powers are constants. The highest coefficients of $\beta'(\eta)$ and $\beta''(\eta)$ would therefore be divisible by p, and $p\beta_0$ by p^2.

6.2. *In the proximity of every given* n *numbers* $z_1, \cdots, z_n$ *there are systems of roots of an irreducible equation of degree* n *with complex rational coefficients.*

Proof. A polynomial $\beta(\eta)$ with (for example rational) roots near the given numbers is easily found. The roots of $p\beta(\eta)+1$ are near to those of $\beta(\eta)$ if p is large, because of the continuity of the roots of an algebraic equation as functions of the coefficients, so that a sufficiently large complex prime p not dividing β_0 will do (6.1).

If the given numbers are symmetric to the real axis, the polynomials $\beta(\eta)$ and $p\beta(\eta)+1$ may be assumed to be real. If we want the roots near the real z_ν to be real, coinciding real z_ν are first separated as in 5.1.

6.3. *The roots of irreducible equations of a given degree* $n\geqq 2$ *with complex integral coefficients are dense on the whole plane.*

Proof. This immediate consequence of Theorem 3.2 on systems $n-1$ of whose numbers are approximately given follows already from 3.1, without reference to 3.2. This is trivial for $n=2$. Even the existence of a polynomial $\alpha(\zeta)$ of degree 2 with one root near a given number z, whose coefficients are divisible by a complex prime p, follows by means of multiplication by p of a system of two conjugate algebraic integers one of which is near z/p (or also, if 3.1 is proved in the same way as 3.2, from the solubility of the Diophantine inequalities for α, divisible by p). But if $n>2$, then $\alpha(\zeta)\alpha'(\zeta)+p$—where $\alpha'(\zeta)$ is a polynomial of degree $n-2$ with large coefficients divisible by p—has two of its roots near to those of $\alpha(\zeta)$, and is irreducible by Eisenstein's rule.

Likewise Theorem 5.1 implies that the roots of irreducible equations of a given degree $n\geqq 3$ with real rational coefficients are dense on the whole plane, while for $n=2$ they are dense only on the real axis.

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[6] This is Eisenstein's rule, after putting $\zeta=1/\eta$, cf. van der Waerden, *Moderne Algebra*, vol. 1, p. 77.

ON MERSENNE'S NUMBER M_{199} AND LUCAS'S SEQUENCES

H. S. UHLER

On July 27, 1946 the writer finished calculating the 198th remainder of the Lucasian sequence 3, 7, 47, $\cdots$ as applied to the 60-digit Mersenne number $2^{199}-1=80346$ 90221 29495 13777 09810 46170 58130 12611 01496 89139 64176 50687. The result was $r_{198}=8387$ 51186 96313 46717 54322 73509 44243 96183 21834 95333 72125 49353. Since this residual is not zero and since the calculations were performed with great care it follows that M_{199} is composite.

During the course of the work each arithmetical operation was checked with the auxiliary moduli 10^5+1 and 10^8+1. After the date given above all of the work-strips of the entire set were again examined and checked with a convenient modulus. As explained in an earlier paper[1] the essential figures of each of the terms above the 8th of the Lucasian sequence for $p=4n-1$ were multiplied in order by the reciprocal of the chief modulus, M_{199}, in preference to direct division by M_{199}. The approximation to this reciprocal was computed to be $(1/M_{199})_a=0.$(59 zeros) 12446 03055 57222 83414 28812 81075 60248 48118 05043 37442 33426 62022 48719 94705 70653 43858 15449 04227 91658 81751 47907 01374 27370 69153 20476 82353 07955 41810 78545 10066 37179 54983 56718 66249 21399 25125 81295 76504 91223 78627 47138 87. This terminated reciprocal was checked by multiplying it by M_{199}. The product $(M_{199})\times(1/M_{199})_a$ equaled $1+(3.41924\cdots)\times10^{-207}$ which indicates a positive error of about 0.425 of a unit in the last figure (7) of $(1/M_{199})_a$. In the present work only the first nine octads (72 significant figures) of $(1/M_{199})_a$ were required. The remaining figures of this reciprocal were computed in order to cover fully the possibility of repeating the investigation by the alternative method[2] in which all quotients are omitted.

For future work and for comparison with the results of others it may be appropriate to record in this place the values of the ninth terms of the Lucasian sequences corresponding to $p=4n-1$ and $p=4n+1$ ($M=2^p-1$, p prime) as computed by the author. For the first sequence 3, 7, 47, $\cdots$ we have $\sigma_9=100$ 38568 98919 21376 68875 42399 92826 25670 48796 27683 18190 15150 99398 61346 56188 84806 97130 40351 21947 36890 55940 88447. This term would

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[1] H. S. Uhler, *First proof that the Mersenne number M_{157} is composite*, Proc. Nat. Acad. Sci. U.S.A. vol. 30 (1944) pp. 314–316.

[2] Ibid. p. 315.

cover all $(4n-1)$-primes up to, and inclusive of, $p=347$. Strictly speaking[3] Mersenne's numbers end with $p=257$. For the second sequence 4, 14, 194, $\cdots$ we have $s_9=26\ 21634\ 65049\ 27851\ 45260\ 59369\ 55756\ 30392\ 13647\ 87755\ 95245\ 45911\ 90600\ 53495\ 55773\ 83123\ 69350\ 15956\ 28184\ 89334\ 26999\ 30798\ 24186\ 64943\ 27694\ 39016\ 08919\ 39660\ 72975\ 85154$. This term would be applicable to *all*[4] odd primes inclusive of $p=479$.

There now remain just two numbers of the form 2^p-1 in the Mersenne range whose character has not been investigated. These are M_{193} and M_{227}. The writer has begun the study of M_{227} with the sequence 4, 14, 194, $\cdots$.

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[3] R. C. Archibald, ***Mersenne's numbers***, Scripta Mathematica vol. 3 (1935) pp. 112–119.

[4] D. H. Lehmer, ***On Lucas's test for the primality of Mersenne's numbers***, J. London Math. Soc. vol. 9–10 (1934–1935) pp. 162–165.

ON THE FACTORS OF $2^n \pm 1$

D. H. LEHMER

A recent investigation concerning the converse of Fermat's theorem disclosed that the fundamental table of Kraitchik [1][1] giving the exponent of 2 modulo p for $p<3\cdot 10^6$ contains numerous errors[2] in the previously unchecked region above 10^6. Hence it was decided to make an independent examination of primes, considerably beyond 10^6, having small exponents. As a by-product of this search the following new factors of $2^n\pm 1$ $(n\leqq 500)$ were discovered. This list is intended to supplement the fundamental table of Cunningham and Woodall [1]. The entries can be inserted in the blank spaces provided in that table. It is believed that all factors under 10^6 have now been found.[3] Moreover, any further factors of 2^n-1 for $n\leqq 300$ or of 2^n+1 for $n\leqq 150$ lie beyond 4538800. The methods used to obtain these results will be described elsewhere.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A partial list of these will appear shortly in Mathematical Tables and Other Aids to Computation.

[3] Including, of course, the previously published addenda to Cunningham and Woodall [1] which are to be found in Kraitchik [3] and [6].

n	Factor of 2^n-1
113	1868569
115	4036961
123	3887047
135	348031
143	724153
151	2332951
161	3188767
163	704161
167	2349023
173	1505447
181	1164193
187	707983
189	1560007
205	2940521
223	1466449·2916841
225	617401
229	1504073
233	622577
255	949111
277	1121297
279	1437967
291	272959
301	490631
315	870031·983431
335	464311
351	446473
353	931921
359	855857
381	349759
405	537841
441	309583
489	836191

n	Factor of 2^n+1
91	1210483
100	340801·2787601
123	165313
136	383521
139	4506937
147	748819
170	550801
196	1007441
208	928513
214	843589
220	109121
237	647011
238	823481
239	340337
280	557761·736961
285	1101811
286	958673
288	816769
289	1077971
294	540961
297	694387
305	331841
316	504337·994769
327	666427
333	304363
341	647219
357	428401
390	468781
397	321571·476401
412	454849·667441
414	318781·853669
417	441187
420	127681
430	370661
436	598193
438	1013533
441	311347
444	532801·854257
449	194867
450	695701
465	316201

n	Factor of 2^n+1
494	515737
499	825347
500	1074001

The following new complete factorizations of $2^n \pm 1$ result from entries in the above list. The large residual factors, in each case, have been proved prime by a direct test of primality. This includes the large factor of $2^{170}+1$ credited to Kraitchik by Cunningham and Woodall [1].

(1) $2^{91}+1 = 3\cdot 43\cdot 2731\cdot 224771\cdot 1210483\cdot 25829691707$

(2) $2^{100}+1 = 17\cdot 401\ 61681\cdot 340801\cdot 2787601\cdot 3173389601$

(3) $2^{113}-1 = 3391\cdot 23279\cdot 65993\cdot 1868569\cdot 1066818132868207$

(4) $2^{115}-1 = 31\cdot 47\cdot 14951\cdot 178481\cdot 4036961\cdot 2646507710984041$

(5) $2^{123}+1 = 3^2\cdot 83\cdot 739\cdot 165313\cdot 8831418697\cdot 13194317913029593$

(6) $2^{123}-1 = 7\cdot 13367\cdot 3887047\cdot 164511353\cdot 177722253954175633$

(7) $2^{135}-1 = 7\cdot 31\cdot 73\cdot 151\cdot 271\cdot 631\cdot 23311\cdot 262657\cdot 348031$
$\cdot 49971617830801$

(8) $2^{170}+1 = 5^2\cdot 41\cdot 137\cdot 953\cdot 1021\cdot 4421\cdot 26317\cdot 550801\cdot 23650061$
$\cdot 72269$[illegible]$4352843746841.$

Incorrect factorizations of the first and last of these numbers are given in Kraitchik [1]. The composite numbers

$$31266402706564481 = 1210483\cdot 25829691707$$

and

$$13026477248861 = 550801\cdot 23650061$$

are given as primes. These errors are perpetuated in Kraitchik [3, 4, 5] and Cunningham and Woodall [1]. The second factorization is incorrectly given in Kraitchik [6] where the number

$$301477472991078323800 1 = 340801\cdot 2787601\cdot 3173389601$$

is listed as a prime. The factorization (6) might have been completed 20 years ago had not Kraitchik [2] given the residue index of 165313 incorrectly as 96, instead of 672. The results (2) and (7) are due to Paul Poulet and were communicated in January 1946 by letter just before his death.

Eleven of the new factors given above pertain to Mersenne numbers 2^p-1, p a prime not greater than 257. In particular the factors given for $2^{167}-1$ and $2^{229}-1$ confirm positively the composite character of these numbers, whose tests for primality have been announced recently by Uhler and Barker (Bull. Amer. Math. Soc. vol. 51 (1945) p. 389, vol. 52 (1946) p. 178, Mathematical Tables and Other Aids to Computation vol. 2, p. 94). The present state of our knowledge about the 55 Mersenne numbers may be summarized as follows:

p	Character of 2^p-1
2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127	Prime
11, 23, 29, 37, 41, 43, 47, 53, 59, 67, 71, 73, 79, 113	Composite and completely factored
151, 163, 173, 179, 181, 223, 233, 239, 251	Two or more prime factors known
83, 97, 131, 167, 191, 197, 211, 229	Only one prime factor known
101, 103, 109, 137, 139, 149, 157, 199, 241, 257	Composite but no factor known
193, 227	Character unknown

What progress has been made in the last fourteen years may be seen by comparing this table with a similar one in Bull. Amer. Math. Soc. vol. 38 (1932) p. 384.[4]

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6. *Factorisation de* $2^n \pm 1$, Sphinx vol. 8 (1938) pp. 148–150.

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[4] The composite character of $2^{199}-1$ has recently been determined by Uhler, *On Mersenne's number M_{199} and Lucas's sequences*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 163–164.

ON k-TO-1 TRANSFORMATIONS

W. H. GOTTSCHALK

The following results are extensions of certain of the theorems of O. G. Harrold (*Exactly* (k, 1) *transformations on connected linear graphs*, Amer. J. Math. vol. 62 (1940) pp. 823–834).

Let X and Y be compact Hausdorff spaces and let f be a continuous transformation of X onto Y. Let k be a positive integer and let μE denote the cardinal of the set E. We say that f is *at most k-to-1* (or *exactly k-to-1*) in case $y \in Y$ implies $\mu f^{-1}(y) \leqq k$ (or $\mu f^{-1}(y) = k$). Let $o(x)$ denote the order of the point x. That is to say, $o(x)$ is the smallest integer m such that μ bdy $U = m$ for an arbitrarily small open neighborhood U of x, if such exists; otherwise $o(x)$ is ∞.

THEOREM 1. *If f is at most k-to-1 and if the inverse points of $y \in Y$ are $x_1, \cdots, x_n$, then $\sum_{i=1}^{n} o(x_i) \leqq k \cdot o(y)$.*

PROOF. We may suppose $o(y)$ is finite. Let $U_1, \cdots, U_n$ be neighborhoods (open neighborhoods) of $x_1, \cdots, x_n$ whose closures are pairwise disjoint. There exists a neighborhood W of y such that μ bdy $W = o(y)$ and $f^{-1}(W) \subset \bigcup_{i=1}^{n} U_i$. Define $V_i = U_i \cap f^{-1}(W)$. It follows that $k \cdot o(y) = k \cdot \mu$ bdy $W \geqq \mu f^{-1}(\text{bdy } W) \geqq \mu$ bdy $f^{-1}(W) = \mu \bigcup_{i=1}^{n}$ bdy $V_i = \sum_{i=1}^{n} \mu$ bdy V_i. We conclude that each $o(x_i)$ is finite. By taking the U_i sufficiently small, μ bdy $V_i \geqq o(x_i)$. The conclusion follows.

COROLLARY 1. *If X and Y are continua and if f is exactly k-to-1, then each inverse point of an end point of Y is an end point of X.*

Let P denote the property of being a continuum in X on which f is exactly k-to-1.

THEOREM 2. *If X has property* P *irreducibly, then Y has no end point; if moreover $k = 2$, then Y has no cut point.*

PROOF. We prove the first statement. Suppose Y has an end point y. Write $f^{-1}(y) = \bigcup_{i=1}^{k} x_i$. Let $U_1, \cdots, U_k$ be neighborhoods of $x_1, \cdots, x_k$ whose closures are pairwise disjoint. There exists a neighborhood W of y such that μ bdy $W = o(y) = 1$ and $f^{-1}(W) \subset \bigcup_{i=1}^{k} U_i$. Define $V_i = U_i \cap f^{-1}(W)$. As in the proof of Theorem 1 it follows that $k = k \cdot o(y) \geqq \sum_{i=1}^{k} \mu$ bdy V_i. Hence, each bdy V_i consists of a single point and it follows easily that $X - \bigcup_{i=1}^{k} V_i = X - f^{-1}(W)$ is a proper

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subcontinuum of X on which f is exactly k-to-1. This is a contradiction.

We prove the second statement. Suppose y is a cut point of Y and let $Y-y=Y_1\cup Y_2$ be a separation. Then, $X-f^{-1}(y)=f^{-1}(Y_1)\cup f^{-1}(Y_2)$ is a separation and since also $f^{-1}(y)$ consists of only two points, at least one of the sets $f^{-1}(Y_i)\cup f^{-1}(y)$ $(i=1, 2)$ is a continuum. This contradicts the hypothesis of irreducibility.

COROLLARY 2. *No dendrite is a continuous exactly k-to-1, $k>1$, image of a continuum.*

PROOF. Suppose $f(X)=Y$ is exactly k-to-1, $k>1$, where X is a continuum and Y is a dendrite. By use of Zorn's lemma it may readily be seen that there exists a subcontinuum X_0 of X which has property P irreducibly. The nondegenerate continuum $f(X_0)$ is a dendrite and hence has an end point. This is impossible by Theorem 2.

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A CHARACTERIZATION OF A SEMI-LOCALLY-CONNECTED PLANE CONTINUUM

F. BURTON JONES

In 1908 Schoenflies characterized a continuous curve (in the plane) by means of its complement.[1] In view of more recent work,[2] the kernel of his characterization may be stated as follows: In order that a plane, bounded, cyclic continuum be a continuous curve it is necessary and sufficient that (1) each of its complementary domains be simple and (2) the collection of its complementary domains be contracting. In his paper on semi-locally-connected sets,[3] G. T. Whyburn pointed out many similarities between semi-locally-connected sets and continuous curves. In particular (it is an immediate consequence of one of his theorems[4]) every complementary domain of a plane, bounded, cyclic semi-locally-connected continuum is simple. But since such a continuum need *not* be a continuous curve, the collection of its complementary domains need *not* be contracting. It is the purpose of this paper to point out what characteristic property this collection of complementary domains *does* possess: namely, the collection contains no *folded* subcollection. This property, in conjunction with (1) above, characterizes a plane, bounded, cyclic semi-locally-connected continuum. In fact it is shown that a plane, bounded continuum, whether cyclic or not, is semi-locally-connected if and only if its *complement* is *non-folded*. These theorems should prove useful when constructing examples of semi-locally-connected continua which also have certain other properties.

Definitions and notation. Let space be a simple closed surface (that is, a 2-sphere) and let S denote the set of all points of space. A circular region of S whose radius is ϵ and whose center is A will be denoted by $U(A, \epsilon)$. If M is a closed subset of S, each component of $S-M$

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[1] A. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Leipzig, 1908, p. 237.

[2] R. L. Moore, *Concerning the common boundary of two domains*, Fund. Math. vol. 6 (1924) pp. 201–213, Theorem 9.

G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942, Theorem 4.4, p. 113.

[3] G. T. Whyburn, *Semi-locally-connected sets*, Amer. J. Math. vol. 61 (1939) pp. 733–749.

[4] Loc. cit. footnote 3, p. 749. Cf. R. L. Wilder, *Sets which satisfy certain avoidability conditions*, Casopis pro Pestovane Mathematicky a Fysiky vol. 67 (1938) pp. 185–198, Theorem 8a, p. 196.

is a *complementary domain* of M. A complementary domain of a simple closed curve is called a ***simple domain.*** A connected set M is said to be *semi-locally-connected* provided that if P is a point of M lying in an open subset U of M, there exists an open subset V of M such that $U\supset V\supset P$ and $M-V$ is the sum of a finite number of connected sets.[3] If A and B are distinct points of a set M, M is said to be *aposyndetic at A with respect to B* provided there exists an open subset of M which contains A and belongs to a connected and relatively closed subset of M lying in $M-B$. If a set M is aposyndetic at a point A with respect to *each* point of $M-A$, M is said to be *aposyndetic at A*. If a set M is aposyndetic at each of its points, then M is said to be *aposyndetic.*[4]

The following lemmas may be easily established with the help of the literature.

LEMMA 0. *In order that a continuum M in S be semi-locally-connected, it is necessary and sufficient that M be aposyndetic.*

LEMMA 1. *If T is a subcontinuum of an aposyndetic continuum M of S, and P is a point of $M-T$, then $M-P$ contains a continuum K containing T such that $M-K$ has no limit point in T.*

LEMMA 2. *If A and B are distinct points of the continuum M of S, and M is not aposyndetic at A with respect to B, then M plus any finite number of its complementary simple domains is not aposyndetic at A with respect to B.*

DEFINITION. A countably infinite collection $D_1, D_2, D_3, \cdots$ of mutually exclusive simple domains in S is said to be *folded* provided there exist sets $T_1, T_2, T_3, \cdots$ and a point A of S such that (1) for each i, T_i is a spanning arc-segment (open arc) lying in D_i and separating D_i into two components C_i and E_i, (2) $T_1, T_2, T_3, \cdots$ converges to a subset T of $S-A$, and (3) for each number $\epsilon>0$, there exists an integer n, so that both C_i and E_i intersect $U(A, \epsilon)$ when $i>n$.

THEOREM 1. *In order that a nondegenerate set M of S be a semi-locally-connected, cyclic continuum it is necessary and sufficient that the components of $S-M$ form a collection of mutually exclusive simple domains containing no folded subcollection.*

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[4] F. B. Jones, *Aposyndetic continua and certain boundary problems*, Amer. J. Math. vol. 63 (1941), pp. 545–553. For the definition of other terms and phrases the reader is referred to R. L. Moore's *Foundations of point set theory* and G. T. Whyburn's *Analytic topology*, vols. 13 and 28 respectively, of the Amer. Math. Soc. Colloquium Publications, New York.

PROOF. Suppose that M is a semi-locally-connected, cyclic continuum (nondegenerate). Every component of $S-M$ is a simple domain.[4] Now suppose that G, the collection of all the components of $S-M$, contains a folded subcollection $D_1, D_2, D_3, \cdots$. There exists a point A and, for each positive integer i, there exists a spanning arc-segment T_i lying in D_i and separating D_i into two components C_i and E_i such that (1) $T_1, T_2, T_3, \cdots$ converges to a subset T of $S-A$, but (2) if ϵ is a positive number, there exists an integer n, so that for $i>n$, $U(A, \epsilon)\cdot C_i\neq 0$ and $U(A, \epsilon)\cdot E_i\neq 0$. It is clear that T is a continuum lying in M. By Lemma 1, $M-A$ contains a continuum K containing T such that $M-K$ has no limit point in T. Let ϵ be a positive number such that $K\cdot \bar{U}(A, \epsilon)=0$. There exist an arc L and an integer n such that (1) the end points of T_n belong to K, (2) $T_n\cdot U(A, \epsilon)=0$ and (3) L lies in $U(A, \epsilon)$ and contains points of both C_n and E_n. Let H denote an arc of L irreducible from $\bar{C}_n$ to $\bar{E}_n$; let J denote the boundary of D_n; and let Q denote $S-(D_n+J)$. Both H and K lie in $J+Q$. Furthermore the end points of H separate the end points of T_n from each other on J. But since K contains the end points of T_n, K must intersect H. This is a contradiction and establishes the necessity of the condition stated in Theorem 1.

Now suppose that G is a collection of mutually exclusive simple domains which contains no folded subcollection. Let M denote the closed set $S-G^*$.[5] If M were not connected, then there would exist a simple closed curve J such that M would not intersect J but would intersect each complementary domain of J. Consequently neither complementary domain of J would be a subset of the element of G containing J. Since this is impossible, M must be connected. Suppose that M is not aposyndetic at a point A. Then there exists a point B such that if W is a subcontinuum of $M-B$, A is a limit point of $M-W$. Let $U_1, U_2, U_3, \cdots$ denote a monotone descending sequence of circular regions centering on A and converging to A. Let $V_1, V_2, V_3, \cdots$ denote a monotone descending sequence of circular regions centering on B and converging to B such that $\bar{U}_1\cdot\bar{V}_1=0$. Although $M\cdot(S-V_1)$ contains $M\cdot\bar{U}_1$, no component of $M\cdot(S-V_1)$ contains $M\cdot U_1$. Hence $M\cdot(S-V_1)$ is the sum of two mutually exclusive closed sets M_1 and N_1 each of which contains a point of U_1. There exists an arc in U_1 irreducible from M_1 to N_1; let T_1 denote this arc less its end points. The arc-segment T_1 lies in an element D_1 of G and separates it into two domains C_1 and E_1 each of which contains a point of V_1 since no arc in the boundary of D_1 lies in $S-V_1$ and contains

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[5] G^* denotes the sum of the elements of G.

both end points of T_1. Using Lemma 2 it is easy to see that no component of $(M+D_1)\cdot(S-V_2)$ contains $M\cdot U_2$. Hence $(M+D_1)\cdot(S-V_2)$ is the sum of two mutually exclusive closed sets M_2 and N_2 each of which contains a point of U_2. There exists an arc in U_2 irreducible from M_2 to N_2; let T_2 denote this arc less its end points. The arc-segment T_2 lies in an element D_2 of G and separates it into two domains C_2 and E_2 each of which contains a point of V_2. Again no component of $(M+D_1+D_2)\cdot(S-V_3)$ contains $M\cdot U_3$. Hence $(M+D_1+D_2)\cdot(S-V_3)$ is the sum of two mutually exclusive closed sets M_3 and N_3 each of which contains a point of U_3. There exists an arc in U_3 irreducible from M_3 to N_3; let T_3 denote this arc less its end points. The arc-segment T_3 lies in an element D_3 of G and separates it into two domains C_3 and E_3 each of which contains a point of V_3. Continue this process indefinitely. Since the sequence $T_1, T_2, T_3, \cdots$ converges to A and $V_1, V_2, V_3, \cdots$ converges to B, G contains a folded subcollection $D_1, D_2, D_3, \cdots$ contrary to hypothesis. Hence M is aposyndetic at every one of its points and is, by Lemma 0, semi-locally-connected. Now suppose that P is a cut point of M. Then M is the sum of two closed sets H and K such that $H\cdot K=P$. Let AB denote an arc in $S-P$ irreducible from H to K. Since $AB-(A+B)$ lies in an element of G, A and B belong to a simple closed curve lying in M. Hence P does not separate A from B in M. Thus the assumption that M contains a cut point has led to a contradiction, and the sufficiency of the condition stated in Theorem 1 is established.

COROLLARY 1. *In order that a plane, bounded, cyclic continuum M be semi-locally-connected* (*or aposyndetic*) *it is necessary and sufficient that* (1) *each complementary domain of M be simple and* (2) *the collection of complementary domains of M contain no folded subcollection.*

Remarks. Theorem 1 serves in a way as a characterization of a true cyclic element of a semi-locally-connected continuum (in a 2-sphere). In order to characterize a semi-locally-connected continuum (whether cyclic or not) by means of its complement, it is necessary to extend the notion of foldedness to open sets in general.

DEFINITION. If D is an open set, D is said to be *folded* provided that there exist in D mutually exclusive sets $X_1, X_2, X_3, \cdots$ such that (1) for each i ($i=1, 2, 3, \cdots$), X_i is the sum of two arc-segments T_{i1} and T_{i2} having their end points in the boundary of D, (2) for each i, T_{i1} crosses T_{i2}, (3) the sequence $T_{11}, T_{21}, T_{31}, \cdots$ converges to a subset T of $S-D$, and (4) the sequence of end points of $T_{12}, T_{22},$

$T_{32}, \cdots$ converges to a point of $S-T$. If the open set D is *not* folded, D is said to be *non-folded.*[7]

THEOREM 2. *In order that the continuum M of S be semi-locally-connected (or aposyndetic) it is necessary and sufficient that $S-M$ be non-folded.*

PROOF. The condition is sufficient. For suppose, on the contrary, that M is not aposyndetic at a point A of M with respect to a point B of M. Let $U_1, U_2, U_3, \cdots$ denote a monotone descending sequence of circular regions centered on and converging to A, and let $V_1, V_2, V_3, \cdots$ denote a monotone descending sequence of circular regions centered on and converging to B such that $\overline{U}_1 \cdot \overline{V}_1 = 0$. Now the component of $M \cdot (S-V_1)$ which contains A is not open (relative to M) at A. Hence the boundary of U_1 contains an arc-segment T_{11} whose end points, X_1 and Y_1, lie in different components of $M \cdot (S-V_1)$ such that $M \cdot T_{11} = 0$. There exists a simple closed curve which separates X_1 from Y_1 and contains no point of $M \cdot (S-V_1)$; in fact, there exists a simple closed curve J_1 which separates X_1 from Y_1 and contains no point of $M \cdot (S-V_1)$ such that $J_1 \cdot T_{11}$ is connected. In J_1 plus the boundary of V_1 there exists an arc-segment T_{12} which crosses T_{11}, lies in $S-V_1$, and has only its end points in M. Obviously these end points lie in the boundary of V_1. Let F_2 denote the boundary of U_2 and let K_2 denote the sum of the components of $F_2 - F_2 \cdot M$ which intersect T_{12}. Then K_2 is the sum of a finite number of arc-segments. Now since the component of $M \cdot (S-V_2)$ which contains A is not open (relative to M) at A and $T_{11}+T_{12}+K_2$ has only a finite number of limit points in M, the component of $(M+T_{11}+T_{12}+K_2) \cdot (S-V_2)$ which contains A is not relatively open at A. Hence F_2 contains an arc-segment T_{21} whose end points, X_2 and Y_2, lie in M and in different components of $(M+T_{11}+T_{12}+K_2) \cdot (S-V_2)$ such that $(M+T_{11}+T_{12}+K_2) \cdot T_{21} = 0$. There exists a simple closed curve J_2 which separates X_2 from Y_2, and contains no point of $(M+T_{11}+T_{12}+K_2) \cdot (S-V_2)$ such that $J_2 \cdot T_{21}$ is connected. In J_2 plus the boundary of V_2 there exists an arc-segment T_{22} which crosses T_{21}, lies in $S-V_2$, and has only its end points in M. These end points lie in the boundary of V_2. Furthermore, $(T_{11}+T_{12}) \cdot (T_{21}+T_{22}) = 0$. This process may be continued indefinitely. Since $T_{11}, T_{21}, T_{31}, \cdots$ converges to A and the sequence of end points of $T_{12}, T_{22}, T_{32}, \cdots$ converges to B, $S-M$ is folded, contrary to hypothesis. Hence the condition is sufficient.

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[7] The reader should observe that no simple domain is folded and that if G is a collection of mutually exclusive simple domains, G^* is folded if and only if G contains a folded subcollection.

The condition is also necessary. For suppose, on the contrary, that $S-M$ is folded. Then $S-M$ contains an infinite sequence of pairs of arc-segments T_{11}, T_{12}, T_{21}, T_{22}, $\cdots$ such that (1) for each i, T_{i1} and T_{i2} span M, (2) for each i, T_{i1} crosses T_{i2}, (3) for each i and j $(i\neq j)$, $(T_{i1}+T_{i2})\cdot(T_{j1}+T_{j2})=0$, (4) the sequential limiting set T of T_{11}, T_{21}, T_{31}, $\cdots$ is a subset of M, and (5) the sequence of end points of T_{12}, T_{22}, T_{32}, $\cdots$ converges to a point B not belonging to T. By Lemma 1, $M-B$ contains a continuum K and an open subset U of M such that $K\supset U\supset T$. Let V denote a circular region containing B such that $K\cdot\bar{V}=0$. There exists an integer i, such that T_{i1} has its end points in K but $\bar{V}\cdot T_{i1}=0$, and such that T_{i2} has its end points in V. But since T_{i1} crosses T_{i2}, this is impossible.

COROLLARY 2. *In order that a plane, bounded continuum be semi-locally-connected (or aposyndetic) it is necessary and sufficient that its complement be non-folded.*

THEOREM 3. *In order that the boundary of a simply connected domain D be a continuous curve it is necessary and sufficient that D be non-folded.*

PROOF. If D is non-folded, $S-D$ is an aposyndetic continuum M. Hence the boundary of D is a continuous curve.[8] On the other hand, if β, the boundary of D, is a continuous curve, then β is aposyndetic, and $S-\beta$ is non-folded. Hence D is non-folded.

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[8] See footnotes 3 and 5. Cf. R. L. Wilder, *Property S_n*, Amer. J. Math. vol. 61 (1939) pp. 823–832, p. 832 in particular.

OPEN TRANSFORMATIONS AND DIMENSION[1]

J. H. ROBERTS

This paper considers separable metric spaces A and B and open transformations. If, for each $x \in A$, $f(x) \in B$ and the image under f of every open set in A is a set open in B, then f is an *open* transformation. Continuity of f is not assumed. Such transformations have been studied by Rhoda Manning [1].[2]

THEOREM 1. *If* $f(A) = B$ *where* f *is open, then there exists a subset* A_1 *of* A *such that* (1) $f(A_1) = B$, (2) *for* $y \in B$, *the set* $f^{-1}(y) \cdot A_1$ *is countable, and* (3) f, *considered as a transformation of* A_1 *into* B, *is open.*

PROOF. Let $K_1, K_2, \cdots$ denote the elements of a countable base (open sets) for the space A. For every $y \in B$ and each i let P_{yi} be a point of $K_i \, f^{-1}(y)$, provided this set is nonvacuous. Let A_1 be the set of all points P_{yi} so obtained. Properties (1) and (2) are obvious. To prove (3), let V be an open set in A_1, and U an open set in A such that $U \cdot A_1 = V$. Now for every y the set $f^{-1}(y) \cdot A_1$ is dense in $f^{-1}(y)$. Hence if $f^{-1}(y)$ has a point in U then it has a point in $A_1 \cdot U$ so $f(V) = f(U)$ is an open set in B.

THEOREM 2. *There exist countable-fold open mappings*[3] *which increase dimension.*

PROOF. There exist open mappings which increase dimension [2]. Theorem 2 follows by applying Theorem 1 to any such example.[4]

THEOREM 3. *If* $\dim A = n$ *and* $-1 < m \leqq n$, *then there exists a* B *and a transformation* f *such that* (1) $f(A) = B$, (2) f *is open and* 1-1, *and* (3) $\dim B = m$. *In other words, dimension can be lowered at will by a* 1-1 *open transformation.*[5]

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[1] One statement in the abstract (Bull. Amer. Math. Soc. Abstract 59-5-210) is incorrect. Theorem 4 gives the correct statement.

[2] Numbers in brackets refer to the bibliography.

[3] A *mapping* is a continuous transformation.

[4] Alexandroff [5] has proved that if A is *compact* then no countable-fold open mapping can increase dimension.

[5] Compare the following special case of a theorem of Hurewicz [4, p. 91, Theorem VI 7]): "If f is a *closed mapping* of A into B and for each $y \in B$, $f^{-1}(y)$ is zero-dimensional, then $\dim B \geqq \dim A$." In a footnote (loc. cit.) the authors state that it is not known if Theorem VI 7 is true for *open* mappings. The answer is in the negative and their example VI 10 is a counter example, as the mapping f is actually open.

PROOF. It is sufficient to prove the theorem for $m=n-1$. For if $f(A)=B$ and $g(B)=C$, where f and g are open and 1-1, then the product gf is open and 1-1 and transforms A onto C.

The author has shown[6] that there exists a space B_1 of dimension $n-1$, and an at most 2-to-1 mapping ϕ of B_1 onto A. Furthermore the subset of B_1 consisting of all x such that $\phi^{-1}\phi(x)=x$ is $(n-1)$-dimensional. For $y \in A$, if $\phi^{-1}(y)$ is a single point then write $f(y)=\phi^{-1}(y)$. If $\phi^{-1}(y)$ is double-valued, select arbitrarily one point in $\phi^{-1}(y)$ and define $f(y)$ to be this point. Let $f(A)=B$. Then B is of dimension $n-1$ and f, as inverse of a 1-1 continuous function, is open.

Remark. If $f(A)=B$ is open and 1-1, then f^{-1} is continuous and 1-1. Thus Theorem 3 provides examples of arbitrary increases of dimension by 1-1 mappings.

THEOREM 4. *Suppose $f(A)=B$ is open, B is locally compact, and for each $y \in B$ the set $f^{-1}(y)$ is not dense-in-itself. Then* $\dim B \leq \dim A$.[7]

PROOF. Let $K_1, K_2, \cdots$ be an open base for A. For each i let A_i be the set of all $x \in K_i$ such that if $x' \in K_i$ and $f(x')=f(x)$, then $x'=x$, and let $B_i=f(A_i)$. For $y \in B$, $f^{-1}(y)$ contains an isolated point (with respect to $f^{-1}(y)$), so $B=\sum_{i=1}^{\infty} B_i$. We prove next that B_i is closed in the open set $f(K_i)$. For suppose on the contrary that there exists a sequence $y_n \to y$, where y and each y_n are in $f(K_i)$, $y_n \in B_i$ but $y \notin B_i$. Since $y \in f(K_i)$ but not in B_i, there exist distinct points x and x' in K_i such that $f(x)=f(x')=y$. For each n there is a unique $x_n \in K_i$ such that $f(x_n)=y_n$. Since $y_n \to y$ it follows that $\liminf f^{-1}(y_n) \supset f^{-1}(y) \supset (x+x')$. But $f^{-1}(y_n)$ has only the one point x_n in K. This gives a contradiction.

Now let $M_{i1}, M_{i2}, \cdots$ be closed and compact subsets of B with $M_{i1}+M_{i2}+\cdots=f(K_i)$. Write $B_{ij}=B_i \cdot M_{ij}$. Then B_{ij} is a compact space and $B=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}$. Hence if $\dim B_{ij} \leq \dim A$ for all i and j, then $\dim B \leq \dim A$ (see [4, Theorem III 2, p. 30]). Write $A_{ij}=A_i \cdot f^{-1}(B_{ij})$. Over A_{ij}, f is open and 1-1 to B_{ij}. For $y \in B_{ij}$ define $g(y)=x \in A_{ij}$ such that $f(x)=y$. Now g, as the inverse of any open 1-1 transformation, is continuous; thus g is a mapping and $g(B_{ij})=A_{ij}$. Furthermore B_{ij} is compact, so that g is a *closed mapping*. Also g is 1-1. Then[8] $\dim B_{ij} \leq \dim A$, so $\dim B \leq \dim A$.

Remark. In Theorem 4 the assumption that B is locally compact can be replaced by the weaker assumption that every point of B has arbitrarily small neighborhoods with compact boundary. The proof is not given. Some assumption of compactness seems necessary. Consider the following example.

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[6] See [3], especially Theorem 9.1. If the M of this theorem is the space A, then M_1 is the desired space B_1 and ϕ_1 is the desired mapping ϕ.

[7] See footnote 4.

[8] See [4, Theorem VI 7, p. 91].

Example. There exist, in the plane, spaces A and B and an open 1-1 transformation f with $f(A)=B$, with $\dim A=0$ and $\dim B=1$.

The space B is an example due to Sierpiński [6, pp. 81–83]. This space B has the following properties: (1) it is 1-dimensional, (2) it lies in the plane with $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, (3) it contains at most one point (x, y) for a given x, and (4) the set of all points $(x, 0)$, such that for some y the point $(x, y)\in B$, is homeomorphic to the Cantor ternary set. Let A be this set defined by (4). Then A is the projection of B onto the x-axis, and for $(x, 0)\in A$ there is a single point (x, y) in B. Define f as follows: $f(x, 0)=f(x, y)\in B$. Then f has the required properties.

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CURVATURE IN HERMITIAN METRIC

S. BOCHNER

Hermitian metric has the peculiarity of favoring negative curvature over positive curvature. We shall explain this phenomenon by pointing out that in the case of an isometric analytic imbedding the relative curvature is on the whole negative; also, by reduction to a *limiting case* of imbedding we shall explain why an invariant metric in the theory of Fuchsian groups is likely to be hyperbolic; see Hua [6].[1]

However, on the other hand, an Hermitian metric is very rigid, and the possibility of imbedding into a *finitely-dimensional* enveloping space is very remote. The classical conjectures about the possibility of Euclidean imbedding are rendered entirely false, but as a compensation, there are more and better theorems about equivalence and uniqueness.

1. **Hermitian metric.** There are many places in the literature where an introduction to the theory of Hermitian metric can be found. We shall refer to our own summary as given in Bochner [2, chap. 2]. We quote from there that in discussing an Hermitian metric in a space V_n of n complex variables $z_1, \cdots, z_n$, the basic variables are the $2n$ conjugate complex quantities

$$z_1, \cdots, z_n; \bar{z}_1, \cdots, \bar{z}_n \tag{1}$$

which we shall also denote indifferently by

$$t_1, t_2, \cdots, t_{2n}. \tag{2}$$

Italic indices run from 1 to $2n$, Greek indices from 1 to n, and starring an index will add to it the value n if it is not greater than n, and subtract n from it if it is not less than $n+1$. All scalars and components of tensors are power series in (1), and they are always *self-adjoint*, meaning that starring all indices in a component of a tensor will change its value into its conjugate complex. Scalars are real-valued.

The fundamental tensor g_{ij} has the properties

$$g_{ij} = g_{ji}, \qquad g_{\alpha\beta} = g_{\alpha^*\beta^*} = 0$$

in addition to the self-adjointness property $g_{\alpha\beta^*} = \bar{g}_{\beta\alpha^*}$; also $g_{\alpha\beta^*}$ is positive-definite. In particular, we have

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$(3)\qquad ds^2 = g_{ij}dt_i dt_j = 2g_{\alpha\beta^*}dz_\alpha d\bar{z}_\beta.$$

Finally, we have Kaehler's restriction (non-torsion)

$$(4)\qquad \frac{\partial g_{\alpha\gamma^*}}{\partial z_\beta} = \frac{\partial g_{\beta\gamma^*}}{\partial z_\alpha}$$

which is equivalent with

$$(5)\qquad g_{\alpha\beta^*} = \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}$$

for some scalar function $\Phi(z, \bar{z})$, locally. On replacing Φ by $(\Phi+\bar{\Phi})/2$ we can always make it real-valued. If we form Γ^i_{jk} by the usual formula, then only the unmixed components $\Gamma^\alpha_{\beta\gamma}$ and their conjugates can be not equal to 0, and

$$(6)\qquad \Gamma^\alpha_{\beta\gamma} = g^{\alpha\sigma^*}\frac{\partial g_{\sigma^*\beta}}{\partial z_\gamma}.$$

If we form the curvature tensor R_{ijkl} by the usual formula, then it satisfies all classical relations. Furthermore, only the components of the form $R_{\alpha\beta^*\gamma\delta^*}$, $R_{\alpha\beta^*\gamma^*\delta}$, $R_{\alpha^*\beta\gamma\delta^*}$, $R_{\alpha^*\beta\gamma^*\delta}$ can be different from 0. Finally, we have

$$R^\alpha_{\beta\gamma^*\delta} = \frac{\partial}{\partial \bar{z}_\gamma}\Gamma^\alpha_{\beta\delta}$$

and therefore

$$\begin{aligned} R_{\alpha^*\beta\gamma^*\delta} &= g_{\alpha^*\rho}\frac{\partial}{\partial \bar{z}_\gamma}\left(g^{\rho\sigma^*}\frac{\partial g_{\sigma^*\beta}}{\partial z_\delta}\right) \\ &= \frac{\partial}{\partial \bar{z}_\gamma}\left(g_{\alpha^*\rho}g^{\rho\sigma^*}\frac{\partial g_{\sigma^*\beta}}{\partial z_\delta}\right) - g^{\rho\sigma^*}\frac{\partial g_{\sigma^*\beta}}{\partial z_\delta}\frac{\partial g_{\alpha^*\rho}}{\partial \bar{z}_\gamma}. \end{aligned}$$

Hence we have finally

$$(7)\qquad R_{\alpha\beta^*\gamma\delta^*} = \frac{\partial^2 g_{\alpha\beta^*}}{\partial z_\gamma \partial \bar{z}_\delta} - g^{\rho\sigma^*}\frac{\partial g_{\rho\beta^*}}{\partial \bar{z}_\delta}\frac{\partial g_{\sigma^*\alpha}}{\partial z_\gamma}.$$

In particular we have the symmetry relations

$$(8)\qquad R_{\alpha\beta^*\gamma\delta^*} = R_{\gamma\beta^*\alpha\delta^*} = R_{\alpha\delta^*\gamma\beta^*}$$

which are unmatched in the case of real variables.

THEOREM 1. *If two tensors g_{ij} of our description in the same coordinates have the same curvature tensor $R_{\alpha\beta^*\gamma\delta^*}$, then they are equal to within an allowable transformation.*

The conclusion also holds if the components of the type $R_{\alpha}{}^{\beta}{}_{\gamma\delta^*}$ *or of some other type are assumed to be the same.*

PROOF. We take one of the tensors g_{ij} and we introduce the power series

$$\Phi(z_1, \cdots, z_n; \bar{z}_1, \cdots, \bar{z}_n) \tag{9}$$

for which (5) holds. We can obviously omit from (9) all monomials

$$a z_1^{p_1} \cdots z_n^{p_n} \bar{z}_1^{q_1} \cdots \bar{z}_n^{q_n} \tag{10}$$

for which either $p_1 + \cdots + p_n = 0$ or $q_1 + \cdots + q_n = 0$, and thus assume (9) in the form

$$a_{\alpha\beta^*} z_\alpha \bar{z}_\beta + \Phi_3(z, \bar{z}) \tag{11}$$

where $a_{\alpha\beta^*} = g_{\alpha\beta^*}(0)$, and Φ_3 contains only monomials with

$$p_1 + \cdots + p_n + q_1 + \cdots + q_n \geqq 3.$$

An allowable transformation will carry (11) into

$$\sum_{\alpha=1}^{n} z_\alpha \bar{z}_\alpha + \Phi_3(z, \bar{z}), \tag{12}$$

and on making it real, whenever (12) contains a monomial (10) it also contains $\bar{a} \bar{z}_1^{p_1} \cdots \bar{z}_n^{p_n} z_1^{q_1} \cdots z_n^{q_n}$. Therefore, we can put

$$\Phi_3 = \sum_{\alpha=1}^{n} (z_\alpha \overline{f_\alpha(z)} + \bar{z}_\alpha f_\alpha(z)) + \Phi_{2,2}(z, \bar{z}),$$

where each function $f_1, \cdots, f_n(z)$ is a power series in $z_1, \cdots, z_n$ with terms of total degree not less than 2, and $\Phi_{2,2}$ contains only terms with

$$p_1 + \cdots + p_n \geqq 2 \quad \text{and} \quad q_1 + \cdots + q_n \geqq 2.$$

After the allowable transformation $z_\alpha' = z_\alpha + f_\alpha(z)$, our function has the form

$$\Phi = \sum_{\alpha=1}^{n} z_\alpha \bar{z}_\alpha + \psi_{2,2}(z, \bar{z}). \tag{13}$$

If now we introduce the derivatives

$$\frac{\partial^{p_1 + \cdots + p_n + q_1 + \cdots + q_n} \Phi}{\partial z_1^{p_1} \cdots \partial z_n^{p_n} \partial \bar{z}_1^{q_1} \cdots \partial \bar{z}_n^{q_n}}$$

at the origin, then our normalization has fixed their values in case the total order is 1, 2 or 3; and for the total order 4, 5, 6, $\cdots$ if either $p_1+\cdots+p_n\leqq 1$ or $q_1+\cdots+q_n\leqq 1$. Now, if we write equation (7) in the form

$$\frac{\partial^4\Phi}{\partial z_\alpha\partial\bar z_\beta\partial z_\gamma\partial\bar z_\delta} = g^{\rho\sigma^*}\frac{\partial g_{\rho\beta^*}}{\partial\bar z_\delta}\frac{\partial g_{\sigma^*\alpha}}{\partial z_\gamma} + R_{\alpha\beta^*\gamma\delta^*}, \tag{14}$$

and if we remember that each component $g^{\rho\sigma^*}$ is a rational function of the components $g_{\alpha\beta^*}$, then the missing derivatives of order 4 can be computed directly from (14); furthermore, the missing derivatives of higher order can be successively computed by taking all possible successive derivatives of (14). Also these computations are unique, which proves our contention. If the components $R_\alpha{}^\beta{}_{\gamma\delta^*}$ are given, the same conclusion will follow from the equations

$$\frac{\partial^2 g_{\alpha\beta^*}}{\partial z_\gamma\partial\bar z_\delta} = g^{\rho\sigma^*}\frac{\partial g_{\rho\beta^*}}{\partial\bar z_\delta}\frac{\partial g_{\sigma^*\alpha}}{\partial z_\gamma} + g_{\beta^*\epsilon}R^{\epsilon}_{\alpha\gamma\delta^*},$$

and this completes the proof of Theorem 1.

THEOREM 2. *If the metric tensors g_{ij} and h_{ij} have the same geodesics, then their affine connections Γ^l_{ij} and Δ^l_{ij} are the same. Thus a projective collineation is automatically an affine collineation.*

PROOF. As in the real case we have, see [3, p. 132],

$$\Delta^l_{ij} - \Gamma^l_{ij} = \delta^l_i\psi_j + \delta^l_j\psi_i.$$

Putting $l=\alpha$, $i=\alpha$, $j=\gamma^*$, we obtain $\psi_{\gamma^*}=0$, and similarly $\psi_\gamma=0$.

2. **Sectional curvature.** A two-dimensional surface element through the origin is given in the form

$$t_i = \lambda^i x + \mu^i y, \tag{15}$$

that is

$$z_\alpha = \lambda^\alpha x + \mu^\alpha y, \qquad \bar z_\alpha = \overline{\lambda^\alpha} x + \overline{\mu^\alpha} y, \tag{16}$$

where x and y are real parameters, and the systems of complex numbers $\{\lambda^\alpha\}$ and $\{\mu^\alpha\}$ are nonproportional. If we make the decomposition

$$z_\alpha = x_\alpha + (-1)^{1/2}y_\alpha,\quad \lambda^\alpha = \lambda^\alpha_0 + (-1)^{1/2}\lambda^\alpha_1,\quad \mu^\alpha = \mu^\alpha_0 + (-1)^{1/2}\mu^\alpha_1,$$

then we can also write

$$x_\alpha = \lambda^\alpha_0 x + \mu^\alpha_0 y, \qquad y_\alpha = \lambda^\alpha_1 x + \mu^\alpha_1 y. \tag{17}$$

We now set up the number

$$K = \frac{R_{kijh}\lambda^k\mu^i\lambda^j\mu^h}{(g_{kj}g_{ih} - g_{kh}g_{ij})\lambda^k\mu^i\lambda^j\mu^h} \tag{18}$$

and we claim that it is the real-valued curvature for the surface element (17) and the line element (3). In fact, if we denote the real component y_α by $x_{\alpha+n}$, if we write the line element (3) in the real form

$$h_{ij}(x)dx_i dx_j,$$

if we introduce the ordinary curvature tensor corresponding to the latter line element, if we form with that curvature tensor the sectional curvature for the section (17), and if finally we carry out purely formally the transformation of coordinates

$$z_\alpha = x_\alpha + (-1)^{1/2}x_{\alpha+n}, \qquad \bar{z}_\alpha = x_\alpha - (-1)^{1/2}x_{\alpha+n}$$

as if it were an allowable transformation; then by formal properties of invariance the curvature will appear in the form (18).

Theorem 3. *For $n>1$, if at every point the sectional curvature K is the same for all possible two-dimensional sections, then the curvature tensor is identically zero.*

Proof. Relation

$$[R_{kijh} - K(g_{kj}g_{ih} - g_{kh}g_{ij})]\lambda^k\mu^i\lambda^j\mu^h = 0 \tag{19}$$

is fulfilled for $2n$ independent variables λ^α, μ^α and their conjugate values $\bar{\lambda}^\alpha$, $\bar{\mu}^\alpha$. We are now applying a fundamental lemma to the effect that whenever a power series

$$\Phi(u_1, \cdots, u_n; v_1, \cdots, v_n)$$

is zero identically in $u_1, \cdots, u_n$ for $v_k = \bar{u}_k$, $k = 1, \cdots, n$, then it is identically 0 in the independent variables u_k, v_k. If we apply this to (19) then for K independent of λ and μ, we obtain by a classical formal procedure the relation

$$R_{kijh} = K(g_{kj}g_{ih} - g_{kh}g_{ij}). \tag{20}$$

In particular, we obtain

$$R_{\alpha\beta^*\gamma\delta^*} = -Kg_{\alpha\delta^*}g_{\beta^*\gamma},$$

and on applying the important relation (8) we deduce

$$Kg_{\alpha\delta^*}g_{\beta^*\gamma} = Kg_{\gamma\delta^*}g_{\beta^*\alpha}.$$

If we multiply both sides by $g^{\alpha\delta^*}g^{\beta^*\gamma}$ and contract we obtain $n^2K = K$, and hence $K = 0$ for $n>1$.

DEFINITION. We call a section *holomorphic* if it is tangent to an analytically imbedded complex "curve." It is not hard to see that (16) is holomorphic if an only if there exists a *non-real* number ϕ such that $\mu^\alpha = \phi\lambda^\alpha$. If we transform the two real parameters x, y by a suitable nonsingular affine transformation $x' = \phi_1 x + \phi_2 y$, $y' = \phi_3 x + \phi_4 y$ with real coefficients ϕ_1, ϕ_2, ϕ_3, ϕ_4, then we can obtain the normalization

$$\mu^\alpha = (-1)^{1/2}\lambda^\alpha; \tag{21}$$

that is, the section can be written in the form

$$z_\alpha = \lambda^\alpha z; \qquad z = x + (-1)^{1/2} y. \tag{22}$$

THEOREM 4. *For a holomorphic section* (21) *we have*

$$\begin{aligned} K &= -\frac{R_{\alpha\beta^*\gamma\delta^*}\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*}}{g_{\alpha\beta^*}g_{\gamma\delta^*}\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*}} \\ &= -2\,\frac{R_{\alpha\beta^*\gamma\delta^*}\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*}}{(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\gamma\beta^*}g_{\alpha\delta^*})\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*}}. \end{aligned} \tag{23}$$

PROOF. This follows from the fact that for an arbitrary section (16) we have

$$R_{hijk}\lambda^h\mu^i\lambda^j\mu^k = R_{\alpha\beta^*\gamma\delta^*}(\lambda^\alpha\mu^{\beta^*} - \lambda^{\beta^*}\mu^\alpha)(\lambda^\gamma\mu^{\delta^*} - \lambda^{\delta^*}\mu^\gamma) \tag{24}$$

and

$$\begin{aligned} &(g_{hj}g_{ik} - g_{hk}g_{ij})\lambda^h\mu^i\lambda^j\mu^k \\ &\quad = g_{hj}\lambda^h\lambda^j g_{ik}\mu^i\mu^k - g_{hk}\lambda^h\mu^k g_{ij}\mu^i\lambda^j \\ &\quad = g_{\alpha\beta^*}g_{\gamma\delta^*}\big[(\lambda^\alpha\mu^\gamma - \lambda^\gamma\mu^\alpha)(\lambda^{\beta^*}\mu^{\delta^*} - \lambda^{\delta^*}\mu^{\beta^*}) \\ &\qquad + (\lambda^\alpha\mu^{\delta^*} - \lambda^{\delta^*}\mu^\alpha)(\lambda^{\beta^*}\mu^\gamma - \lambda^\gamma\mu^{\beta^*})\big]. \end{aligned} \tag{25}$$

We shall now draw an interesting conclusion.

THEOREM 5. *If at a point all holomorphic sections have the same curvature $K = b$, then at that point we have*

$$R_{\alpha\beta^*\gamma\delta^*} = -\frac{b}{2}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\gamma\beta^*}g_{\alpha\delta^*}). \tag{26}$$

Also, if (26) *holds at every point, then b is a constant.*

PROOF. By assumption we have

$$\left[R_{\alpha\beta^*\gamma\delta^*} + \frac{b}{2}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\beta^*\gamma}g_{\alpha\delta^*})\right]\lambda^\alpha\lambda^{\beta^*}\lambda^\gamma\lambda^{\delta^*} = 0.$$

At first the relation holds whenever λ^{α^*} is conjugate complex to λ^{α}. By our previous argument it holds for independent variables $(\lambda^{\alpha}, \lambda^{\beta^*})$, and relation (26) follows now from the fact that both sides of it are symmetric in the pairs of indices (α, γ) and (β^*, δ^*).

Next, Bianchi's relation specializes to

$$R_{\alpha\beta^*\gamma\delta^*,\epsilon} = R_{\alpha\beta^*\epsilon\delta^*,\gamma}, \tag{27}$$

and if (26) holds at all points we obtain

$$b_{,\epsilon}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\gamma\beta^*}g_{\alpha\delta^*}) = b_{,\gamma}(g_{\alpha\beta^*}g_{\epsilon\delta^*} + g_{\epsilon\beta^*}g_{\alpha\delta^*}).$$

This shows that $b(z, \bar z)$ is independent of the variables z_α. The independence of $\bar z_\alpha$ follows from $R_{\alpha\beta^*\gamma\delta^*,\epsilon^*} = R_{\alpha\epsilon^*\gamma\delta^*,\beta^*}$.

3. **Fubini spaces.** For arbitrary real b we put

$$\Phi = \frac{2}{b}\log\left(1 + \frac{b}{2}\sum_{\alpha=1}^{n} z_\alpha \bar z_\alpha\right) = \frac{2}{b}\log S. \tag{28}$$

Therefore,

$$g_{\alpha\beta^*} = \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar z_\beta} = \frac{\delta_{\alpha\beta^*}}{S} - \frac{b}{2}\,\frac{\bar z_\alpha z_\beta}{S^2}, \tag{29}$$

$$\frac{ds^2}{2} = \frac{\sum_\alpha |dz_\alpha|^2 + \frac{b}{2}\left(\sum_\alpha |z_\alpha|^2 \sum_\beta |dz_\beta|^2 - \left|\sum_\alpha \bar z_\alpha dz_\alpha\right|^2\right)}{\left(1 + \frac{b}{2}\sum_\alpha |z_\alpha|^2\right)^2}. \tag{30}$$

At the origin we have $g_{\alpha\beta^*} = \delta_{\alpha\beta^*}$ and on computing (7) we obtain

$$R_{\alpha\beta^*\gamma\delta^*} = -\frac{b}{2}(\delta_{\alpha\beta^*}\delta_{\gamma\delta^*} + \delta_{\alpha\delta^*}\delta_{\beta\gamma^*}).$$

Thus (26) holds at the origin. The validity at other points follows from the existence of a transitive group of analytic homeomorphisms which leave the line element (3) invariant, see Fubini [4].

Theorem 6. *For given b all spaces of constant holomorphic curvature b are equivalent.*

Proof. The conclusion follows from

$$\frac{\partial^2 g_{\alpha\beta^*}}{\partial z_\gamma \partial \bar z_\delta} = g^{\rho\sigma^*}\frac{\partial g_{\rho\beta^*}}{\partial \bar z_\delta}\frac{\partial g_{\sigma^*\alpha}}{\partial z_\gamma} - \frac{b}{2}(g_{\alpha\beta^*}g_{\gamma\delta^*} + g_{\alpha\delta^*}g_{\gamma\beta^*})$$

as in the case of Theorem 1.

THEOREM 7. *For the Fubini space $b=1$ the general sectional curvature K lies in the interval*

$$1/4 \leqq K \leqq 1. \tag{31}$$

The maximum value $K=1$ is reached only if the section is holomorphic and the minimum value $K=1/4$ is reached, among others, if the section is totally real, that is

$$x_\alpha = \overset{\alpha}{\lambda_0} x + \overset{\alpha}{\mu_0} y, \qquad y_\alpha = 0; \qquad \alpha = 1, \cdots, n. \tag{32}$$

PROOF. At the origin, if we use (24) and (25) and if we put

$$Q(\lambda, \mu) = \sum_\alpha \lambda^\alpha \mu^{\alpha^*},$$

we obtain for K the value

$$\frac{Q(\lambda, \lambda)Q(\mu, \mu) + Q(\lambda, \mu)Q(\mu, \lambda) - Q(\lambda, \mu)^2 - Q(\mu, \lambda)^2}{4Q(\lambda, \lambda)Q(\mu, \mu) - 2Q(\lambda, \mu)Q(\mu, \lambda) - Q(\lambda, \mu)^2 - Q(\mu, \lambda)^2}.$$

If we introduce the quotient

$$t = \frac{Q(\lambda, \mu)}{(Q(\lambda, \lambda)Q(\mu, \mu))^{1/2}} = re^{i\psi},$$

then K has the value

$$\frac{1 + r^2 - 2r^2 \cos 2\psi}{4 - 2r^2 - 2r^2 \cos 2\psi} = 1 - \frac{3}{4}\,\frac{1 - r^2}{1 - ((1 + \cos 2\psi)/2)r^2}. \tag{33}$$

We always have $r \leqq 1$. The extreme value $r=1$ can only be reached if the section is holomorphic, and then (33) has the value 1. For $r<1$, the minimum is reached if t is real-valued, thus if $\sum \lambda^\alpha \mu^{\alpha^*}$ is real-valued. This will in particular occur for a section (32).

The Ricci tensor is defined by

$$R_{ij} = g^{pq} R_{piqj}, \qquad R_{\alpha\beta^*} = g^{\rho^*\sigma} R_{\alpha\rho^*\sigma\beta^*},$$

and it has the important property [2, formula (58)]

$$R_{\alpha\beta^*} = \frac{\partial^2 \log G}{\partial z_\alpha \partial \bar{z}_\beta}, \tag{34}$$

where G is the determinant of the matrix $g_{\alpha\beta^*}$. The Ricci curvature in a given direction λ^i is defined as

$$\kappa = -\frac{R_{ij}\lambda^i\lambda^j}{g_{ij}\lambda^i\lambda^j} = -\frac{R_{\alpha\beta^*}\lambda^\alpha\lambda^{\beta^*}}{g_{\alpha\beta^*}\lambda^\alpha\lambda^{\beta^*}}.$$

THEOREM 8. *For a Fubini space we have*

$$R_{\alpha\beta^*} = -\frac{b}{2}(n+1)g_{\alpha\beta^*};$$

and the average Ricci curvature

$$\frac{\kappa}{2n-1} = b\,\frac{n+1}{4n-4}$$

which is everywhere constant, decreases monotonely towards the smallest possible value 1/4, *as* $n \to \infty$.

The proof is quite obvious.

THEOREM 9. *If for a line element we have*

$$R_{\alpha\beta^*} = b(z, \bar{z})g_{\alpha\beta^*}, \tag{35}$$

then b is constant. In other words, if one of our spaces is an Einstein space at each point, it is so universally.

PROOF. By (34) we have $\partial R_{\alpha\beta^*}/\partial \bar{z}_\gamma = \partial R_{\alpha\gamma^*}/\partial \bar{z}_\beta$. Hence we obtain

$$b_{,\gamma^*}g_{\alpha\beta^*} + b\,\frac{\partial g_{\alpha\beta^*}}{\partial \bar{z}_\gamma} = b_{,\beta^*}g_{\alpha\gamma^*} + b\,\frac{\partial g_{\alpha\gamma^*}}{\partial \bar{z}_\beta}$$

and therefore $b_{,\gamma^*}g_{\alpha\beta^*} = b_{,\beta^*}g_{\alpha\gamma^*}$. This implies $b_{,\gamma^*} = 0$ and similarly we obtain $b_{,\gamma} = 0$.

For real spaces of constant curvature there exist modified line elements in terms of normal coordinates. In order to justify the absence of an analogue for Fubini spaces we shall point out a very general theorem.

THEOREM 10. *For a line element of our description there exist no allowable normal coordinates except in the obvious case of a flat space.*

PROOF. Normal coordinates have the usual consequences. In particular, we have

$$g_{ij}(t)t_j = g_{ij}(0)t_j$$

that is

$$g_{\alpha\beta^*}(z, \bar{z})\bar{z}_\beta = g_{\alpha\beta^*}(0)\bar{z}_\beta,$$

or

$$\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}\bar{z}_\beta = g_{\alpha\beta^*}(0)\bar{z}_\beta.$$

For the function

$$\psi(z, \bar z) = \Phi - g_{\alpha\beta^*}(0) z_\alpha \bar z_\beta$$

we then obtain

$$\frac{\partial^2\psi}{\partial z_\alpha \partial \bar z_\beta} \bar z_\beta = 0, \qquad \alpha = 1, \cdots, n,$$

or

$$\frac{\partial\psi}{\partial \bar z_\beta} \bar z_\beta = F(\bar z).$$

On putting $\bar z_\beta = \bar a_\beta \zeta$, $H(z, \bar a, \zeta) \equiv \psi(z, \bar a \zeta)$, we then obtain $\zeta \partial H/\partial \zeta = F(\bar a \zeta)$; and since $F(\bar z)$ has no constant term we obtain

$$H(z, \bar a, \zeta) = R(z, \bar a) + S(\bar a \zeta).$$

Therefore

$$\psi(z, \bar z) = R(z, \bar z/\zeta) + S(\bar z).$$

But the left side is independent of ζ, and so we have finally

$$\psi(z, \bar z) = R(z) + S(\bar z)$$

and thus $\partial\psi/\partial z_\alpha \partial \bar z_\beta = 0$. Thus $g_{\alpha\beta^*}(z, \bar z) \equiv g_{\alpha\beta^*}(0)$, and our line element is flat.

However, as was shown in the proof of Theorem 1, there always exists a coordinate system which is *geodesic* at a prescribed point; meaning that the derivatives $\partial g_{\alpha\beta^*}/\partial z_\gamma$ and therefore also the coefficients $\Gamma^\alpha_{\beta\gamma}$ vanish at the point.

4. **Imbedding.** In addition to our space V_n with the metric (3) we consider a space V_m, $m > n$, in the complex variables $w_1, \cdots, w_m$, $m > n$, with a metric

$$2h_{\xi\eta^*}(w, \bar w) dw_\xi d\bar w_\eta \tag{36}$$

where $\xi, \eta, \zeta, \chi = 1, 2, \cdots, m$; and we assume that there exists a complex-analytic transformation

$$w_\xi = f^\xi(z_1, \cdots, z_n), \qquad \xi = 1, \cdots, m, \tag{37}$$

which is an *isometric* map of V_n intc V_m. Thus

$$g_{\alpha\beta^*} = h_{\xi\eta^*} f^\xi_{,\alpha} \overline{f^\eta_{,\beta}} \tag{38}$$

where $f^\xi_{,\alpha} = \partial f^\xi / \partial z_\alpha$.

THEOREM 11. *If we denote the curvature tensor of* V_n *by* $R_{\alpha\beta^*\gamma\delta^*}$ *and of* V_m *by* $S_{\xi\eta^*\zeta\chi^*}$, *then we have*

$$R_{\alpha\beta^*\gamma\delta^*} = S_{\xi\eta^*\zeta\chi^*} f^{\xi}{}_{,\alpha}\overline{f^{\eta}{}_{,\beta}} f^{\zeta}{}_{,\gamma}\overline{f^{\chi}{}_{,\delta}} + h_{\xi\eta^*} f^{\xi}{}_{,\alpha,\gamma}\overline{f^{\eta}{}_{,\beta,\delta}} \tag{39}$$

where $f^{\xi}{}_{,\alpha,\gamma}$ *denotes the second covariant derivative of* $f^{\xi}(z)$ *relative to the metric* (3).

PROOF. We write (7) in the form

$$R_{\alpha\beta^*\gamma\delta^*} = \frac{\partial^2 g_{\alpha\beta^*}}{\partial z_\gamma \partial \bar{z}_\delta} - \Gamma^{\rho}_{\alpha\gamma}\frac{\partial g_{\rho\beta^*}}{\partial \bar{z}_\delta} \tag{40}$$

and similarly

$$S_{\xi\eta^*\zeta\chi^*} = \frac{\partial^2 h_{\xi\eta^*}}{\partial w_\zeta \partial \bar{w}_\chi} - \Delta^{\phi}_{\xi\zeta}\frac{\partial h_{\phi\eta^*}}{\partial \bar{w}_\chi}, \tag{41}$$

where $\Delta^{\phi}_{\xi\zeta}$ is the affine connection pertaining to (36). Take a point z^0 and its image w^0. It can easily be seen that it will suffice to prove (39) under the assumption that the z-coordinates are geodesic at z^0 and the w-coordinates are geodesic at w^0. In this case all quantities

$$\frac{\partial g_{\alpha\beta^*}}{\partial z_\gamma}, \qquad \Gamma^{\alpha}_{\beta\gamma}, \qquad \frac{\partial h_{\xi\eta^*}}{\partial w_\zeta}, \qquad \Delta^{\xi}_{\eta\zeta}$$

and their conjugates are 0, and we easily obtain (39) by substituting (38) in the right side of (40) and carrying out the differentiation.

We now take a fixed point z^0 in V_n, and we introduce an arbitrary surface element

$$z_\alpha - \overset{0}{z}_\alpha = \lambda^{\alpha} x + \mu^{\alpha} y \tag{42}$$

and its image

$$w_\xi - \overset{0}{w}_\xi = L^{\xi} x + M^{\eta} y, \tag{43}$$

where

$$L^{\xi} = f^{\xi}{}_{,\alpha}\lambda^{\alpha}, \qquad M^{\xi} = f^{\xi}{}_{,\alpha}\mu^{\alpha}. \tag{44}$$

If we multiply both sides of (39) by

$$(\lambda^{\alpha}\mu^{\beta^*} - \lambda^{\beta^*}\mu^{\alpha})(\lambda^{\gamma}\mu^{\delta^*} - \lambda^{\delta^*}\mu^{\gamma}) \tag{45}$$

and add up over α, β, γ, δ we obtain the following theorem.

THEOREM 12. *The sectional curvature*

$$\frac{R_{abcd}\lambda^a\mu^b\lambda^c\mu^d}{(g_{ac}g_{bd} - g_{ad}g_{bc})\lambda^a\mu^b\lambda^c\mu^d} \tag{46}$$

in V_n *is the sum of the corresponding sectional curvature*

$$\frac{S_{ijkl}L^iM^jL^kM^l}{(h_{ik}h_{jl}-h_{il}h_{jk})L^iM^jL^kM^l} \tag{47}$$

in V_m *and of the relative curvature*

$$\frac{h_{\xi\eta^*}f^{\xi}{}_{,\alpha,\gamma}\overline{f^{\eta}{}_{,\beta,\delta}}(\lambda^{\alpha}\mu^{\beta^*}-\lambda^{\beta^*}\mu^{\alpha})(\lambda^{\gamma}\mu^{\delta^*}-\lambda^{\delta^*}\mu^{\gamma})}{(g_{ac}g_{bd}-g_{ad}g_{bc})\lambda^a\mu^b\lambda^c\mu^d}. \tag{48}$$

For a holomorphic section (22) *the relative curvature has the value*

$$-\frac{h_{\xi\eta^*}f^{\xi}{}_{,\alpha,\gamma}\lambda^{\alpha}\lambda^{\gamma}\overline{f^{\eta}{}_{,\beta,\delta}}\lambda^{\beta^*}\lambda^{\delta^*}}{(g_{\alpha\beta^*}\lambda^{\alpha}\lambda^{\beta^*})^2}. \tag{49}$$

It is therefore negative or 0, *and it has the value* 0 *only if the section is such that*

$$f^{\xi}{}_{,\alpha,\gamma}\lambda^{\alpha}\lambda^{\gamma}=0, \qquad \xi=1,\cdots,m. \tag{50}$$

Condition (50) *formally corresponds to directions of asymptotic lines in the real case.*

For the Ricci tensor we have

$$R_{\alpha\beta^*}=S_{\xi\eta^*}f^{\xi}{}_{,\alpha}\overline{f^{\eta}{}_{,\beta}}+h_{\xi\eta^*}g^{\delta^*\gamma}f^{\xi}{}_{,\alpha,\gamma}\overline{f^{\eta}{}_{,\beta,\delta}} \tag{51}$$

and the Ricci curvature $-R_{ad}\lambda^a\lambda^d/g_{ad}\lambda^a\lambda^d$ *is the sum of* $-S_{ij}L^iL^j/h_{ij}L^iL^j$ *and of the relative Ricci curvature*

$$-\frac{h_{\xi\eta^*}g^{\delta^*\gamma}f^{\xi}{}_{,\alpha,\gamma}\lambda^{\alpha}\overline{f^{\eta}{}_{,\beta,\delta}}\lambda^{\beta^*}}{g_{\alpha\beta^*}\lambda^{\alpha}\lambda^{\beta^*}}. \tag{52}$$

The last term is negative or 0, *and it is* 0 *if and only if*

$$f^{\xi}{}_{,\alpha,\gamma}\lambda^{\alpha}=0, \qquad \xi=1,\cdots,m;\ \gamma=1,\cdots,n, \tag{53}$$

which is a stronger condition than (50).

If V_m has the Euclidean metric

$$dw_1d\bar{w}_1+\cdots+dw_m\overline{dw_m}, \tag{54}$$

then (48) is the entire amount of the sectional curvature of V_n. Now, put $m=3$, $n=2$, and

$$\overset{1}{f}(z)=z_1, \qquad \overset{2}{f}(z)=z_2, \qquad \overset{3}{f}(z)=2^{-1}(z_1^2+z_2^2).$$

The metric of V_2 is then $dz_1d\bar{z}_1+dz_2d\bar{z}_2+(z_1dz_1+z_2dz_2)(\bar{z}_1d\bar{z}_1+\bar{z}_2d\bar{z}_2)$. At the origin $z_1=z_2=0$, the z-system is geodesic, and the numerator in (48) has at that point the value

$$\sum_{\alpha,\beta=1}^{n} (\lambda^\alpha \mu^{\beta^*} - \lambda^{\beta^*} \mu^\alpha)^2. \tag{55}$$

Since the denominator of (48) is always positive, see (25), the algebraic sign of (48) is that of (55). If, for instance, all components λ^α, μ^β are real, that is if our section is totally real, then (55) is positive, and thus we see that the relative sectional curvature may be positive for a nonholomorphic section.

THEOREM 13. *For $n=1$, if a line element*

$$ds^2 = g(z, \bar{z})dzd\bar{z} \tag{56}$$

of the special form

$$g(z, \bar{z}) = \sum_{p=0}^{\infty} a_p z^p \bar{z}^p \tag{57}$$

can be isometrically imbedded in V_m, with the Euclidean line element (54), *for some finite m, then the power series* (57) *is a finite polynomial in $z, \bar{z}$. In particular, the line element*

$$ds^2 = \frac{dzd\bar{z}}{(1 - z\bar{z})^2} = \sum_{p=0}^{\infty} (p + 1) z^p \bar{z}^p dzd\bar{z}, \tag{58}$$

although strictly hyperbolic, cannot be so imbedded.

For $n \geqq 1$, if in the line element

$$g_{\alpha\beta^*}(z, \bar{z})dz_\alpha d\bar{z}_\beta \tag{59}$$

in the power series of all $g_{\alpha\beta^*}$ only those monomials

$$z_1^{p_1} \cdots z_n^{p_n} \bar{z}_1^{q_1} \cdots \bar{z}_n^{q_n} \tag{60}$$

occur for which

$$p_1 + \cdots + p_n = q_1 + \cdots + q_n, \tag{61}$$

and if (59) *can be isometrically imbedded in a V_m, then all $g_{\alpha\beta^*}(z, \bar{z})$ are finite polynomials. In particular, the line elements arising in all types of matrix spaces*[3] *cannot be so imbedded.*

PROOF. For $n=1$, our imbedding means the existence of functions

$$w_k = f_k(z) = \sum_{p=1}^{\infty} c_{k,p} z^p, \qquad k = 1, \cdots, m, \tag{62}$$

such that

---

[3] See Siegel [7] and Hua [5] and [6].

$$\sum_k \left|\frac{df_k(z)}{dz}\right|^2 = \sum_p a_p z^p \bar{z}^p.$$

Therefore

$$pq\sum_{k=1}^{m} c_{k,p}\overline{c_{k,q}} = \delta_{pq}a_{p-1}. \tag{63}$$

For each p, the system of numbers

$$C_p = \{c_{k,p}\}, \qquad k = 1, \cdots, m,$$

is a vector in complex m-space. By (63) any two of the vectors are orthogonal, and therefore all but at most m of them must be identically 0. Therefore at most m^2 among the numbers $c_{k,p}$ can be not equal to 0. Each $f_k(z)$ is a polynomial, and so is $g(z, \bar{z})$.

For $n \geqq 1$, if there exists an imbedding, then we write the mapping functions in the form

$$f_k(z_\alpha) = \sum_{p=1}^{\infty} c_{k,p}(z_\alpha) \tag{64}$$

where $c_{k,p}$ is a homogeneous polynomial in the variables z_α of degree p, or identically zero. Now, put

$$z_\alpha = \zeta_\alpha t, \qquad \bar{z}_\alpha = \bar{\zeta}_\alpha \bar{t}, \tag{65}$$

where $\{\zeta_\alpha\}$ are arbitrary parameters in the neighborhood of the origin and t is the complex variable proper. Thus, (65) is a family of complex curves. For each one we derive from (59) a line element

$$g(\zeta, \bar{\zeta}; t, \bar{t})dtd\bar{t},$$

where by our assumptions on (59),

$$g(\zeta, \bar{\zeta}; t, \bar{t}) = \sum_p a_p(\zeta, \bar{\zeta})t^p\bar{t}^p.$$

On substituting (65) into (64), we obtain

$$f_k(\zeta_\alpha t) = \sum_p c_{k,p}(\zeta_\alpha)t^p,$$

and by our previous argument, corresponding to each special point $\{\zeta_\alpha^0\}$, only m^2 among the numbers $c_{k,p}(\zeta_\alpha^0)$ can be not equal to 0. Since each $c_{k,p}(\zeta)$ is a polynomial, it follows easily that all but m^2 among them must vanish, and thus $f_k(z_\alpha)$ is a polynomial. This proves the theorem.

Thus far we have been discussing imbedding into a V_m, for *finite* m.

The situation is radically different if we allow the imbedding space to become infinitely-dimensional. One possibility is to view certain constructions as limiting processes of ordinary imbedding; this will be our method of interpreting groups of automorphisms. However, another possibility is to actually perform an imbedding into a space of countably many variables $w_1, w_2, w_3, \cdots$ with a suitable metric

$$h_{\xi\eta^*}(w, \bar{w})dw_\xi\overline{dw_\eta}, \qquad \xi, \eta = 1, 2, 3, \cdots .$$

The most immediate line element would be the Hilbert metric

$$dw_1 d\bar{w}_1 + dw_2 d\bar{w}_2 + \cdots .$$

In this case the hyperbolic line element (58) can very well be imbedded, namely by the transformation

$$w_k = z^k/(k+1)^{1/2}, \qquad k = 1, 2, \cdots .$$

Furthermore, our previous theorems remain valid, and thus for instance an elliptic line element cannot be isometrically imbedded in flat Hilbert space. Conceivably every finite-dimensional Hermitian line element without torsion could be imbedded into some universal countably-dimensional space with a fixed elliptic metric. It is interesting to note in this connection that the method of S. Bergman [1] for constructing an Hermitian metric on a domain of several complex variables consists precisely in constructing an analytic imbedding of the domain into such a universal enveloping space, the enveloping domain of Bergman being the countably-dimensional Fubini space with positive curvature $b=1$.

We shall not take up this problem in any way; however, we shall point out the following curious little theorem, which apparently is not true for finite m.

THEOREM 14. *Whenever a V_n can be isometrically imbedded in flat Hilbert space it can also be isometrically imbedded in infinitely-dimensional Fubini space with $b=1$.*

PROOF. By assumption we have relation (5) with

$$\Phi(z, \bar{z}) = \sum_{k=1}^{\infty} | f_k(z_\alpha) |^2.$$

On putting $\Phi = \log \psi$, that is, $\psi = e^\Phi$, we obtain

$$\psi = 1 + \sum_{k=1}^{\infty} | g_k(z_\alpha) |^2$$

for suitable functions $g_k(z_\alpha)$, and this is the substance of the theorem.

5. Group invariance. We consider a domain D in the space of the complex variables $z_1, \cdots, z_n$, and an arbitrary topological set Γ, and we consider a continuous function

$$f(\theta; z_\alpha) \tag{66}$$

in $\Gamma \times D$ which is analytic in D. We further assume that there exists some kind of integral in Γ such that

$$\Phi(z, \bar{z}) = \int_\Gamma f(\theta; z)\overline{f(\theta; z)}\,d\theta \tag{67}$$

exists in D, and that in the neighborhood of any point z^0 it is a uniform limit of an approximating sum

$$\sum_{r=1}^{s} \Delta_r f(\theta_r; z)\overline{f(\theta_r; z)}, \qquad \Delta_r > 0.$$

If we now form

$$g_{\alpha\beta^*} = \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}, \tag{68}$$

then our restriction (4) is met; also $g_{\alpha\beta^*}$ is non-negative, and we add the *explicit* assumption that it is strictly positive definite. The approximating tensor

$$h_{\alpha\beta^*} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \sum_{r=1}^{s} \left| \Delta_r^{1/2} f(\theta_r; z) \right|^2 \tag{69}$$

may be also assumed positive definite, and may be looked upon as resulting from an imbedding of D in Euclidean V_s. If we apply Theorem 12, and make the limiting passage from (69) to (68), we see that the resulting line-element (59) on D has nonpositive holomorphic and Ricci curvature.

We shall now vary the set-up slightly. We consider not a family of functions, but a family of analytic homeomorphisms $w_k = f_k(\theta; z_\alpha)$, $k = 1, \cdots, n$, of D into itself, and we assume that Γ is a locally compact group of such homeomorphisms. Correspondingly, we put

$$\Phi(z, \bar{z}) = \int_\Gamma \left(\sum_{k=1}^{n} \left| f_k(\theta; z) \right|^2 \right) d\theta \tag{70}$$

where $d\theta$ represents integration with respect to a Haar-measure, say a left-invariant one. Thus we have

$$
\begin{aligned}
g_{\alpha\beta^*} dz_\alpha d\bar{z}_\beta &= \int_\Gamma \left(\sum_{k=1}^{n} \frac{\partial f_k(\theta, z)}{\partial z_\alpha} \overline{\frac{\partial f_k(\theta, z)}{\partial z_\beta}} \right) d\theta dz_\alpha d\bar{z}_\beta \\
&= \int_\Gamma \left(\sum_{k=1}^{n} \left| dw_k(\theta, z) \right|^2 \right) d\theta.
\end{aligned} \tag{71}
$$

Now, if θ' is a fixed element of Γ, and if we put $z'_\alpha = f_\alpha(\theta'; z) = \theta' z_\alpha$, then we have $w_k(\theta\theta'; z) = w_k(\theta; z')$, and we find that the line element is group invariant. We can now state the following very conditional result.

THEOREM 15. *If Γ is a transitive group of homeomorphisms of a domain D, if the general homeomorphism can be written in the form*

$$w_k = f_k(\lambda; a, \bar{a}; z_\alpha) \tag{72}$$

where $a = (a_1, \cdots, a_n)$ is an arbitrary point of D and λ is a set of parameters which describe the stability subgroup Γ_0 of Γ, if $d\lambda$ designates Haar-measure on Γ_0, if there is given in D a volume element

$$dv_a = \rho(a, \bar{a}) \prod_{\alpha=1}^{n} da_\alpha d\bar{a}_\alpha$$

which is invariant under all transformations of Γ, and if the integrals

$$g_{\alpha\beta^*} = \int_{\Gamma_0 \times D} \left(\sum_{k=1}^{n} \frac{\partial f_k(\lambda; a, \bar{a}; z)}{\partial z_\alpha} \overline{\frac{\partial f_k(\lambda; a, \bar{a}; z)}{\partial z_\beta}} \right) dv_a d\lambda$$

converge uniformly in a neighborhood of every point z of D, then the corresponding Hermitian metric has no torsion and is group-invariant, and the curvature of holomorphic sections and the Ricci curvature are all nonpositive.

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A CHARACTERIZATION OF MINKOWSKIAN GEOMETRY

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Menger called a finitely-compact metric space in which any two distinct points are contained in exactly one set congruent to a euclidean straight line a straight-line space (S.L. space) [1].[1] He observed that in such spaces a segment (or vector) addition can be defined in a very simple way (see §2 of the present paper) which satisfies all requirements for an abelian group except, in general, the associative law. Menger then put the question of determining those geometries in which the associative law holds. It is the purpose of the present note to show that the Minkowskian geometries furnish the answer.[2]

1. **Straight-line spaces.**[3] By a straight-line (S.L.) in a metric space R is understood a continuous curve which may be mapped congruently onto the real axis, that is, which admits of a parametrization $P(\tau)$, $-\infty<\tau<\infty$, with $P(\tau_1)P(\tau_2)=|\tau_1-\tau_2|$. The numbers τ are the *isometric coordinates* of the points of the S.L. A *segment* joining two points $P(\tau_1)$, $Q(\tau_2)\subset R$, written (PQ), is a congruent image of the closed interval between τ_1 and τ_2.

An *S.L. space* is one in which any pair of its points X, Y is contained in a unique S.L. $[XY]$. The following conditions are sufficient for an S.L. space: the space is (1) metric, (2) finitely-compact, (3) convex, (4) externally convex, (5) if (XYZ_1), (XYZ_2), and $YZ_1=YZ_2$, then $Z_1=Z_2$. We shall suppose, hereafter, that these conditions hold.

If $X_n\to X$, $Y_n\to Y$ and $X\neq Y$ then $[X_nY_n]\to[XY]$.

According to Menger M is an *internal center* of X and Y, written $M(X, Y)$, if (XMY) and $XM=MY$. Also, if (EXY) and $EX=XY$, E is an *external center* of X and Y, written $E(X, Y)$ or $(Y, X)E$. In an S.L. space every point-pair has a unique internal center and two external centers.

2. **Addition of segments in S.L. spaces.** Let (OX), (OY) be two arbitrary segments in R having the end point O in common. We shall call the segment (OP) a *sum* of the two given segments, written

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The above problem, together with a conjecture as to the answer, was suggested by H. Busemann.

[3] For explanations of the notations used here, as well as proofs of the statements of this section, see Busemann [2, chaps. I and III].

$(OP)=(OX)+(OY)$, if $M(O, P)=M(X, Y)$. Since O will remain fixed throughout, we shall write simply $P=X+Y$. This relation has the following properties:

1. There exists exactly one sum $P=X+Y$. For if $M(X, Y)$, and $P=(O, M)E$, then $M=M(O, P)$, so that the sum exists. On the other hand if $P'=X+Y$, then $M(O, P')=M(X, Y)=M(O, P)$, hence $P'=(O, M)E$, so that, by the convexity of the space, $P'=P$.
2. Addition is commutative. For $M(X, Y)=M(Y, X)$.
3. If O, X, Y are contained in the same S.L., then $X(\tau_1)+Y(\tau_2)=P(\tau_1+\tau_2)$, that is, the isometric coordinates of the sum are the sum of the isometric coordinates.
4. $P=X+Y$ is continuous in X and Y. For $M(X, Y)$ is continuous in X and Y and $P=(O, M)E$ is continuous in M.
5. $O+X=X$.
6. $M(O, X)+M(O, X)=X$, and we may write $M=X/2$.
7. There exists exactly one point Y such that $X+Y=P$. For if $M=M(O, P)$ and $Y=(X, M)E$, then Y is the (unique) required point, and we may write $Y=P-X$.
8. We shall write $O-Y$ as $-Y$. $Y+(-Y)=O$.

The set of segments through O satisfy all the group axioms for addition except the associative law.[4] The following is an example where it fails to hold: Consider the representation of the hyperbolic (h) plane inside the euclidean (e) unit circle K in which the h.S.L. are the segments of the e.S.L. inside K. Let g, h be two S.L. through the center O and $X\subset g$, $Y\subset h$ two points h-equidistant from O (hence also e-equidistant from O). The h mid-point M of (XY) is also the e mid-point. Calling $Z=X+Y$ in the sense of the h-metric, we have $e(MZ)<e(OM)$, where $e(\ )$ denotes the e-distance. Hence the intersection $P=(Z(-Y))\cdot(OX)$ is such that $e(OP)<e(PX)$, hence $h(OP)\neq h(PX)$, so that P cannot be the h mid-point of (OX). Thus

$$X+[Y+(-Y)]=X;\quad (X+Y)+(-Y)=Z+(-Y)\neq X.$$

3. **The main theorem.** *An S.L. space R has associative segment-addition for every triple of segments around a point if and only if its geometry is Minkowskian.*

We show first that any two S.L. through O are contained in a two-dimensional Minkowski space.

We shall suppose an S.L. space R in which the associative law of segment-addition holds around a point O.

I. Let g and g' (hereinafter fixed in the discussion) be two arbitrary S.L. through O. The totality of points $Z\subset R$ such that $X+Y=Z$,

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[4] Such systems, called *loops*, have been studied by Albert. See [3].

$X \subset g$, $Y \subset g'$ will be called the *plane* p determined by g and g', written $p(g, g')$. p contains g and g'.

II. *The sum and difference of two points* $Z_1 \subset p$, $Z_2 \subset p$ *is also in* p. For $Z_1 = X_1 + Y_1$, $Z_2 = X_2 + Y_2$, $X_1, X_2 < g$, $Y_1, Y_2 \subset g'$ and $Z_1 + Z_2 = (X_1 + Y_1) + (X_2 + Y_2) = (X_1 + X_2) + (Y_1 + Y_2)$. Since $X_1 + X_2 \subset g$ and $Y_1 + Y_2 \subset g'$, $Z_1 + Z_2 \subset p$.

III. *If* Z_1, $Z_2 \subset p$ *then* $M(Z_1, Z_2)$, $E(Z_1, Z_2)$ *and* $(Z_1, Z_2)E$ *are also in* p. First we note that if $Z = X + Y$, then $M(O, Z) = X/2 + Y/2$, since $2(X/2 + Y/2) = (X/2 + Y/2) + (X/2 + Y/2) = (X/2 + X/2) + (Y/2 + Y/2) = X + Y = Z$. We may thus write $M(O, Z) = Z/2$. Using this notation we have

$$M(Z_1, Z_2) = Z_1/2 + Z_2/2, \qquad E(Z_1, Z_2) = 2Z_1 - Z_2, \tag{1}$$
$$(Z_1, Z_2)E = 2Z_2 - Z_1.$$

In the same way we can show that $mZ/n = mX/n + mY/n$. Finally since Z is continuous in X and Y, letting $m/n \to k$, we have $kZ = kX + kY$, where k is any real number.

IV. *A plane* p *is finitely-compact.* Suppose a bounded set $\{Z\}$ in p. Since R is finitely-compact there exists a point $Z \subset R$ and a sequence $Z_n \subset \{Z\}$ such that $Z_n \to Z$. Let $Z_n = X_n + Y_n$. It must be shown that X_n and Y_n are bounded. Otherwise subsequences, which we again denote by X_n, Y_n, would tend to infinity. Let $P \subset g$ $(P \neq 0)$ be a fixed point and let $k = OX_n/OP$. Then

$$Z_n' = Z_n/k = X_n/k + Y_n/k = P + Y_n/k,$$

or $Z_n' - Y_n/k = P$. Since $k \to \infty$, $Z_n' \to 0$, and since $Y_n/k \subset g'$, their difference cannot be a point of g, for n sufficiently large. Thus, for some subsequence, since R is finitely-compact, $X_n \to X$, $Y_n \to Y$, so that $Z = X + Y$, and hence $Z \subset p$.

V. *Every point* $Z \subset p$ *is uniquely represented as a sum* $Z = X + Y$, $X \subset g$, $Y \subset g'$. For if $Z = X_1 + Y_1 = X_2 + Y_2$, $X_1, X_2 \subset g$, Y_1, $Y_2 \subset g'$, then $Z - Y_1 = (X + Y_1) - Y_1 = (X_2 + Y_2) - Y_1$ which becomes, by the associative law, $X_1 + (Y_1 - Y_1) = X_2 + (Y_2 - Y_1)$, or $X_1 = X_2 + (Y_2 - Y_1)$. Since $X_2 \subset g$ and $Y_2 - Y_1 \subset g'$ their sum cannot be a point of g unless $Y_2 - Y_1 = O$. But then $Y_2 = Y_1$ and $X_2 = X_1$.

VI. By virtue of V there may be uniquely associated with every point $Z = X + Y$ $(X \subset g,\ Y \subset g')$ of p the isometric coordinates x, y of X and Y respectively. Since, conversely, to every coordinate-pair (x, y) there exists uniquely a corresponding point $Z \subset p$, a one-to-one mapping may thus be set up between p and the euclidean plane e.

It follows from statement 4, §2, and IV that this mapping is a homeomorphism.

VII. *The S.L. of p are mapped into the S.L. of e.*[5] For, by (1), we may write

$$\begin{aligned} M(Z_1, Z_2) &= (X_1 + Y_1)/2 + (X_2 + Y_2)/2 \\ &= (X_1 + X_2)/2 + (Y_1 + Y_2)/2, \\ E_1 = E(Z_1, Z_2) &= 2(X_1 + Y_1) - (X_2 + Y_2) \\ &= (2X_1 - X_2) + (2Y_1 - Y_2), \\ E_2 = (Z_1, Z_2)E &= 2(X_2 + Y_2) - (X_1 + Y_1) \\ &= (2X_2 - X_1) + (2Y_2 - Y_1), \end{aligned}$$

so that the coordinates in e corresponding to M, E_1, E_2 are respectively $(x_1/2+x_2/2,\ y_1/2+y_2/2)$, $(2x_1-x_2,\ 2y_1-y_2)$, $(2x_2-x_1,\ 2y_2-y_1)$. Thus the centers of any pair of points in R map into the corresponding e-centers. The statement is then proved by repeatedly taking centers and their limit points,[6] and using the fact that the mapping is a homeomorphism. A consequence of this and the fact that the whole of e is covered by the mapping is that the e parallel axiom holds.

VIII. *p admits of a simply-transitive abelian group of translations.* We shall prove that the translations

$$P:(x, y) \rightarrow P^T:(x + a, y + b), \qquad T = T_{ab}, \tag{2}$$

of the euclidean plane are motions for the given metric. It is well known and very easy to prove that the metric of p is then Minkowskian.[7]

1. If, for a set of points $P_0, P_1, \cdots, P_n, \cdots$ on an S.L. $g \subset p$ we have $P_{i-1}P_i = P_iP_{i+1}$, then $P^T_{i-1}P^T_i = P^T_iP^T_{i+1}$, $i=1, \cdots, n, \cdots$. For we have $P_i = M(P_{i-1}P_{i+1})$, which, by VII, has as image in e the corresponding e-center. Since this, in turn, remains center under the mapping (2), and then under the inverse mapping of e into p, the statement is proved.

2. If $P^T, P^{T^2}, \cdots, P^{T^n}, \cdots$ are the images of P under $T, T^2, \cdots, T^n, \cdots$ then $PP^T = P^{T^n}P^{T^{n+1}}$, $n=0, 1, \cdots$. For let $P \rightarrow (x, y)$, then $P^{T^n} \rightarrow (x + na,\ y + nb)$, and since this point is the e-center of $(x+(n-1)a,\ y+(n-1)b)$ and $(x+(n+1)a,\ y+(n+1)b)$, P^{T^n} must be the center of $P^{T^{n-1}}$ and $P^{T^{n+1}}$, that is, $P^{T^{n-1}}P^{T^n} = P^{T^n}P^{T^{n+1}}$, $n=0, 1, \cdots$.

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[5] When such a mapping is possible we say that the metric of p is *Desarguesian*.

[6] This process is exemplified in [1, p. 87].

[7] See [2, Theorem II, 1.17a] (invariance under translation is equivalent to dependence on $x-y$).

3. If, for a set of points $P_0, P_1, \cdots, P_n, \cdots$, on an S.L. $g \subset p$, $P_{i-1}P_i = P_iP_{i+1}$, then P_0, P_1^T, $P_2^{T^2}, \cdots, P_n^{T^n}$ lie on an S.L. and $P_0P_i^T = P_{i-1}^{T^{i-1}}P_i^{T^i}$, $i = 1, \cdots, n, \cdots$. This follows from the fact that the corresponding relations hold in e.

4. We shall now prove that if $P, Q \subset p$ then $PQ = P^TQ^T$. Set $P = P_0$, $Q = P_1$ in the notation of the three previous steps. Then $nPQ = P_0P_n \leqq P_0P_0^T + P_0^TP_n^T + P_n^TP_n$. But by step 1 $P_0^TP_n^T = nP^TQ^T$ and by step two $P_nP_n^T = P_nP_n^{T^n}/n$, hence $n|PQ - P^TQ^T| \leqq P_nP_n^{T^n}/n + P_0P_0^T \leqq P_0P_1 + P_0P_1^T + P_0P_0^T$, since by step 3, $P_nP_n^{T^n} \leqq P_0P_n + P_0P_n^{T^n} = nP_0P_1 + nP_0P_1^T$. Since n can be made arbitrarily large, $PQ = P^TQ^T$. Thus T is proved to be a congruence, and since there are no fixed points, it is a translation. As the translations of e are simply-transitive and abelian, so are those of p.

This completes the proof that any three points O, X, Y of R are contained in a Minkowski plane. Note that we have proved that this is a consequence of associative segment addition merely with respect to a single point as origin. From the existence of translations it follows that addition is associative with respect to every point. The main theorem then follows immediately for S.L. spaces of dimension not less than 2. For the space is Desarguesian (see [2, Theorem III, 6.3]), and, using the same methods as above, the transitive Abelian group of the form $X \rightarrow X + h$ can be introduced. That these transformations are motions follows immediately from the fact that X, Y, $X \rightarrow h$, $Y \rightarrow h$ lie in a plane, hence $XY = (X+h)(Y+h)$. The finite-compactness of the space together with its linear operation imply its finite-dimensionality.

Thanks are due to H. Busemann K. Menger, and his associates for helpful suggestions and criticisms.

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GRATINGS AND HOMOLOGY THEORY

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1. Introduction. The relation of an abelian group to its character group is merely one example of a relation between an algebraic system Γ and a topological system X, each invariantively associated with the other. Of the two systems Γ and X, the first can be described in pseudo-combinatory terms, while the second involves the use of more sophisticated notions, such as passage to the limit, geometrical continuity, and so on. Accordingly, problems which are ordinarily expressed in terms of the system X can often be treated more simply by restating them in terms of the system Γ.

In this paper we shall be dealing with a class of algebraic systems Γ, called *gratings*. The theory of gratings will aim, among other things, to describe the topological properties of a space X in terms of the ways in which the space can be expressed as the union of two subspaces A and C, $A \cup C = X$. The theory will be pseudo-combinatory in character, in the sense that it will have to do with combinatory operations applied to an unlimited number of abstract elements, or symbols. It will acquire a geometrical significance only when the symbols are attached to appropriate geometrical entities. The theory will be applicable both to ordinary topological spaces and to spaces of a more general type, such as the ones determined by distributive lattices, which last need not be assumed to possess atomic elements. With the aid of gratings, we shall be able to reformulate a variety of problems in topology, differential geometry, potential theory, and so on, in pseudo-combinatory terms.

The present paper will be devoted almost exclusively to the elementary formal theory of gratings. Further developments, along with a number of typical applications, will be considered subsequently.[1]

2. Cuts, elements, gratings, cells. A *cut* will be any ordered triad $\gamma = (a, b, c)$ consisting of three different *abstract elements* a, b, and c. The element a will be called the *negative face*, the element b the *edge*, the element c the *positive face* of the cut.

A *grating* $\Gamma = [\gamma]$ will be any (finite or infinite) set of cuts, such that no two of the cuts have an element in common. The cuts γ will be

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[1] The terminology adopted in this paper differs quite radically from that used by the author in an earlier paper on gratings: *A theory of connectivity in terms of gratings*, Ann. of Math. vol. 39 (1938) pp. 883–912.

called the *cuts* of the grating, their elements will be called the *elements* of the grating. Given a cut $\gamma_i=(a_i, b_i, c_i)$, it will be convenient to introduce a symbol z_i to designate an unspecified element of γ_i, $z_i=a_i$, b_i, or c_i.

A *cell* A of a grating Γ will be any finite sequence of elements of Γ,

$$A = z_1, z_2, \cdots, z_m \qquad (z_i = a_i, b_i, \text{ or } c_i). \tag{2.1}$$

The *degree* $m(A)$ of the cell will be the number of terms of the sequence, the *rank* $\rho(A)$ the number of terms in edges b_i, rather than in faces a_i or c_i. Thus we shall always have $0\leqq\rho(A)\leqq m(A)$. A cell of rank ρ will be called a *ρ-cell*. The rank of a cell will be indicated by a superscript, $A=A^\rho$.

A cell will be called a *type-cell* if all its terms are negative faces, $z_i=a_i$. The *type* of a cell $A=z_1, z_2, \cdots, z_m$ will be the type-cell

$$\alpha(A) = a_1, a_2, \cdots, a_m \tag{2.2}$$

of the same degree as A, such that the ith term of the sequence $\alpha(A)$ is the negative face a_i belonging to the same cut as the ith term z_i of A, $i=1, 2, \cdots, m$. Two cells A and B will be said to be of the same *type* if their type-cells are identical. The number of cells of any given type $\alpha=\alpha(A)$ will, of course, be finite (3^m, to be exact). The type of a cell will be indicated by a subscript, $A=A_\alpha$. Thus, a symbol of the form A^ρ_α will denote a ρ-cell of type α. A cell A will be said to be *regular*, or of *regular type*, if its type $\alpha=\alpha(A)$ is composed of m different terms a_i; it will be said to be *singular*, or of *singular type*, if its type α has any repeated terms. A cell A without repeated terms may, of course, be singular. For example, the cell $A=a_i, c_i$ is singular, since its type $\alpha(A)=a_i, a_i$ is singular.

For the sake of greater formal simplicity, we shall assume that there is an *empty sequence* of elements with no terms at all. Accordingly, we shall have a unique cell E_ϵ of degree $m=0$. The cell E_ϵ will be called the *unit cell*. It will be of rank $\rho=0$ and of type $\epsilon=E_\epsilon$. The type ϵ will be called the *unit type*. The unit type will be regular.

Every element z is paired with a cell z of degree $m=1$ consisting of a single term in the element z. Our notation will make no distinction between the element z and the cell z. We shall call the cell z the *elementary cell* determined by the element z. The type a of an elementary cell z, $z=a$, b, or c, will also be said to be *elementary*.

The *product* AB of a cell $A=z_1, z_2, \cdots, z_m$ and a cell $B=w_1, w_2, \cdots, w_n$ will be the cell

$$AB = z_1, z_2, \cdots, z_m, w_1, w_2, \cdots, w_n \tag{2.3}$$

obtained by merely appending the sequence B to the sequence A. Thus, the product AB will be of degree $m+n$, of rank $\rho(A)+\rho(B)$, and of type $\alpha\beta$, where α is the type of A and β the type of B. Cell products will be *associative*, but *noncommutative*. The unit cell E_e will be the unit of cell multiplication,

$$E_eA = A = AE_e \qquad (A \text{ arbitrary}). \tag{2.4}$$

Generally speaking, it will be preferable to think of a cell A as a product of elementary cells, rather than as a sequence of elements. We shall therefore replace the sequential notation (2.1) by the product notation

$$A = z_1z_2 \cdots z_m. \tag{2.5}$$

The cells A_α of any fixed type $\alpha=a_1a_2\cdots a_m$ can all be represented schematically in the space Ξ^n of n real variables $\xi_1, \xi_2, \cdots, \xi_n$, $n\geqq m$, by making the following construction. Corresponding to each elementary factor z_i of a cell $A_\alpha=z_1z_2\cdots z_m$ we write one of the three relations $\xi_i\leqq 0$, $\xi_i=0$, $\xi_i\geqq 0$, the first if we have $z_i=a_i$, the second if we have $z_i=b_i$, the third if we have $z_i=c_i$. The m relations thus obtained are regarded as the *determining relations* of the representation of A_α. The *representation* itself consists of all points $\xi=\xi_1, \xi_2, \cdots, \xi_n$ of the space Ξ^n such that their first m coordinates satisfy the determining relations. In Fig. 1, we have represented the three cells a, b, and c of type $\alpha=a$ on the line Ξ^1; in Fig. 2, the same three cells in the plane Ξ^2. In Fig. 3, we have represented the nine cells z_1z_2 of type $\alpha=a_1a_2$ in the plane Ξ^2.

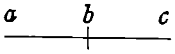

FIG. 1

FIG. 2

FIG. 3

The representation of a ρ-cell A^ρ in the n-space Ξ^n is evidently a point set of dimensionality $n-\rho$, since the determining relations of the representation consist of ρ independent equations and $n-\rho$ inequalities. The representation of the unit cell E_e in the space Ξ^n is the entire space Ξ^n, since the representation of E_e has no determining relations at all.

Given two *different* cells $A = z_1 z_2 \cdots z_m$ and $B = w_1 w_2 \cdots w_m$ of the same type α, we shall say that the cell A is *on the boundary* of the cell B if, for every i, the factor w_i of B is equal either to the corresponding factor z_i of A or to the edge b_i associated with the factor z_i,

$$A \operatorname{bd} B \leftrightarrow A \neq B, \quad z_i = w_i \text{ or } b_i, \qquad (i = 1, 2, \cdots, m). \tag{2.6}$$

Clearly, the cell A will be on the boundary of the cell B if, and only if, the representation of the cell A in the space Ξ^m is made up entirely of points on the geometrical boundary of the representation of the cell B in Ξ^m.

3. **Chains.** Now, let Γ be a grating, let $[A_\alpha]$ be the set of all cells of Γ of some fixed type α, and let $[\lambda]$ be an arbitrary ring of coefficients without divisors of zero. (In most problems, the ring $[\lambda]$ will either be the ring of all the integers or the ring of all the real numbers.) By a *chain* K of the grating Γ we shall mean any mapping of the set $[A_\alpha]$ into the ring $[\lambda]$. The chain K will be said to be of *type* α, where α is the type of the cells A_α. It will be said to be *regular* or *singular* according as its type is regular or singular. We shall write $K = K_\alpha$ when we want to indicate explicitly that the chain K is of type K_α.

In accordance with one of the standard notational procedures, we shall represent a chain K_α by a linear form in the cells of type α,

$$K_\alpha = \textstyle\sum_i \lambda_i A_{\alpha i}. \tag{3.1}$$

For every i, the coefficient of the ith term of the form will be the image λ_i of the cell $A_{\alpha i}$ in the mapping K_α. As suggested by the notation, the *sum* and *difference* of a chain $K_\alpha = \sum_i \lambda_i A_{\alpha i}$ and a chain $L_\alpha = \sum_i \mu_i A_{\alpha i}$ will be the chains $K_\alpha + L_\alpha = \sum_i (\lambda_i + \mu_i) A_{\alpha i}$ and $K_\alpha - L_\alpha = \sum_i (\lambda_i - \mu_i) A_{\alpha i}$ respectively. Moreover, a *linear combination* of a finite set of chains $K_{\alpha j} = \sum_i \lambda_{ij} A_{\alpha i}$, $j = 1, 2, \cdots, n$, will be any chain of the form $\sum_j \mu_j K_{\alpha j} = \sum_{ij} \mu_j \lambda_{ij} A_{\alpha i}$, where the coefficients μ_j and λ_{ij} belong to the ring $[\lambda]$. Of course, sums, differences, and linear combinations will not be defined unless all the chains involved are of the same type α.

The cells *of* a chain K_α will be the cells $A_{\alpha i}$ such that their coefficients λ_i in the linear form (3.1) are different from 0. To each type α there will correspond a chain 0_α with no cells at all, such that the coefficients of all the cells $A_{\alpha i}$ are 0. The chain 0_α will be called the *zero* of type α. It will, of course, be a different entity from the zero 0_β of type β, $\beta \neq \alpha$.

To simplify the exposition, we shall assume in the sequel that the ring $[\lambda]$ has a unit element. (The assumption can easily be dispensed with by introducing symbolic chains, in the manner outlined in Remark 3, below.) If there is a unit element $\lambda=1$, every cell A_α of type α will determine a chain $1\cdot A_\alpha$ of type α, such that the coefficient of the cell $A_{\alpha i}=A_\alpha$ in (3.1) is 1 and that the coefficients of all the other cells $A_{\alpha i}$ are 0. We shall adopt the standard abbreviation $A_\alpha=1\cdot A_\alpha$, and shall say that the chain A_α is the ***cell-chain*** determined by the cell A_α. Here again, our notation will make no distinction between the cell A_α and the cell-chain A_α. The cell-chain z determined by an elementary cell z will be called an ***elementary chain***; the cell-chain E_e determined by the unit cell E_e will be called the ***unit chain***. According to (3.1), every chain K_α will be a uniquely determined linear combination of cell-chains $A_{\alpha i}$.

Given two chains $K_\alpha=\sum_i\lambda_iA_{\alpha i}$ and $L_\beta=\sum_j\mu_jB_{\beta j}$ of arbitrary types α and β respectively, we shall say that their *product* is the chain of type $\alpha\beta$ determined by the linear form

$$K_\alpha L_\beta = \sum_{ij}\lambda_i\mu_j\,(A_{\alpha i}B_{\beta j}). \tag{3.2}$$

(The symbol $A_{\alpha i}B_{\beta j}$ in the right-hand member of (3.2) represents the cell product of the cells $A_{\alpha i}$ and $B_{\beta j}$.) Chain multiplication will be associative, and two-way distributive with respect to chain addition. However, it will be noncommutative, since cell multiplication is noncommutative. Clearly, the unit chain E_e will be the unit of chain multiplication,

$$E_eK_\alpha = K_\alpha = K_\alpha E_e. \tag{3.3}$$

Moreover, we shall have

$$0_\alpha K_\beta = 0_{\alpha\beta} = L_\alpha 0_\beta. \tag{3.4}$$

A chain K_α will be assigned a definite ***degree*** m, equal to the degree of its type α. In general, the chain will not be assigned a rank ρ. However, we shall say that a chain K_α is a ***ρ-chain***, or that K_α is of ***rank*** ρ, if no cell of K_α is of any other rank than ρ. According to this definition, a zero 0_α will be of all ranks ρ simultaneously, $\rho=0, 1, 2, \cdots$, while a cell-chain A_α will be of the same rank as its determining cell A_α. By collecting terms of like rank in (3.1), we shall be able to express a chain K_α in a uniquely determined manner as a sum of the form

$$K_\alpha = K_\alpha^0 + K_\alpha^1 + \cdots + K_\alpha^m \qquad (m = m(K_\alpha)), \tag{3.5}$$

where K_α^ρ, $\rho=0, 1, 2, \cdots, m$, is a ρ-chain. We shall extend the meaning of the symbol K^ρ_α to higher determinations of the rank ρ by writing $K^\rho_\alpha = 0_\alpha$, for $\rho > m$. The chain K^ρ_α will be called the ρth *constituent* of the chain K_α. We shall treat the superscript ρ as the symbol for an operator—the *rank operator* ρ—transforming a chain K_α into the ρth constituent K^ρ_α of K_α. The operator ρ will, of course, be linear,

$$\left(\sum_i \lambda_i K_{\alpha i}\right)^\rho = \sum_i \lambda_i K^\rho_{\alpha i}, \tag{3.6}$$

and we shall have

$$\begin{aligned} (K^\rho_\alpha)^\rho &= K^\rho_\alpha, \\ (K^\rho_\alpha)^\sigma &= 0_\alpha \qquad (\sigma \neq \rho). \end{aligned} \tag{3.7}$$

Obviously, the product $K^\rho L^\sigma$ of a ρ-chain and a σ-chain will be a $(\rho+\sigma)$-chain.

To avoid confusion, we shall never use the symbol X^ρ, as applied to a cell or chain, to denote the ρ-fold product $XX \cdots X$. The symbol will merely indicate that the cell or chain to which it refers is of rank ρ. One further matter of notation: Instead of expressing a chain K_α as a linear form (3.1) in the cells A_α we can express it as a form of degree m in the elementary cells s by treating each cell A_α as a product of m elementary factors; cf. (2.5). The order of the factors in the terms of the form must, of course, be taken into consideration.

Remark 1. We shall use the standard notation $\gamma \in \Gamma$ to indicate that a cut γ is a member of grating Γ. Moreover, by extending the meaning of the symbol $\in$, we shall write $s \in \Gamma$, $A \in \Gamma$, $K \in \Gamma$ to indicate that s is an element, A a cell, and K a chain of Γ.

Remark 2. As a heuristic device, we can construct a schematic model of a chain K_α by representing the cells $A_{\alpha i}$ of type α in a space Ξ^n, after the manner of §2, and by "weighting" or "labelling" the representation of each cell $A_{\alpha i}$ with the coefficient of $A_{\alpha i}$ in the expression (3.1).

Remark 3. The discussion can obviously be extended to the case in which the ring $[\lambda]$ has no unit elements by immersing the ring $[\lambda]$ in an appropriate ring $[\mu]$ with a unit element. For instance, to state matters rather loosely, we can let $[\mu]$ be the set of all linear expressions $\mu = \lambda + n$ such that λ is a member of the ring $[\lambda]$ and n an integer. The set $[\mu]$ becomes a ring when we combine the expressions μ by addition and multiplication in the manner suggested by the notation $\mu = \lambda + n$. Each element λ of $[\lambda]$ can be identified with the element $\mu = \lambda + 0$ of $[\mu]$. The element $\mu = 0 + 1$ is the unit element of $[\mu]$. A chain K_α is written in the form $K_\alpha = \sum_i \mu_i A_{\alpha i}$, $\mu_i \in [\mu]$. However, it is treated as purely *symbolic* unless all its coefficients μ_i are identified with members of $[\lambda]$, and so on.

4. **The conjugation and boundary operators.** As a formal device for taking care of signs, we shall now define a type-preserving operator, called the *conjugation operator*, acting on the chains of a grating Γ. The conjugation operator will transform the chain K_α represented in (3.5) into the chain

$$K_\alpha^* = \overset{0}{K_\alpha} - \overset{1}{K_\alpha} + \cdots + (-1)^m \overset{m}{K_\alpha}, \tag{4.1}$$

formed by merely changing the sign of all constituents of odd ranks. The chain K_α^* will be called the *conjugate* of the chain K_α. The conjugation operator will evidently be linear,

$$\left(\sum_i \lambda_i K_{\alpha i}\right)^* = \sum_i \lambda_i K_{\alpha i}^*, \tag{4.2}$$

and we shall have

$$K_\alpha^{**} = K_\alpha, \qquad (K_\alpha L_\beta)^* = K_\alpha^* L_\beta^*, \qquad (K_\alpha')^* = (K_\alpha^*)'. \tag{4.3}$$

Next, we shall define a more significant type-preserving operator, called the *boundary operator*. The boundary operator will transform a chain K_α into a chain K_α', called the *boundary* of K_α. We shall require that the boundary operator be linear,

$$\left(\sum_i \lambda_i K_{\alpha i}\right)' = \sum_i \lambda_i K_{\alpha i}'. \tag{4.4}$$

It will therefore be sufficient for us to define the boundaries of the cell-chains. The boundary of a cell-chain A_α will be defined by induction on the degree $m = m(A_\alpha)$ of A_α. As the hypothesis of the induction—which will begin with the value $m = 2$—we shall assume that we know the boundaries of all chains of degree $m-1$.

Case $m = 0$. The only cell-chain of degree 0 is the unit chain E_ϵ. We shall write

$$E_\epsilon' = 0_\epsilon. \tag{4.5}$$

Case $m = 1$. The cell-chains of degree 1 are the elementary chains, which are of the forms $z = a$, b, or c. We shall write

$$a' = b, \qquad b' = 0_\alpha, \qquad c' = -b. \tag{4.6}$$

The zero 0_α will, of course, be the zero of the same type α as the chains a, b, and c.

Case $m>1$. A cell-chain A of degree m, $m>1$, can be written in a uniquely determined manner as the product Bz of a cell-chain B of degree $m-1$ and an elementary chain z. We shall define the boundary of A by the recursion formula

$$A' = B'z + B^*z'. \tag{4.7}$$

The boundary of a general chain K of degree m will then be fully determined, by linearity.

We can easily verify, by induction, that the boundary of a chain is of the same type α as the chain itself, and that the boundary of a ρ-chain is a $(\rho+1)$-chain. Therefore, by (3.5), we can write

$$(K'_\alpha)^0 = 0_\alpha, \qquad (K'_\alpha)^\rho = (K^{\rho-1}_\alpha)' \qquad (\rho > 0). \tag{4.8}$$

It is also clear, by (4.1) and (4.8), that the boundary operator anticommutes with the conjugation operator,

$$(K^*)' = -(K')^*. \tag{4.9}$$

The formula for the boundary of a chain product KL is

$$(KL)' = K'L + K^*L'. \tag{4.10}$$

Proof. Since the boundary and conjugation operators are both linear, and since chain multiplication is distributive with respect to chain addition, the proof reduces, at once, to the case in which the factors K and L are both cell-chains. We shall therefore assume that K and L are cell-chains, and shall proceed by induction on the degree m of the second factor L.

The case $m(L)=0$ is trivial. The cell-chain L can only be the unit chain $L=E_\epsilon$. Therefore, we have $L'=0_\epsilon$, and both members of (4.10) reduce to K'.

The case $m(L)=1$ is covered by the assumed recursion formula (4.7).

We treat the case $m(L)>1$ by expressing the cell-chain L as a product $L=AB$, where A and B are cell-chains of degrees less than $m(L)$. By the hypothesis of the induction, we have at once

$$\begin{aligned}(KL)' = (KA\cdot B)' &= (KA)'B + (KA)^*B' \\ &= K'AB + K^*A'B + K^*A^*B' \\ &= K'AB + K^*(AB)' \\ &= K'L + K^*L',\end{aligned}$$

which is what we set out to prove.

The explicit formula for the boundary of a product $K_1K_2 \cdots K_n$ of n factors K_i is

$$(4.11)\qquad (K_1K_2 \cdots K_n)' = \sum_i K_1^*K_2^* \cdots K_{i-1}^*K_i' K_{i+1} \cdots K_n,$$

as may be proved directly by induction on n. As a special case of (4.11) we have the formula for the boundary of a cell-chain $A = z_1z_2 \cdots z_m$,

$$(4.12)\qquad A' = (z_1z_2 \cdots z_m)' = \sum_i z_1^*z_2^* \cdots z_{i-1}^*z_i'z_{i+1} \cdots z_m.$$

This last formula has a simple geometrical interpretation. The ith term on the right vanishes if we have $z_i = b_i$ (whence, $z_i = 0$) and is of one of the two forms $\pm z_1z_2 \cdots z_{i-1}b_iz_{i+1} \cdots z_m$ if we have $z_i = a_i$ or c_i. Therefore, the cells *of* the boundary of a ρ-cell A are precisely the $(\rho+1)$-cells B *on* the boundary of A, in the sense of §2.

The boundary of the boundary of a chain K is the zero of the same type α as K,

$$(4.13)\qquad K'' = 0 \qquad \text{(that is, } K_\alpha'' = 0_\alpha\text{)}.$$

PROOF. Again, the proof reduces, by linearity, to the case in which the chain K is a cell-chain. We therefore assume that K is a cell-chain, and proceed by induction on the degree m of K.

The case $m=0$ and $m=1$ are both trivial, since we have $E_\epsilon'' = 0_\epsilon$, by (4.5), and $a'' = b'' = c'' = 0_\epsilon$, by (4.6).

We treat the case $m>0$ by expressing the cell-chain K as a product $K=AB$, where A and B are cell-chains of degrees less than m. According to (4.10), we have

$$K' = (AB)' = A'B + A^*B', \qquad K'' = A''B + A^{*\prime}B' + A^{*\prime}B' + A^{**}B'.$$

The two outer terms on the right are zeros, by the hypothesis of the induction; the two inner terms cancel, by (4.9).

5. Refinors, refinements, pseudo-sums. The sum of all the cell-chains of type α and rank 0 will be called the *refinor* E_α of type α. The refinor of unit type ϵ will thus be the unit chain E_ϵ, since the only cell of unit type is the cell E_ϵ. The refinor of type $\alpha = a_1a_2 \cdots a_m$, $m>0$, will be the product

$$(5.1)\qquad E_\alpha = (a_1 + c_1)(a_2 + c_2) \cdots (a_m + c_m).$$

Indeed, if we expand the product on the right we clearly obtain the sum of all cell-chains of the form $A_\alpha = z_1z_2 \cdots z_m$, such that each factor z_i has one of the two determinations a_i or c_i. According to (5.1), the product of two refinors is a refinor,

$$E_\alpha E_\beta = E_{\alpha\beta}. \tag{5.2}$$

Moreover, we have

$$\begin{aligned} E_\alpha^0 &= E_\alpha; & E_\alpha^\rho &= 0_\alpha & (\rho > 0);\\ E_\alpha^* &= E_\alpha; & E'_\alpha &= 0_\alpha. \end{aligned} \tag{5.3}$$

Formulae (5.3) are obvious, with the possible exception of the last one. To prove that the boundary of E_α vanishes, we have only to note that the boundary of each factor (a_i+c_i) vanishes, $(a_i+c_i)' = b_i - b_i = 0_{a_i}$, and to apply (4.11) for the boundary of a product.

A *refinement* of a chain K will be any product of the form KE, where E is a refinor. In view of the identity $K = KE_1$, a chain K is to be regarded as a refinement of itself. The geometrical meaning of a refinement is as follows. Suppose we construct schematic models of the chains K_α and $K_\alpha E_\beta$ in the same n-space Ξ^n, $n \geqq m(\alpha)+m(\beta)$, after the manner of §3. Then each ρ-cell $A_\alpha^\rho B_\beta^0$ of the refinement $K_\alpha E_\beta$ will be represented by a portion of the geometrical domain representing the corresponding ρ-cell A_α^ρ of K_α. For heuristic purposes, we can think of the chain $K_\alpha E_\beta$ as the one obtained by "subdividing each ρ-cell A_α^ρ of K_α into the subcells $A_\alpha^\rho B_\beta^0$." Thus, in particular, an *elementary refinement* $K(a+c)$ of a chain K will be obtained by "cutting each cell A of K along the edge b, thereby separating the cell A into a pair of subcells Aa and Ac which touch along the edge b."

We can think of a refinor E as an operator transforming a chain K into the refinement KE of K. The operator is linear,

$$\left[\sum_i \lambda_i K_i\right] E = \sum_i \lambda_i (K_i E), \tag{5.4}$$

since chain multiplication is distributive with respect to chain addition. Moreover, by (5.3), the operator E commutes with the rank, conjugation, and boundary operators,

$$(KE)^\rho = K^\rho E, \qquad (KE)^* = K^* E, \qquad (KE)' = K'E. \tag{5.5}$$

Of course, we can also write

$$(EK)^\rho = EK^\rho, \qquad (EK)^* = EK^*, \qquad (EK)' = EK'. \tag{5.6}$$

The sum of two chains is not defined unless the chains are of the same type. To compensate for this defect it will be convenient (though not strictly essential) to introduce a new law of combination

$$K_\alpha \oplus L_\beta = K_\alpha E_\beta + E_\alpha L_\beta, \tag{5.7}$$

which will be applicable to any pair of chains K_α and L_β. The chain $K_\alpha \oplus L_\beta$ will be called the *pseudo-sum* of K_α and L_β. It will be of the same type $\alpha\beta$ as the product $K_\alpha L_\beta$. Similarly, we shall write

$$(5.8)\qquad K_\alpha \ominus L_\beta = K_\alpha \oplus (-L_\beta) = K_\alpha E_\beta - E_\alpha L_\beta,$$

and shall call the chain $K_\alpha \ominus L_\beta$ the *pseudo-difference* of K_α and L_β. A chain of the form

$$(5.9)\qquad \begin{aligned} &\lambda_1 K_1 \oplus \lambda_2 K_2 \oplus \cdots \oplus \lambda_n K_n \\ &\qquad = \sum_i \lambda_i E_1 E_2 \cdots E_{i-1} K_i E_{i+1} \cdots E_n \end{aligned}$$

will be called a *pseudo-combination* of the chains K_i.

Pseudo-addition will be *associative*,

$$(5.10)\qquad \begin{aligned} &K_\alpha \oplus (L_\beta \oplus M_\gamma) = (K_\alpha \oplus L_\beta) \oplus M_\gamma \\ &\qquad (= K_\alpha E_\beta E_\gamma + E_\alpha L_\beta E_\gamma + E_\alpha E_\beta M_\gamma), \end{aligned}$$

but *noncommutative*. In general, the pseudo-difference $K_\alpha \ominus K_\alpha$ will not be zero. According to (5.4), (5.5), and (5.6), the rank, conjugation, and boundary operators will be "pseudo-linear" with respect to pseudo-addition:

$$(5.11)\qquad \begin{aligned} &(K \oplus L)^\rho = K^\rho \oplus L^\rho, \qquad (K \oplus L)^* = K^* \oplus L^*, \\ &\qquad (K \oplus L)' = K' \oplus L'. \end{aligned}$$

The following identities are worth keeping in mind:

$$(5.12)\qquad K_\alpha E_\beta = K_\alpha \oplus 0_\beta, \qquad E_\alpha K_\beta = 0_\alpha \oplus K_\beta;$$

also the identity

$$(5.13)\qquad \begin{aligned} &(K_\alpha + L_\alpha) \oplus (M_\beta + N_\beta) = (K_\alpha \oplus M_\beta) + (L_\alpha \oplus N_\beta) \\ &\qquad [= (K_\alpha + L_\alpha) E_\beta + E_\alpha (M_\beta + N_\beta)]. \end{aligned}$$

This last is analogous to a distributive law.

6. Permutators and permutations. Let

$$(6.1)\qquad \pi = (i_1, i_2, \cdots, i_n)$$

be the permutation on the integers $1, 2, \cdots, n$ which transforms the integer s into the integer i_s $(s=1, 2, \cdots, n)$. Corresponding to the permutation π we shall define a linear operator, or *permutor*—also to be denoted by π—which will transform every chain K of degree m, $m \geqq n$, into a chain πK, called the *permutation* πK of K. Since the operator π is to be linear,

$$\pi\Big(\sum_i \lambda_i K_{\alpha i}\Big) = \sum_i \lambda_i(\pi K_{\alpha i}) \qquad (m(\alpha) \geqq n), \tag{6.2}$$

it will be sufficient for us to define the permutations $\pi(A)$ of the cell-chains A of degrees m greater than or equal to n. If A is the cell-chain $A = z_1z_2 \cdots z_m$, $m \geqq n$, we shall write

$$\pi A = (-1)^\sigma(z_{i_1}z_{i_2} \cdots z_{i_n})(z_{n+1}z_{n+2} \cdots z_m), \tag{6.3}$$

where the integer σ which determines the sign of the expression is to be calculated in the following manner. We examine the product $z_1z_2 \cdots z_m$ and construct all pairs (z_i, z_j) consisting of two different factors $z_i = b_i$ and $z_j = b_j$ of rank 1. (Factors $z_i = a_i$ or c_i of rank 0 are simply ignored.) The integer σ is equal to the number of pairs (z_i, z_j) such that the relative order of z_i and z_j in the product $z_1z_2 \cdots z_m$ is different from the relative order of z_i and z_j in the product $z_{i_1}z_{i_2} \cdots z_{i_n}z_{n+1} \cdots z_m$. A few examples will make our meaning clear:

(i) Let $A = a_1b_2b_3$ be a cell-chain, and let $\pi = (2, 1, 3)$ and $\kappa = (3, 2, 1)$ be permutations on the integers 1, 2, 3. We write $A = b_2a_1b_3$, with the *plus* sign, since we have not disturbed the relative order of the factors b_2 and b_3, but we write $\kappa A = -b_3b_2a_1$, with the *minus* sign, since we have reversed the order of the factors b_2 and b_3.

(ii) Let $A = bb$ be a singular cell of singular type aa, and let $\pi = (2, 1)$ be the permutation which inverts the integers 1 and 2. We write $\pi A = -bb$, with the *minus* sign, since we have reversed the order of the two factors b. (N.B. A term of a sequence, or a factor of a non-commutative product, is not fully determined unless we know both its *value* and its *position*. When we speak, rather loosely, about identical terms or factors, we are referring, of course, to terms or factors with identical values.)

(iii) Let C be the product of a ρ-cell $A^\rho = z_1z_2 \cdots z_m$ and a σ-cell $B^\sigma = z_{m+1}z_{m+2} \cdots z_n$, and let κ be the cyclic permutation $(m+1, m+2, \cdots, n, 1, 2, \cdots, m)$. Then we must obviously write

$$\kappa(A^\rho B^\sigma) = (-1)^{\sigma\rho}B^\sigma A^\rho, \tag{6.4}$$

since each of the ρ factors $z_i = b_i$ of A^ρ has undergone an inversion relative to each of the σ factors $z_j = b_j$ of B^σ. By linearity, a similar formula applies to the cyclic permutation κ of the product $K^\rho_\alpha L^\sigma_\beta$ of a ρ-chain K^ρ_α and a σ-chain L^σ_β,

$$\kappa(K^\rho_\alpha L^\sigma_\beta) = (-1)^{\rho\sigma}L^\sigma_\beta K^\rho_\alpha \qquad (\kappa\cdot\alpha\beta = \beta\alpha). \tag{6.5}$$

No such simple formula exists for the permutation $\pi(KL)$ of two general chains K and L. However, if either of the chains K or L is a

0-chain we can, of course, write

$$\kappa(KL) = LK \qquad (K = K^0 \text{ or } L = L^0), \tag{6.6}$$

without any change of sign. In consequence of (6.6), it is clear that the pesudo-sum $L_\beta \oplus K_\alpha = L_\beta E_\alpha + E_\beta K_\alpha$ is a cyclic permutation of the pseudo-sum $K_\alpha \oplus L_\beta = K_\alpha E_\beta + E_\alpha L_\beta$,

$$L_\beta \oplus K_\alpha = \kappa(K_\alpha \oplus L_\beta) \qquad (\kappa \cdot \alpha\beta = \beta\alpha). \tag{6.7}$$

A permutor (6.1) obviously transforms a chain K_α of type $\alpha = a_1 a_2 \cdots a_m$, $m > n$, into a chain $K_{\pi\alpha}$ of type

$$\beta = \pi(\alpha) = (a_{i_1} a_{i_2} \cdots a_{i_n})(a_{n+1} a_{n+2} \cdots a_m). \tag{6.8}$$

We shall call the type β the *permutation* $\pi(\alpha)$ of the type α.

Remark. It would be natural to define the permutation of a general cell $A = z_1 z_2 \cdots z_m$ by the formula $\pi(A) = (z_{i_1} z_{i_2} \cdots z_{i_n})(z_{n+1} z_{n+2} \cdots z_m)$. However, we prefer to define *only* the permutations of the type-cells α. It would be confusing to have one formula for a permutation of the cell A and a different formula, involving at times a different sign, for a permutation of the corresponding cell-chain A. Formula (6.8) leads to no confusion, whether we interpret the product $a_1 a_2 \cdots a_m$ as the symbol for a type-cell or as the symbol for a cell-chain.

We can, of course, write the identity

$$(\pi K_\alpha) L_\beta = \pi(K_\alpha L_\beta), \tag{6.9}$$

provided the degree of K_α is such that the chain πK_α is defined. Similarly, we can write

$$K_\alpha(\pi L_\beta) = \omega(K_\alpha L_\beta) \qquad (\omega \cdot \alpha\beta = \alpha \cdot \pi\beta), \tag{6.10}$$

provided the permutation πL_β is defined. The permutor π obviously commutes with the rank and conjugation operators, so that we are justified in using the simplified notations

$$\pi K^\rho = \pi(K^\rho) = (\pi K)^\rho, \quad \pi K^* = \pi(K^*) = (\pi K)^*. \tag{6.11}$$

The permutor π also commutes with the boundary operator,

$$\pi K' = \pi(K') = (\pi K)', \tag{6.12}$$

though, in this case, a formal proof is necessary because of minor difficulties in regard to signs. Since we are dealing with linear operators, the proof of (6.12) reduces to the case in which K is a cell-chain, $K = z_1 z_2 \cdots z_m$. Moreover, every permutation on the integers $1, 2, \cdots, m$ is the resultant of inversions which interchange consecutive integers i and $i+1$. Therefore, the proof reduces, in the last analysis, to the case in which K is a cell-chain of the form $K=AzwB$, and in which π is the permutor which interchanges the adjacent elementary factors z and w. Now, in the case left for consideration, we have

$$\pi K = (-1)^{\rho\sigma} AwzB,$$

where ρ and σ are the ranks of the elementary factors z and w respectively ($\rho=0$ or 1, $\sigma=0$ or 1). Therefore we can write

$$(\pi K)' = (-1)^{\rho\sigma}[A'wzB + A^*w'zB + A^*w^*z'B + A^*w^*z^*B'].$$

On the other hand, we can also write

$$K' = A'zwB + A^*z'wB + A^*z^*w'B + A^*z^*w^*B',$$

whence

$$\pi(K') = (-1)^{\sigma\rho}[A'wzB + (-1)^{\sigma}A^*wz'B + (-1)^{\rho}A^*w'z^*B + A^*w^*z^*B'),$$

since z' and w' are of ranks $\rho+1$ and $\sigma+1$ respectively. To conclude the proof, we have only to note that the expression for $\pi(K')$ is equivalent to the expression for $(\pi K)'$, since we have $z^*=(-1)^{\rho}z$ and $w^*=(-1)^{\sigma}w$.

7. Grating representations; loci. A *representation* (Γ, f, X) of a grating $\Gamma=[\gamma]$ on a point set $X=[x]$ will consist of the grating Γ, the set X, and an arbitrary function $f(\gamma, x)$ of a variable cut of Γ and a variable point of X, such that the only allowable values of the function are -1, 0, and 1. The set X will be called the *carrier*, and the function $f(\gamma, x)$ the *determining function* of the representation. If we have a representation (Γ, f, X) of Γ on X, every cut $\gamma_i=(a_i, b_i, c_i)$ of the grating Γ will determine a separation of the carrier X into a trio of disjoint sets A_i, B_i, and C_i, consisting of the points x at which the determining function $f(\gamma_i, x)$ takes on the values -1, 0, and 1 respectively. We shall call the sets A_i, B_i, and C_i the *representatives* of the elements a_i, b_i, and c_i respectively.

Example A. Let X be a topological space, and let $[\gamma]$ be the set of all real, continuous functions $\gamma(x)$ of a variable point x of X. We treat each member γ_i of $[\gamma]$ as the symbol for a cut consisting of three abstract elements a_i, b_i, and c_i. The cuts γ_i form a grating $\Gamma=[\gamma]$. Moreover, we obtain a representation (Γ, f, X) of the grating Γ on the carrier X by letting the determining function $f(\gamma, x)$ be the function which takes on the value -1, 0, or 1 according as the function $\gamma(x)$ is negative, zero, or positive at x. The representatives of the elements a_i, b_i, and c_i of γ_i are, of course, the point sets determined by the relations $\gamma_i<0$, $\gamma_i=0$, and $\gamma_i>0$ respectively. Other significant representations will be given in §13.

Given a representation (Γ, f, X) of a grating Γ on a set X, there is a natural way of assigning to each chain K of Γ an appropriate subset $|K|$ of X. The subset $|K|$ will be called the *locus* of the chain K. It will be defined in the following manner:

To the unit chain E_e, we shall assign the entire set X,

$$|E_e| = X. \tag{7.1}$$

To the elementary chains $z_i=a_i$, b_i, and c_i, we shall assign the sets

$$|a_i| = A_i \cup B_i, \qquad |b_i| = B_i, \qquad |c_i| = C_i \cup B_i \tag{7.2}$$

respectively, where A_i, B_i, and C_i are the representatives of the elements a_i, b_i, and c_i respectively. Thus, in Example A above, the loci $|a_i|$, $|b_i|$, and $|c_i|$ will be the sets determined by the conditions $\gamma_i\leqq 0$, $\gamma_i=0$, and $\gamma_i\geqq 0$ respectively. According to (7.2), we shall have

$$|a_i| \cup |c_i| = X, \qquad |a_i| \cap |c_i| = |b_i|. \tag{7.3}$$

The locus of a cell-chain $A=z_1z_2\cdots z_m$ of degree m greater than 1 will be defined to be the intersection of the loci of the elementary factors of A,

$$|A| = |z_1| \cap |z_2| \cap \cdots \cap |z_m|. \tag{7.4}$$

The formula for the product of two cell-chains A and B will thus be

$$|AB| = |A| \cap |B|. \tag{7.5}$$

Moreover, we shall have

$$A \text{ bd } B \rightarrow |A| \subseteq |B|. \tag{7.6}$$

Indeed, if the cell $A=z_1z_2\cdots z_m$ is on the boundary of the cell $B=w_1w_2\cdots w_m$, we can write $z_i=w_i$ or b_i, by (2.6); whence $|z_i| \subseteq |w_i|$, by (7.3); whence, finally, $|A|=\cap_i|z_i|\subseteq\cap_i|w_i|=|B|$, by (7.4). The locus of a multiple λA of a cell-chain A will be said to be identical with the locus of A, provided the coefficient λ does not vanish. If the coefficient vanishes, the locus will be said to be the empty set,

$$|\lambda A| = |A| \quad (\lambda \neq 0); \qquad |0\cdot A| = 0. \tag{7.7}$$

The following identities will be useful in the sequel:

$$|ac| = |ab| = |bc| = |bb| = |b|. \tag{7.8}$$

They are immediate consequences of (7.2) and (7.4).

To complete the definition of the locus, we shall say that the locus of a general chain $K=\sum_i\lambda_i A_i$ is the union of the loci of the terms of K,

$$(7.9)\qquad |K| = |\lambda_1 A_1| \cup |\lambda_2 A_2| \cup \cdots \cup |\lambda_n A_n|.$$

We can evidently write the identities

$$(7.10)\qquad |K| = |K^*| = |\pi K|,$$

since corresponding terms of K, K^*, and πK can differ, at most, in sign and in the arrangement of the elementary factors z_i within the terms. We can also write

$$(7.11)\qquad |\lambda K + \mu L| \subseteq |K| \cup |L|,$$

since every nonvanishing term of $\lambda K+\mu L$ corresponds to a similar nonvanishing term of one, at least, of the chains K and L. Again, we can write

$$(7.12)\qquad |KL| = |K| \cap |L|,$$

since the nonvanishing terms of KL are the products of the nonvanishing terms of K and the nonvanishing terms of L. The locus of a zero is, of course, empty.

Remark. In proving the equality (7.12), we make use, for the first time, of the assumption that the ring of coefficients $[\lambda]$ has no zero divisors. If the ring were allowed to have zero divisors, the product of a nonvanishing term of K and a nonvanishing term of L might be a vanishing term of KL, so that the equality (7.12) would have to be replaced by the inequality $|KL| \subseteq |K| \cap |L|$.

According to (7.3) and (7.9), the locus of an elementary refinor a_i+c_i is $|a_i+c_i| = X$. Therefore, by (7.12), we can write

$$(7.13)\qquad |E| = X, \qquad |KE| = |EK| = |K|,$$

where E is an arbitrary refinor and K an arbitrary chain. As a corollary to (7.11) and (7.13), we have

$$(7.14)\qquad |\lambda K \oplus \mu L| \subseteq |K| \cup |L|.$$

Finally, by (7.6) and (7.9),

$$(7.15)\qquad |K'| \subseteq |K|,$$

since every cell of K' is on the boundary of some cell of K.

In the applications to topology, the carrier of a grating Γ will ordinarily be a topological space. A representation of a grating Γ on a topological space X will be said to be ***continuous*** provided the representatives A_i and C_i of all the faces of Γ are open sets. If the representation is continuous, the representatives B_i of the edges must be closed sets, since the sets B_i are the complements of the open sets $A_i \cup C_i$. The loci of the chains K of Γ are also closed sets, as may be verified immediately by considering successively the loci (7.1), (7.2), (7.4), and (7.9). The representation (Γ, f, X) in Example A above is evidently continuous.

Remark. A representation (Γ, f, X) of a grating Γ on a point set X induces a topology on the set X. When we speak of the ***natural topology*** induced by (Γ, f, X) on X we shall mean the coarsest topology such that the representatives of all the faces of Γ are open sets—that is, the coarsest topology such that the representation of Γ on X is continuous. With reference to the natural topology induced on X, we can construct a basic set of neighborhoods consisting of all sets of the form $Z_{i_1} \cap Z_{i_2} \cap \cdots \cap Z_{i_n}$ (n finite, but not fixed), such that the Z_i's are the representatives of faces of Γ, $Z_i = A_i$ or C_i. In other words, we can construct a sub-basic set of neighborhoods consisting of the representatives Z_i of all the faces of Γ.

8. **The intrinsic representation of a grating.** An ***intrinsic point*** y of a grating Γ will be any single-valued function $y = y(\gamma)$ of the cuts of Γ, such that the only allowable values of the function are -1, 0, and 1. The set $Y = [y]$ of all intrinsic points of Γ will be called the ***intrinsic carrier*** of the grating. The ***intrinsic representation*** of the grating will be the representation (Γ, h, Y) of Γ on Y determined by a function $h(\gamma, y)$, such that the value of the function, for every fixed determination of γ and y, is equal to the value of the function $y(\gamma)$ for the given determination of γ, $h(\gamma, y) = y(\gamma)$. The representative of an element z of Γ in relation to the intrinsic representation (Γ, h, Y) will be called the ***intrinsic representative*** of z, the locus of a chain K will be called the ***intrinsic locus*** of K. Clearly, the intrinsic representatives of the elements $z_i = a_i$, b_i, and c_i will be the sets made up of all functions (intrinsic points) $y(\gamma)$ such that we have $y(\gamma_i) = -1$, 0, and 1 respectively. The intrinsic locus of a chain K will be denoted by a special symbol $\|K\|$.

The intrinsic representation of a grating is of some theoretical interest, in that it enables us to describe the properties of the grating in pseudo-combinatory language, without reference to limits and continuity. However, it can ordinarily be dispensed with in problems involving a prescribed representation of the grating on a point set X. When we want to talk about the intrinsic representation in topological terms, we shall assign the natural topology to the intrinsic carrier Y (cf. §7). Thus, the intrinsic representation will be continuous.

The chains K of a grating Γ will be partially ordered in terms of their intrinsic loci, as follows. If the intrinsic locus $\|K\|$ of a chain K is contained in the intrinsic locus $\|L\|$ of a chain L, $\|K\| \subseteq \|L\|$, we shall say that K is ***dominated*** by L and that L ***dominates*** K. The relations of dominating and of being dominated are evidently transitive, since the relation of inclusion is transitive. However, each of two different chains may dominate the other (cf., for example, the chains ac, ab, and so on, in (7.8)), so that the relations are not anti-symmetrical.

Remark The question of whether or not a prescribed chain K is dominated by a prescribed chain L can always be answered effectively, by a finite process. Consider, first, the case in which the chains are cell-chains, $K = A$, $L = B$. The locus $\|A\|$ consists of all intrinsic points $y(\gamma)$, such that the functions $y(\gamma)$ satisfy a finite set of relations, each of one of the forms $y(\gamma_i) \leqq 0$, $y(\gamma_i) = 0$, or $y(\gamma_i) \geqq 0$. A similar remark applies to the locus $\|B\|$. Therefore, by comparing the two sets of relations, for $\|A\|$ and for $\|B\|$, we can determine, by inspection, whether or not there exists an intrinsic point belonging to the locus $\|A\|$, but not to the locus $\|B\|$. Consider, next, the general case. The chain K has only a finite number of cells A, the chain L a finite number of cells B. Therefore, since we know how to handle the special case, we can always determine effectively whether or not there exists an intrinsic point belonging to some $\|A\|$, but not to any $\|B\|$. In other words, we can determine effectively whether or not we have $\|K\| \subseteq \|L\|$.

[8.1] THEOREM. *A representation* (Γ, f, X) *of a grating* $\Gamma = [\gamma]$ *on a space* X *induces a mapping* $y = \mu(x)$ $[= f(\gamma, x)]$ *of the space* X *into the intrinsic carrier* Y *of* Γ. *Moreover, the mapping of* X *into* Y *is continuous if the representation* (Γ, f, X) *is continuous.*

PROOF. The first part of the theorem is obvious if the notation is clearly understood. For a fixed determination $\bar{x}$ of x, the determining function $f(\gamma, x)$ of the representation (Γ, f, X) becomes a function $\bar{y} = f(\gamma, \bar{x})$ of γ alone, with values limited to -1, 0, and 1. Thus, the function $\bar{y}$ is an intrinsic point of Γ, by the very definition of the intrinsic points. By the transformation $y = \mu(x)$, we mean the one which carries the point $\bar{x}$ into the point $\bar{y}$. To verify that the transformation is continuous when (Γ, f, X) is continuous, we let z_i be an element of Γ, Z_X the representation of z_i in X, and Z_Y the representation of z_i in Y. The set Z_Y consists of all intrinsic points $y = y(\gamma)$ such that we have $y(\gamma_i) = k$ ($k = -1$, 0, or 1 according as we have $z_i = a_i$, b_i, or c_i). The set Z_X consists of all points x of X such that we have $f(\gamma_i, x) = \bar{k}$.

In other words, the set Z_X consists of all points of X which are carried into points of Z_Y by the transformation $y=\mu(x)$ $[=f(\gamma, x)]$. The continuity of the transformation is now obvious. Indeed, the intrinsic representations Z_Y of the *faces* z_i of Γ constitute a sub-basic set of neighborhoods of Y (cf. §7); the inverse images $Z_X=\mu^{-1}(Z_Y)$ of the Z_Y's are open, since the representation (Γ, f, X) is continuous; therefore, by a well known criterion for continuity, the transformation $y=\mu(x)$ is continuous.

[8.2] COROLLARY. *Let* (Γ, f, X) *be a representation of a grating* Γ *on a point set* X, *and let* K *and* L *be chains of* Γ. *Then if the chain* K *is dominated by the chain* L *the locus of* K *is contained in the locus of* L,

$$\|K\| \subseteq \|L\| \rightarrow |K| \subseteq |L|. \tag{8.1}$$

PROOF. In the notation of the theorem, we have $|E_\epsilon| = \mu^{-1}\|E_\epsilon\|$, by (7.1), $|z| = \mu^{-1}\|z\|$, by (7.2), $|A| = \mu^{-1}\|A\|$, by (7.4), and finally $|K| = \mu^{-1}\|K\|$, by (7.9). Thus, every point x of $|K|$ is mapped into a point $y=\mu(x)$ of $\|K\|$. Now suppose we have $\|K\|\subseteq\|L\|$. Then the image y of x must also be a point of $\|L\|$. Therefore, the point x must be a point of $|L|$. In other words, we must have $|K| \subseteq |L|$.

9. **Grating ideals. Cycles and bounding cycles.** An *ideal* Φ of a grating Γ will be any set of chains of Γ with the following three properties:

(i) The zero 0_ϵ of unit type ϵ is a member of the set.

(ii) The sum $K_\alpha+L_\alpha$ of two members of the set of like type α is a member of the set.

(iii) Every chain K dominated by a member of the set is a member of the set.

[9.1] THEOREM. *Let* Φ *be an ideal of a grating* Γ. *Then*:

(α) *All the zeros* 0_α *of* Γ *are members of* Φ.

(β) *If* K *is a member of* Φ, *so also are* λK, K^ρ, K^*, πK, KE, *and* K'. *Moreover, the cell-chains* A *determined by the cells of* K *are members of* Φ.

(γ) *If* K *is a member of* Φ *and* L *an arbitrary chain of* Γ, *the chains* KL *and* LK *are members of* Φ.

(δ) *Every linear combination of members of* Φ *is a member of* Φ; *every pseudo-combination of members of* Φ *is a member of* Φ.

The proof is trivial. Statement (α) follows from (i) and (iii), since $\|0_\alpha\| = \|0_\epsilon\|$ is the empty set. Statement (β) follows from (iii), by (7.9), (7.10), (7.13), and (7.15). Statement (γ) follows from (iii), by (7.12). Finally, Statement (δ) follows from (ii) and (β).

Given a grating Γ, there is a smallest ideal of Γ consisting of the chains 0_α only, also a largest ideal consisting of all the chains of Γ. We shall denote the smallest ideal by the symbol Ω, and the largest ideal by the same symbol Γ as the grating itself. Obviously, the intersection $\bigcap_i \Phi_i$ of the members Φ_i of an arbitrary family of ideals of Γ is an ideal of Γ. Thus, if $[K]$ is any set of chains of Γ, there is a smallest ideal Φ of Γ such that we have $[K] \subseteq \Phi$. The ideal Φ is, of course, the intersection of all the ideals Φ_i of Γ such that we have $[K] \subseteq \Phi_i$. The *join* of a family of ideals Φ will be the smallest ideal Φ such that we have $\Phi_i \subseteq \Phi$, for all members Φ of the family.

A chain K of Γ will be said to be a *cycle, modulo* an ideal Φ, if the boundary K' of K is a member of Φ,

$$K' = Z \qquad (Z \in \Phi). \tag{9.1}$$

If a chain K belonging to an ideal Ψ is a cycle, modulo an ideal Φ, we shall say that K is a *cycle* of Ψ, *modulo* Φ. A chain K will be called an *absolute cycle* if its boundary is a zero,

$$K' = 0. \tag{9.2}$$

The theory of absolute cycles does not require special treatment, since an absolute cycle is merely a cycle, modulo the ideal Ω consisting of the zeros only.

[9.2] THEOREM. *Let Φ be an ideal of a grating Γ. Then:*

(α) *All the zeros 0_α and all the refinors E_α of Γ are cycles, modulo Φ.*

(β) *If K is a cycle, modulo Φ, so also are λK, $K^{\bullet}$, K^*, πK, KE, and K'.*

(γ) *If K and L are cycles, modulo Φ, so also is their product KL.*

(δ) *Every linear combination of cycles, modulo Φ, is a cycle, modulo Φ.*

(ϵ) *Every pseudo-combination of cycles, modulo Φ, is a cycle, modulo Φ.*

The theorem follows, at once, from the theorems on loci in §7. We leave the routine verification to the reader.

Now, let Φ and Ψ be ideals of a grating Γ. A chain K of Γ will be said to *bound* in Ψ, *modulo* Φ, if it can be written in the form

$$K = W' + Z \qquad (W \in \Psi, Z \in \Phi). \tag{9.3}$$

By applying the boundary operator to the two members of (9.3), we obtain an expression of the form (9.1), since the chain W'' is always a zero. Therefore, a chain of Ψ which bounds in Ψ, modulo Φ, is always a cycle of Ψ, modulo Φ.

[9.3] THEOREM. *Let Φ and Ψ be ideals of a grating Γ. Then:*

(α) *All the zeros 0_α of Γ bound in Ψ, modulo Φ.*

(β) *If* K *is a cycle which bounds in* Ψ, *modulo* Φ, *so also are* λK, K^ρ, K^*, πK, KE, *and* K'.

(γ) *If* K *is a cycle which bounds in* Ψ, *modulo* Φ, *and if* L *is an arbitrary cycle of* Γ, *modulo* Φ, *the products* KL *and* LK *are cycles bounding in* Ψ, *modulo* Φ.

(δ) *Every linear combination of cycles bounding in* Ψ, *modulo* Φ, *is a cycle bounding in* Ψ, *modulo* Φ.

(ϵ) *Every pseudo-combination of cycles bounding in* Ψ, *modulo* Φ, *is a cycle bounding in* Ψ, *modulo* Φ.

Again, the verification of the theorem is immediate. We shall merely give the proof of Statement (γ) which involves a minor difficulty.

According to (9.3) and (9.1), the bounding cycle K can be written in the form $K=W'+Z$, and the boundary of the cycle L in the form $L'=\overline{Z}$ ($Z, \overline{Z}\in\Phi$, $W\in\Psi$). We can therefore write

$$KL = W'L + ZL = (WL)' - W^*\overline{Z} + ZL.$$

Now, by [9.1], the chain WL is necessarily a chain of Ψ, since W is a chain of Ψ. Moreover, the chain $-W^*\overline{Z}+ZL$ is a chain of Φ, since $\overline{Z}$ and Z are chains of Φ. Therefore, the chain KL is of the form (9.3). In other words, the chain KL bounds in Ψ, modulo Φ. The proof that the chain LK bounds in Ψ, modulo Φ, is treated in a similar manner.

We shall introduce the notation

$$K \approx 0, \qquad (\Psi/\Phi), \tag{9.4}$$

to indicate that K is a chain which bounds in Ψ, modulo Φ. If a pseudo-combination bounds in Ψ, modulo Φ,

$$\lambda_1K_1 \oplus \lambda_2K_2 \oplus \cdots \oplus \lambda_nK_n \approx 0, \qquad (\Psi/\Phi), \tag{9.5}$$

we can rearrange the terms of the left-hand member of (9.5) in any order we please, since the effect of rearranging the terms is merely to transform the pseudo-combination $\lambda_1K_1\oplus\lambda_2K_2\oplus\cdots\oplus\lambda_nK_n$ into a permutation of itself.

10. **A permutation theorem.** According to §6, the permutors π are linear operators which commute with the rank, conjugation, and boundary operators. It would therefore be feasible to identify a chain K with its permutations.[3] We prefer, however, to let matters stand as they are, and to distinguish between the chains K and πK unless,

---

[3] A similar identification is made in the classical theory of complexes of simplexes, where an oriented simplex is treated as a skew-symmetrical function of its $n+1$ vertices.

of course, they happen to be formally identical. It is of some theoretical interest that, in view of the following theorem and its corollary, we can develop a satisfactory homology theory without having to make the identifications.

[10.1] THEOREM. *Let K be a chain of a grating Γ, and let πK be any permutation of K, such that the chains K and πK are of the same type α. Then the difference between the chains K and πK can always be expressed in the form*

$$K - \pi K = W' + Z \qquad (\|W\| \subseteq \|K\|, \|Z\| \subseteq \|K'\|), \tag{10.1}$$

where W is a chain dominated by K and Z a chain dominated by the boundary of K.

PROOF. The theorem is trivial unless the chain K is singular. Indeed, if K is of regular type α, the only permutation of K of type α is K itself. Therefore, the difference $K-\pi K$ can be put in the form (10.1) by writing $W=Z=0_\alpha$.

If the type α of K is singular, the theorem reduces, without difficulty, to the special case in which the type of K is of the form $\alpha=aa\beta$, and in which the permutor π is the *invertor* (2, 1) which merely interchanges the first two elementary factors of $aa\beta$. Let us therefore first dispose of this special case. By grouping together the terms of K with the same first two elementary factors, we can express K in the form

$$\begin{aligned} K = {} & aaK_{11} + abK_{12} + acK_{13} \\ & + baK_{21} + bbK_{22} + bcK_{23} \\ & + caK_{31} + cbK_{32} + ccK_{33}, \end{aligned} \tag{10.2}$$

where the K_{ij}'s are chains of type β. Moreover, in the same notation, we can write

$$\begin{aligned} \pi K = {} & aaK_{11} + abK_{21} + acK_{31} \\ & + baK_{12} - bbK_{22} + bcK_{32} \\ & + caK_{13} + cbK_{23} + ccK_{33}. \end{aligned} \tag{10.3}$$

(Note the sign of the term $-bbK_{22}$.) The chain $K-\pi K$ is therefore of the form

$$K - \pi K = (ab - ba)P + (ac - ca)Q + (bc - cb)R + 2bbS, \tag{10.4}$$

where P, Q, R, and S are appropriate linear combinations of the K_{ij}'s ($P=K_{12}-K_{21}$, and so on). We shall prove that Condition (10.1) is fulfilled when W and Z are chosen in the following manner:

$$(10.5)\quad W = (ab - ba)S + (ac - ca)R, \qquad Z = K - \pi K - W',$$

where the symbols S and R have the same meaning as in (10.4). From the very form of the chain Z, it is clear that W and Z satisfy the *equation* (10.1). What we still have to prove is that W is dominated by K and that Z is dominated by the boundary of K.

Now, it is evident, by inspection, that no two of the chains abP, $-baP$, acQ, $-caQ$, and so on, in the right-hand member of (10.4) can have a cell in common, since the cells of abP all have the left-hand factor ab, the cells of $-baP$ the left-hand factor ba, and so on. We can therefore write the *equality*

$$\|K - \pi K\| = \|abP\| \cup \|-baP\| \cup \|acQ\| \cup \cdots \cup \|2bbS\|.$$

By a similar argument, applied to (10.5), we can also write the equality

$$\|W\| = \|abS\| \cup \|-baS\| \cup \|acR\| \cup \|-caR\|.$$

Furthermore, in view of (7.8) and (7.4), the terms abS and $-baS$ of W are dominated by the term $2bbS$ of $K-\pi K$, and the terms acR and $-caR$ of W are dominated by the term bcR of $K-\pi K$,

$$\|abS\| = \|-baS\| = \|2bbS\| = \|b\| \cap \|S\|,$$
$$\|acR\| = \|-caR\| = \|bcR\| = \|b\| \cap \|R\|.$$

We can therefore write

$$\|W\| \subseteq \|K - \pi K\| \subseteq \| K \| \cup \|\pi K\| \subseteq \|K\|,$$

which proves that the chain W is dominated by the chain K. (*Remark*. The proof makes use of the fact that the ring of coefficients $[\lambda]$ has no zero divisors, since we have to assume $\|2bbS\| = \|bbS\|$.)

To complete the proof for the special case under consideration, we have only to show that the chain Z is dominated by the chain K'. According to (10.5), we obtain, by direct calculation,

$$(10.6)\quad \begin{aligned} W' = &- (ab - ba)(R + S') + (ac - ca)R' \\ &+ (bc - cb)R + 2bbS. \end{aligned}$$

Therefore, by (10.4) and (10.6), the chain Z is of the form

$$\begin{aligned} Z = K - \pi K - W' &= (ab - ba)M + (ac - ca)N \\ (10.7)\qquad\qquad &(M = P + S' + R,\ N = Q - R'). \end{aligned}$$

Next, we construct the boundary Z' of Z. Since the expression (10.7) for Z is similar in form to the expression (10.5) for W, we can write

$$
\begin{aligned}
Z' = &-(ab - ba)(N + M') + (ac - ca)N' \\
&+ (bc - cb)N + 2bbM,
\end{aligned}
\tag{10.8}
$$

without formal computation, by comparison with the expression (10.6) for W'. Finally, by an argument similar to the one made a moment ago, we note that the terms abM and $-baM$ of Z are dominated by the term $2bbM$ of Z', and that the terms acN and $-caN$ of Z are dominated by the term bcN of Z'. Therefore, we can write

$$\|Z\| \subseteq \|Z'\| = \|K' - \pi K'\| \subseteq \|K'\| \cup \|\pi K'\| \subseteq \|K'\|.$$

This completes the proof for the special case under consideration.

The rest of the argument is trivial. We next consider the somewhat more general case in which the type α of K is of the form $\alpha = \beta a \gamma a \delta$, with two identical factors a, and in which the permutor π is the invertor which merely interchanges the two factors a. Let κ be a permutor which transforms the type α into the type $\kappa\alpha = aa\beta\gamma\delta$, and let ω be the invertor (2, 1). Then, according to the case already considered, we can write

$$\kappa K - \omega(\kappa K) = W' + Z \qquad (\|W\| \subseteq \|\kappa K\|, \|Z\| \subseteq \|\kappa K'\|).$$

Thus, we have, at once,

$$K - \pi K = \kappa^{-1}[\kappa K - \omega(\kappa K)] = \kappa^{-1}[W' - Z],$$

$$(\|\kappa^{-1}W\| = \|W\| \subseteq \|\kappa K\| = \|K\|, \|\kappa^{-1}Z\| = \|Z\| \subseteq \|\kappa K'\| = \|K'\|),$$

as required.

The proof of the theorem for the completely general case follows directly from the fact that every permutor π which leaves the type α invariant is the resultant of invertors of the kind just considered.

[10.2] COROLLARY. *Let Φ and Ψ be ideals of a grating Γ and let K be any cycle of Ψ, modulo Φ. Then if πK is any permutation of K the pseudo-difference $K \ominus \pi K$ bounds in Ψ, modulo Φ:*

$$K \ominus \pi K \approx 0, \qquad (\Psi/\Phi). \tag{10.9}$$

PROOF. The left-hand member of (10.9) can be written $K \cdot \pi E - E \cdot \pi K$, where E is the refinor of the same type type as K. Moreover, the chain $E \cdot \pi K$ is a permutation of the chain $K \cdot \pi E$. Therefore, by the theorem, we can write

$$
\begin{aligned}
&K \cdot \pi E - E \cdot \pi K = W' + Z, \\
&(\|W\| \subseteq \|K \cdot \pi E\| = \|K\|, \|Z\| \subseteq \|(K \cdot \pi E)'\| = \|K'\|).
\end{aligned}
\tag{10.10}
$$

Now, the chain W belongs to the ideal Ψ, since it is dominated by the member K of Ψ, and the chain Z belongs to the ideal Φ, since it is dominated by the boundary K' of the cycle K of Ψ, modulo Φ, which boundary K' is, of course, a chain of Φ. Thus, the desired relation (10.9) is implied by (10.10).

11. Homologies, homology groups, and homology rings. The notion of homology is closely related to the notion of bounding. Let Φ and Ψ be ideals of a grating Γ, and let K_α and L_β be chains of Γ. We shall say that the chain K_α is *homologous* to the chain L_β in Ψ, *modulo* Φ,

$$(11.1)\qquad K_\alpha \sim L_\beta, \qquad (\Psi/\Phi),$$

if there exists a refinement of the pseudo-difference $K_\alpha \ominus L_\beta$ such that the refinement bounds in Ψ, modulo Φ,

$$(11.2)\qquad (K_\alpha E_\beta - E_\alpha L_\beta)E_\gamma = K_\alpha \ominus L_\beta \oplus 0_\gamma \approx 0, \qquad (\Psi/\Phi).$$

The relation of homology is clearly *symmetrical*,

$$(11.3)\qquad K_\alpha \sim L_\beta \rightarrow L_\beta \sim K_\alpha, \qquad (\Psi/\Phi),$$

since the relation (11.2) goes over into the relation $L_\beta \ominus K_\alpha \oplus 0_\gamma \approx 0$, by a permutation and a change of sign. In order that K_α be homologous to L_β it is *sufficient*, though not necessary, that the pseudo-difference $K_\alpha \ominus L_\beta$ bound, since we can then write

$$(11.4)\qquad K_\alpha \ominus L_\beta = K_\alpha \ominus L_\beta \oplus 0_\epsilon \approx 0, \qquad (\Psi/\Phi).$$

Another useful condition, which is both necessary and sufficient, is that there be a chain Y_γ of Γ satisfying the joint relations

$$(11.5)\qquad K_\alpha \ominus L_\beta \oplus Y_\gamma \approx 0, \qquad Y_\gamma \approx 0, \qquad (\Psi/\Phi).$$

Indeed, the fulfillment of (11.2) implies the fulfillment of (11.5) with Y_γ set equal to 0_γ. On the other hand, the fulfillment of the first part of (11.5) implies that we have

$$(11.6)\qquad (K_\alpha \ominus L_\beta \oplus 0_\gamma) + (0_\alpha \oplus 0_\beta \oplus Y_\gamma) \approx 0, \qquad (\Psi/\Phi),$$

by (5.13). Moreover, the fulfillment of the second part of (11.5) implies the relation $0_\alpha \oplus 0_\beta \oplus Y_\gamma \approx 0$, since $0_\alpha \oplus 0_\beta \oplus Y_\gamma$ is a permutation of a refinement of Y_γ. By combining this relation with (11.6), we obtain (11.2).

In the remainder of this section, we shall confine our attention to cycles $K, L, M, \cdots$ of Ψ, modulo Φ, as a result of which the theory will assume a particularly simple form. In the first place, it is clear that a cycle K of Ψ, modulo Φ, is homologous to all its permutations,

$$K_\alpha \sim \pi K_\alpha, \qquad (\Psi/\Phi), \tag{11.7}$$

by (10.9) and (11.4). Thus, in particular, the homology relation, as applied to cycles of Ψ, modulo Φ, is *reflexive*,

$$K_\alpha \sim K_\alpha, \qquad (\Psi/\Phi). \tag{11.8}$$

We can further verify that the homology relation, as applied to cycles, is *transitive*,

$$K_\alpha \sim L_\beta,\ L_\beta \sim M_\gamma \rightarrow K_\alpha \sim M_\gamma, \qquad (\Psi/\Phi). \tag{11.9}$$

Indeed, the relations $K_\alpha \ominus L_\beta \oplus 0_\delta \approx 0$ and $L_\beta \ominus M_\gamma \oplus 0_\epsilon \approx 0$ together imply the relation $K_\alpha \ominus M_\gamma \oplus (L_\beta \ominus L_\beta \oplus 0_\delta \oplus 0_\epsilon) \approx 0$, by pseudo-addition and permutation. Moreover, the refinement $L_\beta \ominus L_\beta \oplus 0_\delta \oplus 0_\epsilon$ of $L_\beta \ominus L_\beta$ is a bounding cycle, by (10.9). Therefore, we have $K_\alpha \sim M_\gamma$, by (11.5).

In view of (11.8), (11.3), and (11.9), the cycles K of Ψ, modulo Φ, fall into mutually exclusive *homology classes* $[K]$, such that two cycles belong to the same class if, and only if, they are homologous in Ψ, modulo Φ,

$$[K] = [L] \leftrightarrow K \sim L, \qquad (\Psi/\Phi). \tag{11.10}$$

The permutations πK of a cycle K all belong to the same class $[K]$, by (11.7). So also do the refinements $KE = K \oplus 0$ of K, since we have $K \ominus (K \oplus 0) \approx 0$, by (10.9). The class $[0_\epsilon]$, which we shall call the *zero class*, will consist of all cycles K_α such that some refinement $K_\alpha \oplus 0_\gamma$ of K_α bounds in Ψ, modulo Φ. To verify this, we only have to write $L_\beta = 0_\epsilon$ in (11.2). In particular, the class $[0_\epsilon]$ will include all the zeros 0_α of Γ and all the chains of $\Phi \cap \Psi$.

The *constituents*, *conjugate*, and *boundary* of a class $[K]$ will be defined by the formulae

$$[K]^\rho = [K^\rho], \quad [K]^* = [K^*], \quad [K]' = [K'] = [0_\epsilon]. \tag{11.11}$$

To justify the definitions, we must be sure that the classes $[K^\rho]$, $[K^*]$, and $[K']$ are independent of the choice of the cycle K within its class $[K]$,

$$K \sim \overline{K} \rightarrow K^\rho \sim \overline{K}^\rho, \qquad K^* \sim \overline{K}^*, \qquad K' \sim \overline{K}', \qquad (\Psi/\Phi). \tag{11.12}$$

The verification is immediate, since the relation $K \ominus \overline{K} \oplus 0 \approx 0$ implies the relations $(K \ominus \overline{K} \oplus 0)^\rho = K^\rho \ominus \overline{K}^\rho \oplus 0 \approx 0$, and so on. The class $[K']$ is, of course, the zero class, since the boundary K' of a cycle of Ψ, modulo Φ, is a chain of Φ.

The *sum* of two homology classes will be defined by the formula

$$[K] + [L] = [K \oplus L]. \tag{11.13}$$

Here again, we must be sure that the class $[K \oplus L]$ is independent of the choice of the chains K and L within their respective classes,

$$(11.14) \qquad K \sim \bar{K}, \quad L \sim \bar{L} \rightarrow K \oplus L \sim \bar{K} \oplus \bar{L}, \qquad (\Psi/\Phi).$$

The proof consists merely in noting that the relations $K \ominus \bar{K} \oplus 0_\mu \approx 0$ and $L \ominus \bar{L} \oplus 0_\nu \approx 0$ imply the relation $(K \oplus L) \ominus (\bar{K} \oplus \bar{L}) \oplus 0_{\mu\nu} \approx 0$, by pseudo-summation and permutation. The sum (11.13) is *associative*, since pseudo-summation is associative; it is commutative, since the chains $K \oplus L$ and $L \oplus K$ differ, at most, by a permutation; the class $[0_\epsilon]$ is the *zero* with respect to class addition,

$$(11.15) \qquad [K] + [0_\epsilon] = [K \oplus 0_\epsilon] = [K];$$

the class $[-K]$ is the *negative* of the class $[K]$,

$$(11.16) \qquad [K] + [-K] = [K \ominus K] = [0_\epsilon],$$

by (10.9). In other words, *the homology classes form an additive group*.

The following two theorems will enable us to operate more freely with homologies. Let K_α and L_α be cycles of Ψ, modulo Φ, both of the same type α.

Then we have

$$(11.17) \qquad L_\alpha \approx 0 \rightarrow K_\alpha + L_\alpha \sim K_\alpha, \qquad (\Psi/\Phi)$$

and

$$(11.18) \qquad K_\alpha \oplus L_\alpha \sim K_\alpha + L_\alpha, \qquad (\Psi/\Phi).$$

The proof of (11.17) is as follows. According to (11.4), the relation $(K_\alpha + L_\alpha) \ominus K_\alpha \approx 0$ is a sufficient condition that the homology $K_\alpha + L_\alpha \sim K_\alpha$ be true. With the aid of (5.13), we write the relation in the alternative form $(K_\alpha \ominus K_\alpha) + (L_\alpha \oplus 0_\alpha) \approx 0$. The first term on the left bounds, by (10.9); the second term bounds, since $L_\alpha \oplus 0_\alpha$ is a refinement of the bounding chain L_α. Thus, the relation is verified, and (11.17) is established. The second theorem, (11.17), follows immediately from the identity

$$K_\alpha \oplus L_\alpha = [(K_\alpha + L_\alpha) \oplus 0_\alpha] - [L_\alpha \ominus L_\alpha]$$

which is also obtained from (5.13). The chain $[L_\alpha \ominus L_\alpha]$ bounds, by (10.9). Therefore, we have

$$K_\alpha \oplus L_\alpha \sim (K_\alpha + L_\alpha) \oplus 0_\alpha \sim K_\alpha + L_\alpha, \qquad (\Psi/\Phi),$$

by (11.17). As a corollary to (11.18), we can write

$$(11.19) \qquad [K_\alpha] + [L_\alpha] = [K_\alpha + L_\alpha].$$

The *product* of two homology classes will be defined by the formula

$$(11.20) \qquad [K][L] = [KL].$$

To justify the definition, we must, of course, prove the following:

$$(11.21) \qquad K \sim \bar{K}, L \sim \bar{L} \rightarrow KL \sim \bar{K}\bar{L}, \qquad (\Psi/\Phi),$$

where all the chains involved are cycles of Ψ, modulo Φ. We begin with a lemma. Let K_α and L_β be cycles of Ψ, modulo Φ, and let M_γ be an arbitrary cycle of Γ, modulo Φ. Then we have

$$(11.22) \qquad K_\alpha \sim L_\beta \rightarrow \begin{cases} K_\alpha M_\gamma \sim L_\beta M_\gamma, \\ M_\gamma K_\alpha \sim M_\gamma L_\beta, \end{cases} \qquad (\Psi/\Phi).$$

Proof. The homology $K_\alpha \sim L_\beta$ implies a relation of the form (11.2), which can be written in the alternative form

$$K_\alpha E_\beta E_\pi - E_\alpha L_\beta E_\pi \approx 0, \qquad (\Psi/\Phi),$$

by using the product notation for a refinement. Therefore, since M_γ is a cycle, modulo Φ, we can write

$$(K_\alpha E_\beta E_\pi - E_\alpha L_\beta E_\pi) M_\gamma \approx 0, \qquad (\Psi/\Phi),$$

from which we can infer

$$K_\alpha E_\beta E_\pi M_\gamma \sim E_\alpha L_\beta E_\pi M_\gamma, \qquad (\Psi/\Phi).$$

Thus, by refinement and permutation, we have

$$\begin{aligned} K_\alpha M_\gamma \sim K_\alpha M_\gamma E_\beta E_\pi \sim K_\alpha E_\beta E_\pi M_\gamma \sim E_\alpha L_\beta E_\pi M_\gamma \\ \sim L_\beta M_\gamma E_\alpha E_\pi \sim L_\beta M_\gamma, \end{aligned} \qquad (\Psi/\Phi).$$

The second half of the lemma is proved in a similar manner. The desired theorem (11.21) is an immediate consequence of the lemma, since we can now write $KL \sim K\bar{L} \sim \bar{K}\bar{L}$.

Class multiplication is *associative*, since chain multiplication is associative. We shall prove that it is *two-way distributive* with respect to chain addition:

$$(11.23) \quad K(L \oplus M) \sim KL \oplus KM, \quad (L \oplus M)K \sim LK \oplus MK, \quad (\Psi/\Phi).$$

Proof. We have the identity

$$K_\alpha(L_\beta \oplus M_\gamma) = K_\alpha L_\beta E_\gamma + K_\alpha E_\beta M_\gamma.$$

Now, by (11.18), the expression on the right is homologous in Ψ, modulo Φ, to

$$K_\alpha L_\beta E_\gamma \oplus K_\alpha E_\beta M_\gamma,$$

which last is homologous, by (11.14), to

$$K_\alpha L_\beta \oplus K_\alpha M_\gamma,$$

since we have $K_\alpha L_\beta \sim K_\alpha L_\beta E_\gamma$ and $K_\alpha M_\gamma \sim K_\alpha M_\gamma E_\beta \sim K_\alpha E_\beta M_\gamma$.

In view of the properties of class addition and class multiplication, it is clear that the homology classes form an ordinary (noncommutative) ring. We shall call the ring the ***homology ring*** $\mathcal{H}(\Psi/\Phi)$ of the ideal Ψ, modulo the ideal Φ.

The ρth ***homology group*** $\mathcal{H}^\rho(\Psi/\Phi)$ of Ψ, modulo Φ, will be the additive group formed by the ρth constituents $[K]^\rho$ of the homology classes $[K]$. In view of (3.5) and (11.19), we shall have

$$[K] = [K]^0 + [K]^1 + \cdots + [K]^m, \tag{11.24}$$

and, by (3.7),

$$\begin{aligned} [K]^{\rho\rho} &= [K]^\rho, \\ [K]^{\rho\sigma} &= [0_\alpha] \qquad (\rho \neq \sigma). \end{aligned} \tag{11.25}$$

In other words, the additive group formed by the members of the homology ring $\mathcal{H}(\Psi/\Phi)$, will be the (finite) direct sum of the homology groups $\mathcal{H}^\rho(\Psi/\Phi)$, $\rho = 0, 1, 2, \cdots$. In general, the structure of the homology ring $\mathcal{H}(\Psi/\Phi)$ will not be fully determined by the structure of the homology groups $\mathcal{H}^\rho(\Psi/\Phi)$. In other words, the ring $\mathcal{H}(\Psi/\Phi)$ will be a more powerful invariant than the combined groups $\mathcal{H}^\rho(\Psi/\Phi)$. Two homology rings $\mathcal{H}$ and $\mathcal{G}$ will be said to be ***completely isomorphic*** if, and only if, the members $[K]$ of $\mathcal{H}$ are paired in an isomorphism with the members $[L]$ of $\mathcal{G}$ in such a manner that the ρth constituent $[K]^\rho$ of $[K]$ is always paired with the ρth constituent $[L]^\rho$ of the corresponding $[L]$, $\rho = 0, 1, 2, \cdots$. The homology rings $\mathcal{H}$ and $\mathcal{G}$ will be regarded as equivalent if, and only if, they are *completely* isomorphic. In other words, a homology ring $\mathcal{H}$ will be treated as a ring with the operators ρ.

12. **The homology groups and ring of an ideal.** Let Γ be a grating, and let Ω be the ideal made up of the zeros 0_α of Γ. By the ***homology ring*** $\mathcal{H}(\Phi)$ of an ideal Φ of Γ, we shall mean the homology ring of Φ, modulo the ideal Ω,

$$\mathcal{H}(\Phi) = \mathcal{H}(\Phi/\Omega). \tag{12.1}$$

Similarly, by the ***homology groups*** $\mathcal{H}^\rho(\Phi)$ of Φ, we shall mean the homology groups

$$\mathcal{H}^\rho(\Phi) = \mathcal{H}^\rho(\Phi/\Omega). \tag{12.2}$$

The structure of the homology ring of an ideal Φ is simpler than that of a general homology ring $\mathcal{H}(\Psi/\Phi)$, as will be seen by the following analysis.

[12.1] THEOREM. *Every absolute* 0-*cycle* K *of a grating* Γ *is a multiple of a refinor,*

$$K' = 0 \rightarrow \text{some } K = \lambda E \qquad (\rho(K) = 0). \tag{12.3}$$

Every absolute ρ-cycle K of Γ, $\rho > 0$, is a boundary,

$$K' = 0 \rightarrow \text{some } K = R' \qquad (\rho(K) > 0). \tag{12.4}$$

PROOF. We prove the theorem by induction on the degree m of the cycle K. The case $m=0$ is trivial, since the only chains of degree 0 are the 0-cycles λE_ϵ, and since the only ρ-cycle of degree 0, $\rho > 0$, is the chain 0, which is its own boundary. The case $m > 0$ is treated by expressing the type α of K in the form $\alpha = a\beta$, where the degree of the type β is $m-1$. The ρ-cycle K can be written in the form

$$K = aL + bM + cN, \tag{12.5}$$

where L and N are ρ-chains of β, and where M is a $(\rho-1)$-chain of β. Therefore, the boundary of K is of the form

$$K' = aL' + b(L - M' - N) + cN' = 0, \tag{12.6}$$

from which we infer

$$L' = 0, \qquad L - M' - N = 0, \qquad N' = 0. \tag{12.7}$$

Let us first consider the case in which the rank of the cycle K is 0. In this case, the term bM in (12.5) is missing, and (12.7) reduces to

$$L' = 0, \qquad L - N = 0, \qquad N' = 0.$$

We conclude that L and N are identical absolute 0-cycles of rank $m-1$. Thus, by the hypothesis of the induction, we can assume

$$L = N = \lambda E_\beta,$$

whence,

$$K = aL + cN = \lambda(a + c)E_\beta = \lambda E_\alpha,$$

as required.

Finally, we consider the case in which the rank of K is greater than 0. According to (12.7), the chain L must be an absolute ρ-cycle of degree $m-1$. Therefore, by the hypothesis of the induction, we can express L as a boundary, $L = P'$. Moreover, again by (12.7), we can write $N=L-M'=P'-M'$. By eliminating L and N from (12.5), we obtain

$$K = aP' + bM + c(P' - M') = [(a+c)P - cM]',$$

which is of the desired form $K=R'$.

[12.2] Corollary. *The homology ring of the maximal ideal* Γ *of a grating* Γ *is made up of homology classes of rank* 0 *and is isomorphic with the ring of coefficients* $[\lambda]$.

Proof. We have

$$K \sim K^0 + K^1 + \cdots + K^m \sim \lambda E_\epsilon \qquad (K^\rho \sim 0,\ \rho > 0),$$

so that each homology class is of the form $[\lambda E_\epsilon]$. Thus, to complete the proof, we have only to show that the classes $[\lambda E_\epsilon]$ and $[\mu E_\epsilon]$ are distinct whenever λ and μ are distinct. Now, if the classes were not distinct, we would have $[(\lambda-\mu)E_\epsilon]=[0_\epsilon]$, $\lambda \neq \mu$, which would imply that some refinement $(\lambda-\mu)E_\alpha$ of $(\lambda-\mu)E_\epsilon$ was a boundary. But the only chains of rank 0 which can bound are the zeros, since there are no chains of negative rank.

[12.3] Corollary. *Let* Φ *be any proper ideal of a grating* Γ, $\Phi \neq \Gamma$. *Then the product* KL *of two absolute cycles* K *and* L *of* Φ *always bounds in* Φ.

Proof. The chain K is an absolute cycle of Φ; therefore, *a fortiori*, it is an absolute cycle of Γ. Suppose we resolve the cycle K into its constituents,

$$K = K^0 + K^1 + \cdots + K^m.$$

Then, according to the theorem, the 0th constituent K^0 must be a multiple of a refinor, $K^0=\lambda E$, while the other constituents K^ρ must be boundaries. Moreover, the coefficient λ of the 0th constituent $K^0=\lambda E$ must vanish, otherwise λE would be a chain of Φ; whence all chains of Γ would be chains of Φ, since they would all be dominated by λE; whence Φ would not be a proper ideal of Γ, as is being assumed. We conclude that the chain K is a boundary, $K=R'$. We further conclude that the product KL is of the form $KL=R'L=(RL)'-R^*L'=(RL)'$, since the boundary L' of the absolute cycle L is a zero. To complete the argument, we have only to observe that the chain RL belongs to the ideal Φ, since L belongs to Φ. Since the product KL bounds the chain RL of Φ, the theorem is proved.

The homology ring of an ideal Φ is the analogue of the classical intersection ring of a region of the space Ξ^n of n real variables.

13. **Invariants of topological spaces.** We shall now prove two obvious lemmas which will have significant topological applications.

[13.1] LEMMA. *Let* (Γ, f, X) *be a representation of a grating* Γ *on a point set* X. *Then every subset* X_i *of* X *determines an ideal* Φ_i *of* Γ *consisting of all chains* K *of* Γ *such that their loci are made up of points of* X_i, $|K| \subseteq X_i$.

PROOF. The set Φ_i is an ideal of Γ if it fulfills Conditions (i), (ii), and (iii) at the beginning of §9. We verify, by inspection, that the set Φ_i fulfills Conditions (i) and (ii). It also fulfills Condition (iii), since $\|K\| \subseteq \|L\|$ implies $|K| \subset |L|$, by (8.1); whence $|L| \in X_i$ and $\|K\| \subseteq \|L\|$ together imply $|K| \subset |L| \in X_i$.

According to the lemma, there is an invariant ideal Φ_0 of the representation (Γ, f, X) consisting of all chains of Γ with empty loci. Thus, the homology rings $\mathcal{H}(\Gamma/\Phi_0)$ and $\mathcal{H}(\Phi_0)$, and the corresponding homology groups $\mathcal{H}^p(\Gamma/\Phi_0)$ and $\mathcal{H}^p(\Phi_0)$, are invariants of the representation.

[13.2] LEMMA. *Let* (Γ, f, X) *be a continuous representation of a grating* Γ *on a topological space* X. *Then there exists an ideal* Ψ_c *of* Γ *consisting of all chains* K *of* Γ *such that their loci are compact.*[1]

PROOF. The locus of the zero 0_c is the empty set, which is compact; therefore, Condition (i) is satisfied. The union of two closed, compact sets is closed, compact; therefore, Condition (ii) is satisfied. Every closed subset of a closed, compact set is closed, compact; therefore, Condition (iii) is satisfied, since $\|K\| \subseteq \|L\|$ implies $|K| \subset |L|$.

According to the lemmas, there are two significant ideals of a continuous representation (Γ, f, X): the ideal Φ_0 consisting of the chains of Γ with empty loci, and the ideal Ψ_c consisting of the chains with compact loci. We can therefore obtain the following invariant rings of the representation: $\mathcal{H}(\Gamma/\Psi_c)$, $\mathcal{H}(\Gamma/\Phi_0)$, $\mathcal{H}(\Psi_c/\Phi_0)$, $\mathcal{H}(\Psi_c)$, $\mathcal{H}(\Phi_0)$. Among these invariants, the most significant one is the ring $\mathcal{H}(\Psi_c/\Phi_0)$. We shall call this ring the ***homology ring*** of the representation. Moreover, we shall call the associated groups $\mathcal{H}^p(\Psi_c/\Phi_0)$ the ***homology groups*** of the representation.

Given a topological space X, there are certain grating representations (Γ, f, X) on X which are themselves topological invariants of the space. The invariant groups and rings of these representations are, of course, invariants of the space. The representation described in Example A of §7 is clearly a topological invariant of the space X. Other significant invariant representations are the following:

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[1] By a *compact* set, we shall mean a set satisfying the *complete* Heine-Borel-Lebesgue condition. In the terminology of Alexandroff and Hopf, such a set would be said to be *bicompact*.

Example B. Let X be a topological space, and let $[\gamma]$ be the set of all closed subsets of X. We treat each member γ_i of $[\gamma]$ as the symbol for a cut consisting of three abstract elements a_i, b_i, and c_i. The cuts γ_i form a grating $\Gamma=[\gamma]$. Moreover, there is a continuous representation (Γ, f, X) of the grating Γ on the space X, such that the determining function of the representation is the function $f(\gamma, x)$ which takes on the values -1, 0, or 1 according as the point x is interior to, on the frontier of, or exterior to the closed set γ.

Example C. Let X be a topological space, and let $[\gamma]$ be the set of all ordered pairs $\gamma_i=(A_i, C_i)$ of disjoint open subsets A_i and C_i of X. We treat each member γ_i of $[\gamma]$ as the symbol for a cut with the elements a_i, b_i, and c_i. The cuts γ_i form a grating $\Gamma=[\gamma]$. Again, there is a continuous representation (Γ, f, X) of Γ on X, such that the determining function $f(\gamma, x)$ is the one which takes on the value -1 when x is a point of A, the value 1 when x is a point of C, and the value 0 when x is a point of the complement of $A\cup C$.

In a continuation of this paper, which will deal specifically with the the applications of grating theory to hausdorff spaces, we shall prove the following theorem:

Given a locally compact hausdorff space X, the homology ring $\Gamma(\Psi_0/\Phi_0)$ of the continuous representation (Γ, f, X) described in Example A of §7 is completely isomorphic with the homology rings of the representations described in Examples B and C. Moreover, if the space X is an (open or closed) manifold, the homology ring $\Gamma(\Psi_0/\Phi_0)$ is completely isomorphic with the classical intersection ring of the manifold, provided the classical cycles of dimensionality k are interpreted as of rank $\rho=n-k$.

If X is a space of unrestricted generality, the homology ring of the representation in Example A need not be completely isomorphic with the homology rings of the representations in Examples B and C.

INSTITUTE FOR ADVANCED STUDY

THE NOVEMBER MEETING IN AMES

The four hundred twentieth meeting of the American Mathematical Society was held at the Iowa State College of Agriculture and Mechanical Arts, Ames, Iowa, November 29–30, 1946. About one hundred twenty-five persons attended, among whom were the following eighty-six members of the Society:

E. S. Allen, N. R. Amundson, E. W. Anderson, K. J. Arnold, R. W. Babcock, J. W. Beach, H. W. Becker, R. H. Bing, L. M. Blumenthal, F. E. Bortle, W. C. Brenke, Foster Brooks, G. W. Brown, R. H. Bruck, L. E. Bush, R. H. Cameron, R. E. Carr, E. W. Chittenden, H. E. Clarkson, B. H. Colvin, M. E. Daniells, M. D. Donsker, H. P. Doole, W. G. Dyer, John Dyer-Bennet, G. M. Ewing, J. S. Frame, M. G. Gaba, B. E. Gillam, Cornelius Gouwens, Edison Greer, F. S. Harper, Charles Hatfield, A. S. Henriques, G. A. Herr, Archie Higdon, J. J. L. Hinrichsen, D. L. Holl, Ralph Hull, F. B. Jones, G. K. Kalisch, L. M. Kelly, J. C. Koken, Solomon Lefschetz, A. T. Lonseth, C. T. McCormick, J. V. McKelvey, M. M. McKelvey, Kenneth May, W. L. Mitchell, Vladimir Morkovin, Ivan Niven, M. J. Norris, E. N. Oberg, E. B. Ogden, J. M. H. Olmsted, G. B. Price, Tibor Rado, P. V. Reichelderfer, P. R. Rider, R. F. Rinehart, R. E. Roberson, D. H. Rock, E. H. Rothe, W. T. Scott, C. E. Sealander, M. E. Shanks, Marlow Sholander, E. R. Smith, G. W. Smith, H. F. Smith, R. D. Specht, C. E. Springer, R. H. Stark, Rothwell Stephens, E. B. Stouffer, L. W. Swanson, H. P. Thielman, C. J. Thorne, W. J. Thron, Gerhard Tintner, H. L. Turrittin, V. J. Varineau, Bernard Vinograde, S. E. Warschawski, Roscoe Woods.

On Friday at 4:00 P.M. Professor S. E. Warschawski of the University of Minnesota, under the joint auspices of the Society and the Research Council of Iowa State College, delivered an address entitled *Conformal mapping of nearly circular regions*. Professor Solomon Lefschetz presided. On Saturday at 9:30 A.M. Professor P. V. Reichelderfer of the University of Wisconsin spoke on *The essential part of a surface*, with Professor Tibor Rado presiding.

There was a session for short papers Friday morning. Two such sessions were held Friday afternoon, and one on Saturday morning. Presiding officers at these sessions were, respectively, Professor P. V. Reichelderfer, Professors E. W. Chittenden and H. P. Thielman, and Professor R. H. Cameron.

One hundred sixty-four members and friends of the Society were entertained at a very excellent dinner Friday night, held in the Memorial Union of Iowa State College. Professor D. L. Holl was toastmaster. Following the Brahms Horn Trio, played by Professors E. S. Allen, G. W. Brown and R. G. Wendell, President C. E. Friley welcomed the Society on behalf of Iowa State College and discussed the impact of increased enrollments upon American colleges. Dean E. B. Stouffer replied for the Society and went on to point out the new problems of college teaching. At the suggestion of the toastmaster everyone rose and stood in silent tribute to the memory of the late Professor Dunham Jackson, who died on November 6, 1946. The next speaker was the Associate Secretary, who pointed out that the decision to return to the practice of a movable Thanksgiving meeting was the result of a widespread canvass of opinion at universities and colleges in the midwest. He suggested that, despite difficulties of transportation, the Ames meeting had amply proved the wisdom of this decision. The program concluded with an illustrated lecture by Dean Buchanan of Iowa State College, describing his recent trip through Arabia. The dinner was followed by an informal evening, with bridge and bowling for those who were interested.

The committee on arrangements consisted of Professors E. R. Smith (Chairman), R. H. Cameron, E. W. Chittenden, H. P. Thielman, and Bernard Vinograde. Professor Smith and his committee are to be congratulated on the excellence of the arrangements both for the meetings and for the dinner.

Titles and cross references to the abstracts of papers read follow below. Papers presented by title are indicated by the letter "*t*." Papers numbered 1 to 6 were presented Friday morning; papers 7 to 16, in two sections Friday afternoon; papers 17 to 21, Saturday morning. Papers 22 to 44 were read by title. Paper 9 was presented by Professor Greer, and paper 21 was read for Professor Bourgin by Professor Lonseth. Professor Haskell was introduced by Professor Craig; Mr. Langenhop and Mr. Stone by Professor Thielman; Professor Mood, by Professor Bruck; Professor Stiles by Professor Holl; and Sister Petronia Van Straten by Professor Menger.

1. H. L. Turrittin: *Stokes multipliers for asymptotic solutions of a certain differential equation.* (Abstract 53-1-79.)

2. W. B. Stiles: *Bending of clamped plates.* (Abstract 53-1-76.)

3. R. E. Carr: *Enantiomorphism in mathematical models of organic molecules.* Preliminary report. (Abstract 53-1-67.)

4. J. W. Beach: *Flow of slow viscous fluid between rotating cylinders.* (Abstract 53-1-63.)

5. F. E. Bortle: *Boundary problem in partially clamped rectangular plates.* (Abstract 53-1-64.)

6. G. W. Brown: *The stability of feedback solutions of simultaneous linear equations.* (Abstract 53-1-66.)

7. Bernard Vinograde: *Radicals associated with equivalent semi-simple residue systems.* (Abstract 53-1-23.)

8. Ralph Hull: *Generators of the rational integral symplectic groups.* (Abstract 53-1-17.)

9. Edison Greer and P. O. Bell: *A study of analytic surfaces by means of a projective theory of envelopes.* (Abstract 52-11-390.)

10. C. E. Springer: *Union torsion of a curve on a surface.* (Abstract 53-1-85.)

11. L. M. Blumenthal: *Superposability in elliptic space.* II. (Abstract 53-1-80.)

12. H. W. Becker: *Circuit algebra.* (Abstract 52-11-377.)

13. Gerhard Tintner: *The statistical estimation of the dimensionality of a given set of observations.* (Abstract 53-1-91.)

14. Kenneth May: *Technological change as a functional variation.* (Abstract 53-1-74.)

15. R. H. Bing: *Some characterizations of a simple closed curve.* (Abstract 53-1-95.)

16. F. B. Jones: *Concerning non-aposyndetic continua.* (Abstract 53-1-100.)

17. H. P. Thielman: *On continuous and discontinuous solutions of generalized Cauchy functional equations.* (Abstract 53-1-53.)

18. C. E. Langenhop: *Analytic functions of matrices.* (Abstract 53-1-37.)

19. W. M. Stone: *On Samuelson's generalized Laplace transformation.* (Abstract 53-1-49.)

20. W. J. Thron: *Periodic iterated fractions.* (Abstract 53-1-54.)

21. D. G. Bourgin: *Representations of compositions.* (Abstract 53-1-65.)

22. A. A. Albert: *A structure theory for Jordan algebras.* (Abstract 53-1-1-*t*.)

23. N. R. Amundson: *Unsymmetrically loaded orthotropic thin plates on elastic foundations.* (Abstract 53-1-62-*t*.)

24. Reinhold Baer: *Direct decompositions.* (Abstract 53-1-3-*t*.)

25. Reinhold Baer: *Endomorphism rings of operator loops.* (Abstract 53-1-4-*t*.)

26. Reinhold Baer: *Splitting endomorphisms.* (Abstract 53-1-5-*t*.)

27. H. W. Becker: *Generalizations of the Epstein-Fourier series.* (Abstract 53-1-25-*t*.)

28. H. W. Becker: *Rooks and rhymes.* (Abstract 52-11-395-*t*.)

29. R. H. Bing: *Solution of a problem proposed by R. L. Wilder.* (Abstract 53-1-94-*t*.)

30. Evelyn Frank: *On continued fraction expansions of analytic functions.* (Abstract 53-1-29-*t*.)

31. R. E. Gaskell: *An extension of the finite Fourier transformation.* (Abstract 52-11-379-*t*.)

32. R. N. Haskell: *Mean values and biharmonic polynomials.* (Abstract 52-9-294-*t*.)

33. M. R. Hestenes: *An indirect sufficiency proof for the problem of Bolza in nonparametric form.* (Abstract 53-1-34-*t*.)

34. W. N. Huff: *On the type of the polynomials generated by* $f(x\tau)\phi(\tau)$. (Abstract 52-11-367-*t*.)

35. A. M. Mood: *On Hotelling's weighing problem.* (Abstract 53-1-89-*t*.)

36. W. D. Munro: *Orthogonal trigonometric sums with auxiliary conditions.* (Abstract 53-1-43-*t*.)

37. L. J. Paige: *A note on finite abelian groups.* (Abstract 53-1-20-*t*.)

38. P. V. Reichelderfer: *On the definition of the essential multiplicity for continuous transformations in the plane.* (Abstract 52-11-372-*t*.)

39. H. J. Ryser: *Rational vector spaces.* Preliminary report. (Abstract 53-1-21-*t*.)

40. W. T. Scott and H. S. Wall: *On the convergence and divergence of continued fractions.* (Abstract 52-11-374-*t*.)

41. Marlow Sholander: *On the existence of the inverse operation in certain spaces.* (Abstract 52-9-278-*t*.)

42. M. H. Stone: *Pseudo-norms and partial orderings in abelian groups.* (Abstract 53-1-48-*t*.)

43. M. H. Stone: *Remarks on metrizability.* (Abstract 53-1-103-*t*.)

44. Sister Petronia Van Straten: *Toroidal and non-toroidal graphs.* (Abstract 53-1-104-*t*.)

R. H. BRUCK,
Associate Secretary

THE NOVEMBER MEETING IN LOS ANGELES

The four hundred twenty-first meeting of the American Mathematical Society was held at the University of California at Los Angeles on November 30, 1946. The total attendance was a little over sixty, including the following fifty-two members of the society:

H. M. Bacon, E. F. Beckenbach, Clifford Bell, H. F. Bohnenblust, F. A. Butter, Jr., W. D. Cairns, J. W. Calkin, F. M. Clarke, L. M. Coffin, Myrtie Collier, E. L. Crow, J. H. Curtiss, R. P. Dilworth, H. J. Ettlinger, G. E. Forsythe, J. W. Green, William Gustin, H. J. Hamilton, Olaf Helmer, P. G. Hoel, W. C. Hoffman, D. H. Hyers, C. G. Jaeger, Glen James, P. B. Johnson, H. L. Langhaar, Hans Lewy, F. R. Morris, T. E. Overbeck, R. I. Piper, W. T. Puckett, W. C. Randels, Mina Rees, A. C. Schaeffer, G. E. F. Sherwood, Ernst Snapper, I. S. Sokolnikoff, R. H. Sorgenfrey, D. C. Spencer, J. D. Swift, Gabor Szegö, A. E. Taylor, E. F. Tyler, S. E. Urner, F. A. Valentine, L. F. Walton, Morgan Ward, J. G. Wendel, P. A. White, R. L. White, František Wolf, E. R. Worthington.

Contributed papers were presented during the morning session, at which Professor I. S. Sokolnikoff presided. During the afternoon session the remaining contributed papers were presented, and Professor D. C. Spencer delivered the invited hour address entitled *Outstanding problems in conformal mapping*. Professor W. T. Puckett presided during the afternoon session.

Titles and cross references to the abstracts of papers read at the meeting follow below. Papers where abstract numbers are followed by the letter "t" were read by title. Paper 3 was read by Dr. Forsythe.

1. J. D. Swift: *Periodic functions over a finite field*. (Abstract 52-11-354.)

2. R. P. Dilworth: *Decomposition of relatively complemented lattices*. (Abstract 53-1-12.)

3. Cornelius Lanczos and G. E. Forsythe: *Approximation of the eigen-vectors of a nonsymmetric matrix*. Preliminary report. (Abstract 53-1-73.)

4. František Wolf: *On a decomposition of functions*. (Abstract 53-1-60.)

5. W. T. Puckett: *On a problem in connected finite closure algebras*. (Abstract 53-1-102.)

6. P. B. Johnson: *A contribution to parallel displacement theory in Riemannian space*. (Abstract 53-1-82.)

7. Gabor Szegö: *On an inequality due to P. Turan concerning Legendre polynomials*. (Abstract 53-1-52.)

8. A. E. Taylor: *A geometric theorem and its application to biorthogonal systems*. (Abstract 53-1-86.)

9. R. H. Sorgenfrey: *On the topological product of paracompact spaces.* (Abstract 52-7-261-*t*.)

10. Alfred Horn: *The asymptotic behavior of systems of Volterra integral equations.* (Abstract 52-11-366-*t*.)

11. A. D. Michal: *An existence and uniqueness theorem for a nonlinear differential equation in Banach spaces.* (Abstract 53-1-39-*t*.)

12. A. D. Michal: *The solutions of systems of linear differential equations as entire analytic functionals of the coefficient functions.* (Abstract 53-1-41-*t*.)

13. A. D. Michal: *Global groups of motions in some infinitely dimensional Riemannian spaces.* (Abstract 53-1-40-*t*.)

14. A. D. Michal, R. C. James, and Max Wyman: *Topological Abelian groups with ordered norms.* (Abstract 53-1-42-*t*.)

15. J. C. C. McKinsey: *On the representation problem for projective algebras.* (Abstract 52-7-227-*t*.)

16. B. A. Bernstein: *Weak definitions of field.* (Abstract 53-1-8-*t*.)

17. Paul Civin: *Mean values of periodic functions.* (Abstract 53-1-28-*t*.)

A. C. SCHAEFFER,
Associate Secretary

THE ANNUAL MEETING OF THE SOCIETY

The fifty-third Annual Meeting of the American Mathematical Society was held at Swarthmore College, Swarthmore, Pennsylvania, Thursday to Sunday, December 26–29, 1946, in conjunction with the Annual Meeting of the Mathematical Association of America. Over five hundred persons registered, including the following four hundred three members of the Society:

J. C. Abbott, C. R. Adams, R. B. Adams, Louise Adams, V. W. Adkisson, R. P. Agnew, C. B. Allendoerfer, Warren Ambrose, R. F. Arens, H. E. Arnold, Emil Artin, Silvio Aurora, M. C. Ayer, Frank Ayres, W. L. Ayres, R. P. Bailey, D. H. Ballou, J. D. Bankier, E. W. Barankin, Joshua Barlaz, P. T. Bateman, F. W. Beal, H. P. Beard, T. J. Benac, A. A. Bennett, Theodore Bennett, Stefan Bergman, Felix Bernstein, Lipman Bers, M. T. Bird, Z. W. Birnbaum, W. A. Blankinship, W. E. Bleick, Henry Blumberg, M. L. Boas, R. P. Boas, T. A. Botts, S. G. Bourne, J. G. Bowker, H. W. Brinkmann, Foster Brooks, R. H. Bruck, C. T. Bumer, R. S. Burington, A. W. Burks, Herbert Busemann, Hobart Bushey, J. H. Bushey, K. E. Butcher, S. S. Cairns, E. J. Camp, R. C. Campbell, H. H. Campaigne, Achille Capecelatro, P. A. Caris, F. E. Carner, H. C. Carter, W. C. Carter, W. B. Carver, J. E. Case, Omar Catunda, F. L. Celauro, Subrahmanyan Chandrasekhar, J. O. Chellevold, W. F. Cheney, K. L. Chung, Alonzo Church, Randolph Church, Edmund Churchill, R. V. Churchill, J. A. Clarkson, R. F. Clippinger, A. B. Coble, Nathaniel Coburn, E. A. Coddington, I. S. Cohen, L. W. Cohen, Nancy Cole, Esther Comegys, J. A. Cooley, T. F. Cope, Richard Courant, W. H. H. Cowles, V. F. Cowling, G. F. Cramer, A. R. Craw, H. B. Curry, J. H. Curtiss, E. H. Cutler, James Elmer Davis, L. J. Deck, F. F. Decker, C. H. Denbow, A. H. Diamond, L. L. Dines, H. L. Dorwart, T. L. Downs, F. W. Dresch, Arnold Dresden, D. M. Dribin, Roy Dubisch, R. J. Duffin, Nelson Dunford, J. C. Durand, John Dyer-Bennet, E. D. Eaves, Nat Edmonson, J. E. Eaton, W. F. Eberlein, Samuel Eilenberg, Wade Ellis, Paul Erdös, W. H. Fagerstrom, Herbert Federer, William Feller, F. G. Fender, F. A. Ficken, N. J. Fine, L. R. Ford, Tomlinson Fort, M. C. Foster, R. M. Foster, F. H. Fowler, R. H. Fox, J. S. Frame, M. R. Freundlich, R. E. Fullerton, A. S. Galbraith, David Gale, C. S. Gardner, G. N. Garrison, L. L. Gavurin, H. M. Gehman, B. H. Gere, F. J. Gerst, J. H. Giese, B. P. Gill, Wallace Givens, A. M. Gleason, R. E. Gleason, Casper Goffman, V. D. Gokhale, Michael Goldberg, H. H. Goldstine, R. A. Good, A. W. Goodman, R. O. Goodman, S. H. Gould, A. A. Grau, Lewis Greenwald, George Grossman, C. C. Grove, E. J. Gumbel, Margaret Gurney, W. W. Gutzman, M. R. Hafner, Theodore Hailperin, R. W. Hamming, E. S. Hammond, G. H. Handelman, O. G. Harrold, A. E. Heins, M. H. Heins, Erik Hemmingsen, Aaron Herschfeld, T. H. Hildebrandt, Einar Hille, G. P. Hochschild, D. L. Holl, T. R. Hollcroft, E. M. Hove, E. M. Hull, M. G. Humphreys, W. A. Hurwitz, H. D. Huskey, W. R. Hutcherson, M. H. Ingraham, W. H. Ingram, S. B. Jackson, Herbert Jehle, E. D. Jenkins, Fritz John, Madeline Johnsen, R. F. Johnson, B. W. Jones, Bjarni Jónsson, Mark Kac, M. L. Kales, Hyman Kamel, L. H. Kanter, Irving Kaplansky, William Karush, M. E. Kellar, D. E. Kibbey, J. R. Kline, E. G. Kogbetliantz, E. R. Kolchin, Horace Komm, C. F. Kossack, H. L. Krall, R. R. Kuebler, H. W. Kuhn, Wouter van der Kulk, M. E. Ladue, H. G. Landau, G. A. Larew, V. V. Latshaw, J. A. Larrivee,

C. G. Latimer, P. D. Lax, Solomon Lefschetz, Joseph Lehner, Marguerite Lehr. R. A. Leibler, Walter Leighton, B. A. Lengyel, C. A. Lester, Howard Levi, Harry Levy, A. L. Lewis, J. V. Lewis, F. W. Light, A. T. Lonseth, Lee Lorch, E. D. McCarthy, S. L. McDonald, D. L. McDonough, S. S. McNeary, C. C. MacDuffee, Saunders MacLane, H. F. MacNeish, Ingo Maddaus, J. D. Mancill, H. B. Mann, Murray Mannos, R. H. Marquis, M. H. Martin, W. T. Martin, P. R. Masani, F. I. Mautner, A. E. Meder, Emanuel Mehr, A. S. Merrill, H. L. Meyer, A. N. Milgram, B. L. Miller, D. D. Miller, L. I. Mishoe, E. E. Moise, Deane Montgomery, Lillian Moore, K. A. Morgan, Vladimir Morkovin, Richard Morris, A. P. Morse, I. R. Moses, B. H. Mount, C. W. Munshower, W. R. Murray, D. S. Nathan, C. A. Nelson, O. E. Neugebauer, Morris Newman, C. V. Newsom, P. B. Norman, C. O. Oakley, L. F. Ollmann, Isaac Opatowski, F. W. Owens, H. B. Owens, J. C. Oxtoby, J. S. Oxtoby, S. T. Parker, G. W. Patterson, E. K. Paxton, L. G. Peck, A. M. Peiser, Anna Pell-Wheeler, B. J. Pettis, H. R. Phalen, C. R. Phelps, R. S. Phillips, Joel Pitcairn, Everett Pitcher, G. B. Price, A. L. Putnam, Hans Rademacher, John Raleigh, J. F. Randolph, S. E. Rasor, G. E. Raynor, F. V. Reno, C. N. Reynolds, D. P. Richardson, R. G. D. Richardson, P. R. Rider, F. D. Rigby, R. F. Rinehart, L. A. Ringenberg, E. K. Ritter, J. B. Robinson, R. M. Robinson, R. E. Root, J. B. Rosenbach, L. J. Rosenbaum, R. A. Rosenbaum, J. H. Rosenbloom, P. C. Rosenbloom, J. B. Rosser, C. W. Saalfrank, Raphael Salem, Hans Samelson, R. G. Sanger, Arthur Sard, S. S. Saslaw, A. T. Schafer, R. D. Schafer, Robert Schatten, Lowell Schoenfeld, I. J. Schoenberg, I. E. Segal, Wladimir Seidel, H. N. Shapiro, Jacob Sherman, Seymour Sherman, D. T. Sigley, M. H. Slud, L. L. Smail, C. V. L. Smith, W. M. Smith, Ernst Snapper, A. K. Snyder, Andrew Sobczyk, F. W. Sohon, E. S. Sokolnikoff, A. H. Sprague, E. R. Stabler, E. P. Starke, F. H. Steen, H. W. Steinhaus, Rothwell Stephens, F. M. Stewart, C. N. Stokes, E. C. Stokes, M. H. Stone, H. F. Story, W. J. Strange, Walter Strodt, Irving Sussman, R. L. Swain, J. L. Synge, Otto Szász, Gabor Szegő, Alexander Tartler, R. L. Taylor, Feodor Theilheimer, R. M. Thrall, E. W. Titt, Leonard Tornheim, Peter Treuenfels, W. J. Trjitzinsky, C. A. Truesdell, A. W. Tucker, J. W. Tukey, J. L. Vanderslice, H. E. Vansant, C. W. Vickery, G. L. Walker, R. J. Walker, A. D. Wallace, J. L. Walsh, J. B. Walton, W. R. Wasow, C. W. Watkeys, G. C. Webber, M. S. Webster, J. V. Wehausen, Alexander Weinstein, M. J. Weiss, E. T. Welmers, L. R. White, G. W. Whitehead, A. L. Whiteman, E. A. Whitman, P. M. Whitman, G. T. Whyburn, L. S. Whyburn, W. M. Whyburn, J. E. Wilkins, S. S. Wilks, K. P. Williams, W. L. Williams, Clement Winston, E. E. Witmer, H. A. Wood, F. M. Wright, M. A. Wurster, Bertram Yood, J. W. Young, J. W. T. Youngs, Daniel Zelinsky, Leo Zippin, Antoni Zygmund.

Sessions for reading contributed papers were held at 3:15 P.M., Friday, at 10:45 A.M. and 3:15 P.M., Saturday, and at 10:00 A.M., Sunday. The presiding officers for these sessions were Dean R. G. D. Richardson, Professor J. L. Walsh, Professor B. W. Jones, Dr. W. M. Whyburn, Professor R. H. Bruck, Professor W. L. Ayres, and Professor R. V. Churchill.

The twentieth Josiah Willard Gibbs lecture was given on Thursday evening by Professor Subrahmanyan Chandrasekhar of the Yerkes Observatory, University of Chicago, on *The transfer of radiation in stellar atmospheres*. President T. H. Hildebrandt presided.

As a token of respect for the late Professor Dunham Jackson and an appreciation of his great and continued interest in mathematics, at the beginning of the general session at 3:15 P.M., Friday, President T. H. Hildebrandt read the abstract of the paper, *The boundedness of orthonormal polynomials on certain curves of the fourth degree*, presented by title on the program of this meeting.

On Friday afternoon with Vice President Saunders MacLane presiding, Professor A. P. Morse of the University of California gave an address entitled *Derivatives and their integrals*.

Saturday afternoon with President Elect Einar Hille presiding, Professor Harald Cramér of the University of Stockholm gave an address entitled *Some aspects of the theory of stochastic processes*.

Sessions of the Mathematical Association of America were held on Thursday afternoon and Friday morning.

At the dinner on Friday evening for mathematicians and their guests, the toastmaster was Professor C. O. Oakley of Haverford College. President J. W. Nason of Swarthmore College gave an address of greeting. Professor W. T. Martin of Massachusetts Institute of Technology spoke on the present state and future prospects of graduate study in mathematics. Professor C. G. Latimer of the University of Kentucky presented resolutions expressing the thanks and appreciation of the members of the Society and of the Association to the President, the local committee, the Department of Mathematics and the staff of Swarthmore College for the excellent arrangements.

The annual business meeting and election of officers was held on Saturday morning, President T. H. Hildebrandt presiding. The proceedings are included later in this report. At the close of the business session President Hildebrandt announced the award of the Cole Prize in Theory of Numbers to Professor H. B. Mann of Ohio State University. Professor Mann gave a brief address on the subject of the prize winning paper, *A proof of the fundamental theorem on the density of sums of sets of positive integers*, Ann. of Math. (2) vol. 43 (1942) pp. 523–527.

Tea was served in Bond Memorial Hall on Thursday, Friday, and Saturday afternoons. A program of chamber music was given on Friday evening. On Saturday evening a number of those attending the meeting attended a concert by the Philadelphia Symphony Orchestra at the Academy of Music in Philadelphia.

At the meeting of the Board of Trustees at 6:00 P.M. on December 26, 1946, in Parrish Hall, there was no quorum present and the Board adjourned to December 30, 1946. The Council met at 9:20 P.M. on

December 26 and at 8:30 P.M. on December 27, 1946, in the Board Room of Clothier Memorial.

The Secretary announced the election of the following fifty-three persons to ordinary membership in the Society:

Mr. Max Beberman, Nome Territorial High School, Nome, Alaska;
Professor Ferdinand Pierre Beer, Williams College;
Mr. Howard Curtis Bennett, Syracuse University;
Mr. Joseph Blum, U. S. Geological Survey, Arlington, Va.;
Mr. Alan Cathcart, New York, N. Y.;
Professor Omar Catunda, University of São Paulo, São Paulo, Brazil;
Dr. Jeremiah Certaine, Naval Research Laboratory, Washington, D. C.;
Professor John Olson Chellevold, Lehigh University;
Mr. Benjamin Frederic Cheydleur, Naval Ordnance Laboratory, Washington, D. C.;
Mr. Kai-lai Chung, Princeton University;
Dr. Francis M. Crist, Collingswood, N. J.;
Mr. Stephen Peter Diliberto, Princeton University;
Dr. Wallace John Eckert, International Business Machines, New York, N. Y.;
Mr. Thomas Anthony Elkins, Gulf Research and Development Company, Pittsburgh, Pa.;
Mr. David Owen Ellis, University of Missouri;
Mr. Alvaro Ferlini, St. Joseph College, West Hartford, Conn.;
Mr. Emanuel Fischer, College of the City of New York;
Dr. Cleota G. Fry, Purdue University;
Mr. Edwin Culbert Gras, U. S. Naval Academy;
Mr. Arthur Armstrong Hauser, Jr., Rensselaer Polytechnic Institute;
Professor Donald Milton Hester, New Mexico School of Mines;
Mr. Joseph Hilsenrath, Lieutenant Commander, U.S.N.R., Naval Ordnance Laboratory, Washington, D. C.
Dr. Madeline Johnsen, Purdue University;
Professor D. D. Kosambi, Tata Institute of Fundamental Research, Bombay, India;
Mr. Roy Raymond Kuebler, Jr., Dickinson College, Carlisle, Pa.;
Mr. Chester William Leaman, Carnegie Institute of Technology;
Mr. George Bernard LeBeau, The Thompson and Lichtner Company, Inc., Boston, Mass.;
Mr. Harold Fletcher Mathis, Northwestern University;
Mr. William James McCallion, McMaster University;
Professor Luna Isaac Mishoe, Delaware State College, Dover, Del.;
Mr. Benjamin Harrison Mount, Jr., University of Pittsburgh;
Mr. Chester Ray Morris, Indiana University;
Mr. Wilburn Avery Nicholson, Jr., American Telephone and Telegraph Company, Charlotte, N. C.;
Professor Joseph Donald Novak, University of South Carolina;
Dr. Christian Yvon Pauc, Faculté des Sciences de Marseille;
Mr. Donald Knight Pease, University of Connecticut;
Mr. Roy I. Pepper, University of Tennessee;
Professor Balthasar van der Pol, Delft University and N. V. Philips' Laboratory, Eindhoven, Holland;
Mollie A. Rubenstein (Mrs. Herbert), Columbia University;

Mr. Hans Rudberg, Iron Refining Co., Haellefornaes, Sweden;
Mr. Pierre Samuel, Princeton University;
Mr. Albert Schild, University of Pennsylvania;
Professor Emil Schoenbaum, Charles University, Prague, Czechoslovakia;
Mr. Harold N. Shapiro, Princeton University;
Mr. David Singer, College of the City of New York;
Mr. Robert Sips, Union Chemique Belge, Brussels, Belgium;
Mr. Robert Stock, University of Montreal;
Professor Nellie R. Swanson, State Teachers College, Minot, N. D.;
Mr. Edward Arthur Trabant, Purdue University;
Professor Hsu-Shen Tsien, Massachusetts Institute of Technology;
Professor Gilbert Ulmer, University of Kansas;
Professor Philon Vassiliou, University of Engineering Science, Athens, Greece;
Mr. Joseph Lawrence Zemmer, Jr., Tulane University of Louisiana.

It was reported that the following had been elected to membership on nomination of institutional members as indicated:

Brown University: Messrs. Maynard Goodwin Arsove and Joel Saul Isenberg, Dr. Wouter van der Kulk, Mr. Robert Whittaker Schmied;
Bryn Mawr College: Miss Frances Elizabeth Crook;
University of California: Mr. Roy Bergh Leipnik;
University of California at Los Angeles: Messrs. Harold Luxenberg and Harold Shniad;
California Institute of Technology: Messrs. Richard William Davies, Irving Stoy Reed, Carl Hertz Savit, and George Finlay Simmons;
University of Cincinnati: Messrs. Louis Francis Doty and William Henry Spragens, Jr.;
Columbia University: Mr. Lester Lipman Gavurin, Miss Mary Stuart Gwathmey, Mr. Alex Heller;
Duke University: Mr. Frank Eugene Clark;
Harvard University: Messrs. Rowland Edward Cross, Clifford Speer Gardner, Ivan Raymond Hershner, Jr., and William Alonzo Hosier;
Johns Hopkins University: Reverend Richard Ernest Ingram, Messrs. Hans Jakob Reiter and Robert Peter Rich;
University of Illinois: Messrs. Dwight Benjamin Goodner, I. Henson Harris, Selmer Martin Johnson, Max S. Kramer, and George Francis Leger, Miss Vivian Rose Nuess, Messrs. John Abraham Schumaker, Henry Schutzberger, Laurie James Snell, Joseph Jean Stipanowich and Fred William Thalgott;
Illinois Institute of Technology: Messrs. Maurice Lee Anthony, Robert Richard Bentley, Herbert John Curtis, Richard Harbin Edwards, Newton Seymour Hawley, Sheldon Leonard Levy and Alexander Medina, Dr. Anatol Rapoport, Mrs. Mary Katherine Rapp, Professor Eli Sternberg;
Immaculata College: Sister Mary Raphael Hafner;
Indiana University: Lee Detner Pruett (Mrs. J. R.);
Institute for Advanced Study: Professors Karol Borsuk, Li-Fu Chiang, Loo-Keng Hua, and George Lawrence Kreezer, Dr. Shih-Tsun Ma, Mr. Friederich Ignaz Mautner;
Iowa State College: Mr. William Matthewson Stone;
University of Kansas: Mr. William Calvin Foreman;
Lehigh University: Messrs. Leon Benson and Robert Harry Spohn;

Massachusetts Institute of Technology: Messrs. Hsieh Chang, James Dugundji, Joe Reeder Foote, Elmer Jacob Frey, Edgar Nelson Gilbert, and Raymond Moos Redheffer;
University of Michigan: Miss Winifred Kip Burroughs, Messrs. George Richard Costello, David Gale, Luang-Feng Hsieh, Gustave Rabson, Ward Conrad Sangren, Arnold Samuel Shapiro, Robert Lawrence Taylor, and Martin T. Wechsler;
University of Minnesota: Mr. Bernard W. Lindgren;
Northwestern University: Mr. G. Melvin Bloom, Miss Dorothy Marie DeWitt;
Ohio State University: Messrs. Landon A. Colquitt and Harold Ellsworth Tinnappel;
University of Oregon: Miss Shirley Kathryn Anderson;
University of Pennsylvania: Messrs. Hyman Kamel and Ching-Tsiin Loo;
Purdue University: Sister M. Agnes Hatke, Messrs. Paul Eugene Irick, Maurice M. Lemme and Fred Charles Leone, Dr. Walter Phillip Reid;
Swarthmore College: Dr. Bernard Lewis Miller;
Syracuse University: Mr. Daniel Resch;
Texas Technological College: Mr. Bob Parker;
University of Virginia: Mr. Wesley Emory Pace, Professor Lucille Enid Whyburn;
Williams College: Mr. Chester Wallace Jordan;
Yale University: Mr. Alfred Edward Roberts, Jr.

The Secretary announced that Professor Heinz Hopf of Eigenössische Technische Hochschule, Zurich, Switzerland, had been admitted to the Society in accordance with the reciprocity agreement with the Swiss Mathematical Society and that the following had been admitted in accordance with the reciprocity agreement with the London Mathematical Society: Professor Selig Brodetsky, University of Leeds; Professor Harald Cramér, University of Stockholm, Dr. Bernard Hermann Neumann, University College of Hull.

The Secretary announced the election by mail of Lehigh University, Illinois Institute of Technology and Rice Institute to institutional contributing membership in the Society. The following were elected to institutional membership at the meetings of the Council: University of Alabama, University of Arizona, University of Georgia, Kenyon College, University of Maryland, Oklahoma Agricultural and Mechanical College, University of Saskatchewan, Temple University, Department of Pure Mathematics, University of Texas.

The Secretary is pleased to report at this time that the ordinary membership in the Society is now 3,233, including 299 nominees of institutional members and 59 life members. There are also 101 institutional members. The total attendance at all meetings in 1946 was 1,925; the number of papers read was 477; there were 14 hour addresses, 2 symposium addresses, 1 Gibbs Lecture, 4 Colloquium Lectures, and 1 Retiring Presidential Address; the number of members attending at least one meeting was 1,079.

The following appointments by President T. H. Hildebrandt of representatives of the Society were reported: Professor Morgan Ward at the inauguration of Arthur Gardiner Coons as President of Occidental College on October 3, 1946; Professor H. R. Brahana at the Centennial Celebration of MacMurray College on October 8–10, 1946; Professor J. B. Rosenbach at the inauguration of James Herbert Case, Jr., as President of Washington and Jefferson College on October 25, 1946; Professor W. B. Fite at the inauguration of Sarah Gibson Blanding as President of Vassar College on October 11, 1946; Professor G. W. Hess at the Celebration of the Fiftieth Anniversary of the Founding of Alabama College on October 12–14, 1946; Professor R. P. Agnew at the inauguration of Richard Leighton Greene as President of Wells College on November 1, 1946; Professor V. H. Tingey at the inauguration of Albert Ray Olpin as President of The University of Utah on October 15–16, 1946; Professor W. M. Smith at the inauguration of Martin Dewey Whitaker as President of Lehigh University on October 20, 1946; Professor Gillie A. Larew at the inauguration of Martha Lucas as President of Sweet Briar College on November 1, 1946; Professor G. A. Hedlund at the inauguration of George Matthews Modlin as President of the University of Richmond on November 15–16, 1946; Professor Dorothy McCoy at the inauguration of Fred Tom Mitchell as President of Mississippi State College on November 16, 1946; Professor Arnold Dresden at conference of Commission for International Educational Reconstruction on November 22–23, 1946; Professor J. L. Synge on the Policy Board of the abstracting journal in applied mechanics to be published by the American Society of Mechanical Engineers.

The following additional appointments by the President were reported: Professors J. I. Tracey (Chairman), E. G. Begle, B. H. Camp, Nelson Dunford, T. R. Hollcroft, and B. W. Jones as a Committee on Arrangements for the 1947 Summer Meeting at Yale University; Professors Tomlinson Fort (Chairman), D. F. Barrow, T. R. Hollcroft, B. W. Jones, H. A. Robinson, J. M. Thomas, and J. A. Ward as a Committee on Arrangements for the 1947 Annual Meeting at the University of Georgia; Professors C. C. MacDuffee (Chairman), B. P. Gill, and Solomon Lefschetz as a committee to confer with the editors of the Annals of Mathematics regarding publication problems; Professors J. L. Synge (Chairman), R. V. Churchill, Richard Courant, G. C. Evans, W. T. Martin, John von Neumann, and J. W. Tukey as a committee to discuss the Society's relations with applied mathematicians; Professors Arnold Dresden (Chairman), P. A. Smith, and Leo Zippin as a committee to supervise the distribution of material to devastated libraries; Professors M. H. Martin (Chairman), J. H. Taylor and G. T. Whyburn as a committee to study material regarding the undergraduate origins of Ph.D.'s in Mathematics, collected by the Office of Scientific Personnel; Professors P. A. Caris and Mary-Elizabeth Ladue, Dr. E. R. Kolchin as tellers for the election at the 1946 Annual Meeting; Dean E. B. Stouffer (Chairman), Professors L. M. Graves and J. R. Kline as a committee to nominate representatives of the Society on the Policy Committee for Mathematics; Professor A. W. Tucker as a member of the Committee on the Role of the Society in Mathematical Publication; Professor J. L. Walsh (Chairman), Dr. R. P. Boas, Professors B. P. Gill, M. R. Hestenes, Dean M. H. Ingraham, Professors J. R. Kline, R. E. Langer, Morgan Ward and J. W. T. Youngs as a committee to study all possible sources of additional revenue, in particular the possibility of increasing membership dues; Professor D. V. Widder as an additional member of the Emergency Committee for the International Congress of Mathematicians; Professor Oscar Zariski as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings for the period 1947–1948 (committee now consists of Professors J. R. Kline, Chairman, R. P. Agnew, and Oscar Zariski); Professor C. B. Morrey as a member of the Committee to Select Hour Speakers for Far Western Sectional Meetings for the period 1947–1948 (committee now consists of Professors A. C. Schaeffer, Chairman, C. B. Morrey, and Morgan Ward); Professor W. T. Martin as a member of the Committee to Select Hour Speakers for Eastern Sectional Meetings for the period 1947–1948 (committee now consists of Professors T. R. Hollcroft, Chairman, W. T. Martin, and A. W. Tucker); Professor Ralph Hull as a member of the Committee to Select Hour Speakers for Western Sectional Meetings for the period 1947–1948 (committee now consists of Professors R. H. Bruck, Chairman, Ralph Hull, and N. E. Steenrod).

The Council voted to continue Professor T. H. Hildebrandt as a member of the Emergency Committee for the International Congress of Mathematicians.

The Council appointed Professor Solomon Lefschetz as a representative of the Society on the Council of the American Association for the Advancement of Science for the year 1947.

Professor R. E. Langer was elected by the Council to serve as the Society's representative on the Policy Committee for Mathematics, for a period of four years beginning January 1, 1947. (Professor Langer succeeds Professor G. C. Evans.)

It was reported that Professor P. M. Morse of Massachusetts Institute of Technology had accepted the invitation to deliver the twenty-first Josiah Willard Gibbs Lecture at the 1947 Annual Meeting. It was also reported that Professor T. Y. Thomas had found it impossible to accept the invitation to deliver a series of Colloquium Lectures at the 1948 Summer Meeting.

The Secretary reported that the Rockefeller Foundation had extended the period of its grant to the Society for the International Congress of Mathematicians to December 31, 1950.

The Council adopted the following resolution on the death of Professor Dunham Jackson:

> After long suffering Dunham Jackson died November 6, 1946, at the age of fifty-eight. He is survived by his wife and two daughters.
>
> Jackson obtained his Ph.D. degree at Göttingen in 1911 under Landau. In 1919 he left Harvard University to become a professor at the University of Minnesota, where he remained until his death.
>
> This is not the place to review his important scientific achievements which are represented by many articles and by books and by a large number of dissertations written under his direction.
>
> Between the years 1918 and 1931 Jackson rendered valuable services to the Society as a member of the Council, as a member of the Transactions Editorial Committee, and as a Vice President. He was at one time President of the Mathematical Association, and Vice President and Secretary of Section A of the Association for the Advancement of Science.
>
> Each one of the large number of papers which he presented to the Society revealed his mastery of subject matter as well as his perfection of presentation. In clearness and elegance of exposition he was unsurpassed. This was recognized by the award of the Chauvenet Prize in 1935 for the most excellent expository article in advanced mathematics during the period from 1932 to 1934.
>
> In rare cases it is given to a man to defy long and torturing illness by maintaining creative ability and vital interest in the progress of his students to the very door of death. Such a man was Dunham Jackson. During the last months of his life he approved four dissertations, one of them only weeks before the end. A posthumous paper adorns the present program of the Society.
>
> An occurrence at the beginning of his career illustrates the character of Jackson. As a student at Göttingen he won in open competition an academic prize for a valuable mathematical contribution. The monetary award he turned over to the University of Göttingen for the improvement of American-German student relations.
>
> On behalf of the Society, the Council expresses its deep sorrow at the untimely death of this beloved and distinguished member.

At the annual election which closed on December 28, and at which 704 votes were cast, the following officers were elected:

President, Professor Einar Hille.

Vice President, Professor P. A. Smith.

Secretary, Professor J. R. Kline.

Associate Secretary, Professor T. R. Hollcroft.

Treasurer, Professor B. P. Gill.

Member of the Editorial Committee of the Bulletin, Professor R. E. Langer.

Member of the Editorial Committee of the Transactions, Professor Antoni Zygmund.

Member of the Editorial Committee of the Colloquium Publications, Professor R. L. Wilder.

Member of the Editorial Committee of Mathematical Reviews, Professor M. H. Stone.

Member of the Editorial Committee of Mathematical Surveys, Professor Nelson Dunford.

Representative on the Editorial Board of the American Journal of Mathematics, Professor Hassler Whitney.

Members at large of the Council, Dr. H. W. Bode, Professors Samuel Eilenberg, B. O. Koopman, Gordon Pall, and J. W. Tukey.

Members of the Board of Trustees, Professors T. H. Hildebrandt, W. R. Longley, and G. W. Mullins, Dean R. G. D. Richardson, Dr. Warren Weaver.

The Librarian reported the following additions to the Library: 218 bound volumes of periodicals, 62 books, 67 pamphlets (including 40 dissertations), and 2 dissertations on microfilm. He reported that the collection of doctoral dissertations is not yet complete, that exchanges with foreign countries had been re-established to a large extent, that a list of additions to the library since the completion of the catalogue manuscript would be published in the January number of the Bulletin.

In an appendix to this report are excerpts from the report of the Treasurer for the fiscal year 1946 as verified by the auditors. A copy of the complete report will be sent, on request, to any member.

The American Journal of Mathematics reported that it had published 688 pages in 1946. The Society contributes an annual subvention of $2,500 to the American Journal.

Times and places for meetings of the Society in 1947 were set as follows: September 2–5, Summer Meeting at Yale University; November 29 at California Institute of Technology; December 29–31, Annual Meeting at University of Georgia. It was reported that the October, 1947, meeting in New York City would be held at Hunter College. On recommendation of the Committee on Places of Meetings, the Council voted to hold the 1948 Summer Meeting at the University of Wisconsin, the 1948 Annual Meeting in New York City (in connection with the Annual Meeting of the American Association for the Advancement of Science), and the 1949 Summer Meeting at the University of Colorado.

Certain invitations to give addresses were announced: Professor N. H. McCoy for the February, 1947, meeting in New York City;

Professor S. S. Wilks for the 1947 Summer Meeting at Yale University.

The Bulletin Editorial Committee reported an increase in the number of papers offered for publication in the Bulletin and the Council and Trustees authorized the publication of an additional one hundred pages in 1947. Professors Herbert Busemann, Maurice Heins, and R. M. Thrall were reported as new Assistant Editors of the Bulletin.

The Transactions Editorial Committee reported that a total of 1106 pages had been published in the 1946 volumes and that Volume 60 had been designated as the Birkhoff Memorial Volume. It was reported that Dr. Irving Kaplansky had been named Associate Editor.

The Mathematical Reviews Editorial Committee reported that the 1946 volume would contain 540 pages, as compared with 284 pages in 1945. It was further reported that a large number of persons, particularly from abroad, had been added to the reviewing staff, in order to take care of the increased amount of material to be reviewed.

The Mathematical Surveys Editorial Committee reported that it expected to receive two or three manuscripts for publication in that series during 1947.

A committee appointed to study the question of revising the By-Laws in such a manner that the terms of office of all members of the Board of Trustees would not expire in the same year recommended that Article V, Section 1, of the By-Laws be amended to read as follows:

ARTICLE V. ELECTIONS AND TERMS OF OFFICERS

Section 1. The term of office shall be two years in the case of the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. In every case, however, the officials specified in Articles I, II, III, and IV shall continue to serve until their successors have been duly elected and qualified.

The term of office in the case of the trustees to be elected at the annual meeting to be held between the fifteenth of December, 1948 and the fifteenth of January, 1949, shall be three years in the case of two of the trustees to be elected and shall be two years in the case of three of the trustees to be elected. Upon the expiration of the terms of office of the trustees elected at the annual meeting to be held between the fifteenth of December, 1948 and the fifteenth of January, 1949, the term of office of all trustees shall be two years.

The Council and Trustees approved this change and voted that it be acted upon by the Society at the February meeting.

Professor Arnold Dresden, chairman of a committee to distribute mathematical material to devastated libraries, reported that a small amount of material had been received in response to the notice tipped into the August issue of the Bulletin. He further reported that the American Book Center would aid in shipping material to libraries which are to receive aid.

The Council voted to accept the recommendation of the Committee on the Role of the Society in Mathematical Publication that the Annals of Mathematics be granted a temporary subvention of $1,000 for the year 1947 and that the possibility of securing funds for the publication of foreign papers be investigated.

An invitation from Harvard University to hold the International Congress of Mathematicians at that institution in 1950 was referred to the Emergency Committee.

Professors Gordon Pall and D. J. Struik were appointed representatives of the Society on the Editorial Board of the Duke Mathematical Journal for a period of three years beginning January 1, 1947.

The Council authorized the appointment of a committee to study the question of cooperation with the Pan American Congress of Mathematicians to be held in Mexico City in November, 1947, under the auspices of the Mexican Mathematical Society.

The Council voted to adopt a report of the Birkhoff Memorial Committee, providing for the publication of the complete works of the late Professor G. D. Birkhoff by means of the photo-offset process and authorized the appointment of a Finance Committee to handle the solicitation of subscriptions and contributions and an Editorial Committee to edit the collected papers and to decide upon other details such as arrangement, format, and other essential matters.

On the basis of a report of the Special Committee on Applied Mathematics (Professor J. L. Synge, Chairman), the Council adopted the following recommendations: (1) That the Committee on the Role of the Society in Mathematical Publication study the question of financial support to The Journal of Mathematics and Physics and The Quarterly of Applied Mathematics; (2) That a Committee on Applied Mathematics be appointed to organize programs in applied mathematics and to make recommendations to the Council in all matters pertaining to the interests of applied mathematics; (3) That the Society establish an annual symposium on applied mathematics, the Committee on the Role of the Society in Mathematical Publication being requested to study the possibility of the publication of the proceedings of the symposium in book form.

The Council adopted the following recommendations of the Committee on Additional Revenue (Professor J. L. Walsh, Chairman):

(1) That the By-Laws be amended to effect a change in ordinary membership dues, beginning January 1, 1948, to the following rates: $6.00 per year for the first two years of membership, $8.00 per year for the next two years, and $10.00 per year thereafter; (2) That the regular subscription price of the Bulletin be increased to $11.00 per year beginning January 1, 1948; (3) That the customary discount to agents be discontinued after December 31, 1947, for periodicals destined for the United States and Canada. Other recommendations of this committee were referred to the Board of Trustees.

Titles and cross references to papers read at the meeting follow below. Papers 1 to 5 were read Friday afternoon, papers 6 to 16 Saturday morning, papers 17 to 27 Saturday afternoon, papers 28 to 36 Sunday morning, and papers 37 to 71, whose abstract numbers are followed by the letter *t*, were read by title. Dr. Glenn was introduced by Professor T. R. Hollcroft, Dr. Kober was introduced by Professor Einar Hille, and Professor Su was introduced by Professor Oscar Zariski. Paper number 5 was presented by Professor Weinstein, number 7 by Mr. Chung, number 18 by Professor Martin, number 29 by Professor Montgomery, and number 26 by Professor Diamond.

1. Herbert Jehle: *Hydrodynamical self-consistent fields in stellar dynamics*. (Abstract 53-1-72.)

2. J. L. Synge: *The general problem of antenna radiation*. (Abstract 53-1-77.)

3. Nathaniel Coburn: *Application of the Kármán-Tsien relation to the two-dimensional supersonic flow of fluids in jets*. (Abstract 53-1-68.)

4. A. E. Heins: *The scattering and transmission properties of a pair of semi-infinite parallel plates*. (Abstract 53-1-71.)

5. J. B. Díaz and Alexander Weinstein: *On an extremal property of the torsional rigidity*. (Abstract 53-3-141.)

6. Walter Strodt (National Research Fellow): *On the principal solution of a difference equation*. Preliminary report. (Abstract 53-1-50.)

7. K. L. Chung and Paul Erdös: *On the lower limit of sums of independent random variables*. (Abstract 53-1-27.)

8. Casper Goffman: *Fubini interval functions*. (Abstract 53-1-33.)

9. Robert Schatten: *An evaluation of the bound for certain functionals*. (Abstract 53-1-46.)

10. W. F. Eberlein: *Weak compactness in Banach spaces*. (Abstract 52-11-365.)

11. Daniel Zelinsky: *Nonassociative valuations*. (Abstract 53-1-24.)

12. R. H. Bruck: *An extension theory for Moufang loops*. (Abstract 53-1-11.)

13. Isaac Opatowski: *Simple Markoff chains with reverse transitions: the time moments.* (Abstract 53-1-90.)

14. E. J. Gumbel: *The asymptotic distribution of the range.* (Abstract 53-1-88.)

15. R. M. Thrall: *On ahdir algebras.* Preliminary report. (Abstract 53-1-22.)

16. Irving Kaplansky: *Semi-simple algebraic algebras.* (Abstract 53-1-18.)

17. J. D. Mancill: *Unilateral variations with variable end points.* (Abstract 53-1-38.)

18. R. H. Cameron and W. T. Martin: *Fourier-Wiener transforms of functionals belonging to* L_2 *over the space C.* (Abstract 53-1-26.)

19. W. J. Trjitzinsky: *Singular integral equations with complex-valued kernels.* (Abstract 53-1-56.)

20. J. W. T. Youngs: *Remarks on the isoperimetric inequality for closed Fréchet surfaces.* (Abstract 53-1-61.)

21. Leonard Tornheim: *Harmonic series in two variables.* (Abstract 53-1-55.)

22. A. M. Gleason: *A theorem on Banach spaces.* Preliminary report. (Abstract 53-1-31.)

23. Wallace Givens: *Parametric solution of linear homogeneous Diophantine equations.* (Abstract 53-1-14.)

24. Gabor Szegö: *The capacity of a circular plate-condenser.* (Abstract 53-1-78.)

25. J. S. Frame: *Group decomposition by double coset matrices.* (Abstract 53-1-13.)

26. A. H. Diamond and J. C. C. McKinsey: *A relation between algebras and their subalgebras.* (Abstract 53-3-116.)

27. R. J. Duffin: *Function classes invariant under the Fourier transform.* (Abstract 53-3-127.)

28. E. E. Moise: *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua.* (Abstract 53-1-101.)

29. Salomon Bochner and Deane Montgomery: *Groups on analytic manifolds.* (Abstract 52-11-397.)

30. Samuel Eilenberg: *Simplicial products of simplicial complexes.* (Abstract 53-1-97.)

31. C. A. Truesdell: *On a class of differential-difference equations.* (Abstract 53-1-57.)

32. Otto Szász: *On closed sets of rational functions.* (Abstract 53-1-51.)

33. I. J. Schoenberg: *On totally positive functions, Laplace integrals*

and entire functions of the Laguerre-Pólya-Schur type. (Abstract 53-1-47.)

34. P. C. Rosenbloom: *Studies on the heat equation.* (Abstract 53-1-45.)

35. O. E. Glenn: *A differential equation that plays a part in a mathematical theory of biological variation.* (Abstract 53-1-32.)

36. V. F. Cowling: *A generalization of a theorem of LeRoy and Lindelöf.* (Abstract 52-11-362.)

37. R. F. Arens: *Pseudo-normed algebras.* (Abstract 53-1-93-*t*.)

38. R. F. Arens: *Representation of rings in which* $x^p = x$. (Abstract 53-1-2-*t*.)

39. Reinhold Baer: *The double chain condition in cyclic operator groups.* (Abstract 53-1-6-*t*.)

40. J. D. Bankier: *Extended regular continued fractions.* (Abstract 53-1-7-*t*.)

41. Stefan Bergman: *On functions satisfying certain partial differential equations of elliptic type and their representation.* I. (Abstract 53-3-122-*t*.)

42. Stefan Bergman: *On functions satisfying certain partial differential equations of elliptic type and their representation.* II. (Abstract 53-3-123-*t*.)

43. Stefan Bergman and Menahem Schiffer: *A representation of Green's and Neumann's functions in the theory of partial differential equations of second order.* Preliminary report. (Abstract 53-3-124-*t*.)

44. Garrett Birkhoff: *Notes on lattices.* (Abstract 53-1-9-*t*.)

45. Garrett Birkhoff: *Subdirect representation of certain algebras.* (Abstract 53-1-10-*t*.)

46. Claude Chevalley and Samuel Eilenberg: *Cohomology theory of Lie groups and Lie algebras.* (Abstract 53-1-96-*t*.)

47. H. S. M. Coxeter: *The product of three reflections.* (Abstract 53-1-69-*t*.)

48. John DeCicco: *Union-preserving transformations of higher surface elements.* (Abstract 52-11-389-*t*.)

49. Bernard Epstein: *A method for the approximate solution of the Dirichlet problem for certain types of domains.* Preliminary report. (Abstract 53-1-70-*t*.)

50. C. J. Everett: *Representation of a sequence of finite sets.* (Abstract 53-1-98-*t*.)

51. G. E. Forsythe: *On Nörlund summability of random variables to zero.* (Abstract 53-1-87-*t*.)

52. Evelyn Frank: *On methods for the location of the zeros of complex polynomials.* (Abstract 53-1-30-*t*.)

53. Edwin Hewitt: *Rings of unbounded continuous functions.* (Abstract 53-1-99-*t*.)

54. L. K. Hua: *Geometries of matrices.* II. *Study of involutions in the geometry of symmetric matrices.* (Abstracts 53-1-15-*t*.)

55. L. K. Hua: *On the automorphisms of the symplectic group over any field.* Preliminary report. (Abstract 53-1-16-*t*.)

56. Dunham Jackson: *The boundedness of orthonormal polynomials on certain curves of the fourth degree.* (Abstract 53-1-35-*t*.)

57. Nathan Jacobson: *On the theory of primitive rings.* (Abstract 53-3-158-*t*.)

58. Edward Kasner and John DeCicco: *Rational harmonic curves.* (Abstract 53-1-84-*t*.)

59. Herman Kober: *Approximation of non-bounded functions by integral functions of finite order.* (Abstract 53-1-36-*t*.)

60. Kenneth May: *The aggregate effect of technological changes in a two-industry model.* (Abstract 53-1-75-*t*.)

61. B. E. Meserve: *Division sequences by cross-multiplication.* (Abstract 53-1-19-*t*.)

62. M. O. Reade: *On areolar monogenic functions.* (Abstract 53-1-44-*t*.)

63. Buchin Su: *Descriptive collineations in spaces of K-spreads.* (Abstract 53-3-155-*t*.)

64. Y. W. Tschen: *Existence of minimal surfaces with a simple pole and normal derivative equal to zero on boundary.* Preliminary report. (Abstract 53-3-136-*t*.)

65. Alexander Weinstein: *On the generalized Stokes-Beltrami equations.* (Abstract 53-1-58-*t*.)

66. D. W. Western: *Inequalities of the Markoff and Bernstein type for integral norms.* (Abstract 53-3-137-*t*.)

67. J. E. Wilkins: *The growth of the solutions of a certain linear integro-differential equation.* (Abstract 53-1-59-*t*.)

68. Jacob Wolfowitz: *On the efficiency of unbiased sequential estimates.* (Abstract 53-1-92-*t*.)

69. G. S. Young: *On compact fiberings of the plane.* (Abstract 53-1-105-*t*.)

70. G. S. Young: *On converging sequences of 2-manifolds.* (Abstract 53-1-106-*t*.)

T. R. HOLLCROFT,
Associate Secretary

EXCERPTS FROM REPORT OF TREASURER

December 13, 1946

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1946. The following comments may be of interest.

Investment Portfolio

On November 30, 1946, the market value of securities held for Invested Funds exceeded book value by $13,240, and the market value of securities held for Current Funds exceeded book value by $417. In view of these figures and of the large amount of U. S. Government bonds in the portfolio, the reserves held in accounts "Reserve for Investment Losses" and "Profit on Sales of Securities" may still be considered adequate protection against contingent depreciation in market value.

The following is a summary of the changes in security holdings made on authorization of the Board of Trustees during the year.

Acquired by Purchase:

$39,000	U. S. Savings Ser. G. Reg. $2\frac{1}{2}$s 1958
$ 5,000	Texas and Pacific Rwy. Gen. and Ref. Ser. E $3\frac{1}{8}$s 1985
100 shares	Aetna Insurance Co.
50 shares	Allied Chemical and Dye Corp.
50 shares	Bethlehem Steel Corp. 7% Cum. Pfd.
65 shares	Radio Corp. of America $3.50 Cum. Pfd.

Acquired by Exchange:

50 shares	Northern States Power Co. of Minnesota $3.60 Cum. Pfd. for 50 shares Northern States Power Co. of Minnesota 5% Cum. Pfd.

Sold or Recalled:

$6,000	U. S. Treasury 2s 1952–50
$8,000	U. S. Treasury 2s 1954–52
$3,500	U. S. Treasury $2\frac{1}{2}$s 1969–64
$5,000	Columbia Gas and Electric Corp. 5s 1961

$5,000	Consolidated Edison Co. of New York 3½s 1948
$5,000	Gas and Electric Co. of Bergen County Gen. 5s 1954
$1,000	New England Telephone and Telegraph Co. 1st B 4½s 1961–58
$5,000	New York, Chicago and St. Louis RR Ref. 3¾s 1975
$3,000	Oregon Washington RR and Navigation Co. Ref. A 3s 1960
50 shares	Crane Co. 3¾% Cum. Pfd.
100 shares	Continental Can Co. Inc.
50 shares	Public Service Corp. of New Jersey 7% Cum. Pfd.
25 rights	American Telephone and Telegraph Co.

The investment portfolio, valued at market November 30, 1946, now includes Government bonds 36.6 per cent, other bonds 8.2 per cent, preferred stock 12.4 per cent, common stock 36.1 per cent, cash in savings bank 6.7 per cent.

Income from Investments

Income received during the year from investment of Current Funds amounted to $1,644, exclusive of $95 earmarked for International Congress. This represents a return of just under 2.4 per cent computed on average book value of investments. Income on Invested Funds amounted to $7,317, representing a return of just under 3.8 per cent. Total investment income from all sources was $9,056, representing a return just under 3.37 per cent.

Income from the Henderson Estate was $4,800; in 1945 it was $4,750.

Increase in Surplus and Assets

Surplus account shows an increase of $9,971, while the net increase in assets for the year was $7,816. This last figure is at once surprising and somewhat illusory. Noticeably increased costs of publication were to a considerable extent offset by dues payments from members reinstated during the year and from dues and initiation fees from an unexpectedly large number of new members. Investment income amounted to some $400 more than in the previous year. Sales of all publications were extremely heavy, the Colloquium series for example showing receipts more than $2,500 in excess of 1945. Sales of securities netted profits of over $1,000.

But some $7,000 of expenses for which provision was made in the year's estimates were not disbursed during the year because of various delaying conditions. Nearly all of this represents costs of new Colloquium volumes now in process of publication, and of necessary reprinting of exhausted issues of the Bulletin and Transactions. In addition, some $1,300 of advance dues for 1947 were paid by institutional members. If it had not been for these two facts, operations for the year would have shown a small net loss. On the other hand intensive activity by Mathematical Reviews in reducing "backlog" accumulated during the war years resulted in a reduction of over $6,250 in the net assets which has been charged against the balance built up.

I trust that these more than usually detailed remarks will make it clear that it is extremely difficult this year to describe in any general way the outcome of the year's activities without careful analysis of individual items in the schedules and the exhibits submitted herewith.

Respectfully submitted,
BENNINGTON P. GILL,
Treasurer

BALANCE SHEET

Assets

	November 30, 1946	November 30, 1945
CURRENT FUNDS:		
Cash	$ 26,130.60	$ 24,868.41
Investments	60,837.08	54,433.13
	$ 86,967.68	$ 79,301.54
INVESTED FUNDS:		
Cash	$ 652.40	$ 5,833.06
Investments	192,874.73	187,543.57
	$193,527.13	$193,376.63
TOTAL ASSETS	$280,494.81	$272,678.17

Liabilities

CURRENT FUNDS:		
Mathematical Reviews	$ 7,389.34	$ 13,637.34
Colloquium	9,495.90	9,310.80
Mathematical Surveys	1,796.87	1,250.95
Library Catalogue		1,868.68
Reprinting Funds	5,780.67	3,310.24
Prize Funds and Other Special Funds Accumulated Income	5,591.42	4,518.11
International Congress	6,321.92	6,331.71
Policy Committee	913.34	143.25
Sinking Fund	960.65	820.91
Profit on Sales of Securities	982.60	413.36
Miscellaneous	159.24	91.60
Surplus	47,575.73	37,604.59
	$ 86,967.68	$ 79,301.54
INVESTED FUNDS:		
Endowment Fund Principal	$ 71,000.00	$ 71,000.00
Prize Funds and Other Special Funds	31,033.22	31,033.22
Life Membership and Subscription Reserve	3,563.40	3,866.88
Colloquium	5,000.00	5,000.00
Mathematical Reviews	65,000.00	65,000.00
Reserve for Investment Losses	4,385.89	4,385.89
Profit on Sales of Securities	13,544.62	13,090.64
	$193,527.13	$193,376.63
TOTAL LIABILITIES	$280,494.81	$272,678.17

SUMMARY STATEMENT OF INCOME AND EXPENDITURES

1945–1946

	1946		1945	
	Receipts	*Disbursements*	*Receipts*	*Disbursements*
GENERAL RECEIPTS:				
Dues—Ordinary Memberships	$20,761.05		$18,346.37	
Dues—Contributing Memberships	951.54		978.64	
Dues—Institutional Memberships	8,362.60		7,013.32	
Initiation Fees	1,139.22		445.00	
Investment Income	10,130 97		9,694.54	
Miscellaneous	290.79		261.74	
Grant from Rockefeller Foundation for Policy Committee	3,000.00		2,500.00	
GENERAL DISBURSEMENTS:				
Secretaries		$ 7,372.96		$ 6,277.52
Treasurer		1,200.37		1,153.96
Officers' Traveling		629.56		386.69
Furniture and Fixtures		601.14		
Library		1,473.46		1,015.67
Policy Committee		2,229.91		4,076.16
Printing Contracts Committee		.59		
Committee on Aid to Devastated Libraries		41.20		
Emergency		120.65		15.82
Total	$44,636.17	$13,669.84	$39,239.61	$12,925.82
Excess of General Receipts	$30,966.33		$26,313.79	
PUBLICATION:				
Bulletin	$ 2,498 87	$16,425 94	$ 2,195.05	$18,194.69
Bulletin Reprinting	928.75		936.16	1,629.45
Transactions	7,114.07	10,215.58	6,592.76	7,918.76
Transactions Reprinting	1,541 68		491.75	253.48
Colloquium	5,829.18	5,644 08	3,266.38	1,323.73
Mathematical Reviews	17,365 28	23,613.28	15,684 45	13,983.20
Mathematical Surveys	621.51	75.59	220.49	205.64
Semicentennial Publications	147.71	7.97	53.09	1.14
American Journal		2,500.00		2,500.00
Library Catalogue		1,868.68	2,500.00	631.32
Total	$36,047.05	$60,351.12	$31,940.13	$41,641.41
Excess Cost of Publication		$24,304.07		$ 9,701.28
OTHER				
Special Funds (including Congress)	$ 1,267.91	$ 204.39	$ 1,283.68	
Profit/Loss on Sales of Securities	1,023.22			$ 280.25
Miscellaneous	67.64		19.04	
Surplus:				
Mathematical Reviews		1,000.00		1,000.00
Library Catalogue				2,500.00
War Policy Committee			334.17	
Total	$ 2,358.77	$ 1,024.39	$ 1,636.89	$ 3,780.25
Difference	$ 1,154.38			$ 2,143.36
Net Change in Assets	$ 7,816.64		$14,469.15	
ASSETS BEGINNING OF YEAR	$272,678.17		$258,209.02	
ASSETS END OF YEAR	$280,494 81		$272,678.17	

BOOK REVIEWS

Les systèmes différentiels extérieurs et leurs applications géométriques. (Actualités Scientifiques et Industrielles, no. 994.) By Elie Cartan. Paris, Hermann, 1945. 214 pp. 450 fr.

This book gives a revised account of lectures delivered in 1936–1937 at the Sorbonne. The content is based almost exclusively on the author's own outstanding contributions to the subject at the beginning of the century. In developing many of these old results, however, the author's present viewpoint is new as well as stimulating. The whole treatment, even at the few points where it touches upon contemporary work by other writers, bears the stamp of the author's individuality.

The fundamental calculus employed is Grassmann algebra. Cartan's manner of regarding this discipline might be described as follows. The indeterminates are differentials and the polynomials are forms. Interest centers in skew-symmetric (necessarily multilinear) forms, which can be written

$$F = a_{i_1 \cdots i_p} u_1^{i_1} \cdots u_p^{i_p}, \tag{1}$$

where the coefficients a are skew-symmetric in every pair of indices and where the summation convention is being used. An equivalent expression for F is

$$F = \frac{1}{p!} a_{i_1 \cdots i_p} \begin{vmatrix} u_1^{i_1} & \cdots & u_1^{i_p} \\ \cdot & \cdot & \cdot \\ u_p^{i_1} & \cdots & u_p^{i_p} \end{vmatrix},$$

where the a's are the same as in (1). The exterior product of F by the form $G = b_{i_{p+1} \cdots i_{p+q}} u_{p+1}^{i_{p+1}} \cdots u_{p+q}^{i_{p+q}}$ is the skew-symmetric form defined by

$$[FG] = c_{i_1 \cdots i_{p+q}} u_1^{i_1} \cdots u_{p+q}^{i_{p+q}},$$

where the c's are found by applying all $(p+q)!$ signed permutations to the subscripts on $a_{i_1 \cdots i_p} b_{i_{p+1} \cdots i_{p+q}}$ and adding. The definition of product applied to $u_1^{i_1}, \cdots, u_p^{i_p}$ gives

$$[u_1^{i_1} \cdots u_p^{i_p}] = \begin{vmatrix} u_1^{i_1} & \cdots & u_1^{i_p} \\ \cdot & \cdot & \cdot \\ u_p^{i_1} & \cdots & u_p^{i_p} \end{vmatrix},$$

so that F may be written

$$F = \frac{1}{p!} a_{i_1 \cdots i_p} [u_1^{i_1} \cdots u_p^{i_p}],$$

or, if the subscripts on the u's are omitted, as

$$F = \frac{1}{p!} a_{i_1 \cdots i_p} [u^{i_1} \cdots u^{i_p}] \tag{2}$$

with the understanding that the interchange of two u's is to be compensated by multiplying by -1. When a skew-symmetric form is written in the symbolic notation (2), it is called an *exterior form*.

In this way, Cartan is able to construct simultaneously for his symbols two types of product, both of which can be evaluated for real values of the indeterminates. To distinguish between them a notation such as the brackets is necessary. As in the case of the vector product the use of brackets has its disadvantages: in the latter half of the book brackets are frequently used also as parentheses. Other writers, for example, Goursat in his *Problème de Pfaff*, regard the matter differently. One set of indeterminates (say, derivatives) is subjected solely to ordinary multiplication and another (say, differentials) solely to exterior multiplication. This not only makes the brackets unambiguously available for their normal use as fences but has manipulative advantages as well.

Besides developing Grassmann algebra along lines familiar to readers of Cartan's *Invariants intégraux*, Chapter I has several noteworthy features, which we proceed to mention.

On page 10 there is a theorem neatly contrasting an ordinary quadratic form F with an exterior quadratic form G, namely,

$$u^i \frac{\partial F}{\partial u^i} = 2F, \qquad u^i \frac{\partial G}{\partial u^i} = 0,$$

$$\left[u^i \frac{\partial F}{\partial u^i}\right] = 0, \qquad \left[u^i \frac{\partial G}{\partial u^i}\right] = 2G.$$

The theorem on linear dependence (p. 11) is also cast in this contrasting form.

The condition that a form be monomial (that is, have independent linear factors equal in number to its degree) is discussed at length. One might be tempted to dismiss the matter after saying that the associated system must have rank equal to the degree of the form. Cartan, however, finds in simple and elegant manner polynomials whose vanishing expresses the condition. It is interesting that the polynomials are quadratic. In the case of a quadratic exterior form, the facts could also be inferred from two familiar results: (i) a skew-symmetric matrix has rank at most two if and only if all principal determinants of order four are zero and (ii) a skew-symmetric determinant of order four is the square of a quadratic in its elements. If the theorem on matrices corresponding to Cartan's general result has been stated, it is at least not well known.

Instead of defining k-planes, as formerly, by the components of k independent vectors Cartan now employs the analogue of Plücker's line coördinates. At least as far as the applications in the present book are concerned, the increase in the number of coördinates and the necessity of satisfying the relations among them more than offset the advantages of their introduction. For example, brackets being omitted, the system (p. 27) $u^1u^3=u^1u^4=u^1u^2-u^3u^4=0$ is at once seen to imply $u^1=0$ or $u^3=au^1$, $u^4=bu^1$, $u^3u^4=0$, $u^1u^2=0$ without appealing to Plücker coördinates and the quadratic relation among them. Conceivably, these coördinates may be elegant means for further developments in the theory. An extended treatment of their properties has been made by J. W. Givens, *Tensor coördinates of linear spaces*, Ann. of Math. (2) vol. 38 (1937) pp. 355–385.

The definition (p. 30) of the associated system for a system of equations is a noteworthy accomplishment, although it is not the ultimate because the rank gives the minimum number of indeterminates in an algebraically (defined on p. 28) equivalent system rather than simply in an equivalent system. The author remarks (p. 56) that the characteristic systems (which are associated systems) for two equivalent differential systems are not necessarily equivalent.

Chapter II discusses the differential dF of an exterior form F, its geometrical significance, and its connection with Stokes' theorem.

Chapters III, IV, V and VI give a systematic account of the author's theory of Pfaffian systems. Although most of the exposition follows established lines, it is decidedly advantageous to have this material united in one volume, particularly for the purpose of the geometrical applications in the last 90 pages.

The gist of this theory is as follows. The system considered consists of a finite number of exterior forms F_β in the differentials of n variables x^i with $0 \leqq \deg F_\beta \leqq n$. A solution is a set of functions $f_\alpha(x^1, \cdots, x^n)$ such that the vanishing of f_α, df_α implies the vanishing of F_β. An immediate consequence of the definition is that the system F_β is equivalent to the system F_β, dF_β, which is called *closed*. Attention accordingly can be confined to closed systems. From this standpoint

f_α constitutes a solution of F_β if and only if the closed system f_α, df_α implies the closed system F_β, dF_β.

Of fundamental importance are the linear Pfaffian system, for which each F_β is linear, and the completely integrable type of such a linear system, which is equivalent to df_β. The properties of these systems are developed in Chapter III. The condition for a completely integrable linear system is stated (p. 49) in terms of the "anneau" defined on p. 26; unfortunately the author has not followed Ore's lead (*L'Algèbre abstraite*, p. 6) in reserving "anneau" as a translation for "ring." Mayer's method (integration by means of ordinary equations) is treated in §37; its generalization (integration by means of equations in $k-1$ fewer independent variables) is indicated later (p. 78) for systems having k characters equal to zero. In reading §37 it may be helpful to insert the stipulation "of $n-r$ dimensions" into the statement of the theorem, to supply the arguments $z^1, \cdots, z^r$ on the right of (2), and to by-pass the question of the hypotheses on which the discussion rests by taking refuge in the general hypothesis of real, analytic functions (p. 45) and in Cauchy's theorem for ordinary differential equations. Later (p. 50, Remark IV) the assumption of the existence of first derivatives shows that (2) satisfies the Lipschitz condition even when its right members contain the z's. The well known integration of a single partial differential equation is expounded as an application of the theory of completely integrable systems.

Cartan's existence theorem, which is the backbone of the subject, involves the notion of *k-tangent* (or integral k-plane). If the form F_β has degree p, let its indeterminates be $u_1^{i_1}, \cdots, u_p^{i_p}$. A k-plane is defined by k independent vectors v_λ^i, $\lambda=1, \cdots, k$. This k-plane is a k-tangent for the system if and only if the set of forms F_α vanishes whenever each indeterminate vector u_α is identified with an arbitrary one of the vectors v_λ. If we start with a k-tangent and seek a $(k+1)$-tangent containing it, the system Σ_{k+1} upon the components of the $(k+1)$th vector v_{k+1}^i is linear and homogeneous. A point x^i is an *ordinary* 0-tangent if the forms of degree zero in F_β vanish at x^i and if the matrix of the linear equations in F_β has maximum rank at x^i. By induction, a k-tangent is defined as ordinary if it contains an ordinary $(k-1)$-tangent and if for it the system Σ_{k+1} has maximum rank. The notation E_k will be reserved for an ordinary k-tangent.

An integral variety of k dimensions is an *ordinary solution* if the k vectors tangent to its parametric curves constitute an E_k for every point. The notation V_k will be reserved for an ordinary solution of k dimensions.

The maximum dimension g for an ordinary k-tangent is called the *genus*. Cartan previously gave for the genus of a linear system a lower bound which is not repeated or generalized here. It is

$$(n-r)/(r+1) \leqq g,$$

where n is the number of variables and r is the number of equations.

The existence theorem asserts there is a V_k if and only if k does not exceed the genus. More precisely, it states that if $k<g$, through V_k there passes a V_{k+1} depending on specified arbitrary elements. This is proved by an appeal to the Cauchy-Kowalevsky theorem.

A system is in involution with respect to a set of k variables $x^1, \cdots, x^k$ if it possesses a V_k on which $x^1, \cdots, x^k$ are independent, that is, on which $[dx^1 \cdots dx^k] \neq 0$. Necessary and sufficient conditions for a system to be in involution are given in terms of the characters, which are non-negative integers determined by the ranks of the systems Σ_i used to define E_k.

The final chapter on existence theorems describes the author's method of *prolongation*, whose purpose (p. 112) is to make each solution of a given system appear as an ordinary solution of a system in involution. As the author states (p. 120), singular solutions escape the theoretical discussion of prolongation, but for particular systems prolongation can be effected without this drawback. It is therefore recognized that the general process of prolongation and the field of its applicability have still to be rendered precise.

There are two fundamental requirements for a theory of differential systems: (1) a normal form for which a precise theorem can be rigorously proved; (2) a process for reduction to normal form. The process in (2) has two aspects: (2.1) formation of integrability conditions; (2.2) application of an implicit function theorem. Cartan's normal form is essentially a set of Cauchy-Kowalevsky systems and is less inclusive than Riquier's orthonomic system. Cartan's operation (2.1) involves prolongation, whereas Riquier's is precise. As for (2.2), the theories are subject to the same difficulties, but for algebraic systems these have been removed from Riquier's theory by Ritt.

The last two chapters give applications to the differential geometry of surfaces (Chapter VII) and of certain higher spaces (Chapter VIII). In addition to furnishing excellent illustrations for the preceding theory, they bring out in novel fashion the dependence of the solutions of the problems on the arbitrary elements. Conformal and isometric correspondences, Weingarten surfaces, isothermic surfaces and triply orthogonal systems are among the score of topics treated.

Schlaefli's theorem that every Riemannian space of n dimensions can be immersed in Euclidean space of $n(n+1)/2$ dimensions is discussed in great detail (pp. 199–210) for the particular case $n=3$ and reference is made to the proofs of Janet and Cartan for the general n. In this connection, it might be remarked that the detailed discussion given by Janet (Annales de la Société Polonaise de Mathématique vol. 5 (1926) pp. 39–40) for the case $n=2$ is convincing, whereas the counter-example given for the same case by Forsyth (*Intrinsic geometry of ideal space*, vol. 1, pp. 231–233) is not. A relatively simple proof of the theorem would be highly desirable. Cartan's discussion of the case $n=3$ may help in that direction.

J. M. THOMAS

Tables of fractional powers. Prepared by Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1946. 489+30 pp. $7.50.

The tables here printed yield the values of A^x and x^a. For example, there are tables of A^x for $A=10, \pi, 10^{-3}P$ (where P is a prime between 100 and 1000), as well as for other values. Thus 10^x is given to 15 decimals for $0.001 \leqq x \leqq 1.000$ with x advancing in intervals of .001. The function x^a is computed for the values $a=\pm 1/2, \pm 1/3, \pm 2/3, \pm 1/4$ with $0 \leqq x \leqq 9.99$ in intervals of .01. There is a bibliography with 76 titles and an introduction by Dr. Lowan in which the method of computation of the tables is explained and the accuracy of interpolation is illustrated by examples.

E. R. LORCH

Tables of the modified Hankel functions of order one-third and of their derivatives. Cambridge, Harvard University Press; London, Oxford University Press, 1945. 36+235 pp. $10.00.

This set of tables is the first to be published by the Computation Laboratory of Harvard University. The functions here considered are solutions of Stokes' differential equation $d^2u/dz^2+zu=0$ and were needed in connection with the work of the Radiation Laboratory on diffraction and refraction of waves. Solutions to Stokes' equation are $h_1(z)=(k/i\pi)\int_{L_1}e^{zt+t^3/3}dt$ (where k is a constant and L_1 is an infinite broken line in the complex plane) and $h_2(z)$, which has a similar expression. It is the functions $h_i(z)$ and their derivatives $h_i'(z)$ which are tabulated. The tables give the real and imaginary parts to eight decimal places for $z=x+iy$ with $|x+iy| \leqq 6$ and x, y progressing in intervals of 0.1. The functions $h_i(z)$ are related to the Hankel functions of order 1/3 by the equations $h_i(z)=((2/3)z^{3/2})^{1/3}H^{(i)}_{1/3}((2/3)z^{3/2})$,

$i=1, 2$. Although the $H^{(i)}_{1/3}(x)$ are triple-valued functions, the $h_i(z)$ are single-valued.

There is a thorough discussion of the properties of the functions. The computation time was forty-five days. It is stated that without the calculator years would have been necessary for this task. This fact gives some indication of the capacity of the machine.

E. R. LORCH

A manual of operation for the automatic sequence controlled calculator. Cambridge, Harvard University Press; London, Oxford University Press, 1946. 6+561 pp. $10.00.

The "analytical engine" of Babbage has finally been realized in the Harvard Calculator described in this book. This device, a brilliant engineering achievement, was constructed and donated by the International Business Machines Corporation.

The Calculator is designed to carry out any sequence of calculations. A tape is punched in accordance with a certain code and this is inserted in the sequence control mechanism, a complicated arrangement of relays. Under the control of the sequence mechanism are a large number of storage counters which also function as adders, a "multiply-divide" unit, and various function units for $\log_{10} x$, 10^x, and $\sin x$. The sequence mechanism arranges for the transfer of a number in one unit to another, directs the operation of the various units and appropriately stores the result. The machine will print the tables it calculates.

The present volume is intended as a manual of operation for this device. Accordingly, we find a complete set of coding and plugging instructions and a chapter devoted to the solution of examples.

There are however a number of aspects of the present volume which are of more general interest. For instance, the descriptions of the relay controls, the multiply-divide unit, the storage counters and the function units would certainly appeal to many who are not concerned with the operation of the calculator.

In addition there is a long bibliography on numerical methods. The historical introduction in Chapter 1 is quite interesting and undoubtedly serves the present purpose, but it is not a complete history of even the direct ancestors of the present device.

Thanks are due to Professor Aiken and his associates for the preparation of this manual in a form which will permit a wide circulation.

F. J. MURRAY

NOTES

The University of Toronto is planning a Summer Seminar on Algebra and Theory of Numbers from August 15 to September 14, 1947. It is expected that there will also be courses of lectures on several subjects somewhat related to the above topics. Fuller details will be sent to the members of the Society at a later date.

A conference on the Problems of mathematics was held at Princeton University on December 17–19, 1946. It is planned to issue a descriptive pamphlet, and later a more complete monograph, covering the work of the conference.

A Graduate Department of the History of Mathematics has been established at Brown University.

The Harry Bateman Research Fellowship for postdoctoral research in pure mathematics has been established at the California Institute of Technology. The stipend is $3,000.00 for the academic year.

The French Academy of Sciences has announced the following awards for 1946: Carrière Prize, to Mr. Claude Chabauty of the University of Strasbourg; Francoeur Prize, to Mr. Laurent Schwartz of the University of Nancy; Émile Picard Medal, to Professor Maurice Fréchet of the University of Paris; Leonard Eugene Dickson Prize, to Professor Georges Bourion of the University of Alger; Montyon Prize, to Professor Jean Ville of the University of Lyon; Grand Prize in Mathematical Science, to Professor Jacques Dufresnoy of the University of Bordeaux; Général Muteau Prize, to Mr. Joseph-Georges Darpas and Mr. Osée Marcus; Houllevigue Prize, to Dr. Gustave Choquet; Fondation Jerôme Ponti, to Professor Daniel Dugué of the University of Alger.

The Sylvester Medal of the Royal Society has been awarded to Professor G. N. Watson of the University of Birmingham in recognition of his distinguished contributions to mathematical analysis.

The Duodecimal Society of America has conferred its Annual Award for 1947 upon Mr. R. H. Beard of the New York Telephone Company.

Professor Marston Morse of the Institute for Advanced Study and Professor P. A. M. Dirac of the University of Cambridge have been awarded honorary degrees by the University of Paris.

Professor J. P. Everett of Western Michigan College of Education, Kalamazoo, Michigan, has retired with the title emeritus.

Additional information concerning certain mathematicians in Holland has become available. Professor W. van der Woude of the University of Leiden has retired. At Amsterdam University, Professor D. van Dantzig of the Technical School of Delft has been appointed to a professorship; Dr. E. W. Beth has been appointed to an associate professorship of logic and the philosophy of science; Dr. G. H. A. Grosheide and Dr. J. Haantjes have been appointed to lectureships at the Free University of Amsterdam; Professor G. Mannoury has been awarded an honorary Ph.D. degree. At the Technical School of Delft, Dr. N. G. de Bruyn, Dr. S. C. van Veen, and Dr. C. Visser have been appointed to professorships. At Groningen University, Dr. J. H. C. Gerretsen and Dr. C. S. Meyer have been promoted to professorships. At the University of Utrecht, Dr. H. Freudenthal has been appointed to a professorship.

Dr. Leonidas Alaoglu has accepted an appointment with the Office of Operations Analysis, Washington, D. C.

Professor J. V. Atanasoff of Iowa State College of Agriculture and Mechanical Arts has been appointed Chief of the Acoustics Division at the Naval Ordnance Laboratory, Silver Spring, Maryland.

Associate Professor J. J. Barron of Marquette University has been appointed to a professorship at Marshall College, Huntington, West Virginia.

Mr. A. H. Bowker has been appointed to an assistant professorship at Stanford University.

Professor Gregory Breit of the University of Wisconsin has been appointed to a professorship at Yale University.

Mr. N. A. Brigham has been appointed Senior Research Engineer with Republic Aviation Corporation, Farmingdale, New York.

Mr. Leon Brillouin has been appointed to a professorship of applied mathematics at Harvard University.

Dr. F. W. Brown has accepted a position as design specialist with North American Aviation, Inc., Los Angeles, California.

Dr. J. W. Calkin of the California Institute of Technology has been appointed to an associate professorship at Rice Institute.

Professor Tobias Dantzig of the University of Maryland has retired.

Professor J. L. Doob of the University of Illinois has been appointed to a visiting professorship at Columbia University.

Professor J. A. H. Duffie of the University of Western Ontario has been appointed to a professorship of chemistry at the University of Ottawa.

Adjunct Professor Mary L. Foster of the University of South Carolina has been appointed to an associate professorship at Henderson State College, Arkadelphia, Arkansas.

Mr. J. E. Freund has been appointed to an assistant professorship at Alfred University.

Dr. B. E. Gatewood has been appointed to an associate professorship at the Army Air Force Institute of Technology, Wright Field, Dayton, Ohio.

Dr. M. A. Girshick has been appointed statistician with the Bureau of the Census.

Dr. A. D. Hestenes has accepted a position as research engineer with the Franklin Institute in Philadelphia.

Associate Professor Grace M. Hopper of Vassar College has accepted a position in the Cruft Research Laboratory at Harvard University.

Dr. M. L. Kales has accepted a position as radio engineer with the Naval Research Laboratory, Washington, D. C.

Professor C. C. MacDuffee of the University of Wisconsin will be visiting professor at the University of Puerto Rico.

Associate Professor A. N. Milgram of the University of Notre Dame has been appointed to an associate professorship at Syracuse University.

Associate Professor E. K. Paxton of Washington and Lee University has resigned.

Miss Charlotte B. Shapira has accepted a position at the Naval Ordnance Laboratory, White Oak, Maryland.

Dr. R. W. Shepard of the University of California has been appointed to an assistant professorship at New York University.

Dr. F. C. Smith of the Lincoln National Life Insurance Company, Fort Wayne, Indiana, has been appointed to an associate professorship at the College of St. Thomas.

Professor H. S. Vandiver of the University of Texas will be visiting professor at the University of Indiana during the present term.

Assistant Professor L. B. Williams of Hamilton College has been appointed to an assistant professorship at Reed College.

The following promotions are announced:

L. V. Ahlfors, of Harvard University, to a professorship.

Garrett Birkhoff, of Harvard University, to a professorship.

R. W. Cowan, of the University of Alabama, to an assistant professorship.

Saunders MacLane, of Harvard University, to a professorship.

G. J. O'Boyle, of Catholic University of America, to an assistant professorship.

Henry E. Smith, of Dickinson College, Carlisle, Pennsylvania, to an assistant professorship.

The following appointments to instructorships are announced: Catholic University of America: Mr. R. W. Moller, Mr. S. J. Rosenfeld; Hofstra College: Mr. Frank Hawthorne, Mr. Bernard Mason; Hunter College: Mr. Paul Brock; Michigan State College: Mr. Cornelius Groenwoud; University of Missouri: Dr. P. B. Burcham; University of Nebraska: Mrs. Margaret Ehlers, Mr. W. T. Lenser, Mrs. T. A. Pierce, Miss F. E. Pool, Mr. G. D. Rippe; University of Oregon: Mrs. Ethel Lawrence, Mr. W. G. Scobert; Queens College: Miss Louise H. Ercolano, Miss Florence E. Gerhardt, Miss Bernice J. Lehrman, Miss Aida Kalish, Mrs. Charlotte L. R. Knag; Rider College, Trenton, New Jersey: Mr. Emanuel Levine; Syracuse University, Mr. J. B. Greeley.

Information has been received which confirms the deaths of the Dutch mathematicians M. M. Biedermann and L. W. Nieland during the German occupation.

Professor H. C. Plummer, formerly of the Military College of Science, Woolwich, died September 30, 1946, at the age of seventy years.

Dr. S. A. Lepeshkin, a former member of the Society, died August 20, 1946, at the age of fifty-nine years.

Professor G. B. Sweazy of Westminster College, Fulton, Missouri, died August 10, 1946.

Professor C. C. Wagner of Pennsylvania State College died on May 23, 1946. He had been a member of the Society since 1930.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

107. Iacobo Barsotti: *Structure theorems for algebraic algebras without a finite basis.*

A few elementary properties are established, and a complete structure theorem for division algebras of type 2 (locally finite) and a countable basis is given (Theorem 17). A few properties connecting arithmetic with the structure of algebras over p-adic fields are found. Theorem 22 reduces the main unsolved problem to a simpler one, when the underlying field is of a particular kind, including algebraic and local fields. Theorem 24 analyzes the structure of algebras of type 1 with a noncountable basis. Such a structure will be treated in another paper. (Received December 26, 1946.)

108. Iacobo Barsotti: *Valuations in division algebras without a finite basis.*

The author studies arithmetic in infinite division algebras, based on nonarchimedean valuations. Algebras satisfying a certain condition are called valuable, and a relation is found between this condition and the algebraic character of the algebra. Ideal theory in valuable algebraic algebras is studied. Algebraic algebras over local fields are shown to be valuable under a very general assumption. Three *types* of algebras are introduced, according to the existence of particular finite subalgebras. A principal theorem on the structure of unramified infinite local fields is found, and another one on the structure of local algebras of type 2. At the end of the paper an unsolved problem is stated together with some clues about its tentative solution. A new proof is given of a particular case of a known theorem about algebras of finite degree. (Received December 26, 1946.)

109. P. T. Bateman: *Modular subgroups of finite index.* Preliminary report.

The modular group (Modulgruppe) is the group of linear fractional transformations $\tau' = (a\tau + b)/(c\tau + d)$, where a, b, c, d are integers such that $ad - bc = 1$. The subgroups of the modular group most frequently encountered are those which can be defined by means of certain arithmetical congruence relations on a, b, c, d with respect to some positive integer, called the Stufe of the subgroup. By using the geometry of the transformations (cf. Ford, *Automorphic functions*, New York, 1929, chap. 3, in particular Theorem 22) it is proved here that these congruence subgroups are the only subgroups of finite index. (Received February 1, 1947.)

110. Richard Brauer: *Applications of induced characters.*

By application of a theorem on induced characters proved in an earlier paper it is shown that if G is a group of finite order and if q is the least common multiple of the orders of the elements of G, then every representation of G can be written in the field of the qth roots of unity. This improves a result given by the author (Amer. J. Math. vol. 67 (1945) pp. 461–471). As a further application of the theorem on induced characters it is shown that the characters of a group G of finite order are uniquely determined provided that the following information is given: (I) The linear characters of those subgroups H of G which are direct products of cyclic groups and groups of prime power order. (II) The manner in which the classes of conjugate elements of G break up into classes of H, if only elements of H are considered (H denoting the subgroups of the same type as in (I)). (Received January 27, 1947.)

111. Richard Brauer: *On Artin's L-series with general group characters.*

It is shown that if G is a group of finite order, every character of G can be expressed as a linear combination with integral rational coefficients of characters ω^* such that every ω^* is a character of G induced by a linear character ω of a subgroup of G. This result had been conjectured by E. Artin (Abh. Math. Sem. Hamburgischen Univ. vol. 3 (1924) pp. 89–108; vol. 8 (1931) pp. 292–306). It now follows that Artin's L-series with general group characters are meromorphic functions. (Received January 27, 1947.)

112. Richard Brauer: *On the zeta-functions of algebraic number fields.*

It is shown that if the algebraic number field K is normal over the subfield k, the quotient $\zeta(s, K)/\zeta(s, k)$ of the corresponding zeta-functions is an integral function of the complex variable s. A proof is given for the following theorem, conjectured by C. L. Siegel and proved by him in special cases: Consider all algebraic number fields k of a fixed degree n. Let d be the discriminant of k, let h be the number of classes of ideals in k, and let R be the regulator of k. Then $\log (hR) \sim \log(|d|)^{1/2}$ for $|d| \to \infty$. (Received January 27, 1947.)

113. Leonard Carlitz and E. L. Cohen: *Divisor functions of polynomials in a Galois field.*

The main result given in this paper is concerned with the number of solutions of the equation $(\alpha_1 + \cdots + \alpha_t)F = \alpha_1 X_1 Y_1^e + \cdots + \alpha_t X_t^e Y_t$ in primary polynomials of $GF[p^n, x]$ of degree k, ek respectively, where $e \geq 1$ is fixed. For $e = 1$, the result reduces to a known one. The proof is considerably simpler than that used previously. (Received January 28, 1947.)

114. Dr. P. W. Carruth: *Generalised power series fields.*

Kaplansky (Duke Math. J. vol. 9 (1942) pp. 303–321 and vol. 12 (1945) pp. 243–248) has shown that if the characteristic of a field that is maximal with respect to a valuation is the same as that of its residue class field, under certain conditions the maximal field is analytically isomorphic to a power series field. In this paper, a generalized power series field is constructed. Criteria not dependent on the characteristics of a maximal field and its residue class field are stated for a maximal field to be analytically isomorphic to this generalized power series field. (Received January 23, 1947.)

115. E. L. Cohen: *Sums of an even number of squares in* $GF[p^n, x]$.

The problem discussed in this paper is an extension of one considered by Carlitz (Trans. Amer. Math. Soc. vol. 35 (1933) pp. 397–410). Suppose $\alpha_1, \cdots, \alpha_{2s}$ are nonzero elements of $GF(p^n)$, $p\neq 2$, $\epsilon=\alpha_1+\cdots+\alpha_m$ $(2s\geqq m\geqq 1)$, and let F be a polynomial of $GF[p^n, x]$. The problem is to find the number of solutions of $\alpha F=\sum_1^{2s}\alpha_i X_i^2$ in primary polynomials $X_1, \cdots, X_m$ of degree k and arbitrary polynomials $X_{m+1}, \cdots, X_{2s}$ of degree less than k, for the cases in which (1) $\alpha=\epsilon\neq 0$, F primary of degree $2k$, and (2) $\alpha=1$, $\epsilon=0$, F arbitrary of degree less than $2k$. The number of solutions in either case is given by the function $\rho_{s-1}(F, \lambda)$, defined in the paper referred to above, where $\lambda=+1$ or -1 according as $(-1)^s\alpha_1\cdots\alpha_{2s}$ is or is not a square. (Received January 28, 1947.)

116. A. H. Diamond and J. C. C. McKinsey: *A relation between algebras and their subalgebras.*

Certain sets of postulates for Boolean algebra do not contain equations in more than three variables. It obviously follows that if an algebra has the property that every subalgebra generated by three elements is a Boolean algebra then the algebra is itself a Boolean algebra. The question arises whether the number of elements generating the subalgebra can be reduced to two. This question is answered in the negative by giving an example of an algebra having the property that every subalgebra generated by two elements is a Boolean algebra while the algebra itself is not. A corollary is that every set of equations defining Boolean algebra must include at least one equation in three variables. Analogous results are established for groups, rings, and other algebras. (Received December 5, 1946.)

117. R. J. Levit: *The nonexistence of a certain type of odd perfect number.*

Let $n=a_0a_1\cdots a_t$ be an integer with prime power factors $a_i=p_i^{\alpha_i}$, where the p_i are distinct primes, $i=0, 1, \cdots, t$. For n to be perfect it is necessary that exactly one of the prime powers, say a_0, have an even divisor sum $\sigma(a_0)$. Then, if $\sigma_0=\sigma(a_0)/2$, $\sigma_i=\sigma(a_i)$, $i=1, 2, \cdots, t$, the condition for n to be perfect may be written $n=a_0a_1\cdots a_t$ $=\sigma_0\sigma_1\cdots\sigma_t$. All even perfect numbers are known to be of the form $n=2^{q-1}(2^q-1)$, where 2^q-1 is prime, so that the σ_i are the a_i in reverse order. The question presents itself whether there may exist odd perfect numbers for which analogously the σ_i are a mere rearrangement of the a_i. In this paper it is proved by elementary methods that no odd perfect numbers of this type can exist. (Received January 17, 1947.)

118. E. G. Straus: *On the existence of square-free numbers of the form* ax^2+b.

It is shown that there are infinitely many square-free integers among the numbers of the form ax^2+b, $x=1, 2, \cdots$, where a, b are integers whose g.c.d. (a, b) is square-free. An elementary counting process used in the proof yields an asymptotic formula for a number $S(n)$ of square-free numbers of the form ax^2+b, $x \leqq n$. The method of proof can be used for several generalizations. (Received January 7, 1947.)

119. R. D. Wagner: *The generalized Laplace equations in a function theory for commutative algebras.*

The author considers the generalization of the theory associated with the Laplace equation $\partial^2 u/\partial x^2+\partial^2 u/\partial y^2=0$ in the theory of functions of a complex variable to that of functions in a commutative linear associative algebra with a principal unit. The usual theory extends readily if the algebra is a Frobenius algebra. Using matric methods introduced by Ward (Duke Math. J. vol. 7 (1940)), the author obtains the generalized Cauchy-Riemann equations as necessary and sufficient conditions for analyticity of a function in the algebra A of order n. The generalized Laplace equations are then obtained as necessary and sufficient conditions that a function u of n variables shall be a component of an analytic function. One form of the generalized Laplace equations is equivalent to the statement that the Hessian matrix of u shall be the parastrophic matrix of some number of A. (Received December 10, 1946.)

ANALYSIS

120. Richard Bellman: *On the boundedness of solutions of nonlinear differential and difference equations.*

The purpose of the author is to discuss the behavior of solutions of the nonlinear system of differential equations: $dx_i/dt=\sum_{j=1}^{N} a_{ij}x_j+f_i(x_1, x_2, \cdots, x_N, t)$, $i=1, 2, \cdots, n$, as $t \rightarrow \infty$, under various restrictions upon the matrix (a_{ij}), the functions $f_i(x, t)$, and the initial values. The more general case where the right-hand side contains the derivatives dx_i/dt is also considered. The three methods used are the method of successive approximations, the fixed-point method due to Birkhoff and Kellogg, and the method of approximating to a differential equation by a difference equation. Analogous results are derived for nonlinear difference equations. (Received January 22, 1947.)

121. Stefan Bergman: *Functions satisfying linear partial differential equations and their properties.*

The author investigates functions $\psi(\theta, \mathrm{H})$ of two real variables which satisfy the equation $(-\mathrm{H})^s\psi_{\theta\theta}+\psi_{\mathrm{HH}}=0$ for $\mathrm{H}\leqq 0$, and $(-\mathrm{H})^s\psi_{\theta\theta}+\psi_{\mathrm{HH}}=0$, for $\mathrm{H}\geqq 0$, $s>-2$, $\psi_{\theta\theta}=(\partial^2\psi/\partial\theta^2)$, $\cdots$. In this paper the initial-value problem is considered. To this end $\psi(\theta, \mathrm{H})$ is expressed in terms of $T^{(0)}(\theta)$ and $T^{(1)}(\theta)$, the prescribed values of ψ and $\partial\psi/\partial\mathrm{H}$ respectively on the line $\mathrm{H}=0$. It is shown that the function ψ satisfying the conditions $\psi(\theta, 0)=0$, $\psi_{\mathrm{H}}(\theta, 0)=[\partial(\theta, \mathrm{H})/\partial\mathrm{H}]_{\mathrm{H}=0}=T^{(1)}(\theta)$, $\theta^{(0)}\leqq\theta\leqq\theta^{(1)}$, where $T^{(1)}(\theta)$ is an analytic function of the real variable θ, can be written in the form $2\pi^2 i(2+s)\psi(\theta, \mathrm{H})=\mathrm{H}\int_{\chi=-1}^{1}\int_{t=0}^{\pi}\int_{C}(1-\chi)^{(s+2)^{-1}}T^{(1)}(\vartheta)\,dt\,d\chi\,d\vartheta/(\vartheta-\theta)\,[1+4(2+s)^{-2}(\vartheta-\theta)^{-2}(-\mathrm{H})^{s+2}\chi\sin^2 t]$, $\mathrm{H}<0$, and a similar expression holds for $\mathrm{H}>0$. C is a simple closed curve in the regularity domain of $T^{(1)}(\vartheta)$ (considered as a function of the complex variable $\vartheta=\theta+i\Theta$), which curve includes the interval $\theta^{(0)}\leqq\theta\leqq\theta^{(1)}$ of the real axis. An analogous formula holds for the solution ψ satisfying the condition $\psi(\theta, 0)=T^{(0)}(\theta)$, $\psi_{\mathrm{H}}(\theta, 0)=0$. Using these formulae, the author investigates the connections which exist between the location and the nature of the singularities of $T^{(0)}(\vartheta)$ and

$T^{(1)}(\vartheta)$ on the one hand, and the properties (in the large) of $\psi(\theta, \mathrm{H})$ on the other. (Received January 30, 1947.)

122. Stefan Bergman: *On functions satisfying certain partial differential equations of elliptic type and their representation.* I.

The author considers function pairs (ϕ, ψ) satisfying the system of equations: $\phi_x = l\psi_y$, $\phi_y = -l\psi_x$, where $l = l(x, y)$ is a differentiable function which is positive in a domain $\mathfrak{P}$. Let $\mathfrak{B}$, $\overline{\mathfrak{B}} \subset \mathfrak{P}$, be a connected domain bounded by $(n+1)$ curves $\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_n$. The author defines single-valued functions ϕ and ψ which possess logarithmic singularities and singularities of nth order respectively at a prescribed point (x_0, y_0). In analogy to the case of analytic functions of a complex variable, he introduces integrals $(\phi^{(k,s)}, \psi^{(k,s)})$, $k = 1, 2, \cdots, n$, of the first kind and sth type, $s = 1, 2$. Here $\phi^{(k,1)}$ and $\psi^{(k,2)}$ are single-valued and regular in $\mathfrak{B}$, and are equal to 1 on $\mathfrak{a}_k$ and vanish on the remaining $\mathfrak{a}_\nu$. Integrals $(\phi^{(L,s)}, \psi^{(L,s)})$ of the third kind are then defined. $\phi^{(L,1)}$ and $\psi^{(L,2)}$ are single-valued functions and possess at the prescribed point $(x_0, y_0) \in \mathfrak{B}$ a logarithmic singularity. It is proved that $\phi^{(k,1)}(x_0, y_0) = (-1/2\pi) \oint_{\mathfrak{a}_k} \phi^{(L,1)}(x, y; x_0, y_0) ds$; various other relations between the periods of the above integrals are derived. (Received December 23, 1946.)

123. Stefan Bergman: *On functions satisfying certain partial differential equations of elliptic type and their representation.* II.

The author considers functions satisfying the equation $L(\phi) = \phi_{xx} + \phi_{yy} + 4P\phi = 0$, where $P = P(x, y)$ is nonpositive in the domain $\mathfrak{B}$ under consideration. Two solutions $\phi^{(\nu)}, \phi^{(\mu)}$ are said to be orthonormal in $\mathfrak{B}$ with respect to L if $\{\phi^{(\nu)}, \phi^{(\mu)}\}_{L\mathfrak{B}} = \iint_{\mathfrak{B}} [\phi_x^{(\nu)} \phi_x^{(\mu)} + \phi_y^{(\nu)} \phi_y^{(\mu)} - 4P\phi^{(\nu)}\phi^{(\mu)}] dx dy = \delta_{\nu\mu}$. ($\delta_{\nu\mu}$ is the Kronecker delta.) Let $(\phi^{(\nu)})$, $\nu = 1, 2, \cdots$, be a system of solutions of L which are orthonormal in $\mathfrak{B}$ with respect to L. It is proved that $\sum_{\nu=1}^{\infty} [\phi^{(\nu)}(x, y)]^2$, $\sum_{\nu=1}^{\infty} [\phi_x^{(\nu)}(x, y)]^2$, $\sum [\phi_y^{(\nu)}(x, y)]^2$ converge uniformly and absolutely in every closed subdomain which lies in $\mathfrak{B}$. If the system $(\phi^{(\nu)})$ is complete with respect to the class of solutions ϕ of L for which $\{\phi, \phi\}_{L\mathfrak{B}} < \infty$, then every solution ϕ can be represented in the form $\phi = \sum_{\nu=1}^{\infty} a_\nu \phi_\nu$, $a_\nu = \oint_{\mathfrak{c}} \phi (\partial \phi^{(\nu)} / \partial n) ds - \oint_{\mathfrak{c}} \phi^{(\nu)} (\partial \phi / \partial n) ds$. Here n is the inward normal and ds a line element of the boundary $\mathfrak{c}$ of $\mathfrak{B}$. (Received December 23, 1946.)

124. Stefan Bergman and Menahem Schiffer: *A representation of Green's and Neumann's functions in the theory of partial differential equations of second order.* Preliminary report.

Generalizing results by Schiffer (Duke Math. J. vol. 13 (1946) pp. 529–540) which permit the representation of Green's function of Laplace's equation in terms of orthonormal analytic functions of one complex variable, and using orthogonal functions introduced in the previous abstract, the authors consider the Green's function $G(Z; \zeta)$ and Neumann's function $N(Z; \zeta)$, $Z = (x, y)$, $\zeta = (\xi, \eta)$, of differential equations $L(\phi) = \Delta\phi + 4P\phi = 0$, where $P \leqq 0$ in the domain under consideration. Let $\mathrm{K}(Z; \zeta) = \sum_{\nu=1}^{\infty} \phi^{(\nu)}(Z) \phi^{(\nu)}(\zeta)$ be the kernel function of a complete system of functions which are orthonormal in $\mathfrak{B}$ with respect to L, that is, for which $\{\phi^{(\nu)}, \phi^{(\mu)}\}_{L\mathfrak{B}} = \delta_{\nu\mu}$. (See the previous abstract.) Every solution ϕ of $L(\phi) = 0$ satisfies the integro-differential equation $\phi(Z) = \{\mathrm{K}(Z; \zeta), \phi(\zeta)\}_{L\mathfrak{B}}$. From this result it follows that $\mathrm{K}(Z; \zeta) = (2\pi)^{-1}(G(Z; \zeta) - N(Z; \zeta))$. Let $S(Z; \zeta)$ be a fundamental solution of $L(S) = 0$. Then $G(Z; \zeta) = S(Z; \zeta) + \{S(Z, T), \mathrm{K}(T; \zeta)\}_{L\mathfrak{B}}$. The variation $\delta\mathrm{K}(Z; \zeta)$, when $\mathfrak{B}$

varies, is given by $\oint_c [K(Z;T)K(T;\zeta)]\delta n ds$, $T=(u, v)$, $[\phi, \psi]=\phi_u\psi_u+\phi_v\psi_v-4P\phi\psi$, n being the inward normal, c the boundary curve of $\mathfrak{B}$, ds a line element of c. The same formulae hold for G and N. Analogous results can be obtained for much more general equations in two or n, $n>2$, variables. (Received December 23, 1946.)

125. V. F. Cowling: *Some results for Dirichlet series.*

Let $f(s)=\sum_{n=1}^{\infty} a_n e^{-(\log n)s}$ with abscissa of convergence $\sigma_c<\infty$. Let $l-1<h<l$ where l is integral and positive but otherwise arbitrary. Denote by D the domain in the w-plane $\psi_2\leqq \operatorname{Arg}(w-h)\leqq\psi_1$ where $0<\psi_1\leqq\pi/2$ and $-\pi/2\leqq\psi_2<0$. Let $a(w)$ be a function regular in D, with the possible exception of the point at infinity, for which $a(n)=a_n$, $n=0, 1, 2, \cdots$. Suppose, for $w=h+Re^{i\psi}$ in D and $R\geqq R_1$, that $|a(h+Re^{i\psi})|<R^K$ for some K. Then $f(s)=g(s)+h(s)$ where $g(s)$ is an integral function of s and $h(s)$ is regular for s in any closed bounded domain contained in $\mathfrak{R}(s)>K+1$. If for $w=h+Re^{i\psi}$ in D and $R\geqq R_1$, $|a(h+Re^{i\psi})|<R^K e^{-LR\sin\psi}$ for some K and some $0<L<2\pi$, then $f(s)$ defines an integral function of s. (Received January 24, 1947.)

126. J. B. Díaz and H. J. Greenberg: *Upper and lower bounds for the solution of the general biharmonic boundary value problem.*

The authors consider the equation $\Delta\Delta w=p(x, y)$ in a domain D, subject to boundary conditions on w and $\partial w/\partial n$. Let U and V be the sets of all functions satisfying the boundary conditions and the differential equation, respectively. Integral expressions involving arbitrary functions of U and V are found which provide upper and lower bounds for the solution w at any given point of D. The method uses Fourier type expansions for w in functions of U and V separately to yield Bessel inequalities bounding an integral of w from above and below. (These inequalities are interpretable as complementary variational principles for w; one principle extending over the class U, the other over the class V.) Introduction of a boundary value problem, connected with the fundamental solution of the biharmonic equation, leads to bounds directly on w at the given point of D. The bounds are successively improved as additional Fourier coefficients are introduced. Conditions are established for convergence of the bounds to the desired value of w. (Received January 22, 1947.)

127. R. J. Duffin: *Function classes invariant under the Fourier transform.*

A function $f(x)$ belongs to class J if: (a) $(-xd/dx)^n f(x)\geqq 0$; $0<x<\infty$ $n=0, 1, 2, \cdots$, (b) $f(x)\to 0$ as $x\to\infty$, (c) $xf(x)\to 0$ as $x\to 0$. It is shown that the Fourier sine transform of $f(x)$ also belongs to the class J. For example, $x^{-1/2}$ is obviously of class J and its transform is $cx^{-1/2}$. A self-contained proof is given which hinges on a lemma which asserts that $x^{-1}f(x^{-1})$ is of class J. With a slight modification a similar theorem is valid for the cosine transform. Another invariant class K is characterized essentially by the condition $(-d/xdx)^n f(x)\geqq 0$. (Received December 23, 1946.)

128. J. J. Gergen and F. G. Dressel: *A minimal problem for harmonic functions.*

Let $\{\theta_k\}$, $\{h_k\}$, $k=1, 2, \cdots, n$, be arbitrary sets of n real numbers, $0\leqq\theta_1<\theta_2<\cdots<\theta_n<2\pi$. Let $\lambda_1(\theta)$, $\lambda_2(\theta)$ be step functions of period 2π determined by $\lambda_1(\theta)=\min(h_k, h_{k+1})$, $\lambda_2(\theta)=\max(h_k, h_{k+1})$, $\theta_k\leqq\theta<\theta_{k+1}$. Let Γ be the class of functions,

$U(r, \theta)$, each of which is harmonic in the interior S of the circle $r=1$ and satisfies $\lambda_1(\theta) \leqq \liminf U(r, \theta)$, $\limsup U(r, \theta) \leqq \lambda_2(\theta)$, $r \to 1-$, for almost all $0 \leqq \theta < 2\pi$. The authors consider minimizing in Γ the functional $A[U] = \iint_S |\nabla U|^2 dS$, where $|\nabla U|$ is the magnitude of the gradient of U. The chief results are: (1) The lower bound A_0 of $A[U]$ for $U \subset \Gamma$ is finite. (2) $A[U] = A_0$ at least once in Γ. (3) $A[U] = A_0$ at most once in Γ unless all the h_k of even index are greater than, or are less than, all those of odd index. (4) If $U \in \Gamma$, $A[U] = A_0$, and if $F(z)$ is analytic for $|z| < 1$ and has U as real part, then $[zF'(z)]^2 = i\sum_{k=1}^{n} B_k(z+z_k)/(z-z_k)$, $z_k = e^{i\theta_k}$, where B_k are real constants such that $\sum_{k=1}^{n} B_k = \sum_{k=1}^{n} B_k/z_k = 0$. For $n=4$, A_0 and F are determined in terms of θ_i and h_i. The analysis is based on Douglas' paper (Trans. Amer. Math. Soc. vol. 33 (1931) pp. 263–321). (Received December 12, 1946.)

129. Irving Gerst: *Meromorphic functions with simultaneous multiplication and addition theorems.*

This paper is devoted to the characterization of all meromorphic functions $f(z)$ for which there exist simultaneously relations of the form $f(mz) = R[f(z)]$, $f(z+h) = S[f(z)]$, where m and h are complex numbers and R and S are rational functions. When $|m| > 1$, the case of chief interest, it is shown that if $f(z)$ is transcendental it must be periodic. Then, if results of Ritt (Trans. Amer. Math. Soc. vol. 23 (1922) pp. 16–25) on periodic functions with multiplication theorems are utilized, it is found that except for $f(z)$ a linear function of e^{az}, h, when not a period, must be a suitable half or third of a period. The corresponding functions and their addition theorems are given explicitly. Other results are: (1) If $f(z)$ is rational it must be a linear function of z. (2) The linear function of z is the only type possible when $|m| \leqq 1$ except when m is a rational root of unity. (Received January 11, 1947.)

130. R. N. Haskell and W. H. Bradford: *Sub-biharmonic functions.* Preliminary report.

A definition of this class of functions is made by use of areal derivatives as defined by Ridder (Nieuw Archief voor Wiskunde (2) vol. 16 (1929–1930) p. 63). Let $D_{x,y}\theta_B(J)$ denote the areal derivative of a biharmonic function $B(x, y)$ and let $u(x, y)$ be of class c'. If $D_{x,y}\theta_u(J)$ is subharmonic in a domain G and if $D_{x,y}\theta_u(J) \leqq D_{x,y}\theta_B(J)$ and also $u(x, y) \leqq B(x, y)$ on boundary of any subdomain G' of G, then $u(x, y) \leqq B(x, y)$ in interior of G' and $u(x, y)$ is sub-biharmonic. A characterization of sub-biharmonic functions is made in terms of areal and peripheral means. Let $L(u; x_0, y_0; r)$ and $A(u; x_0, y_0; r)$ represent the ordinary peripheral and areal means respectively on a circle with center at (x_0, y_0) and radius r. Further let L_1 and A_1 be defined by the relations $2\pi(1-\mu^2)L_1(u; x_0, y_0; \mu r, r) \leqq \int_0^{2\pi}[u(\mu r, \theta) - \mu^2 u(r, \theta)]d\theta$ and $\pi r^2(1-\mu^2)A_1(u; x_0, y_0; \mu r, r) \leqq \int_0^r\int_0^{2\pi}[u(\mu s, \theta) - \mu^2 u(s, \theta)]s\,ds\,d\theta$ where $0 < \mu < 1$ but fixed. It is shown that $L_1 \geqq A_1$ and both are increasing functions of r. Necessary and sufficient conditions that $u(x, y)$ shall be sub-biharmonic in G are that for all circles in G either $u(x_0, y_0) \leqq 2A - L$, $u(x_0, y_0) \leqq L_1$, or $u(x_0, y_0) \leqq A_1$. (Received December 26, 1946.)

131. Tibor Rado: *On two-dimensional concepts of bounded variation and absolute continuity.*

Let $x = x(u, v)$, $y = y(u, v)$ be continuous functions in the unit square $Q: 0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$. Then these equations define a continuous mapping T. For such a mapping, the author defined an essential multiplicity function $\mu(x, y)$ (See Duke Math. J. vol. 4 (1938) pp. 189–221). During the war, analogous researches were carried on in Italy by L. Cesari, whose work is based upon a certain multiplicity function $\Psi(x, y)$ termed the characteristic function of the mapping T (see Atti R. Accademia d'Italia vol. 13 (1943) pp. 1323–1481). The main purpose of the present paper is to show that $\Psi(x, y) = \mu(x, y)$ with the exception of a countable set. As a consequence, it follows that the concepts of bounded variation and of absolute continuity developed by Cesari for continuous mappings T agree with concepts introduced and studied by the author and by P. V. Reichelderfer (see Trans. Amer. Math. Soc. vol. 49 (1941) pp. 258–307 and vol. 53 (1943) pp. 251–291). (Received January 1, 1947.)

132. L. B. Robinson: *Generation of an infinite system of functions with the unit circle as singular line from the Garvin series.*

Let $G(x)$ be a Garvin series. Let $u(x) = \int_0^x G(x_1)u(x_1^p)dx_1 + u_0$ whose adjoint is $u(x) = \int_{\infty}^{x} G(x_1)u(x_1^p)dx_1 + v_0$. A solution $u_1(x)$ of the above has the unit circle as singular line like the Garvin series. In $u(x)$ replace $G(x)$ by $u_1(x)$. The solution $u_2(x)$ has the unit circle as singular line. Continue in this way until from the Garvin series is generated an infinite system of functions $u_1(x), u_2(x), \cdots, u_n(x), \cdots$ all of which admit the unit circle as singular line. (Received December 2, 1946.)

133. H. E. Salzer: *The approximation of numbers as sums of reciprocals.*

Every number x, $0<x<1$, admits a unique expansion of the form $x = (1/a_1) + (1/a_2) + \cdots$, where a_i are integers so chosen that after i terms, when the sum x_i has been obtained, a_{i+1} is the least integer such that $x_i + 1/a_{i+1}$ does not exceed x (denoted as an R-expansion). Similarly, to x_i one might add either $1/a_{i+1}$ or $1/(a_{i+1}-1)$, depending upon which addend gives a small value to $|x - x_{i+1}|$ (denoted as an $\bar{R}$-expansion). It is shown that rational numbers have finite R- and $\bar{R}$-expansions. Properties investigated include rapidity of convergence, including the closeness of approximation of x_i when written as p_i/q_i, and necessary and sufficient conditions for a given sum of reciprocals to be either an R- or $\bar{R}$-expansion. These properties are applied to prove the irrationality of certain infinite sums S, where the criterion of $|S - p/q| < 1/q^2$ for an infinitude of integers p and q is not applicable. Both the R- and $\bar{R}$-expansions are applied to the solution of algebraic and transcendental equations by a well known variant of the Newton-Raphson method, with the amount of digital work considerably less than in the usual procedures. (Received December 4, 1946.)

134. I. E. Segal: *Postulates for general quantum mechanics.*

A set of postulates for the observables in a physical system is given. On the basis of these it is shown that for any two observables there is a pure state (in essentially the sense of von Neumann and Weyl) of the system in which they have different expectation values, a spectral resolution for each observable is obtained, probability distributions of simultaneously measurable observables are defined, and an indeterminacy principle is derived. The postulates are partly algebraic (the observables form a real linear space in which each element generates an associative algebra) and partly metric (each observable has a maximum numerical value which is a Banach norm and has certain special properties). Mathematical examples of systems satisfying the postulates are the self-adjoint elements of a uniformly closed self-adjoint operator-algebra, with the bound as norm, and the real-valued continuous functions on a compact Hausdorff space, with the maximum as norm. (Received January 6, 1947.)

135. A. E. Taylor: *The inverse of a polynomial function of a closed operator.*

Let T be a closed distributive operator with domain D and range both in the complex Banach space X. Further let the resolvent set R of T be nonempty. If $n \geqq 1$, let D_n be the set of elements x of X such that $x, Tx, \cdots, T^{n-1}x$ lie in D. The author considers the problem: given a polynomial $P(\lambda)$ of degree n, and an element y in X, to find an element x in D_n such that $P(T)x = y$. Under the assumption that all the zeros of $P(\lambda)$ lie in R the author proves that the solution x exists and is unique. It is given by $x = By$, where B is a linear operator with domain X and range D_n, defined by $2\pi i B = \int \{1/P(\lambda)\}(T-\lambda I)^{-1}d\lambda$, the integration being extended counterclockwise over a set of nonoverlapping circles, one each about the zeros of $P(\lambda)$, each circle and its interior lying in R. Thus, if the inverse operator $(T-\lambda I)^{-1}$ is known explicitly, B may be calculated by the calculus of residues. The proof employs spectral methods developed previously by the author for the case in which T is assumed to be continuous, with domain X. (Received January 30, 1947.)

136. Y. W. Tschen: *Existence of minimal surfaces with a simple pole and normal derivative equal to 0 on boundary.* Preliminary report.

The purpose of the author is to solve the problem: given a simple closed curve C in the xy-plane to find a minimal surface $z(x, y)$ with $z = (\text{const.}) \cdot x + \cdots$ at infinity and $\partial z/\partial n = 0$ on C. In parametric form the problem is: to find three harmonic functions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ defined in the interior of the unit circle $u^2+v^2<1$, which represent a minimal surface with a simple pole at (0, 0) and x', y' representing C, $\partial z'/\partial r = 0$ on $u^2+v^2=1$. This problem in the parametric form is solved by using direct methods of calculus of variations on the minimum problem for two functions. Conditions are studied under which x', y' are differentiable on the boundary and $\partial(x',y')/\partial(u, v) \neq 0$ in the interior. As soon as these two conditions are fulfilled, the function $z(x, y)$ is obtained. (Received December 13, 1946.)

137. D. W. Western: *Inequalities of the Markoff and Bernstein type for integral norms.*

Let $P(z)$ be a polynomial of degree n; A, constants independent of n and $P(z)$; C, a simple closed Jordan curve of length $L(C)$. Denote the unit circle by Γ, its closed interior by $\bar{\Gamma}$, the points of C satisfying $|z-k| > m$ for k on C by C^m. The curvilinear norm is defined by $\|P(z)\|_{p,C} = \{(1/L(C))\int_C |P(z)|^p |dz|\}^{1/p}$. The results follow. For smooth C, $\|P'(z)\|_{p,C} \leqq An\|P(z)\|_{p,C}$ for $p \geqq 1$. There is a constant B depending only on $p \geqq 1$ and C such that $\|F'(z)\|_{p,C} \geqq Bn\|F(z)\|_{p,C}$ for $F(z)$ the Faber polynomial of degree n associated with C, C analytic. If C consists of smooth arcs meeting in a finite number of corners of exterior angles $k_i\pi$ where $0<k_i \leqq k<2$, $k \geqq 1$, then $\|P'(z)\|_{p,C} \leqq An^{k-1}\|P(z)\|_{p,C}$ for $p \geqq 1$; and $\|P'(z)\|_{p,C} \leqq An^{k(1+1/p)}\|P(z)\|_{p,C}$ for $p>0$. If $P^{(\alpha)}(z)$ denotes the generalized derivative of the Riemann-Liouville definition then for $\alpha>0$ and $p \geqq 1$, $\|P^{(\alpha)}(z)\|_{p,\Gamma^m} \leqq An^{\alpha}\|P(z)\|_{p,\Gamma}$ where m may satisfy $m = 1/n$ and $k = -1$; and $\|P^{(\alpha)}(x)\|_{p,(-1+m,1)} \leqq An^{2\alpha}\|P(x)\|_{p,(-1,1)}$ for $p>1$, $0<m<1$; and $\|P^{(\alpha)}(z)\|_{p,C^m} \leqq An^{\alpha}\|P(z)\|_{p,C}$ for $p>1$ and smooth C. Results of a similar nature are obtained in particular cases for area norms correspondingly defined, for $p = 2$, for certain functions not polynomials and for homogeneous polynomials of degree n in two real variables. (Received December 24, 1946.)

138. Bertram Yood: *On ideals in operator rings over Banach spaces.* Preliminary report.

Let R be the ring of all continuous linear operators on a Banach space X. For any ideal A in R let $M \subset X$ be the set of all x in X for which $T(x) = 0$ for every T in A, and let N be the linear manifold in X generated by the ranges of T in A. It is shown that if A is a left (right) ideal in R then the closure of A in both the weak and strong topology of operators is the set of all T in R which vanish on all of M (set of all T in R whose range is contained in $\overline{N}$). Thus for ideals the topologies are equivalent and the closed ideals characterized. These facts are used to show that if S is a subset of R, the left (right) annihilator of the right (left) annihilator of S in R is the smallest weakly closed left (right) ideal containing S. (Received January 6, 1947.)

139. Bertram Yood: *Regular and singular elements in the ring of operators on a Banach space.* Preliminary report.

The sets G and S of regular and singular elements respectively of the ring R of all bounded linear operators defined on an infinite-dimensional Banach space X are studied in the uniform, strong and weak topologies of R. In the uniform topology, S is decomposed into disjoint sets with topological properties in such a way that the properties of an operator T being an isomorphism or not, having X as its range or not, and the existence or absence of certain projections are correlated with the ring properties of being a left or right generalized null-divisor or not, and the possession or lack of a left or right inverse. If G can be dense in R then an open question of Banach (*Théorie des opérations linéaires*, p. 246) on linear dimensions has a negative answer. Banach's question is further discussed in terms of the concepts of this paper. In both the strong and weak topologies of R, it is shown that G and S are each dense and not open. (Received January 31, 1947.)

140. H. J. Zimmerberg: *Definite integral systems.*

In this paper the notions of definitely self-conjugate adjoint integral systems of Wilkins (Duke Math. J. vol. 11 (1944) pp. 155–166) and those definite systems of the author (Bull. Amer. Math. Soc. Abstract 52-3-76) are extended to integral systems written in matrix form $y(x) = \lambda\int_a^b K(x, t)y(t)dt$, where no restriction is made on the form of the kernel $K(x, t)$. These definite integral systems include the definite systems previously treated as subclasses. With the exception of expansion theorems, most of the properties of the definite integral systems of Wilkins and the author are preserved. (Received January 8, 1947.)

APPLIED MATHEMATICS

141. J. B. Díaz and Alexander Weinstein: *On an extremal property of the torsional rigidity.*

The torsional rigidity S of a beam with simply or multiply connected cross section can be given by either of the following formulas, which have been hardly explicitly mentioned in the literature: (*) $S = P - D(\phi)$, (**) $S = P - D(\psi)$, where P is the polar moment of inertia, ϕ and ψ are the warping function and its conjugate, and D denotes the Dirichlet integral. From these formulas follows $S<P$, and $S=P$ only for circular or annular sections. Upper and lower bounds for S are obtained from (*) for any section. For simply connected sections estimates are also obtained from (**). (Received December 9, 1946.)

142. Howard Eves: *Some consequences of a simple theorem on torque.*

The simple theorem on torque is: *Let F_i ($i=1, \cdots, n$) be a set of n coplanar forces. Then the locus of a point P in the plane of the forces such that the sum of the torques of the forces about P is a constant k is a straight line. As we vary k the line moves parallel to itself.* From this theorem a number of physical and geometrical theorems and constructions are derived. Consideration is given to the envelope of the diametral lines of a given system of lines for a given system of multiples, and of the envelope of the diametral planes of a given system of planes for a given system of multiples. Another application leads to a simple construction locating the point about which a system of coplanar forces exerts minimum inertia. This last is a generalization of the symmedian point of a triangle and leads, in turn, to some new theorems in the modern geometry of the symmedian point. (Received January 27, 1947.)

143. Howard Eves: *Systems of particles with a common centroid.*

In general if the members of a system of n weighted particles be independently displaced in space to new positions, the centroid of the new arrangement will not coincide with that of the original arrangement. This paper develops a number of theorems and constructions concerning some rearrangements of n weighted particles for which the centroid remains invariant. New and elementary proofs are supplied for some known theorems due to Brocard, M'Cay, and Weill. Sollerstinsky's elegant construction for the equicenter of two triangles, and Neuberg's analogous construction for the equicenter of two tetrahedra, are very simply established. (Received January 27, 1947.)

144. H. J. Greenberg: *The determination of upper and lower bounds for the solution of the Dirichlet problem.*

Let w be the solution of the Dirichlet problem, that is, harmonic in a domain D and assuming assigned boundary values. A method is given for determining upper and lower bounds for the value of w at any given point of D. The bounds are integral expressions involving arbitrary functions of two classes, U and V; U consists of all functions satisfying the boundary conditions, V is the class of functions harmonic in D. The method is based on the application of two complementary variational principles for w: the first (the Dirichlet principle) applies to U, the complementary principle applies to V. This yields inequalities for $\int_D(w_x^2+w_y^2)dA$. The introduction of an auxiliary boundary value problem leads to bounds for w. By employing sequences of functions in U and V, successively improved bounds are obtained. The method applies to the plane or 3-space and can be made to yield bounds directly for derivatives of the solution. Also treated is the homogeneous boundary value problem associated with Poisson's equation. As an example, bounds are obtained for the central deflection of a square elastic membrane under uniform loading. (Received January 22, 1947.)

145. M. Z. Krzywoblocki: *On certain cases of simple solutions of flow equations in compressible fluid with particular conditions.*

In the most general case of flow in a compressible fluid the density, viscosity and heat conductivity are functions of pressure and temperature. The physical conditions superimpose certain boundary values on velocity, density, pressure and temperature. Besides that one has to take into account a thermodynamic condition expressing usually the relationship between heat added, intrinsic energy and external work. This picture shows that at the present time the difficulties in the solution of equations are probably insurmountable. But it is possible to select cases satisfying only a part of the boundary conditions, which, although uninterpretable from a physical standpoint, are solvable from a mathematical point of view. In the present papers three such cases are solved under the conditions that the coefficients of viscosity and thermal conductivity are constant, that only one boundary condition is superimposed, namely, the one concerning the velocity at infinity and that all other restrictive conditions are neglected. The cases are: source, circular vortex, and spiral vortex. (Received January 30, 1947.)

146. H. E. Salzer: *Table of coefficients for interpolating in functions of two variables.*

When the double Gregory-Newton interpolation formula for functions of two variables is rewritten in terms of the tabular entries $f_{s,t}$, one obtains (1) $f(x+ph_1, y+qh_2) = \sum_{s+t=0}^{n} C_{s,t}(n, p, q) f_{s,t}$, where $C_{s,t}(n, p, q) = {}_{n-p-q}C_{n-s-t}\,{}_pC_s\,{}_qC_t$. Another proof of the corresponding general theorem for any number of variables is given. (See Bull. Amer. Math. Soc. vol. 51 (1945) pp. 279–280.) Formula (1) approximates $f(x, y)$ by a binary n-ic and is particularly applicable whenever Taylor's theorem for two variables holds in a suitable neighborhood of $f(x_0, y_0)$, or when interpolating separately for the real or imaginary part of an analytic function of degree n in $z = x+iy$. Also (1) has the advantages of involving the fewest numbers of points $f_{s,t}$ (that is, 6, 10, or 15 for $n = 2$, 3, or 4), and of being applicable near the beginning or end of a table. The present table gives the exact values of $C_{s,t}(n, p, q)$ for $n = 2$, 3, or 4, that is, in quadratic, cubic, or quartic interpolation, for all values of s, t whose sum ranges from 0 to n, and for p and q each ranging from 0.1 to 0.9 at intervals of 0.1. The saving in labor by avoiding partial differences or the double Lagrangian formula (latter requiring 12, 20, or 30 multiplications for $n = 2$, 3, or 4) is indicated. (Received December 27, 1946.)

147. P. A. Samuelson: *A generalized Newton iteration.*

An unknown simple root, A, of the equation $f(x) = 0$ can be approximated by a general iteration of the form $x_{i+1} = H(x_i)$, defined by the implicit relation $y(x_{i+1}) = G[x_{i+1} - x_i; f(x_i), f'(x_i), \cdots, f^{(n)}(x_i)] = 0$ where y is a solution of some differential equation system $y^{(n+1)} = g[y, y', \cdots, y^{(n)}]$ and $y^{(i)}(x_i) = f^{(i)}(x_i)$ for $i \leqq n$. It can be shown that $H(A) = A$ and $H^{(i)}(A) = 0$, $i \leqq n$, so that the iteration not only converges to the correct root for all sufficiently near initial approximations, but does so at an extremely rapid rate. Special cases include the Newton iteration (when $y'' = g = 0$), numerous suggested high-order osculatory extrapolations, truncated Taylor series, inverse interpolation by means of derivatives, and other methods proposed in the literature—often without proof of convergence. (Received January 15, 1947.)

148. P. A. Samuelson: *Generalization of the Laplace transform for any operator.*

Assume given any admissible operator k, such that the first-order linear equation $(k-s)y(t) = f(t)$ has a unique solution $F(t, s, b; f)_k$ for $t \geqq t_0$, for arbitrary initial condition $y(t_0) = b$, and for $f(t)$ a sufficiently limited function. Examples of admissible operators are d/dt, E, td/dt, Δ, $5+\Delta$, and so on, but not d^2/dt^2. By the generalized

Laplace transform, $L(s;f)_k$, is meant a linear functional of f with the fundamental property that $L(s;kf)_k=sL(s;f)_k-f(t_0)$. Clearly $\int_{t_0}^{\infty}\exp -s(t-t_0)f(t)dt, \sum_{t_0}^{\infty}f(t)s^{-t-1+t_0}$, $\int_{t_0}^{\infty}t^{-s-1}t_0^{s}f(t)dt, \sum_{t_0}^{\infty}f(t)(1+s)^{-t+t_0-1}$ are generalized L.T.'s for d/dt, E, td/dt, and Δ respectively. It may be verified that the appropriate expression is in every case given formally by $\lim_{t\to\infty}F(t, s, 0; f)F(t, s, 1; 0)^{-1}$. Tables of transform pairs may be set up for practical use precisely like those of the usual Laplace transform. Regions of convergence must be determined for each different operator; however, in practice it is often possible to get the correct answer by proper "interpretation" even when convergence fails. (Received January 15, 1947.)

149. Fred Supnick: *Cooperative phenomena.* II. *Structure of the two-dimensional Ising model.*

Let a distribution of A's and B's be made over the vertices of a linear graph G in which any two vertices are joined by at most one edge. Associate with each edge a $(+1)$ or a (-1) accordingly as its end points are the same or different. Denote the sum of the numbers on the edges by E (the energy). The (physical) partition function is obtained by putting E into the Boltzman exponential and summing over all possible states In this paper the author examines from a combinatorial point of view the structure for the case where G is a portion of a (plane or cylindrical) rectangular grating. A method is obtained for constructing all those distributions which have the same energy. An examination of the structure of the three-dimensional model is also made. It is pointed out that the problem of reducing "end effects" is equivalent to certain problems in the topology of sphere clusters. (See Bull. Amer. Math. Soc. Abstracts 52-9-323 and 52-11-386 by the author.) (Received January 17, 1947.)

GEOMETRY

150. John DeCicco: *An extension of Euler's theorem on homogeneous functions.*

The author determines the partial differential equation of order r to be obeyed by a function $\phi(x, y)$ which is the sum of r homogeneous functions with degrees n, $n-1$, $n-2, \cdots, n-r+1$. It is observed that such a function $\phi(x, y)$ may be said to be of degree n and is a generalization of a polynomial. This is related to the problem of determining all the algebraic curves C_n of degree n such that the $(n-r)$ polars C_r of degree r all pass through a fixed point O. This point O is a singularity of C_n of order $(n-r+1)$. For example, if the polar conics all pass through a given point O, then O is a singularity of C_n of order $(n-1)$. This whole theory is extended quite readily to any number of dimensions. (Received January 31, 1947.)

151. John DeCicco: *New proofs of the theorems of Kasner concerning the infinitesimal contact transformations of mechanics.*

The author submits new proofs of the theorems of Kasner concerning the infinitesimal contact transformations of general dynamics. (See *The infinitesimal contact transformations of mechanics*, Bull. Amer. Math. Soc. vol. 16 (1910) pp. 408–412)). The theorems deal with the nature of two dynamical systems of the same number of degrees of freedom for which the commutator or alternant of the associated infinitesimal contact transformations is a point transformation. The main result is that this situation can arise if and only if the expressions for the kinetic energy are the same or differ merely by a factor. The other proposition is that two infinitesimal contact transformations with the same transversality law will have a point transformation for commutator if and only if they are associated with two dynamical systems of the type described above. These theorems are proved for homogeneous contact transformations. The methods are those of tensor calculus. (Received January 8, 1947.)

152. Marshall Hall: *Cyclic projective planes.*

A projective plane π is said to be cyclic with respect to a collineation ϕ, if the cyclic group generated by ϕ is transitive on the points of π. It can be shown that this is equivalent to assuming that the cyclic group generated by ϕ is transitive on the lines of π. Every finite Desarguesian plane is a cyclic plane, but there are also infinite cyclic non-Desarguesian planes. Methods are given for constructing the infinite cyclic planes, and their properties are studied. Methods for constructing the representaion of a finite Desarguesian plane as a cyclic plane have been given by Singer and by the author. Restrictions are found here on finite cyclic planes. For example there is no cyclic plane with $n+1$ points on a line and $n\equiv 1$ (mod 3) unless $n=x^2+xy+y^2$. This excludes $n\equiv 10$ (mod 12). Also every cyclic plane possesses a polarity and results of Baer lead to strong restrictions on even values of n. (Received January 17, 1947.)

153. Edward Kasner and John DeCicco: *Harmonic transformations and velocity systems.*

An arbitrary point transformation of the plane carries the ∞^2 straight lines into the special cubic family $y''=A+By'+Cy'^2+Dy'^3$, studied by R. Liouville. The velocity type $y''=(1+y'^2)[\psi(x, y)-y'\phi(x, y)]$ appears frequently in geometry and dynamics. (See Kasner, *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 3, 1912, and Kasner and DeCicco, *The geometry of velocity systems*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 236–245.) In the present paper, the authors inquire when a harmonic transformation (in which the components obey the Laplace equation but not necessarily the Cauchy-Riemann equations) will convert the ∞^2 straight lines into a velocity family. This problem yields on the one hand a special type of harmonic transformation and, on the other, a special kind of velocity system. The resulting harmonic transformations can be analyzed into the products of conformalities by affinities (this obviously includes the conformalities as a special case). The corresponding velocity families can be identified with the Γ_0 families, namely, the conformal image of the ∞^2 straight lines, that is, the isogonals of an arbitrary isothermal family. (Received December 27, 1946.)

154. Edward Kasner and John DeCicco: *New theorems in the polar theory of a general algebraic curve.*

The authors present some new theorems in the polar theory of a general algberaic curve C_n of degree n. If the rth polar curve C_{n-r}, of degree $n-r$, with respect to the algebraic curve C_n of an ordinary point O on C_n, is constructed, then it is known that C_n and C_{n-r} are initially tangent. It is found that the curvatures at O are in general distinct. The ratio ρ of the curvatures is studied and, in the case of higher order contact p, the ratio ρ of the departure of the polar curve C_{n-r} to that of the algebraic curve C_n from their common tangent line. This ratio ρ is given by a simple rational formula involving only the positive integers (n, r, p). Also the case is considered where the point O is a singularity of the algebraic curve C_n. Both C_n and C_{n-r} have a singularity of the same qualitative nature at O. In this case the ratio ρ depends also on the coefficients of the polynomial defining C_n. The new theorems are essentially theorems of projective differential geometry. (Received January 15, 1947.)

155. Buchin Su: *Descriptive collineations in spaces of K-spreads.*

The author studies descriptive collineations in a space of K-spreads as a generalization of projective collineations in the general geometry of paths. After the equations expressing the conditions for an infinitesimal descriptive collineation which the space of Douglas may admit are derived, the notion of the Lie derivative is utilized in reducing the number of the integrability conditions of these equations. The group property of descriptive collineations is established. (Received December 19, 1946.)

STATISTICS AND PROBABILITY

156. P. L. Hsu and H. E. Robbins: *Complete convergence and the law of large numbers.*

A sequence $\{X_n\}$ of random variables is said to converge to 0 *completely* if for every $\epsilon>0$, $\lim_{n\to\infty}[P(|X_n|>\epsilon)+P(|X_{n+1}|>\epsilon)+\cdots]=0$. Let $\{Y_n\}$ be a sequence of independent random variables with the same distribution function $F(y)=P(Y_n\leq y)$ and such that $\int_{-\infty}^{\infty}ydF(y)=0$, $\int_{-\infty}^{\infty}y^2dF(y)<\infty$. It is proved that the sequence $\{X_n\}=\{n^{-1}(Y_1+\cdots+Y_n)\}$ converges to 0 completely. A partial converse of this theorem is given, and the relation between complete convergence and convergence with probability 1 is clarified. (Received January 18, 1947.)

TOPOLOGY

157. C. H. Dowker: *An imbedding theorem for paracompact metric spaces.*

A space is called paracompact if every covering by open sets has a neighborhood-finite refinement. A Banach space in which every finite-dimensional linear subspace is Euclidean is called a general Euclidean space or a non-separable Hilbert space or a unitary Banach space. It is shown that a metrizable space is paracompact if and only if it is homeomorphic to a subset of a unitary Banach space. (Received January 31, 1947.)

158. Nathan Jacobson: *On the theory of primitive rings.*

In a previous paper a ring $\mathfrak{A}$ has been called *primitive* if it contains a maximal right ideal $\mathfrak{J}$ whose quotient $(\mathfrak{J}:\mathfrak{A})=0$. Here $(\mathfrak{J}:\mathfrak{A})$ is the totality of elements $b\in\mathfrak{A}$ such that $\mathfrak{A}b\subseteq\mathfrak{J}$. An extrinsic characterization of these rings is that $\mathfrak{A}$ is isomorphic to an irreducible ring of endomorphisms. In this paper the author studies the one-sided ideals in primitive rings and defines certain topologies in $\mathfrak{A}$ by using these ideals. Particular attention is given to the primitive rings that contain minimal ideals. For these, the structure theories given previously for simple rings by the author and by Dieudonné are generalized. (Received December 12, 1946.)

159. Everett Pitcher: *Čech homology invariants of continuous maps.*

The effect of a continuous map on the Čech homology (or cohomology) groups under assumptions appropriate to the theory is stated in terms of exact homomorphism sequences. This is done with the aid of new homology invariants of maps, namely homology groups of the Čech type based on chains of the nerve of the domain which vanish and homology groups based on chains of the nerve of the range which are images. The former are analogous to invariants already introduced by the writer (Bull. Amer. Math. Soc. Abstract 46-5-216) for simplicial maps. The whole development is the geometric extension of the algebraic treatment of homomorphisms of chain complexes in the paper *Exact homomorphism sequences in homology theory* (Bull. Amer. Math. Soc. Abstract 52-1-49) by Kelley and the writer, to appear in Annals of Mathematics. Further developments include duality theorems, necessary and sufficient conditions for isomorphism of the homology groups of domain and range and characterizations of critical levels in critical point theory. (Received January 20, 1947.)

160. J. H. Roberts: *A problem in dimension theory.*

In *Dimension theory* by Hurewicz and Wallman, the following question is raised: If M is a noncompact separable metric space of dimension n, is the set of homeomorphisms of M into I_{2n+1} (cube in $(2n+1)$-euclidean space) a G_δ set in the space of all continuous mappings of M into I_{2n+1}? The present paper answers this question in the negative. In fact examples are given such that the set of homeomorphisms is not even a Suslin set. (Received January 3, 1947.)

161. A. E. Ross: *On continuous transformations of a continuous closed curve.*

Let s be a continuous one-to-one transformation of a closed, but not simply closed, curve with a finite number of points of fixed multiplicity l. Then some power s^k of s has fixed points and every set $D_p: p, s(p), s^2(p), \cdots$ either consists of a finite number of points or has a finite number of limit points. (Received December 9, 1946.)

162. A. E. Ross: *On continuous transformations of a Jordan curve.*

Let s be a continuous one-to-one transformation of a Jordan curve (a continuous simply closed curve) into itself. Consider the set $D_p: p, s(p), s^2(p), \cdots$ of the images of a point p. It is proved that (a) if s^k (k integral and positive) is the smallest power of s having fixed points, then the set D_p has exactly k limit points or consists of exactly k distinct points. If, however, no integral power s^k of s has fixed points then either (b_1) for every p the closure $C[D_p]$ of D_p is the whole Jordan curve or (b_2) there exists a point p for which the closure of D_p is a perfect nowhere dense set, and if q is a point of one of the denumerably infinite intervals not containing the points of $C[D_p]$, then D_q consists only of isolated points. The treatment is elementary throughout. In the cases (b_1) and (b_2) it is shown that there is a p such that D_p is dense in itself. This leads to the classification (b_2) and (b_1) ("transitive case" of van Kampen, Amer. J. Math. vol. 57 (1935) p. 145 and Lewis and Wintner, ibid. vol. 56 (1934)) and relates it to equicontinuity of the set of monotonic functions $t_i(x)$ determining s^i $(i=0, 1, \cdots)$ (Received December 9, 1946.)

163. P. A. White: *On the equivalence between avoidability and co-local connectedness.*

Equivalences are found between the avoidability properties defined by R. L. Wilder (*Sets which satisfy certain avoidability conditions*, Casopis pro Pestovani Matematiky a Fysiky vol. 67 (1938) pp. 185–198) and properties involving co-local-connectedness and local co-betti numbers (see E. G. Begle, *Locally connected spaces and generalized manifolds*, Amer. J. Math. vol. 64 (1942) pp. 553–574). In particular for compact spaces S it is shown that the property of being semi-i-connected $(0 \leqq i \leqq n-2)$, $(n-1)-$lc, and completely-i-avoidable $(0 \leqq 1 \leqq n-2)$ at $p \in S$ is equivalent to being i-lc and i-co-lc $(0 \leqq i \leqq n-1)$ at p. Also if $p^n(S)=1$ irreducibly and S is semi-$(n-1)$-connected at $p \in S$, then the properties local-$(n-1)$-avoidability at p and $R_n(p)=1$ are equivalent. (Received December 10, 1946.)

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REMARKS ON METRIZABILITY

M. H. STONE

In connection with those paragraphs of my paper *Applications of the theory of Boolean rings to general topology* (Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481) dealing with regular spaces, I have long been curious to know whether certain results proved there could be used to obtain the well known theorem that a separable (Hausdorff) space is metrizable if (and only if) it is regular. Since a positive answer to the question thus posed may have some interest from a methodological point of view, I communicate a demonstration here. The essential step in this demonstration even has some intrinsic interest, consisting as it does in the proof of new facts about dissection-spaces and the related maps. However, as a proof of the metrizability theorem this discussion is not as simple or as direct as the now classical proof of Tychonoff and Urysohn—which, it may be recalled, consists in showing, first, that a separable regular space is normal[1] and, second, that a separable normal space is metrizable.[2]

As a direct corollary of theorems established in our paper cited above, we may state the following result.

THEOREM. *If $\mathfrak{R}$ is a separable regular space, then $\mathfrak{R}$ has a map $m(\mathfrak{R}, \mathfrak{S}, \mathcal{X})$ where $\mathcal{X}$ is a continuous family of disjoint closed sets in a compact metric space $\mathfrak{S}$, which may be taken as a closed subset of the Cantor discontinuum; in other words, $\mathfrak{R}$ is topologically equivalent to the space obtained by introducing the "weak" topology in $\mathcal{X}$.*

Theorems 26 and 69 of our paper show that the desired map can be constructed with $\mathfrak{S}$ taken to be the Boolean space representing the countable Boolean algebra generated by an arbitrarily chosen countable basis for $\mathfrak{R}$; but Theorems 1, 10, and 13 show that the space $\mathfrak{S}$ is topologically equivalent to a closed subspace of the Cantor discontinuum.

We shall now establish the following result.

THEOREM. *Let $\mathcal{X}$ be a continuous family[3] of mutually disjoint, non-void, compact subsets $\mathfrak{X}$ in a metric space $\mathfrak{S}$. Then the space obtained by*

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[1] Tychonoff, Math. Ann. vol. 95 (1925–1926) pp. 139–142.

[2] Urysohn, Math. Ann. vol. 94 (1925) pp. 309–315.

[3] In the terminology of R. L. Moore, an upper semi-continuous collection.

introduction of the "weak" topology in $\mathcal{X}$ *is metrizable; and, if* $\mathfrak{S}$ *is separable, so is* $\mathcal{X}$.

Since the final statement of the theorem is an immediate corollary of Theorem 21 of our cited paper, we need concern ourselves only with the metrizability of $\mathcal{X}$.

Now the system of neighborhoods assigned in the "weak" topology to an arbitrary $\mathfrak{X}_0$ in $\mathcal{X}$ consists of all the subfamilies $\mathfrak{U}(\mathfrak{X}_0, \mathfrak{G})$ of $\mathcal{X}$ obtainable by the following construction: $\mathfrak{G}$ is chosen arbitrarily as an open neighborhood of $\mathfrak{X}_0$—that is, as an open set in $\mathfrak{S}$ which contains $\mathfrak{X}_0$—and $\mathfrak{U}(\mathfrak{X}_0, \mathfrak{G})$ is then defined as the class of all $\mathfrak{X}$ in $\mathcal{X}$ such that $\mathfrak{X} \subset \mathfrak{G}$. Our first step is to show that this system of neighborhoods can be replaced by an equivalent system of neighborhoods defined in terms of the distance-function ρ given in $\mathfrak{S}$. For this purpose, let $\mathfrak{G}(\mathfrak{X}_0, \epsilon)$, where $\epsilon > 0$, denote the ϵ-neighborhood of $\mathfrak{X}_0$—that is to say, the open set characterized by the fact that each of its points is at distance less than ϵ from some point of $\mathfrak{X}_0$. It is now almost immediate that the neighborhoods $\mathfrak{U}(\mathfrak{X}_0, \epsilon) = \mathfrak{U}(\mathfrak{X}_0, \mathfrak{G}(\mathfrak{X}_0, \epsilon))$, where ϵ runs through any sequence of positive values with limit 0, constitute a system contained in and equivalent to the system originally given. Indeed, the only detail to be verified is that each open neighborhood $\mathfrak{G}$ of $\mathfrak{X}_0$ determines a positive ϵ_0 with the property that $\mathfrak{G}(\mathfrak{X}_0, \epsilon) \subset \mathfrak{G}$ for $\epsilon < \epsilon_0$. If the distance $\rho(\mathfrak{X}_0, \mathfrak{G}')$ between $\mathfrak{X}_0$ and the complement $\mathfrak{G}'$ of $\mathfrak{G}$ is not equal to 0, we can obviously take $\epsilon_0 = \rho(\mathfrak{X}_0, \mathfrak{G}')$. Hence we can obtain the result desired if we can deduce a contradiction from the assumption that $\rho(\mathfrak{X}_0, \mathfrak{G}') = 0$. This assumption implies the existence of sequences $\{\mathfrak{x}_n\}$ in $\mathfrak{X}_0$ and $\{\mathfrak{y}_n\}$ in $\mathfrak{G}'$ with $\lim_{n \to \infty} \rho(\mathfrak{x}_n, \mathfrak{y}_n) = 0$, where by virtue of the compactness of $\mathfrak{X}_0$ the first sequence can be chosen so as to have a limit $\mathfrak{x}$ in $\mathfrak{X}_0$; but then the sequence $\{\mathfrak{y}_n\}$ must also have $\mathfrak{x}$ as its limit and, in view of the fact that $\mathfrak{G}'$ is closed, this limit is in $\mathfrak{G}'$. However, it is clear that there is no $\mathfrak{x}$ common to $\mathfrak{X}_0$ and $\mathfrak{G}'$.

The next step is to interpret in terms of these neighborhoods the hypothesis that the family $\mathcal{X}$ is continuous. In terms of the original system of neighborhoods, the continuity of $\mathcal{X}$ means that to every open neighborhood $\mathfrak{G}$ of $\mathfrak{X}_0$ there corresponds a second open neighborhood $\mathfrak{G}_0$ with the property that every $\mathfrak{X}$ having a point in common with $\mathfrak{G}_0$ is contained in $\mathfrak{G}$. If $\mathfrak{G}$ is here taken to be an ϵ-neighborhood, then $\epsilon_0 = \phi(\mathfrak{X}_0, \epsilon) < \epsilon$ can obviously be so chosen that the ϵ_0-neighborhood of $\mathfrak{X}_0$ has the property required of $\mathfrak{G}_0$: for, if any suitable $\mathfrak{G}_0$ is found corresponding to $\mathfrak{G}$, we may choose ϵ_0 so small that $\mathfrak{G}(\mathfrak{X}_0, \epsilon_0) \subset \mathfrak{G}_0$ and can then replace $\mathfrak{G}_0$ by $\mathfrak{G}(\mathfrak{X}_0, \epsilon_0)$. Since $\mathfrak{X}$ and $\mathfrak{X}_0$ are both compact the distance $\rho(\mathfrak{X}, \mathfrak{X}_0)$ is equal to the distance between two points suitably chosen in the respective sets; and, accordingly, $\mathfrak{X}$ has a point in common with $\mathfrak{G}(\mathfrak{X}_0, e_0)$ if and only if $\rho(\mathfrak{X}, \mathfrak{X}_0) < e_0$. Thus the continuity of $\mathcal{X}$ is seen to imply the existence of a real function $\phi(\mathfrak{X}_0, e)$ with the property that $\mathfrak{X} \in \mathfrak{U}(\mathfrak{X}_0, e)$ whenever $\rho(\mathfrak{X}, \mathfrak{X}_0) < \phi(\mathfrak{X}_0, e)$ and the property that $0 < \phi(\mathfrak{X}_0, e) < e$.

With each $\mathfrak{X}_0$ we can now associate a system of neighborhoods $\mathfrak{U}_n(\mathfrak{X}_0) = \mathfrak{U}(\mathfrak{X}_0, e_n)$ and a system of positive integers $m_n = m(\mathfrak{X}_0, n)$ so that whenever $\mathfrak{U}_m(\mathfrak{X}_0)$ and $\mathfrak{U}_m(\mathfrak{X})$, $m = m(\mathfrak{X}_0, n)$, have an element in common the relation $\mathfrak{U}_m(\mathfrak{X}) \subset \mathfrak{U}_n(\mathfrak{X}_0)$ is satisfied. In order to do so we put $e_1(\mathfrak{X}_0) = 1$, $e_{n+1}(\mathfrak{X}_0) = \phi(\mathfrak{X}_0, 2^{-1}e_n(\mathfrak{X}_0))$, $\mathfrak{U}_n(\mathfrak{X}_0) = \mathfrak{U}(\mathfrak{X}_0, e_n(\mathfrak{X}_0))$ and choose $m_n = m(\mathfrak{X}_0, n)$ as the least positive integer such that $2^{-m} < e_{n+2}(\mathfrak{X}_0)$. We obviously have, by definition and recursion, the inequalities $e_{n+1}(\mathfrak{X}_0) < 2^{-1}e_n(\mathfrak{X}_0)$, $e_n(\mathfrak{X}_0) < 2^{-n}$, and hence conclude that $m_n > n+2$. The system of neighborhoods $\mathfrak{U}_n(\mathfrak{X}_0)$ is, as we proved above, equivalent to that originally given in $\mathcal{X}$. Now let $\mathfrak{X}_1$ be an element common to $\mathfrak{U}_m(\mathfrak{X}_0)$ and $\mathfrak{U}_m(\mathfrak{X})$, where $m = m(\mathfrak{X}_0, n)$; and let $\mathfrak{X}_2$ be an *arbitrary* element of $\mathfrak{U}_m(\mathfrak{X})$. If $\mathfrak{x}_1$ is an arbitrary point of $\mathfrak{X}_1$, there exist points $\mathfrak{x}_0$ and $\mathfrak{x}$ in $\mathfrak{X}_0$ and $\mathfrak{X}$ respectively such that $\rho(\mathfrak{x}_0, \mathfrak{x}_1) < e_m(\mathfrak{X}_0)$, $\rho(\mathfrak{x}, \mathfrak{x}_1) < e_m(\mathfrak{X})$. Hence we see that

$$\begin{aligned} \rho(\mathfrak{X}_0, \mathfrak{X}) &\leqq \rho(\mathfrak{x}_0, \mathfrak{x}) < e_m(\mathfrak{X}_0) + e_m(\mathfrak{X}) \\ &< 2 \cdot 2^{-m} < 2e_{n+2}(\mathfrak{X}_0) \\ &< e_{n+1}(\mathfrak{X}_0) = \phi(\mathfrak{X}_0, 2^{-1}e_n(\mathfrak{X}_0)). \end{aligned}$$

Consequently we see that $\mathfrak{X} \in \mathfrak{U}(\mathfrak{X}_0, 2^{-1}e_n(\mathfrak{X}_0))$. Now let $\mathfrak{x}_2$ be an arbitrary point of $\mathfrak{X}_2$. Since $\mathfrak{X}_2$ is in $\mathfrak{U}_m(\mathfrak{X})$, there exists a point $\mathfrak{x}$ in $\mathfrak{X}$ such that $\rho(\mathfrak{x}_2, \mathfrak{x}) < e_m(\mathfrak{X}) < 2^{-1}e_n(\mathfrak{X}_0)$; and since $\mathfrak{X}$ is in $\mathfrak{U}(\mathfrak{X}_0, 2^{-1}e_n(\mathfrak{X}_0))$ there exists a point $\mathfrak{x}_0$ in $\mathfrak{X}_0$ such that $\rho(\mathfrak{x}, \mathfrak{x}_0) < 2^{-1}e_n(\mathfrak{X}_0)$. Hence we have $\rho(\mathfrak{x}_2, \mathfrak{x}_0) < e_n(\mathfrak{X}_0)$, $\mathfrak{X}_2 \in \mathfrak{U}_n(\mathfrak{X}_0)$, and $\mathfrak{U}_m(\mathfrak{X}) \subset \mathfrak{U}_n(\mathfrak{X}_0)$.

The result of the preceding paragraph permits the application of a theorem of A. H. Frink,[4] according to which the space $\mathcal{X}$ is metrizable.

Combining the two theorems above, we evidently obtain the following result.

THEOREM. *A separable regular space $\mathfrak{R}$ is metrizable.*

For the space $\mathfrak{S}$ of the first theorem and its closed subsets are compact; and the second theorem is therefore applicable.

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[4] A. H. Frink, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 133–142, Theorem 3.

SOME REMARKS ON THE THEORY OF GRAPHS

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The present note consists of some remarks on graphs. A graph G is a set of points some of which are connected by edges. We assume here that no two points are connected by more than one edge. The complementary graph G' of G has the same vertices as G and two points are connected in G' if and only if they are not connected in G.

A special case of a theorem of Ramsey can be stated in graph theoretic language as follows:

There exists a function $f(k, l)$ of positive integers k, l with the following property. Let there be given a graph G of $n \geqq f(k, l)$ vertices. Then either G contains a complete graph of order k, or G' a complete graph of order l. (A complete graph is a graph any two vertices of which are connected. The order of a complete graph is the number of its vertices.)

It would be desirable to have a formula for $f(k, l)$. This at present we can not do. We have however the following estimates:

THEOREM I. *Let* $k \geqq 3$. *Then*

$$2^{k/2} < f(k, k) \leqq C_{2k-2,k-1} < 4^{k-1}.$$

The second inequality of Theorem I was proved by Szekeres,[1] thus we only consider the first one. Let $N \leqq 2^{k/2}$. Clearly the number of different graphs of N vertices equals $2^{N(N-1)/2}$. (We consider the vertices of the graph as distinguishable.) The number of different graphs containing a given complete graph of order k is clearly $2^{N(N-1)/2}/2^{k(k-1)/2}$. Thus the number of graphs of $N \leqq 2^{k/2}$ vertices containing a complete graph of order k is less than

$$C_{N,k} \frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{N^k}{k!} \frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{2^{N(N-1)/2}}{2} \tag{1}$$

since by a simple calculation for $N \leqq 2^{k/2}$ and $k \geqq 3$

$$2N^k < k! 2^{k(k-1)/2}.$$

But it follows immediately from (1) that there exists a graph such that neither it nor its complementary graph contains a complete subgraph of order k, which completes the proof of Theorem I.

The following formulation of Theorem I might be of some interest:

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[1] P. Erdös and G. Szekeres, Compositio Math. vol. 2 (1935) pp. 463–470.

Define $A(n)$ as the greatest integer such that given any graph G of n vertices, either it or its complementary graph contains a complete subgraph of order $A(n)$. Then for $A(n) \geq 3$

$$\frac{\log n}{2 \log 2} < A(n) < \frac{2 \log n}{\log 2} \cdot \tag{2}$$

The proof of (2) follows immediately from Theorem I. ($4^{A(n)} > n$, $2^{A(n)/2} < n$.)

The general theorem of Ramsey will now be stated.

THEOREM (RAMSEY). *Given three positive integers i, k, l, $i \leq k$, $i \leq l$, there exists a function $f(i, k, l)$ with the following property: If $n \geq f(i, k, l)$ and if there is given a collection of combinations of order i of n elements, such that every combination of order k contains at least one combination of order i of the collection, then there exists a combination of order l all of whose combinations of order i belong to the collection.*

Several proofs of this theorem have been published.[1] Szekeres's proof gives the best known limits for $f(i, k, l)$. He proves[1]

$$f(i, k, l) \leq f(i - 1, f(i, k - 1, l), f(i, k, l - 1)) + 1; \tag{3}$$

also clearly $f(1, k, l) = k + l - 1$; $f(i, i, l) = l$; $f(i, k, i) = k$. By the same method we used in the proof of Theorem I we obtain that for sufficiently large k

$$f(i, k, k) > c^{k^{i-1}/a}, \tag{4}$$

or

$$A(i, n) < c_1(\log n)^{1/(i-1)}.$$

(To see this put $k = c_1(\log n)^{1/(i-1)}$ in (4) and observe $f(i, k, k) > n$, for sufficiently large c_1.) Here c and c_1 depend only on n and i, and $A(i, n)$ is the greatest integer with the following property: Split the combinations of order k of n elements into two classes U_1 and U_2 in an arbitrary way. Then there exist $A(i, n)$ elements all whose combinations of order k are either in U_1 or in U_2. The values given by (4) are very much smaller than the values given by (3).

From (3) we obtain[1]

$$f(k, l) = f(2, k, l) \leq C_{k+l-2,k}.$$

Thus

$$f(3, l) \leq C_{l+1,2}.$$

It is possible that

$$f(3, l) = O(l).$$

Our method used in the proof of Theorem I does not enable us to show that $f(3, l)/l \to \infty$.

Before concluding we prove the following theorem.

THEOREM II. *Let there be given* $(k-1)(l-1)+1$ *integers* $a_1 < a_2 < \cdots$. *Then either there exist* k *of them no one dividing the other or* l *of them each a multiple of the previous one.*

Construct a matrix $a_i^{(j)}$ with the following properties: (1) no $a_i^{(j)}$ is a multiple of any $a_i^{(r)}$ with $j \leqq r$; (2) every $a_i^{(r+1)}$ is a multiple of some $a_i^{(r)}$; (3) all the a's occur among the a_i^j once and only once. If any row contains k or more elements we have k a's, no one dividing the other. If not, it clearly follows that the number of rows must be at least l. Now we obtain from (2) that by considering any $a_i^{(l)}$ we obtain a sequence of l a's, each being a multiple of the previous one, which completes the proof. The $(k-1)(l-1)$ integers p_u^v, $1 \leqq u \leqq k-1$; $1 \leqq v \leqq l-1$, p_u primes, show that $(k-1)(l-1)+1$ is best possible.[2]

By the same method we can prove the following theorem.

THEOREM IIa. *Let there be given a graph* G *of* $(k-1)(l-1)+1$ *vertices. Then either* G *contains a complete graph of order* k, *or* G' *contains a directed path of* l *vertices, for every orientation of the edges of* G' *in which there are no directed closed paths.*

Recently very much more general theorems have been proved by Grünwald and Milgram.[3] They in fact proved (among others) that the condition that G' contains no closed directed path is superfluous.

We suppress the proof of Theorem IIa since it is essentially the same as that of Theorem II. (We only remark that a connected to b by a line directed from a to b should be replaced by a divides b.)

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[2] This proof is due to J. Bruninga.

[3] Oral communication.

ON COMPACT FIBERINGS OF THE PLANE

GAIL S. YOUNG

In a recent paper [2],[1] Montgomery and Samelson have raised the question whether there exists, for some n, a compact fibering of Euclidean n-space, and have given some reasons for thinking that no such fibering is possible. The purpose of this note is to provide further evidence for this belief by proving that at least there is no compact fibering of the plane. The theorem I shall prove is in fact somewhat stronger.

THEOREM 1. *If $f(E^2)=A$ is an interior[2] transformation of the plane such that for each two points, x and y, of A, $f^{-1}(x)$ and $f^{-1}(y)$ are homeomorphic, and such that each component of $f^{-1}(x)$ is compact, then no component of $f^{-1}(x)$ separates E^2 and A is a 2-manifold.*

If in addition $f^{-1}(x)$ is compact, then f is monotone and A is a plane.

PROOF. It cannot have escaped notice that a transformation satisfying the conditions placed on f (omitting the homeomorphy condition) can be factored into a monotone closed transformation, $g(E^2)=B$, followed by a light interior transformation, $h(B)=A$, even though such theorems have been stated for compact spaces only. The arguments for Theorems[3] VIII, 4.1, and VIII, 3.1, establish this. The proof of Theorem IX, 2.3, shows that B is homeomorphic to a set obtained by removing some non-cut point from a cactoid. If some component of some $f^{-1}(x)$ separates E^2, then B has a cut point. Since each set $g^{-1}(x)$ is a component of some set $f^{-1}(y)$, y in A, and each two inverses of points of A are homeomorphic, B has uncountably many cut points. Hence some point p of B is of Menger order 2, by Theorem VII, 3.2. There is a point q of B such that the closure, C, of the component of $B-q$ that contains p is compact. Over C, h is continuous, and over $C-q$, h is interior; indeed, $h(C-q)$ is open in A.

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[1] Numbers in brackets refer to the bibliography.

[2] A continuous transformation $f(X)=Y$ is interior provided that the image of every open subset of X is open in Y, and is light provided that every set $f^{-1}(y)$, y in Y, is totally disconnected.

The statement of this theorem is weaker than that in Bull. Amer. Math. Soc. Abstract 53-1-106. In the original formulation, I used a characterization of the possible interior images of a 2-manifold that I had announced in Bull. Amer. Math. Soc. Abstract 52-5-220, but in the proof of which an error has been found.

[3] Theorems referred to in this way are from Whyburn's book [4]. The roman numeral is the chapter number.

Let R be a neighborhood of p whose boundary consists of just two points, and which lies with its boundary in $C-q$. By Theorem VIII, 7.3, $\overline{h(R)}-h(R)$ is contained in $h(\overline{R}-R)$. Since $h(R)$ is open in A, it follows that its boundary is contained in $h(\overline{R}-R)$ and is therefore finite. Hence A contains a point x which separates some connected open subset, D, of A into two separated sets, D_1 and D_2. Let x_1 and x_2 be points of D_1 and D_2, respectively. Let K be a component of $f^{-1}(x)$; K must separate E^2, for suppose that it does not. Then by Theorem VI, 3.11, in $f^{-1}(D)$ there is a 2-cell, C, that contains K, but has no point of $f^{-1}(x)$ in its boundary, and does not have any point of $f^{-1}(x_1+x_2)$ as limit point. Since $f(\overline{C})$ is in $D-(x_1+x_2)$, the boundary of $f(C)$ intersects both D_1 and D_2, and, by Theorem VIII, 7.3, is contained in the image of the boundary of C. But the image of the boundary of C is a connected subset of $D-x$, and so lies wholly in D_1 or D_2. This is impossible, so that K does separate E^2. In passing, notice that this implies that every component of each set $f^{-1}(y)$, y in A, separates E^2, by the hypothesis that each two point-inverses are homeomorphic. One component, E, of E^2-K is bounded. Since $K+E$ is compact, over $K+E$, f may be factored into a monotone transformation, m, followed by a light transformation, from Theorem VIII, 4.1. The continuum $m(K+E)$ has at least two non-cut points, one of which is $m(K)$; let y be any other non-cut point. Then by Theorem VIII, 2.2, $m^{-1}(y)$ cannot separate $K+E$, so cannot separate E, and hence does not separate E^2. This contradiction shows that B is a plane, so that $h(B)$ is a 2-manifold, by Theorem X, 4.4.

It remains to show that if each set $f^{-1}(x)$ is compact, then f is monotone. This is true if the factor h of the previous paragraph is a homeomorphism, so that the result will be established if f is assumed light and proved to be a homeomorphism. Whyburn's Theorem X, 5.1, shows that a light interior transformation on a 2-manifold is locally finite-to-1. Hence there is an integer k such that for each point x of A, $f^{-1}(x)$ consists of exactly k points. Adjoin a point ω to E^2 to make it a 2-sphere, S^2. The collection G whose elements are ω and all sets $f^{-1}(x)$ is continuous, for if the sequence $\{y_n\}$ of points of A converges to the point y of A, then by Theorem VII, 4.32—the Eilenberg condition for interiority—$\{f^{-1}(y_n)\}$ converges to $f^{-1}(y)$ in E^2. The k-to-1 property implies that $\{f^{-1}(y_n)\}$ converges in S^2 to $f^{-1}(y)$. The continuity of G means that f can be extended to a light interior transformation, $g(S^2)=A'$, which will necessarily be a local homeomorphism except at ω. On page 197 of [4] Whyburn has noted that since an interior transformation on a locally connected compact continuum cannot raise coherence, A' is either a sphere, a projective plane, or a closed 2-cell. The last of these is impossible under a local homeomorphism. In Theorem X, 7.3, Whyburn has given a formula connecting the Euler characteristic of a compact 2-manifold with that of its image under a light interior transformation, t. Since A' has no boundary, this reduces here to

$$k\chi(A') - \chi(S^2) = kr - n,$$

where r is the number of points y in A' such that t is not a local homeomorphism at some point in $t^{-1}(y)$ and n is the total number of points in the inverses of all such points. For $t=g$, $r=n=1$; and $\chi(A')$ is either 1 or 2. Hence either $k-2=k-1$, which is impossible, or $2k-2=k-1$, which implies that $k=1$, and that g is a homeomorphism.

COROLLARY 1. *There is no compact fibering of the plane other than a homeomorphism.*

PROOF. Since a compact fibering, $f(E^2)=A$, satisfies all the hypotheses of Theorem 1, it is necessarily monotone, and A is a plane. Let C be an open 2-cell in A, and let F be a fiber. Then $f^{-1}(C)$ is homeomorphic to $C\times F$, if C is sufficiently small. If F is not a point, this implies that $f^{-1}(C)$ is the sum of uncountably many mutually exclusive open 2-cells, an impossibility in E^2.

I do not know of an example of an interior mapping of the plane onto a 2-manifold where each component of the inverse of each point is compact, but where the transformation is not light. However, if the hypotheses of the above theorem are strengthened by local connectivity assumptions, lightness can be proved.

COROLLARY 2. *If, in addition to the hypotheses of the first part of Theorem* 1, *each component of each set $f^{-1}(x)$ is locally connected, then f is light.*

PROOF. By Theorem 1, no component of a set $f^{-1}(x)$ separates E^2, so that no component is a simple closed curve. Hence every such component is either a point, an arc, or contains a simple triod. If one component of one point-inverse contained a simple triod, the condition that each two point-inverses are homeomorphic would establish the existence of an uncountable number of mutually exclusive triods in the plane; this would contradict Theorem 75, p. 254, of [5]. Hence each component of a point-inverse is either an arc or a point. Let T be a nondegenerate component. In the notation of the first paragraph of the proof of Theorem 1, since B is a plane, and h is interior, there exist an integer k and a neighborhood R of $g(T)$ such that $h(R)$ is at most k-to-1, by Theorem X, 5.1. It is easy to see that the Eilenberg condition for interiority implies that there is a point z of $R-g(T)$ such that $g^{-1}(z)$ is an arc, and such that h is locally 1-to-1 in a neighborhood U of z; U may be chosen to be simply connected and so small that each point of U has a nondegenerate inverse under g. Then $g^{-1}(U)$ is open and simply connected. The collection of all sets $g^{-1}(x)$, x in U, is a continuous collection of arcs filling up a set homeomorphic to a plane. J. H. Roberts has proved in [3] that there is not even an upper semi-continuous collection of arcs filling up a plane. Hence f is light.

It is not difficult to see that the arguments for Theorem 1 and Corollary 2 can be combined and extended to prove the following.

THEOREM 2. *An interior transformation of a 2-manifold onto a 2-manifold such that each component of each point-inverse is an arc or a point is necessarily light.*

The mapping of the plane onto a cylinder of radius 1 obtained by identifying all vertical lines whose abscissae are congruent modulo 2π, and the mapping $w=\sin z$ are interior mappings of E^2 for which any two point-inverses are homeomorphic, showing that compactness is necessary in the second part of the theorem. If the requirement of interiority is removed instead, the conclusion of the second part is also false, as the following indication of an example[4] shows: In the Cartesian plane, form undercut steps by making a downward crease along each line $x=\pm 1, \pm 4, \cdots, \pm(3n+1), \cdots$, and by making an upward crease along each line $x=\pm 2, \pm 5, \cdots, \pm(3n+2), \cdots$. "Flatten" the plane along these creases onto a plane A so that each point of A is covered by exactly 3 points of the original plane. I should remark that Civin has proved in [1] that no exactly 2-to-1, closed transformation of E^1, E^2, or E^3 exists.

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[4] This is modified from an example of a 3-to-1 mapping of an arc onto an arc due to Schweigert.

ON THE INTERIOR OF THE CONVEX HULL OF A EUCLIDEAN SET

WILLIAM GUSTIN

In this note we shall prove for each positive integer n the following theorem Δ_n concerning convex sets in an n-dimensional euclidean space.

THEOREM Δ_n. *Any point interior to the convex hull of a set E in an n-dimensional euclidean space is interior to the convex hull of some subset of E containing at most $2n$ points.*

This theorem is similar to the well known result that any point in the convex hull of a set E in an n-dimensional euclidean space lies in the convex hull of some subset of E containing at most $n+1$ points [1, 2].[1] In these theorems the set E is an arbitrary set in the space. The convex hull of E, denoted by $H(E)$, is the set product of all convex sets in the space which contain E.

A euclidean subspace of dimension $n-1$ in an n-dimensional euclidean space will be called a plane. Every plane in an n-dimensional euclidean space separates its complement in the space into two convex open sets, called open half-spaces, whose closures are convex closed sets, called closed half-spaces. If each of the two open half-spaces bounded by a plane L intersects a given set E, then L is said to be a separating plane of E; otherwise L is said to be a nonseparating plane of E.

In order to prove our sequence of theorems we shall make use of the following result: A point i is interior to the convex hull of a set E in an n-dimensional euclidean space if and only if every plane through i is a separating plane of E [1].

We prove our sequence of theorems by induction. The proof of Theorem Δ_1 is trivial and will be omitted. Now suppose that Theorem Δ_{n-1} is true for an integer $n>1$. We shall show that Theorem Δ_n is also true. To this end let i be a point interior to the convex hull of a set E in an n-dimensional euclidean space. We are to demonstrate that i is interior to the convex hull of some subset P of E containing at most $2n$ points.

First we show that i is interior to the convex hull of some finite subset Q of E. Since i is interior to $H(E)$, it is interior to a simplex

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[1] Numbers in brackets refer to the references cited at the end of the paper.

lying in $H(E)$. Consider the $n+1$ vertices q_k ($k=1, \cdots, n+1$) of such a simplex. The vertex q_k lies in $H(E)$ and hence, according to the previously mentioned result of Carathéodory and Steinitz, lies in the convex hull of some subset Q_k of E containing at most $n+1$ points. The set $Q=\sum_k Q_k$ is then a finite subset of E containing at most $(n+1)^2$ points. Evidently the convex hull of this set contains the simplex with vertices q_k and hence contains the point i in its interior.[3]

Since Q is finite, there exists a subset P of Q which contains the point i in the interior of its convex hull and which is irreducible with respect to this property. Let p be a definite point of P. Then i is not an interior point of $H(P-p)$, so some plane L through i is a non-separating plane of $P-p$. Let D be that one of the two open half-spaces bounded by L which is disjoint with $P-p$ and let D' be the other open half-space. Thus $P-p$ lies in the closed half-space $\bar{D}'$ complementary to D.

Since i is an interior point of the convex hull of P, the open half-space D contains a point of P. This point must be p, for D contains no point of $P-p$. Similarly the open half-space D' contains a point p' of P. We shall use this point p' later in the proof.

Consider an arbitrary point x of the closed half-space $\bar{D}'$. Since p lies in the complementary open half-space D, the line segment $H(p+x)$ intersects the boundary L of D in exactly one point which we denote by $\phi(x)$. Thus $\phi(x)$ is the projection of x from p onto L.

The projection ϕ is 1-1 over the subset $P-p$ of the closed half-space $\bar{D}'$. For suppose, to the contrary, that some two points p_1 and p_2 of $P-p$ project into the same point of L. The three points p, p_1, and p_2 are then collinear. Now p does not lie between the other two points, else the open half-space D containing p would contain at least one of these other two points. We may then assume p_1 and p_2 to be so labeled that a linear order of the three points is p, p_1, p_2. Therefore

$$p_1 \subset H(p + p_2) \subset H(P - p_1),$$

so the sets $H(P-p_1)$ and $H(P)$ are identical. The point i is then interior to $H(P-p_1)$ in contradiction to the irreducibility of P.

The projection of the convex hull of a set is the convex hull of the projection of that set, and the projection of an interior point of a convex set is an interior point of the projection of that set [3]. Therefore the point $\phi(i)=i$ is an interior point of the set $\phi(H(P-p))$ $=H(\phi(P-p))$ in the euclidean subspace L of dimension $n-1$. According to Theorem Δ_{n-1} the point i is an interior point in L of the convex hull of some subset P_L of $\phi(P-p)$ containing at most $2n-2$ points. Define

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[3] That i is interior to the convex hull of some finite subset of E may also be proved by the Heine-Borel theorem. I am indebted to the referee for the above proof.

$$P^* = p + P\phi^{-1}(P_L) + p'.$$

Since the projection ϕ is 1-1 over $P-p$, the set $P\phi^{-1}(P_L)$ is a subset of P containing at most $2n-2$ points. Therefore P^* is a subset of P containing at most $2n$ points.

We shall show that i is interior to $H(P^*)$. First we notice that the coplanar set P_L lies in $H(P^*)$. For, if x is an arbitrary point of $P\phi^{-1}(P_L)$, then

$$\phi(x) \subset H(p + x) \subset H(P^*),$$

since both p and x lie in $H(P^*)$. Now consider the pyramid $H(p+P_L)$ whose apex p lies in D and whose base $H(P_L)$ lies in L. The point i is an interior point in L of the base $H(P_L)$ of this pyramid, so some closed hemisphere A with center i and base on L lies in $H(p+P_L)$. Similarly, some closed hemisphere A' with center i and base on L lies in the pyramid $H(p'+P_L)$. Evidently there exists a sphere I with center i such that $I \subset A+A' \subset H(p+P_L)+H(p'+P_L) \subset H(P^*)$. The point i is then interior to the convex hull of the subset P^* of P. From the irreducibility of P it follows that $P^*=P$. Therefore P contains at most $2n$ points.

Thus for every integer $n>1$, Theorem Δ_{n-1} implies Theorem Δ_n. Since Theorem Δ_1 is true, we conclude by induction that Theorem Δ_n is true for each positive integer n.

The following example shows that the number $2n$ in Theorem Δ_n cannot be improved. Let i be the zero point of an n-dimensional vector space. Choose any n linearly independent and hence nonzero points in this space. Let E be the set consisting of these points and their vector negatives; E then contains $2n$ points. It is easy to show that the zero point i is interior to the convex hull of E but is not interior to the convex hull of any proper subset of E.

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ON NÖRLUND SUMMABILITY OF RANDOM VARIABLES TO ZERO

GEORGE E. FORSYTHE

1. **Introduction.** In a previous paper[1] [1], the author considered the Cesàro summability methods $\{C_r\}$ $(0<r<\infty)$ for sequences of independent, real-valued random variables $\{x_k\}$. For summability in probability of $\{x_k\}$ to 0, it was shown that: (i) $r<s$ implies $C_r\subset C_s$; (ii) for $r\geqq 1$ all the methods C_r are essentially equivalent, in contrast to the Cesàro theory for sequences of real numbers. The field of the investigation reported here is the summability in probability of sequences $\{x_k\}$ to 0 by the Nörlund summability methods, which include the Cesàro methods. The objective (attained only in special cases) was to prove that if two Nörlund methods N_p and N_q share the relation $N_p\subset N_q$ over sequences of real numbers, then the analogous relation $N_p\subset N_q$ holds for the summability of sequences of independent, real-valued, symmetric random variables to zero. The converse is, of course, false.

The only sequences $\{x_k\}$ considered here are normal families of independent, real-valued, symmetric random variables. For these $\{x_k\}$ the objective has been attained for three special cases; see Theorems 4, 5, and 6. The earlier theorems are tools: Theorem 1 gives a necessary and sufficient condition for the Nörlund summability of $\{x_k\}$ to 0, while Theorems 2 and 3 give sufficient conditions for the relations $N_p\subset N_q$ and $N_p\equiv N_q$, respectively. Theorem 7 shows that equivalence with C_1 over $\{x_k\}$ extends to a Nörlund method N_p whose counterpart N_p is strictly weaker than C_1 over sequences of real numbers. Such equivalence with C_1 is impossible for Cesàro methods weaker than C_1 over sequences of real numbers.

It is conjectured that Theorems 4, 5, and 6, here proved for normal families only, can be extended without change of statement to arbitrary sequences of independent, real-valued, symmetric random variables. If the x_k are not symmetric there are complications (see [1]), but it is conjectured that Theorems 4, 5, and 6 still hold without essential change.

2. **Nörlund summability of sequences of real numbers.** Let $p=\{p_n\}$ $(n=0, 1, 2, \cdots)$ be a sequence of nonnegative real numbers, with

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$p_0=1$; for each n let $P_n=\sum_0^n p_k$. For any sequence $\{x_k\}$ of real numbers, a transformed sequence $\{y_n\}$ is defined by the relation $y_n=P_n^{-1}\sum_{k=0}^{n} p_{n-k}x_k$ ($n=0, 1, 2, \cdots$). If the sequence $\{y_n\}$ has the limit x', the sequence $\{x_k\}$ is said to be summable-N_p to x', where N_p is the Nörlund summability method corresponding to p. The method N_p is known to be regular (that is, consistent with ordinary convergence, for all convergent sequences $\{x_k\}$) if and only if $p_n=o(P_n)$. There is a substantial known theory of the Nörlund summability methods (see [2, 6, 7, 8, 9]), of which certain results will be quoted in this section for comparison with their analogues for sequences of random variables.

For two summability methods A and B, the statement "$A\subset B$" means that any sequence summable-A to a finite limit is also summable-B to the same limit. The statement "$A\equiv B$" means that $A\subset B$ and $B\subset A$. The negation of "$A\subset B$" is "$A\not\subset B$."

In addition to N_p, defined by $\{p_k\}$, let a second Nörlund summability method N_q be defined by $\{q_k\}$, with $q_0=1$, $q_k\geqq 0$, and $Q_n=\sum_0^n q_k$. The following generating functions and coefficients are defined formally by M. Riesz [7]:

$$p^*(x)=\sum_{n=0}^{\infty}p_nx^n;\qquad q^*(x)=\sum_{n=0}^{\infty}q_nx^n;$$

$$\lambda^*(x)=\frac{q^*(x)}{p^*(x)}=\sum_{n=0}^{\infty}\lambda_n^*x^n;$$

$$\mu^*(x)=\frac{p^*(x)}{q^*(x)}=\sum_{n=0}^{\infty}\mu_n^*x^n.$$

It is assumed in (2-1) to (2-5) that N_p and N_q are both regular.

(2-1) (M. Riesz [7]) *$N_p\subset N_q$ if and only if, as $n\to\infty$, both $\sum_{k=0}^{n}P_k|\lambda_{n-k}^*|=O(Q_n)$ and $\lambda_n^*=o(Q_n)$.*

(2-2) (M. Riesz [7]) *$N_p\equiv N_q$ if and only if $\sum_{n=0}^{\infty}(|\lambda_n^*|+|\mu_n^*|)<\infty$.*

The Cesàro summability methods C_r $(0<r<\infty)$ are of the Nörlund type N_p, where $p^*(x)=(1-x)^{-r}$. If we let N_p be C_1, (2-2) takes the following form ([2], p. 782):

(2-3) *$C_1\subset N_q$ if and only if, as $n\to\infty$, $\sum_{k=0}^{n}(n+1-k)|q_k-q_{k-1}|$ $=O(Q_n)$, where $q_{-1}=0$.*

It follows immediately from (2-3) that:

(2-4) *If $q_n\leqq q_{n+1}$ for all n, then $C_1\subset N_q$.*

It follows from (2-1) that:

(2-5) *If* $p_n \geqq p_{n+1}$ *for all* n, *then* $N_p \subset C_1$.

The author has not seen (2-5) in the literature, but it is undoubtedly known. It can be proved much like the analogous Theorem 6 of the next section.

3. **Nörlund summability of random variables to zero.** Sections 3 and 4 of [1] contain the notation, basic definitions, and references for the summability in probability of a sequence $\{x_k\}$ to 0. We shall here confine ourselves to sequences of independent, real-valued, symmetric random variables which form a normal family.

DEFINITION 1 (P. Lévy [5]). $\{x_k\}$ forms a *normal family* when both (3-1) and (3-2) hold:

(3-1) $E(x_k)=0$ *and* $E(x_k^2)=\sigma_k^2<\infty$, *for all* k;

(3-2) *There exists a random variable* x *with finite* $E(x^2)$ *such that for all* $A>0$ *and all* k

$$\text{Prob}\ \{\,|\,x_k\,|>A\sigma_k\}\ \leqq\ \text{Prob}\ \{\,|\,x\,|>A\}.$$

(Lévy does not require that $E(x_k)=0$.)

DEFINITION 2. With the notation of §2 above, a sequence $\{x_k\}$ is said to be *summable-*N_p *in probability to* 0 when for each $\epsilon>0$, as $n\to\infty$,

$$\text{(3-3)}\qquad \text{Prob}\left\{\left|P_n^{-1}\sum_{k=0}^{n} p_{n-k}x_k\right|>\epsilon\right\}\to 0.$$

The words "in probability" will sometimes be omitted for brevity.

The basic theorem is the following:

THEOREM 1. $\{x_k\}$ *is a normal family of independent, real-valued, symmetric random variables. In order that* $\{x_k\}$ *be summable-*N_p *in probability to* 0, *it is necessary and sufficient that, as* $n\to\infty$,

$$\text{(3-4)}\qquad \sum_{k=0}^{n} p_{n-k}^2\sigma_k^2 = o(P_n^2).$$

Theorem 1 follows immediately from Theorem 5.3 of [1] by letting $p_{n-k}P_n^{-1}=a_{nk}$. The proof of Theorem 5.3 in [1] did not depend on the special character of $\|a_{nk}\|$ as a Cesàro matrix.

The basic tools of the present investigation are Theorems 2 and 3, which are analogues of (2-1) and (2-2). In the following, regularity of N_p for sequences of real numbers is not assumed unless explicitly stated. It is assumed always that $p_n \geqq 0$, $q_n \geqq 0$, $p_0 = q_0 = 1$.

Let the following generating functions and coefficients be defined formally:

$$p(x) = \sum_{n=0}^{\infty} p_n^2 x^n; \qquad q(x) = \sum_{n=0}^{\infty} q_n^2 x^n;$$

$$\lambda(x) = \frac{q(x)}{p(x)} = \sum_{n=0}^{\infty} \lambda_n x^n; \qquad \mu(x) = \frac{p(x)}{q(x)} = \sum_{n=0}^{\infty} \mu_n x^n.$$

THEOREM 2. *Statements* (3-5) *and* (3-6) *together form a sufficient condition for $N_p \subset N_q$ with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0:

$$\sum_{k=0}^{n} P_k^2 |\lambda_{n-k}| = O(Q_n^2); \tag{3-5}$$

$$\lambda_n = o(Q_n^2). \tag{3-6}$$

PROOF. Let $\{\mathbf{x}_k\}$ be any normal family of independent, real-valued, symmetric random variables which is summable-N_p to 0. Suppose (3-5) and (3-6) hold. Let $t_k = P_k^{-2}\sum_{h=0}^{k} p_{k-h}^2 \sigma_h^2$ $(k = 0, 1, 2, \cdots)$. By Theorem 1, $\{t_k\}$ is a null-sequence of non-negative numbers. Let $u_n = Q_n^{-2}\sum_{k=0}^{n} q_{n-k}^2 \sigma_k^2$ $(n = 0, 1, 2, \cdots)$. To prove Theorem 2 it is sufficient to show that $\{u_n\}$ is also a null-sequence and apply Theorem 1 again.

Define $\sigma(x)$ as $\sum_{k=0}^{\infty} \sigma_k^2 x^k$, formally. Now $t_n P_n^2 = \sum_{k=0}^{n} p_{n-k}^2 \sigma_k^2$, by definition of t_n. Hence $\sigma(x)p(x) = \sum_{n=0}^{\infty} t_n P_n^2 x^n$. Similarly, $\sigma(x)q(x) = \sum_{n=0}^{\infty} u_n Q_n^2 x^n$. But $\sigma(x)q(x) = [\sigma(x)p(x)]\lambda(x)$. Hence, by equating coefficients, we see that we may write $u_n = \sum_{k=0}^{n} b_{nk} t_k$, where $b_{nk} = Q_n^{-2} P_k^2 \lambda_{n-k}$. The sequence $\{u_n\}$ is seen to be obtained as the transform of $\{t_k\}$ by the triangular matrix $\|b_{nk}\|$. To prove Theorem 2 it suffices to show that $\|b_{nk}\|$ is null-preserving. By a theorem of Kojima [4], $\|b_{nk}\|$ is null-preserving if and only if:

$$\lim_{n\to\infty} b_{nk} = 0, \qquad \text{for each } k; \tag{3-7}$$

$$\sum_{k=0}^{n} |b_{nk}| \leqq M < \infty, \qquad \text{for all } n. \tag{3-8}$$

But (3-6) is equivalent to (3-7), and (3-5) is equivalent to (3-8). This proves Theorem 2.

Since its members are non-negative, $\{t_k\}$ is not an arbitrary null-sequence; hence this proof cannot yield necessary conditions for

$N_p \subset N_q$. Indeed, (3-5) is not necessary; see Corollary 2, following Theorem 7.

The symbol C_r $(r \geqq 0)$ represents the Cesàro summability method of order r over sequences of random variables.

COROLLARY 1. *Let N_q be a regular Nörlund summability method. Set $q_{-1} = 0$. A sufficient condition for $C_1 \subset N_q$ with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0 *is that*

$$(3\text{-}9) \qquad \sum_{k=0}^{n} (n - k + 1)^2 \left| q_k^2 - q_{k-1}^2 \right| = O(Q_n^2).$$

PROOF. Identify C_1 with N_p in Theorem 2. It is found that $p(x) = (1-x)^{-1}$ and that $\lambda_n = q_n^2 - q_{n-1}^2$. Condition (3-6) is automatically satisfied, since N_q is regular. Condition (3-5) is thus sufficient for $C_1 \subset N_q$. But when N_p is C_1 ,(3-5) takes the form (3-9), proving the corollary.

THEOREM 3. *Suppose $p_n \leqq p_{n+1}$ and $q_n \leqq q_{n+1}$, for all n. Then a sufficient condition that $N_p = N_q$ with respect to the summability in probability to* 0 *of normal families of independent, real-valued, symmetric random variables is that*

$$(3\text{-}10) \qquad \sum_{k=0}^{\infty} (| \lambda_k | + | \mu_k |) < \infty .$$

PROOF. Let $\sum_{k=0}^{\infty} |\lambda_k| = A^2 < \infty$ and $\sum_{k=0}^{\infty} |\mu_k| = B^2 < \infty$. Since $p(x) = q(x)\mu(x)$, and since q_n is nondecreasing, we find that $p_n^2 = \sum_{k=0}^{n} q_k^2 \mu_{n-k} \leqq q_n^2 \sum_{k=0}^{n} |\mu_{n-k}| \leqq B^2 q_n^2$. Thus $p_n \leqq B q_n$, whence it follows that, for all $k \leqq n$, $P_k \leqq B Q_n$. Hence, for all n,

$$Q_n^{-2} \sum_{k=0}^{n} P_k^2 | \lambda_{n-k} | \leqq B^2 \sum_{k=0}^{n} | \lambda_{n-k} | \leqq B^2 A^2 < \infty .$$

Therefore (3-5) is satisfied. Since $\sum |\lambda_n| < \infty$, $\lambda_n = o(1)$ and (3-6) is satisfied. By Theorem 2, $N_p \subset N_q$ for all normal families under consideration. By interchanging p and q it is seen similarly that $N_q \subset N_p$. Hence $N_p = N_q$, proving Theorem 3.

With the strong hypothesis that p_n and q_n are nondecreasing, Theorem 3 is rather weak. In fact, it follows from Theorem 5 that if $p_n \leqq p_{n+1}$ and $q_n \leqq q_{n+1}$, for all n, *and if N_p and N_q are both regular*, then $N_p = N_q$.

THEOREM 4. *Let N_q be any regular Nörlund summability method such* *that* $C_1 \subset N_q$ *with respect to sequences of real numbers. Then* $C_1 \subset N_q$ *with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0.

PROOF. If N_q satisfies the hypothesis, we know by (2-3) that

$$\sum_{k=0}^{n} (n + 1 - k) \left| q_k - q_{k-1} \right| = O(Q_n). \tag{3-11}$$

By Lemma 3, proved in §4, (3-11) implies (3-9). By Corollary 1 above, (3-9) implies that $C_1 \subset N_q$.

Theorem 5 is a second special case in which it is shown that $N_p \subset N_q$ implies $N_p \subset N_q$. Indeed, in this case it is even shown that $N_p = N_q$. Since it is possible that $N_q \not\subset N_p$, it is untrue that $N_q \subset N_p$ implies $N_q \subset N_p$.

THEOREM 5. *Let* N_q *be any regular Nörlund summability method such that* $q_n \leqq q_{n+1}$, *for all* n. *Then* $C_1 = N_q$ *with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0.

PROOF. Given a regular N_q with $q_n \leqq q_{n+1}$ for all n. By (2-4), $C_1 \subset N_q$. Hence, by Theorem 4, $C_1 \subset N_q$. There remains only the proof that $N_q \subset C_1$. This will be given in two steps.

I. Since N_q is regular, for each n we have, as $r \to \infty$, $q_r/Q_{n+r} \leqq q_r/Q_r \to 0$. Let $\delta(n) = \max_{r \geqq 0}(q_r/Q_{n+r})$. We shall prove that

$$n\delta(n) \geqq \beta > 0 \qquad (n = 1, 2, 3, \cdots). \tag{3-12}$$

Let $t_r = q_r/Q_r$. Then

$$\frac{q_r}{Q_{n+r}} = t_r(1 - t_{r+1})(1 - t_{r+2}) \cdots (1 - t_{r+n}).$$

Fix $n > 0$. Since $t_0 = 1$ and $\lim_r t_r = 0$, we may let $r(n)$ be the largest value of r for which $t_r \geqq (n+1)^{-1}$. Then

$$\begin{aligned} \frac{q_{r(n)}}{Q_{n+r(n)}} &> \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n \\ &= \frac{1}{n+1}\left\{\left(1 + \frac{1}{n}\right)^n\right\}^{-1} \cong \frac{e^{-1}}{n+1}. \end{aligned} \tag{3-13}$$

Since $\delta(n) \geqq q_{r(n)}/Q_{n+r(n)}$, it is seen from (3-13) that (3-12) is true.

II. Suppose, for a contrapositive proof, that $\{x_k\}$ is a normal family of independent, real-valued, symmetric random variables which is not summable-C_1 to 0. Let $s_{-1} = 0$; for $n \geqq 0$ let $s_n = \sum_{k=0}^{n} \sigma_k^2 \leqq s_{n+1}$. By

Theorem 1, $\{n^{-2}s_n\}$ is not a null-sequence; we can therefore find $\epsilon>0$ and integers $0<n_1<n_2<\cdots<n_i<n_{i+1}<\cdots$ such that

$$s_{n_i} \geqq \epsilon n_i^2 \qquad (i = 1, 2, \cdots). \tag{3-14}$$

Let $q_{-1}=0$. Let $\phi(n)=Q_n^{-2}\sum_{k=0}^{n}\sigma_k^2 q_{n-k}^2=Q_n^{-2}\sum_{k=0}^{n}s_k(q_{n-k}^2-q_{n-k-1}^2)$. By Theorem 1, $\{\mathbf{x}_k\}$ is summable-$\mathbf{N}_q$ to 0 if and only if $\{\phi(n)\}$ is a null-sequence. To complete the contrapositive proof we show that $\{\phi(n)\}$ is not a null-sequence. Fix any n_i. Since q_n and s_n are both nondecreasing, it is seen for all $n\geqq n_i$ that

$$\begin{aligned}
\phi(n) &\geqq Q_n^{-2}\sum_{k=n_i}^{n} s_k(q_{n-k}^2 - q_{n-k-1}^2)\\
&\geqq Q_n^{-2}s_{n_i}\sum_{k=n_i}^{n}(q_{n-k}^2 - q_{n-k-1}^2)\\
&= s_{n_i}q_{n-n_i}^2 Q_n^{-2}\\
&\geqq \epsilon n_i^2 q_{n-n_i}^2 Q_n^{-2}.
\end{aligned}$$

The last inequality is by (3-14). Hence for some integer n_i' greater than n_i we have

$$\phi(n_i') > \epsilon n_i^2\left\{2^{-1}\max_{n\geqq n_i}(q_{n-n_i}^2 Q_n^{-2})\right\} = 2^{-1}\epsilon n_i^2\{\delta(n_i)\}^2.$$

By (3-12) we see that $\phi(n_i')>2^{-1}\epsilon\beta^2>0$. Thus $\{\phi(n)\}$ cannot be a null-sequence, completing the proof of Theorem 5.

The satisfaction of condition (3-12) is equivalent to regularity for Nörlund summability methods N_p whose counterparts $\mathbf{N}_p$ can sum to 0 a sequence $\{\mathbf{x}_k\}$ of random variables which are not all identically zero. That is, nonregular Nörlund methods which satisfy (3-12) have only a trivial applicability to the summability of random variables.

It was proved in [1, Theorem 5.10] that, for $r>1$, $C_1\equiv C_r$ with respect to the summability to 0 of *arbitrary* sequences $\{\mathbf{x}_k\}$ of independent, real-valued, symmetric random variables. When we further restrict $\{\mathbf{x}_k\}$ to normal families, the present Theorem 5 extends identity with C_1 to a wide class of Nörlund summability methods including all methods C_r for $r>1$.

Theorem 6 is the third special case in which it is shown that $N_p\subset N_q$ implies $\mathbf{N}_p\subset\mathbf{N}_q$.

THEOREM 6. *Let* N_p *be any Nörlund summability method such that* $p_n\geqq p_{n+1}$, *for all* n. *Then* $\mathbf{N}_p\subset C_1$ *with respect to the summability in prob-*

ability of normal families of independent, real-valued, symmetric random variables to 0.

PROOF. Let C_1 be identified with the N_q of Theorem 2. We have $p(x)=\sum_{n=0}^{\infty}p_n^2x^n$ and $q(x)=(1-x)^{-1}$. Letting $a_n=p_{n-1}^2-p_n^2$, we see formally that $\sum_{n=0}^{\infty}\lambda_nx^n=q(x)/p(x)=(1-\sum_{n=1}^{\infty}a_nx^n)^{-1}$. Hence

$$\left(\sum_{n=0}^{\infty}\lambda_nx^n\right)\left(1-\sum_{n=1}^{\infty}a_nx^n\right)=1. \tag{3-15}$$

Equating coefficients in (3-15), we find that $\lambda_0=1$ and

$$\lambda_n=a_1\lambda_{n-1}+a_2\lambda_{n-2}+\cdots+\lambda_0a_n \qquad (n=1,2,\cdots). \tag{3-16}$$

Since $\lambda_0=1$ and since $a_n\geqq 0$, it is seen from (3-16) that $\lambda_n\geqq 0$ for all n. (The non-negativity of $\{\lambda_n\}$ is also a result of Kaluza [3].) Now $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(p_{n-1}^2-p_n^2)=p_0^2=1$. By (3-16), λ_n is a weighted sum of $\{\lambda_0,\lambda_1,\cdots,\lambda_n\}$, with total weight $\sum_{k=1}^{n}a_k\leqq 1$. Therefore $0\leqq\lambda_n\leqq\max_{0\leqq k\leqq n-1}\lambda_k$. Since $\lambda_0=1$, this implies that all $\lambda_n\leqq 1$. But $Q_n=n+1$. Hence $\lambda_n=o(Q_n^2)$, proving (3-6).

Let $R_n=\sum_{k=0}^{n}p_k^2$. We have the following formal identities:

$$\lambda(x)=\frac{q(x)}{p(x)}=\frac{(1-x)^{-1}q(x)}{(1-x)^{-1}p(x)}=\frac{(1-x)^{-2}}{(1-x)^{-1}\sum_n p_n^2x^n}=\frac{\sum_n(n+1)x^n}{\sum_n R_nx^n}.$$

Therefore $\lambda(x)\sum_{n=0}^{\infty}R_nx^n=\sum_{n=0}^{\infty}(n+1)x^n$, formally. Equating coefficients of x^n and remembering that $\lambda_n\geqq 0$, we find that

$$\sum_{k=0}^{n}R_k\left|\lambda_{n-k}\right|=n+1 \qquad (n=0,1,2,\cdots). \tag{3-17}$$

But, by Schwarz's inequality, $P_k^2=(\sum_{j=0}^{k}p_j)^2\leqq(\sum_{j=0}^{k}p_j^2)(\sum_{j=0}^{k}1)=(k+1)R_k$. Hence

$$P_k^2\leqq(n+1)R_k \qquad (k=0,1,2,\cdots,n). \tag{3-18}$$

From (3-17) and (3-18) it is seen that $\sum_{k=0}^{n}P_k^2|\lambda_{n-k}|\leqq(n+1)\sum_{k=0}^{n}R_k|\lambda_{n-k}|=(n+1)^2=Q_n^2$. Hence (3-5) is satisfied. By Theorem 2 the present proof is complete.

Any Nörlund method N_p with nonincreasing p_n is necessarily regular, for $p_n/P_n\leqq p_n/np_n=1/n\rightarrow 0$, as $n\rightarrow\infty$. Moreover, $N_p\subset C_1$.

The Cesàro summability methods C_r $(0<r<1)$ are of the Nörlund type with nonincreasing $\{p_n\}$. Theorem 5.5 of [1] showed that, for $0<r<1$, $C_r\subset C_1$ over *arbitrary* sequences of independent, real-valued, symmetric random variables. Theorem 5.6 of [1] showed that, for $0<r<1$, $C_1 \not\subset C_r$, even over *normal families*. Hence the conclusion of Theorem 6 cannot in general be extended to assert that $\boldsymbol{N}_p \equiv C_1$.

One may ask whether Theorem 4 has a converse. That is, if N_q is regular, and if $C_1 \not\subset N_q$ over real sequences, can one always find a normal family summable-C_1 to 0 but not summable-$\boldsymbol{N}_q$ to 0? When N_q is C_r $(0<r<1)$, we just saw that the answer is "yes." In general, however, the answer is "no," as is shown by the following theorem.

THEOREM 7. *There exists a regular Nörlund summability method* N_p *with the following properties:*

(3-19) $N_p \subset C_1$, *with respect to sequences of real numbers;*

(3-20) $C_1 \not\subset N_p$, *with respect to sequences of real numbers;*

(3-21) $\boldsymbol{N}_p \equiv C_1$, *with respect to the summability in probability of normal families of independent, real-valued, symmetric random variables to* 0.

PROOF. Let $p_n = 1$ (n even); let $p_n = 0$ (n odd). Then $P_n \cong n/2$. Now $p^*(x) = \sum_n p_n x^n = \sum_n p_n^2 x^n = p(x) = (1-x^2)^{-1}$. Let C_1 be identified with the N_q of (2-1), (2-2), and Theorem 2. Then $q_n = 1$ and $Q_n = n+1$, for all n. Furthermore, $q^*(x) = \sum_n q_n x^n = \sum_n q_n^2 x^n = (1-x)^{-1}$. Then $\lambda^*(x) = \lambda(x) = (1-x)^{-1}/(1-x^2)^{-1} = 1+x$, while $\mu^*(x) = \mu(x) = 1/\lambda(x) = (1+x)^{-1}$. Hence $\lambda_0^* = \lambda_0 = \lambda_1^* = \lambda_1 = 1$; $\lambda_n^* = \lambda_n = 0$ $(n \geqq 2)$; $\mu_n^* = \mu_n = (-1)^n$, for all n.

I. Obviously $\lambda_n^* = o(Q_n)$; and $\sum_{k=0}^n P_k |\lambda_{n-k}^*| = P_n + P_{n-1} \cong n = O(Q_n)$. Hence, by (2-1), $N_p \subset C_1$ over sequences of real numbers.

II. Since $\sum_{n=0}^\infty |\mu_n^*| = \infty$, we find from (2-2) that $N_p \not\equiv C_1$ for sequences of real numbers. Since $N_p \subset C_1$, we know that $C_1 \not\subset N_p$.

III. Obviously $\lambda_n = o(Q_n^2)$; and $\sum_{k=0}^n P_k^2 |\lambda_{n-k}| = P_n^2 + P_{n-1}^2 \cong n^2/2 = O(Q_n^2)$. Hence, by Theorem 2, $\boldsymbol{N}_p \subset \boldsymbol{C}_1$ for the $\{\boldsymbol{x}_k\}$ of (3-19).

IV. To prove $\boldsymbol{C}_1 \subset \boldsymbol{N}_p$ we are unable to use Theorem 2 by interchanging p and q and putting $\{\mu_n\}$ into (3-5) and (3-6). However, (3-5) is not a necessary condition, and we shall show directly that $\boldsymbol{C}_1 \subset \boldsymbol{N}_p$. Consider any normal family $\{\boldsymbol{x}_k\}$ which is summable-$\boldsymbol{C}_1$ to 0. By Theorem 1, $n^{-2}\sum_{k=0}^n \sigma_k^2 \to 0$, as $n \to \infty$. Hence $P_n^{-2}\sum_{k=0}^n p_{n-k}^2 \sigma_k^2 \leqq P_n^{-2}\sum_{k=0}^n \sigma_k^2 \cong (4/n^2)\sum_{k=0}^n \sigma_k^2 \to 0$, as $n \to \infty$. Theorem 1, applied to $\boldsymbol{N}_p$, proves that $\{\boldsymbol{x}_k\}$ is summable-$\boldsymbol{N}_p$ to 0. Hence $\boldsymbol{C}_1 \subset \boldsymbol{N}_p$ and so, by III above, $\boldsymbol{C}_1 \equiv \boldsymbol{N}_p$. This completes the proof of Theorem 7.

It is curious that the necessity of condition (3-5) for $\boldsymbol{N}_p \subset \boldsymbol{N}_q$ over normal families $\{\boldsymbol{x}_k\}$ breaks down just for an example where we fail to have $N_p \subset N_q$ over real sequences.

COROLLARY 2. *Condition* (3-5) *is not necessary for* $\boldsymbol{N}_p \subset \boldsymbol{N}_q$ *with re-*

spect to the summability in probability of a normal family of independent, real-valued, symmetric random variables to 0.

4. Lemmas used in proof of Theorem 4. The following Lemmas 1 and 2 are used to prove Lemma 3, which was applied to the proof of Theorem 4. In all three lemmas, $c_{-1}=0$ and $\{c_i\}$ $(i=0, 1, 2, \cdots)$ is an arbitrary sequence of non-negative numbers. The number $|c_i-c_{i-1}|$ is usually abbrevated to h_i.

LEMMA 1. *With the above notation, for each integer* $n=1, 2, 3, \cdots,$ (4-1) *holds*:

$$2\sum_{i=0}^{n-1} | c_i^2 - c_{i-1}^2 | \leqq 2\sum_{i,j=0}^{n-1} h_i h_j. \tag{4-1}$$

PROOF. We use a proof by induction. Since $c_{-1}=0$, (4-1) is true for $n=1$. Suppose that (4-1) holds for $n=N$. Let $c_N \geqq 0$ be arbitrary. Now

$$\begin{aligned}\sum_{i=0}^{N} | c_i^2 - c_{i-1}^2 | &\leqq | c_N^2 - c_{N-1}^2 | + \sum_{i,j=0}^{N-1} h_i h_j \\ &= | c_N^2 - c_{N-1}^2 | + \sum_{i,j=0}^{N} h_i h_j - 2\sum_{i=0}^{N-1} h_i h_N - h_N^2.\end{aligned}$$

Thus to prove (4-1) for $n=N+1$ it is sufficient to prove that

$$| c_N^2 - c_{N-1}^2 | - 2\sum_{i=0}^{N-1} h_i h_N - h_N^2 \leqq 0. \tag{4-2}$$

Now $c_{N-1}=\sum_{i=0}^{N-1}(c_i-c_{i-1}) \leqq \sum_{i=0}^{N-1}|c_i-c_{i-1}|$. Therefore

$$2c_{N-1} \leqq 2\sum_{i=0}^{N-1} h_i. \tag{4-3}$$

Case 1. Suppose $c_N \geqq c_{N-1}$. Adding $c_N-c_{N-1}=h_N$ to both sides of (4-3), we find that

$$c_N + c_{N-1} \leqq 2\sum_{i=0}^{N-1} h_i + h_N. \tag{4-4}$$

Multiplying both sides of (4.3) by $|c_N-c_{N-1}|=h_N$, we see that (4-2) holds, proving the lemma for Case 1.

Case 2. Suppose $c_N<c_{N-1}$. Then from (4-3) we see that $2c_N < 2\sum_{i=0}^{N-1}h_i$. Adding $c_{N-1}-c_N=h_N$ to both sides of the last inequality, we see that (4-4) holds. The proof of the lemma is completed as in Case 1.

LEMMA 2. *With the same notation, for each integer* $n=1, 2, 3, \cdots,$ (4-5) *holds*:

$$
2\sum_{i=0}^{n-1}|c_i^2 - c_{i-1}^2| + |c_n^2 - c_{n-1}^2| \le 2\sum_{i,j=0}^{n-1} h_i h_j + 2h_n\sum_{i=0}^{n-1}(n+1-i)h_i + h_n^2. \tag{4-5}
$$

PROOF. By Lemma 1 it suffices to prove that

$$
|c_n^2 - c_{n-1}^2| \le 2h_n\sum_{i=0}^{n-1}(n+1-i)h_i + h_n^2. \tag{4-6}
$$

If $h_n = |c_n - c_{n-1}| = 0$, (4-6) is trivial. If not, (4-6) is equivalent to (4-7):

$$
c_n + c_{n-1} \le 2\sum_{i=0}^{n-1}(n+1-i)h_i + h_n. \tag{4-7}
$$

To prove (4-7), we start with the inequality

$$
\begin{aligned}
c_{n-1} &\le c_0 + c_1 + \cdots + c_{n-2} + 2c_{n-1}\\
&= \sum_{i=0}^{n-1}(n+1-i)(c_i - c_{i-1}).
\end{aligned}
$$

Hence

$$
2c_{n-1} \le 2\sum_{i=0}^{n-1}(n+1-i)h_i. \tag{4-8}
$$

Now (4-7) follows from (4-8) just as (4-4) followed from (4-3), by use of two cases. Thus the proof is complete.

LEMMA 3. *With the same notation, for each integer* $n=0, 1, 2, \cdots$, (4-9) *holds*:

$$
\sum_{k=0}^{n}(n-k+1)^2|c_k^2 - c_{k-1}^2| \le \left\{\sum_{k=0}^{n}(n-k+1)|c_k - c_{k-1}|\right\}^2. \tag{4-9}
$$

PROOF. Let $\phi_n = \sum_{k=0}^{n}(n-k+1)^2|c_k^2 - c_{k-1}^2|$, for $n=0, 1, 2, \cdots$. Let $\phi_{-1}=0$. Let $\psi_n = \{\sum_{k=0}^{n}(n-k+1)h_k\}^2 = \sum_{i,j=0}^{n}(n-i+1)(n-j+1)h_ih_j$, for $n=0, 1, 2, \cdots$. Let $\psi_{-1}=0$. We must prove that $\phi_n \le \psi_n$, for all n. Since the result is trivial for $n=0$, fix $n \ge 1$.

We take first and second differences of ϕ_n and ψ_n:

$$\Delta\phi_n = \phi_n - \phi_{n-1} = \sum_{k=0}^{n} (2n - 2k + 1)\,|\,c_k^2 - c_{k-1}^2\,|.$$

$$\Delta^2\phi_n = \Delta\phi_n - \Delta\phi_{n-1} = 2\sum_{k=0}^{n-1}|\,c_k^2 - c_{k-1}^2\,| + |\,c_n^2 - c_{n-1}^2\,|.$$

$$\Delta\psi_n = \psi_n - \psi_{n-1} = \sum_{i,j=0}^{n} (2n + 1 - i - j)k_ik_j.$$

$$\Delta^2\psi_n = \Delta\psi_n - \Delta\psi_{n-1} = 2\sum_{i,j=0}^{n-1} k_ik_j + 2k_n\sum_{i=0}^{n-1}(n + 1 - i)k_i + k_n^2.$$

Now by Lemma 2, $\Delta^2\phi_n \leqq \Delta^2\psi_n$ for all integers $n \geqq 1$. Hence $\Delta\phi_n \leqq \Delta\psi_n$ and therefore $\phi_n \leqq \psi_n$ for the same integers. This proves Lemma 3.

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A NOTE ON THE MEAN VALUE OF THE POISSON KERNEL

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In some investigations it is necessary to evaluate the mean value of some power of the Poisson kernel,

$$P(r,\theta) = (1-r^2)/(1-2r\cos\theta+r^2),$$

with respect to θ. This note gives a closed expression for this mean value, and an exact statement of the order of growth as r approaches 1.

THEOREM 1. *If* $x=2r/(1+r^2)$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi} P^{n+1}(r,\theta)\,d\theta = \left(\frac{1-r^2}{1+r^2}\right)^{n+1}\cdot\frac{1}{\Gamma(n+1)}\cdot\frac{d^n}{dx^n}\left(\frac{x^n}{(1-x^2)^{1/2}}\right),\qquad n>-1. \tag{1}$$

If n is not an integer the derivative is to be computed by the formula of Riemann and Liouville[1]

$$\frac{d^n}{dx^n}(f(x)) = \frac{d^m}{dx^m}\,\frac{1}{\Gamma(\rho)}\int_0^x (x-t)^{\rho-1}f(t)\,dt, \tag{2}$$

where m is the smallest integer not less than n and $\rho=m-n$.

The proof consists merely of the comparison of two power series. Clearly

$$P^{n+1}(r,\theta) = \left(\frac{1-r^2}{1+r^2}\right)^{n+1}\left(1-\frac{2r}{1+r^2}\cos\theta\right)^{-(n+1)},$$

and the second parenthesis, with $x=2r/(1+r^2)$, is $1+(n+1)x\cos\theta+(n+1)(n+2)/2!\,x^2\cos^2\theta+\cdots$ by the binomial theorem. Since

$$\int_0^{2\pi}\cos^p\theta\,d\theta = 0 \qquad \text{(if } p \text{ is an odd integer)}$$

$$= \frac{4(p-1)(p-3)\cdots 3\cdot 1}{p(p-2)\cdots 4\cdot 2}\cdot\frac{\pi}{2} \qquad \text{(if } p \text{ is even)}$$

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[1] See, for example, Courant, *Differential and integral calculus*, rev. ed., vol. 2, pp. 339–340.

equation (1) can be written

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} P^{n+1}(r,\theta)d\theta \\
&\quad = \left(\frac{1-r^2}{1+r^2}\right)^{n+1}\left[1+\frac{(n+1)(n+2)}{2!}\cdot x^2\cdot\frac{1}{2}\right. \\
&\qquad \left. + \frac{(n+1)(n+2)(n+3)(n+4)}{4!}\cdot x^4\cdot\frac{1\cdot 3}{2\cdot 4}+\cdots\right] \\
(3)\qquad &\quad = \left(\frac{1-r^2}{1+r^2}\right)^{n+1}\left[1+\sum_{k=1}^{\infty}\frac{(n+1)(n+2)\cdots(n+2k)}{(2k)!}\right. \\
&\qquad \left. \cdot\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k\cdot k!}\cdot x^{2k}\right].
\end{aligned}
$$

Now

$$
\begin{aligned}
(4)\qquad x^n(1-x^2)^{-1/2} = x^n + \frac{1}{2}x^{n+2} + \frac{1\cdot 3}{2^2\cdot 2!}x^{n+4} \\
+\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^{n+6}+\cdots.
\end{aligned}
$$

If n is an integer, this can be differentiated n times to yield

$$
\begin{aligned}
n! + \frac{(n+2)!}{2!}\cdot\frac{1}{2}\cdot x^2 + \frac{(n+4)!}{4!}\cdot\frac{1\cdot 3}{2^2\cdot 2!}x^4 \\
+\frac{(n+6)!}{6!}\cdot\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^6+\cdots.
\end{aligned}
$$

Division by $\Gamma(n+1)$, as required in (1), produces the power series of (3).

If n is not an integer, formula (2) is applied to (4) to yield, on the right, a series of terms containing integrals of the form

$$\frac{1}{\Gamma(\rho)}\int_0^x (x-t)^{\rho-1}t^{n+2p}dt, \qquad p = 0, 1, 2, \cdots.$$

The substitution $t=xu$ changes these to

$$
\begin{aligned}
\frac{x^{n+2p+\rho}}{\Gamma(\rho)}\int_0^1 (1-u)^{\rho-1}u^{n+2p}du &= \frac{x^{n+2p}}{\Gamma(\rho)}\cdot B(\rho, n+1+2p) \\
&= \frac{x^{n+2p}\Gamma(n+2p+1)}{\Gamma(n+2p+1+\rho)},
\end{aligned}
$$

and the mth derivative of one of these terms is

$$\frac{(m+2p)!x^{2p}\Gamma(n+2p+1)}{(2p)!\Gamma(n+2p+1+\rho)}.$$

Hence the right member of (4) becomes

$$\frac{m!\Gamma(n+1)}{\Gamma(n+1+\rho)}+\frac{1}{2}\cdot\frac{(m+2)!\Gamma(n+3)}{2!\Gamma(n+3+\rho)}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}\cdot\frac{(m+4)!\Gamma(n+5)}{4!\Gamma(n+5+\rho)}x^4+\cdots.$$

Since $n+\rho=m$, this reduces to

$$\Gamma(n+1)\left[1+\frac{1}{2}\cdot\frac{(n+1)(n+2)}{2!}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}\cdot\frac{(n+1)(n+2)(n+3)(n+4)}{4!}x^4+\cdots\right],$$

the desired series.

The integration term by term is justified, since $(x-t)^{\rho-1}$ is integrable and the series which it multiplies is uniformly convergent.

The order of growth of this mean value, as r approaches 1, is specified by the following theorem.

THEOREM 2.

$$\lim_{r\to 1-}\frac{(1-r)^n}{2\pi}\int_0^{2\pi}P^{n+1}(r,\theta)d\theta = \frac{(2m)!\Gamma(n+1/2)}{m!2^m\Gamma(n+1)\Gamma(n+1/2+\rho)}, \qquad n>-1/2, \tag{5}$$

where m, n and ρ are related as in Theorem 1.

If n is an integer the right member reduces to $(2n)!/(n!)^2 2^n$.

PROOF. Since $(1-r^2)^2/(1+r^2)^2=1-x^2$, where x has the same meaning as in Theorem 1, and since $(1+r)/(1+r^2)$ will approach unity, it is convenient to replace $(1-r)^n$ by $(1-x^2)^{n/2}$, and prove that the right member of (5) is equal to

$$\lim_{x\to 1-}(1-x^2)^{n/2}(1-x^2)^{(n+1)/2}\frac{1}{\Gamma(n+1)}\cdot\frac{d^n}{dx^n}(x^n(1-x^2)^{-1/2}). \tag{6}$$

If $n=0$, this is obviously true. If n is a positive integer, let

$\phi(x) = (1-x^2)^{-1/2}$, and consider the derivatives of $x^n\phi(x)$.

$$d(x^n\phi(x))/dx = x^{n+1}\phi^3(x) + nx^{n-1}\phi(x);$$
$$d^2(x^n\phi(x))/dx^2 = 3x^{n+2}\phi^5(x) + \text{terms in lower powers of } \phi(x);$$

and by mathematical induction it can readily be shown that

$$d^n(x^n\phi(x))/dx^n = 1\cdot 3\cdot 5 \cdots (2n-1)x^{2n}\phi^{2n+1}(x) + \text{terms in } \phi^{2n-1}, \phi^{2n-3}, \cdots, \phi.$$

Since

$$\lim_{x\to 1-} (1-x^2)^{n+1/2}\phi^p(x) \begin{cases} = 0, & p = 1, 3, 5, \cdots, 2n-1, \\ = 1, & p = 2n+1, \end{cases}$$

the limit in (6) is

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{\Gamma(n+1)} = \frac{(2n)!}{2^n(n!)^2},$$

as required in the theorem.

If n is not an integer,[1]

$$\frac{d^n}{dx^n}(x^n\phi(x)) = \frac{d^m}{dx^m}\frac{1}{\Gamma(\rho)}\int_0^x (x-t)^{\rho-1}t^n(1-t^2)^{-1/2}dt$$
$$= \frac{1}{\Gamma(\rho)}\int_0^x (x-t)^{\rho-1}\frac{d^m}{dt^m}(t^n\phi(t))dt.$$

As before, it is necessary to consider only the first term of the derivative, $1\cdot 3\cdot 5 \cdots (2m-1)x^{m+n}\phi^{2m+1}(x)$, since

$$(1-x^2)^{n+1/2}\int_0^x (x-t)^{\rho-1}t^{m+n-2}\phi^{2m-1}(t)dt$$
$$< (1-x^2)^{n+1/2}\int_0^x \frac{(x-t)^{\rho-1}dt}{(1-x^2)^{m-1/2}}$$
$$= (1-x^2)^{1-\rho}\int_0^x (x-t)^{\rho-1}dt,$$

and this approaches zero. Consequently it is necessary to consider

$$\lim_{x\to 1-} (1-x^2)^{n+1/2}\int_0^x (x-t)^{\rho-1}t^{m+n}\phi^{2m+1}(t)dt.$$

This limit, multiplied by

$$\frac{1\cdot 3\cdot 5\cdots(2m-1)}{\Gamma(\rho)\Gamma(n+1)}, \tag{7}$$

is the result sought.

The substitution $t=x-(1-x)u$ reduces the integral to

$$\begin{aligned} &\frac{x^{2m-\rho}(1-x^2)^{n+1/2+\rho}}{(1+x)^{\rho}(1-x^2)^{m+1/2}} \\ &\qquad\cdot\int_0^{x/(1-x)} \frac{u^{\rho-1}(1-(1-x)u/x)^{2m-\rho}du}{(1+u)^{m+1/2}(1-(1-x)u/(1+x))^{m+1/2}}. \end{aligned}\tag{8}$$

Since $n+\rho=m$, and x is to approach 1 later, the factor outside the integral sign will have the limit $2^{-\rho}$. To evaluate the integral, let a be a number between 0 and 1 (the way to choose a will become clear later; a will depend on n but not on x), and consider

$$\int_{ax/(1-x)}^{x/(1-x)} (\text{same integrand as in (8)})\ du. \tag{9}$$

In the interval of integration, $u\geqq ax/(1-x)$, $1+u\geqq(1-x(1-a))/(1-x)$, and $1-(1-x)u/(1+x)\geqq 1/(1+x)$. Hence the integral in (9) is less than

$$\begin{aligned} &((1-x)/ax)^{1-\rho}(1+x)^{m+1/2}\{(1-x)/(1-x(1-a))\}^{m+1/2} \\ &\qquad\qquad\cdot\int_{ax/(1-x)}^{x/(1-x)}(1-(1-x)u/x)^{2m-\rho}du. \end{aligned}$$

Except for a bounded factor this is $(1-x)^{m+1/2-\rho}x(1-a)^{2m+1-\rho}/(1-x)$, and accordingly approaches zero as x approaches 1; if n is negative, m is zero, and $n=-\rho>-1/2$.

Therefore the desired limit can be found by replacing the upper limit of integration in (8) by $ax/(1-x)$. This new integral will be not less than

$$\int_0^{ax/(1-x)} \frac{u^{\rho-1}(1-a)^{2m-\rho}}{(1+u)^{m+1/2}}du. \tag{10}$$

As x approaches 1 this has the limit $(1-a)^{2m-\rho}B(\rho, m-\rho+1/2)$. Also, the new integral will not be greater than

$$\begin{aligned} &\int_0^{ax/(1-x)} \frac{u^{\rho-1}du}{(1+u)^{m+1/2}(1-ax/(1+x))^{m+1/2}} \\ &\qquad = \left(\frac{1+x}{1+x(1-a)}\right)^{m+1/2}\int_0^{ax/(1-x)}\frac{u^{\rho-1}du}{(1+u)^{m+1/2}}, \end{aligned}\tag{11}$$

which has the limit $2^{m+1/2}/(2-a)^{m+1/2}B(\rho,\ m-\rho+1/2)$. If n is negative, m is zero, and the factor $(1-a)^{2m-\rho}$ will appear in (11) instead of (10).

Since a can be taken as close to zero as is desired, it follows that the limit exists and is $B(\rho,\ m-\rho+1/2)$, or $B(\rho,\ n+1/2)$. If the factor (7) is annexed, the theorem follows.

The theorem cannot be extended to the case $-1<n\leqq -1/2$, for if n is replaced by $-\rho$,

$$\frac{(1-r)^n}{2\pi}\int_0^{2\pi}P^{n+1}(r,\theta)d\theta$$
$$=\frac{(1-r)^{-\rho}}{2\pi}\left(\frac{1-r^2}{1+r^2}\right)^{1-\rho}\int_0^{2\pi}\frac{d\theta}{(1-x\cos\theta)^{1-\rho}}.$$

After a bounded factor is removed, the substitution $u=\cos\theta$ gives

$$(1-x)^{1/2-\rho}\int_{-1}^{+1}\frac{du}{(1-u^2)^{1/2}(1-xu)^{1-\rho}}.$$

Since this integral converges at $u=-1$, it is necessary to consider only the interval from 0 to 1 to show divergence. The change of variable $xu=t(1-x)$ yields

$$(1-x)^{1/2-\rho}\int_0^{x/(1-x)}\frac{dt(1-x)/x}{(1-(1-x)t/x)^{1/2}(1+(1-x)t/x)^{1/2}(1-t(1-x))^{1-\rho}},$$

which is greater than

$$\frac{(1-x)^{3/2-\rho}}{2^{1/2}}\int_0^{x/(1-x)}(1-(1-x)t/x)^{-1/2}dt=(1-x)^{1/2-\rho}2^{1/2}x.$$

Hence, if ρ is greater than 1/2, the integral diverges and no limit exists.

If $n=-1/2$, the theorem fails to hold, but the order of growth of the mean value can be found. Here $m=0$, $\rho=1/2$, and

$$\frac{(1-r)^n}{2\pi}\int_0^{2\pi}P^{n+1}(r,\theta)d\theta$$
$$=\frac{(1-r)^{-1/2}}{2\pi}\left(\frac{1-r^2}{1+r^2}\right)^{1-1/2}\int_0^{2\pi}(1-x\cos\theta)^{-1/2}d\theta.$$

The last integral can be written

$$\int_0^{2\pi}(1-x+2x\sin^2\theta/2)^{-1/2}d\theta,$$

which is greater than

$$\int_0^{2\pi} (1 - x + 2 \sin^2 \theta/2)^{-1/2} d\theta.$$

If $1-x$ is set equal to $2\alpha^2$, the last integral is greater than

$$\frac{1}{2^{1/2}}\int_0^{2\pi} (\alpha^2 + \theta^2)^{-1/2} d\theta = \frac{1}{2^{1/2}} \log\,(2\pi + (4\pi^2 + \alpha^2)^{1/2})/\alpha,$$

which becomes infinite as α approaches zero. Now $|\log \alpha|$ is effectively $|\log\,(1-x)|$ or $|\log\,(1-r)|$ multiplied by a constant. By an estimation similar to the foregoing, the integral can be shown to be less than $|\log\,(1-r)|$ multiplied by a second constant. Hence, if $n = -1/2$, the order of growth of the mean value is $(1-r)^{1/2}|\log(1-r)|$.

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ON AVERAGES OF NEWTONIAN POTENTIALS

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1. **Introduction.** Averages (mean-values) have proved extremely useful in the investigation of properties of potential functions [2, 3, 4, 7, 9, 11];[1] to a great extent this has been due to the fact that averages of potential functions are themselves *smoother* potential functions.

It is the purpose of this note to exhibit the relations between the mass distribution $\sigma(E)$ associated with the potential function $A[U(x, y)]$, which is an average of a potential function $U(x, y)$, and the mass distribution $\mu(E)$ associated with $U(x, y)$. In a general sense it is proved that $\sigma(E)=A[\mu(E)]$ and that the density $D\sigma(x, y)$ of $\sigma(E)$ is the corresponding average of the density $D\mu(x, y)$. Precise statements of these results are contained in §4 below.

It should be noted that Thompson has investigated the problem noted above [11]; except for an error in the statement of his most general result (given without proof), his results are substantially those contained here. However, whereas Thompson's method depends upon a discussion of the interchange of the order of integration in iterated Radon-Stieltjes integrals, the method of this paper depends upon the use of approximations to potentials by means of smoother potentials. Both for the sake of completeness and to point up the difference of the two methods, a proof of Thompson's (corrected) general result (which is Theorem 3 of this note), based upon Thompson's own method, is given in §4 below.

2. **Notation and definitions.** Let F be a closed bounded set in the x, y-plane, and let $\mu(e)$ be an arbitrary *distribution of positive mass* on F, that is, $\mu(e)$ is defined for all Borel sets e in F such that (i) $\mu(e) \geqq 0$, (ii) $\mu(\sum_{i=1}^{\infty} e_i) = \sum_{i=1}^{\infty} \mu(e_i)$, for each sequence $\{e_i\}$ of mutually disjoint Borel sets contained in F [6, p. 25]. The distribution $\mu(e)$ is said to be of *finite total amount* if $\mu(e)$ is uniformly bounded for all e in F.

A distribution $\mu(e)$ may be extended so as to be defined for all Borel sets E in the plane by means of the definition [3, p. 227] $\mu(E) \equiv \mu(E \cdot F)$. It is apparent that $\mu(E)$ satisfies (i) and (ii) above. In this paper it is assumed that all distributions have been thus extended to all Borel sets in the plane, although, strictly speaking, each distribution had

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

been originally defined only for those Borel sets contained in a closed bounded set F.

The *density* of $\mu(E)$ at (x, y) is defined as [10, p. 149]

$$(1) \qquad D_s\mu(x, y) = \lim_{\rho\to 0} \frac{\mu(D_\rho)}{\pi\rho^2}$$

when the limit exists, where D_ρ is the closed circular disc with center at (x, y) and radius ρ. For distributions of the type used in this paper, it is well known that $D_s\mu(x, y)$ exists almost everywhere [10, pp. 115, 149].

The Borel set E is said to be *μ-regular* if and only if $\mu(\bar{E}-E)=0$, where $\bar{E}$ is the point set closure of E [5, p. 9].

The sequence of mass distributions $\{\mu_n(E)\}$ is said to *converge to the mass distribution* $\mu(E)$ if and only if $\mu(E)=\lim_{n\to\infty}\mu_n(E)$ for each *open* μ-regular set E. The sequence $\{\mu_n(E)\}$ is said to be of *uniform finite total amount* if and only if $\mu_n(E)$ is uniformly bounded for all E and for all n.

The *Newtonian potential* at (x, y) of the distribution $\mu(E)$ is defined by the Stieltjes-Radon integral [10, pp. 65–67]

$$(2) \qquad u(x, y) = \int\int_W \log \frac{1}{PQ}\, d\mu(\xi, \eta),$$

where $P=(x, y)$, $Q=(\xi, \eta)$, $PQ=((x-\xi)^2+(y-\eta)^2)^{1/2}$, and where the integral is extended over the whole finite plane W. It should be noted that the integral in (2) is in fact a finite integral since $\mu(E)$ vanishes for all E disjoint with some closed bounded set F; however, the use of W simplifies some of the discussion below.

R will always denote an open *oriented* rectangle with sides parallel to the coordinate axes, while $\mathfrak{R}$ will always denote the open rectangle with vertices $(\pm h, \pm k)$.

If E is a previously assigned Borel set, then $E_{x,y}$, for fixed (x, y), denotes the set of all points $\{(x+x', y+y')\}$ for which (x', y') is in E, and $E^{**}_{x,y}$ denotes the set of all points $\{(x-x', y-y')\}$ for which (x', y') is in E.

The *average* of $U(x, y)$ over rectangles $\mathfrak{R}_{x,y}$ is defined by

$$\begin{aligned} A_{h,k}(x, y) &= A_{h,k}[U(x+\xi, y+\eta)] \\ &= \frac{1}{4hk}\int\int_{\mathfrak{R}} U(x+\xi, y+\eta)\,d\xi d\eta \\ &= \frac{1}{4hk}\int\int_{\mathfrak{R}_{x,y}} U(\xi, \eta)\,d\xi d\eta, \end{aligned}$$

while the average of $U(x, y)$ over open circular discs $D(x, y; \rho)$, with center at (x, y) and radius ρ, is defined by [1, 2]

$$\mathfrak{A}_\rho^{(1)}(x, y) \equiv \mathfrak{A}_\rho^{(1)}[U(x + \xi, y + \eta)] \equiv \frac{1}{\pi\rho^2}\iint_{D(x,y;\rho)} U(\xi, \eta)d\xi d\eta.$$

In this paper, an *iterated* circular average $\mathfrak{A}_\rho^{(3)}(x, y)$ [3, p. 236] will be used, where

$$\mathfrak{A}_\rho^{(2)}(x, y) \equiv \mathfrak{A}_\rho^{(1)}[\mathfrak{A}_\rho^{(1)}(x + \xi, y + \eta)],$$
$$\mathfrak{A}_\rho^{(3)}(x, y) \equiv \mathfrak{A}_\rho^{(1)}[\mathfrak{A}_\rho^{(2)}(x + \xi, y + \eta)].$$

3. **Lemmas.** Some of the important known results in potential theory that will be used here are listed in the form of lemmas. However, the corollary to Lemma 2, Lemma 3, and the remark following Lemma 3 are new.

Let $U(x, y)$ be a Newtonian potential with associated mass distribution $\mu(E)$, of finite total amount; then the following lemmas hold.

LEMMA 1 [3, p. 236]. *$\mathfrak{A}_\rho(x, y) \equiv \mathfrak{A}_\rho^{(3)}(x, y)$ is a Newtonian potential with continuous partial derivatives of the second order, with mass distribution*

$$\mu_\rho(E) \equiv \iint_E \frac{\Delta\mathfrak{A}_\rho(x, y)}{2\pi} dxdy, \tag{3}$$

where Δ is the Laplace operator, such that $\mathfrak{A}_\rho(x, y) \nearrow U(x, y)$ as $\rho \to 0$; moreover, if ρ is bounded, then $\{\mu_\rho(E)\}$ is of uniform finite total amount.

LEMMA 2 [9, p. 351]. *There exists a sequence $\{\rho_n\} \searrow 0$ as $n \to \infty$, such that $\{\mu_{\rho_n}(E)\}$ converges to $\mu(E)$.*

COROLLARY. *If $V(x, y)$ is another Newtonian potential with associated mass distribution $\nu(E)$, of finite total amount, then there exists a sequence $\{\rho_n\} \searrow 0$, as $n \to \infty$, such that $\{\mu_{\rho_n}(E)\}$ and $\{\nu_{\rho_n}(E)\}$ converge to $\mu(E)$ and $\nu(E)$, respectively; here $\nu_{\rho_n}(E)$ is defined by an expression analogous to* (3).

PROOF. The construction of the convergent sequence $\{\mu_{\rho_n}(E)\}$ in Lemma 2, as given by Riesz [9, pp. 351–352], follows a Cantor "diagonal" process which may be extended so as to yield a convergent sequence $\{\nu_{\rho_n}(E)\}$.

LEMMA 3. *If $\{\mu_{\rho_n}(E)\}$ is the convergent sequence noted in Lemma 2, and if R is a fixed rectangle in the plane, then $R_{x,y}$ is μ-regular for almost all points (x, y) in the plane; hence, except for a set of superficial measure* *zero it follows that*

$$\lim_{n\to\infty} \mu_{\nu_n}(R_{x,y}) = \mu(R_{x,y}). \tag{4}$$

PROOF. Since $\mu(E)$ is essentially distributed on a closed bounded set F, there is a sufficiently large rectangle R^* containing all the mass, that is, there is a rectangle R^* containing F such that $\mu(F) = \mu(R^*) < \infty$. Now according to Reichelderfer and Ringenberg [8, p. 235] there are at most a countable number of values of x, $\{\xi_i\}$, and y, $\{\eta_i\}$, such that the mass on the lines $x = \xi_i$ and $y = \eta_i$, $i = 1, 2, \cdots$, is positive; hence, since R has fixed dimensions, it follows that the boundary of $R_{x,y}$ *may* have positive mass if and only if the point (x, y) lies on a certain plane set consisting of a countable number of lines, that is, $R_{x,y}$ is μ-regular except possibly for a plane set of measure zero. The second part of the lemma follows from the definition of convergence of $\{\mu_{\nu_n}(E)\}$ to $\mu(E)$.

Remark. The key to the preceding proof is the result due to Reichelderfer and Ringenberg quoted above. However, that result can be extended to oblique lines as well as to other sufficiently smooth curves, and thus yield more general results. For example, under the hypothesis that $\mu(F) < \infty$, there are at most a countable number of lines with a fixed direction, say α, that have positive mass; hence $R_{x,y}$ in (4) may be replaced by $\Pi_{x,y}$ where Π is a certain fixed polygon.

LEMMA 4 [3, p. 231]. *The limit of a monotone increasing convergent sequence of Newtonian potentials, whose mass distributions are uniformly of finite total amount, is a Newtonian potential with an associated mass distribution of finite total amount.*

4. **Main results.** The proof of Theorem 1 illustrates the general method used in the paper.

THEOREM 1. *If $U(x, y)$ is a Newtonian potential with associated mass distribution $\mu(E)$, of finite total amount, then $A(x, y) \equiv A_{h,k}(x, y)$ is a Newtonian potential with an associated mass distribution $\sigma(E)$ of finite total amount; moreover, $\sigma(E)$ is unique and given by each of the representations*

$$\sigma(E) = \iint_E \frac{\mu(\mathfrak{R}_{x,y})}{4hk}\, dxdy, \tag{5}$$

and

$$\sigma(E) = \frac{1}{4hk} \iint_{\mathfrak{R}} \mu(E_{x,y}) dxdy. \tag{6}$$

PROOF. Since $U(x, y)$ satisfies the hypothesis of Lemma 1, it follows that there is a sequence $\{\rho_n\} \searrow 0$, as $n \to \infty$, such that $\{\mu_{\rho_n}(E)\}$ converges to $\mu(E)$, where $\mu_{\rho_n}(E)$ is defined by (3). Moreover, by Lemma 1, $\mathfrak{A}^{(2)}_{\rho_n}(x, y) \equiv \mathfrak{A}_n(x, y)$ is sufficiently smooth so that Fubini's classic theorem may be used to prove that

$$\begin{aligned} \mathfrak{A}_n A(x, y) &= \mathfrak{A}^{(2)}_{\rho_n}[A_{h,h}(x + \xi, y + \eta)] = A_{h,h}[\mathfrak{A}^{(2)}_{\rho_n}(x + \xi, y + \eta)] \\ &\equiv A\mathfrak{A}_n(x, y) \end{aligned} \tag{7}$$

holds. But $\mathfrak{A}_n(x, y)$ is a Newtonian potential, so that (7) yields

$$\mathfrak{A}_n A(x, y) = \frac{1}{4hk}\iint_R \left[\iint_W \log \frac{1}{PQ} \frac{\Delta\mathfrak{A}_n(s, t)}{2\pi} dsdt\right] d\xi d\eta, \tag{8}$$

where $P \equiv (x+\xi, \ y+\eta)$ and $Q \equiv (s, \ t)$. If $(s, \ t)$ is replaced by $(s+\xi, \ t+\eta)$ then (8) becomes

$$\begin{aligned} &\mathfrak{A}_n A(x, y) \\ &\quad = \frac{1}{4hk}\iint_R \left[\iint_W \log \frac{1}{P'Q} \frac{\Delta\mathfrak{A}_n(s + \xi, t + \eta)}{2\pi} dsdt\right] d\xi d\eta, \end{aligned} \tag{9}$$

where $P' \equiv (x, y)$. If Fubini's theorem is applied to (9), then

$$\begin{aligned} &\mathfrak{A}_n A(x, y) \\ &\quad = \iint_W \log \frac{1}{P'Q} \left[\frac{1}{4hk}\iint_R \frac{\Delta\mathfrak{A}_n(s + \xi, t + \eta)}{2\pi} d\xi d\eta\right] dsdt \end{aligned} \tag{10}$$

holds; now an application of both Leibnitz' rule and Fubini's theorem to (10) yields

$$\mathfrak{A}_n A(x, y) = \iint_W \log \frac{1}{P'Q} \frac{\Delta\mathfrak{A}_n A(s, t)}{2\pi} dsdt. \tag{11}$$

From (11) it follows that $\mathfrak{A}_n A(x, y) = A\mathfrak{A}_n(x, y)$ is a Newtonian potential with mass distribution

$$\sigma_{\rho_n}(E) = \iint_E \frac{\Delta\mathfrak{A}_n A(x, y)}{2\pi} dxdy. \tag{12}$$

By Lemma 1, $\mathfrak{A}_n(x, y) \nearrow U(x, y)$ as $n \to \infty$; hence (7) yields $\mathfrak{A}_n A(x, y) \nearrow A(x, y)$, as $n \to \infty$. Therefore, by Lemmas 1 and 4, $A(x, y)$ is a Newtonian potential with an associated mass distribution (of finite total amount), say $\sigma(E)$.

By the corollary to Lemma 2, there is a sequence $\{\rho_n'\} \searrow 0$ as $n \to \infty$, such that $\{\mu_{\rho_n'}(E)\}$ converges to $\mu(E)$ *and* $\{\sigma_{\rho_n'}(E)\}$ converges to $\sigma(E)$, where $\sigma_{\mu_n}'(E)$ is defined by an expression similar to (3). If the substitutions $\mathfrak{A}^{(1)}_{\mu_n'}(x, y) = a_n(x, y)$, $\mathfrak{A}^{(1)}_{\mu_n'}A(x, y) = a_nA(x, y)$, $\mu_n'(E) = \mu_n(E)$, and $\sigma_{\mu_n}'(E) = \sigma_n(E)$ are made, then it follows that for each R,

$$\mu_n(R) = \iint_R \frac{\Delta a_n(x, y)}{2\pi}\, dxdy$$

and

$$\sigma_n(R) = \iint_R \frac{\Delta a_n A(x, y)}{2\pi}\, dxdy \tag{13}$$

must hold.

Now (13) may be written in the forms

$$\begin{aligned} \sigma_n(R) &= \iint_R \left[\frac{1}{4hk}\iint_{\mathfrak{R}} \frac{\Delta a_n(x + \xi, y + \eta)}{2\pi}\, d\xi d\eta\right] dxdy, \\ &= \iint_R \frac{\mu_n(\mathfrak{R}_{x,y})}{4hk}\, dxdy, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \sigma_n(R) &= \frac{1}{4hk}\iint_{\mathfrak{R}} \left[\iint_R \frac{\Delta a_n(x + \xi, y + \eta)}{2\pi}\, dxdy\right] d\xi d\eta, \\ &= \frac{1}{4hk}\iint_{\mathfrak{R}} \mu_n(R_{\xi,\eta})\, d\xi d\eta. \end{aligned} \tag{15}$$

Since $\{\sigma_n(E)\}$ converges to $\sigma(E)$, it follows from (14) and Lemma 3 that

$$\sigma(R^*) = \lim_{n\to\infty} \sigma_n(R^*) = \iint_{R^*} \frac{\mu(\mathfrak{R}_{x,y})}{4hk}\, dxdy \tag{16}$$

holds for all σ-regular R^*. Since each rectangle R is the point set limit of a monotone increasing sequence of σ-regular rectangles, say $\{R_n^*\} \nearrow R$, such that [8, p. 236] $\lim_{n\to\infty}\sigma(R_n^*) = \sigma(R)$, it follows from (16) that

$$\sigma(R) = \iint_R \frac{\mu(\mathfrak{R}_{x,y})}{4hk} \tag{17}$$

holds for all rectangles R.

Let D be any large open disc that contains F in the interior, such that the distance between the boundaries of F and D exceeds $(h+k)$. The mass distribution defined by

$$\sigma^*(E) = \iint_E \frac{\mu(\mathfrak{R}_{x,y})}{4hk}\,dxdy$$

is an extension of the rectangle function (17) to the class of mass distributions within D; moreover, $\sigma(E)$ is such an extension of (17) too. But Reichelderfer and Ringenberg have shown that the extension of a rectangle function such as (17) to the class of mass distributions defined for all Borel sets within D is unique [8, p. 234], that is, $\sigma(E)=\sigma^*(E)$ for all Borel sets E within D. Since D was an arbitrary, large disc, it follows that $\sigma(E)=\sigma^*(E)$ for all Borel sets in the plane. Hence (5) holds.

In a similar way, (6) may be derived from (15) and the "extension theorem" of Reichelderfer and Ringenberg noted above. This completes the proof.

COROLLARY 1. *Under the hypotheses of Theorem* 1, *the relation*

$$D_s\sigma(x, y) = \frac{\mu(\mathfrak{R}_{x,y})}{4hk}$$

holds almost everywhere.

PROOF. A proof follows at once from (1) and (5).

In keeping with the remark made following Lemma 3, one can state the following theorem and corollary whose proofs would follow the lines of the preceding two proofs.

THEOREM 2. *Let T be a fixed figure in the plane, of superficial measure $|T|\neq 0$, and let $A(x, y)$ be the average of the Newtonian potential $U(x, y)$, defined in Theorem* 1, *over $T_{x,y}$. If the analogue of Lemma* 3 *holds, that is, if $\lim_{n\to\infty}\mu_{p_n}(T_{x,y})=\mu(T_{x,y})$, almost everywhere, then $A(x, y)$ is a Newtonian potential with associated mass distribution*

$$\sigma(E) = \iint_E \frac{\mu(T_{x,y})}{|T|}\,dxdy = \frac{1}{|T|}\iint_T \mu(E_{x,y})dxdy, \tag{18}$$

of finite total amount.

COROLLARY 2. *Under the hypotheses of Theorem* 2,

$$D_s\sigma(x, y) = \frac{\mu(T_{x,y})}{|T|}$$

holds almost everywhere.

It is now a simple matter to obtain the following general result.

THEOREM 3. *Let G be a fixed open set in the plane, with superficial measure $|G| < \infty$, and let $A(G; x, y)$ be the average of the Newtonian potential $U(x, y)$, defined in Theorem 1, over $G_{x,y}$. Then $A(G; x, y)$ is a Newtonian potential with associated mass distribution*

$$\sigma(E) = \iint_E \frac{\mu(G_{x,y})}{|G|}\, dxdy = \frac{1}{|G|} \iint_G \mu(E_{x,y}) dxdy, \tag{19}$$

of finite total amount.

PROOF. Since G is an open set in the plane, $|G| < \infty$, there exists a monotone increasing sequence of open sets $\{T_n\}$, such that each T_n is the sum of a finite number of simple, open polygons, and such that $(T_n)_{x,y} \nearrow G_{x,y}$, as $n \to \infty$, for each x, y. In what follows, T will denote a member of the sequence $\{T_n\}$, and instead of using the symbol "$n \to \infty$," the symbol "$T \nearrow A$" will be used.

It is necessary to approximate the average

$$\begin{aligned} A(G; x, y) &= \frac{1}{|G|} \iint_G U(x + \xi, y + \eta) d\xi d\eta \\ &= \frac{1}{|G|} \iint_{G_{x,y}} U(\xi, \eta) d\xi d\eta. \end{aligned} \tag{20}$$

One such approximation is the average,

$$\begin{aligned} A(T; x, y) &= \frac{1}{|T|} \iint_T U(x + \xi, y + \eta) d\xi d\eta \\ &= \frac{1}{|T|} \iint_{T_{x,y}} U(\xi, \eta) d\xi d\eta. \end{aligned} \tag{21}$$

Now by virtue of the remark following Lemma 3, it follows that Theorem 2 applies to the average (21) because T is the sum of a finite number of simple polygons. Hence $A(T; x, y)$ is a Newtonian potential of the form

$$A(T; x, y) = \iint_W \log \frac{1}{PQ} \frac{\mu(T_{\xi,\eta})}{|T|} d\xi d\eta, \tag{22}$$

where $P = (x, y)$, $Q = (\xi, \eta)$, and where (18) holds. Moreover, since $\mu(E)$ is a distribution of positive mass, it follows that

$$\mu(T_{\xi,\eta}) \nearrow \mu(G_{\xi,\eta}), \qquad \text{as} \quad T \nearrow A, \tag{23}$$

holds for all ξ, η. Now it follows from (20), (21), (22), and (23) that

$$A(G;x,y)=\lim_{T\nearrow A}A(T;x,y)=\iint_{W}\log\frac{1}{PQ}\,\frac{\mu(G_{\xi,\eta})}{|G|}\,d\xi d\eta \tag{24}$$

holds for all (x, y). It now follows from (24) and the definition of a Newtonian potential that $A(G; x, y)$ is a Newtonian potential having a summable density $D\sigma(x, y)$ for which

$$D\sigma(x,y)=\frac{\mu(G_{x,y})}{|G|} \tag{25}$$

holds almost everywhere.

The relation (19) now follows from (18) and (25). This completes the proof.

Another proof of the preceding theorem will now be given; this proof is based upon Thompson's method [11].

From (2) and (20) we have

$$A(G;x,y)\equiv A(x,y)=\frac{1}{|G|}\iint_{G_{x,y}}d\xi d\eta\iint_{W}\log\frac{1}{MQ}\,d\mu(s,t), \tag{26}$$

where $M=(s,t)$, $Q=(\xi,\eta)$. Since the integrand in (26) is a lower semicontinuous function, the order of integration may be interchanged; hence

$$A(x,y)=\frac{1}{|G|}\iint_{W}d\mu(s,t)\iint_{G_{x,y}}\log\frac{1}{MQ}\,d\xi d\eta. \tag{27}$$

But

$$\iint_{G_{x,y}}\log\frac{1}{MQ}\,d\xi d\eta=\iint_{G^{**}_{s,t}}\log\frac{1}{PQ}\,d\xi d\eta;$$

hence (27) becomes

$$\begin{aligned} A(x,y)&=\frac{1}{|G|}\iint_{W}d\mu(s,t)\iint_{G^{**}_{s,t}}\log\frac{1}{PQ}\,d\xi d\eta\\ &=\frac{1}{|G|}\iint_{W}d\mu(s,t)\iint_{W}\log\frac{1}{PQ}\,d\nu_{s,t}(\xi,\eta), \end{aligned} \tag{28}$$

where

$$\nu_{s,t}(E)=|E\cdot G^{**}_{s,t}|. \tag{29}$$

Now it follows from a fundamental result of Thompson [11] that (28) may be written in the form

$$(30)\qquad A(x, y) = \frac{1}{|G|}\iint_W \log\frac{1}{PQ}\, d\left[\iint_W r_{s,t}(\xi, \eta)\,d\mu(s, t)\right].$$

Consider $\sigma(E)$ defined by

$$(31)\qquad \sigma(E) = \iint_W r_{s,t}(E)\,d\mu(s, t).$$

From (29) and (31) it follows that

$$\begin{aligned}
(32)\qquad \sigma(E) &= \iint_W |E\cdot G^{**}_{s,t}|\,d\mu(s, t)\\
&= \iint_W d\mu(s, t)\iint_{E\cdot G^{**}_{s,t}} dudv\\
&= \iint_W d\mu(s, t)\iint_W B(u, v; s, t)dudv\\
(33)\qquad &= \iint_W dudv\iint_W B(u, v; s, t)d\mu(s, t),
\end{aligned}$$

where

$$\begin{aligned}
B(u, v; s, t) &= 1, \qquad &(u, v) \text{ in } E\cdot G^{**}_{s,t},\\
B(u, v; s, t) &= 0, \qquad &(u, v) \text{ not in } E\cdot G^{**}_{s,t}.
\end{aligned}$$

But for each (u, v) in E, $B(u, v; s, t)$ vanishes except for (s, t) in $G_{u,v}$. Hence (33) yields the first part of (19).

Now (19), (29), (30), and (32) yield (24), which is a Newtonian potential with density (25); this latter holds almost everywhere.

Since $\mu(F) < \infty$ and F is bounded, it follows from (31) that $\sigma(E)$ is bounded. This completes the proof.

At this point it should be remarked that Thompson's error consisted of writing $G^{**}_{x,y}$ for $G_{x,y}$ in (19) and (25).

5. **Conclusion.** Additional results may be obtained by considering point averages and averages over curves.

It should be noted that the following interesting result follows from (19):

$$(34)\qquad \iint_E \mu(G_{x,y})dxdy = \iint_G \mu(E_{x,y})dxdy$$

for each open set G and for each set E measurable Borel. It would be interesting to see a direct proof of this equality based upon Fubini's theorem.

Added in proof. Professor H. Federer has informed the author that he has found a direct proof of (34).

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ON THE STRUCTURE OF INTRINSIC DERIVATIVES

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Introduction. The primary purpose of the present paper is to express the **M**th order intrinsic derivative of a higher order absolute tensor such as T^{ab}, T_{ab}, or T^{ab}_{cd} as a contraction of extensors. As a first step, we develop a rule for constructing extensors of the types $E^{\alpha a\cdot\beta b}$, $E_{\alpha a\cdot\beta b}$, $E^{\alpha a\cdot\beta b}_{\gamma c\cdot\delta d}$ from absolute tensors T^{ab}, T_{ab}, T^{ab}_{cd} by repeated differentiation with respect to the curve parameter followed by multiplication by the appropriate coefficients. We then consider the contractions of the various E's with the extended components of connection $L^{a}_{\alpha b}$, $L^{\alpha a}_{b}$ (to be introduced) and prove by induction that these contractions give the **M**th order intrinsic derivatives of the original tensors. In this way we establish a highly satisfactory theory of the algebraic structure of the higher order intrinsic derivatives, for the constituent E's and L's obviously possess an invariance of form and being extensors they are such that other extensors, tensors, and invariants can be built from them and other extensors by simple algebraic procedures—addition, multiplication, and contraction.

1. Notation and preliminaries.

In the present paper we shall employ at most two coordinate systems x and $\bar{x}$ and so far as the quantities that bear indices are concerned, we shall distinguish between them whenever feasible by restricting the choice of indicial letters. Specifically, letters at the first of the alphabet a, b, c, d, e shall serve to denote system x, while r, s, t, u, v, w will be correlated to system $\bar{x}$. Thus x^r is the rth coordinate variable of system $\bar{x}$, while x^a is variable number a of system x. Differentiation with respect to the parameter t of a parameterized arc will be indicated by primes and Greek indices, the latter are enclosed except in certain abridged symbols. To illustrate,

$$x'^{a} = x^{a\prime} = dx^{a}/dt, \qquad x^{(\alpha)a} = x^{a(\alpha)} = d^{\alpha}x^{a}/dt^{\alpha},$$
$$X^{\beta r}_{\alpha a} = X^{(\beta)r}_{(\alpha)a} = \partial x^{(\beta)r}/\partial x^{(\alpha)a}, \qquad X^{\alpha a}_{\beta r} = \partial x^{(\alpha)a}/\partial x^{(\beta)r}.$$

Furthermore, we assume that there is given an affine connection L^{a}_{bc} and let L^{a}_{b}, called the two-index connection, represent $L^{a}_{bc}x'^{c}$.

Summation convention. Repeated lower case Latin indices call for summations 1 to N, while the summations indicated by repeated lower case Greek indices are from zero (unless the contrary is specified) to some terminal value usually **M** or **M**+1. Repeated capital Greek in-

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dices do not generate sums, thus $\binom{\mathbf{M}}{\alpha}U_c^{(M-\alpha)}$ with α not summed would be written $\binom{\mathbf{M}}{\mathbf{A}}U_c^{(M-\mathbf{A})}$.

Extensors. Extensors will be denoted by means of symbols bearing Greek-Latin doublet indices such as ρr or αa. The extensor transformation law is exemplified by the equation

$$T^{\rho r}_{\sigma s\cdot t} = T^{\alpha a}_{\beta b\cdot c}X^{\rho r}_{\alpha a}X^{\beta b}_{\sigma s}X^{c}_{t}.$$

This particular extensor, or more properly tensor-extensor since the component symbols bear a tensor index, may be described by saying it is of the type (1, 1, 0, 1, 0)—the numbers referring to the number of doublet superscripts, doublet subscripts, single superscripts, single subscripts, and the weight. Here the range of the Greek indices is 0 to $\mathbf{M}$; however, in the summations this is effectively reduced since X sup ρr inf αa vanishes whenever α exceeds ρ.

Owing to the algebraic character of the extended coordinate transformation so far as the x''s are concerned, extensors have many properties not held by tensors. In particular there are $\mathbf{M}+1$ contractions instead of one, and a greater variety of quotient laws. In the present paper we shall have need for but two of these laws.

Quotient laws. Let $V^{\alpha c}$ and $U_{\alpha c}$ be defined by

$$V^{c\alpha} = V^{c(\alpha)}, \qquad U_{\alpha c} = \binom{\mathbf{M}}{\mathbf{A}}U_c^{(\mathbf{M}-\mathbf{A})}, \qquad \mathbf{A} = \alpha. \tag{1.1}$$

(1) If for arbitrary U, $U_{\alpha c}T^{\alpha c\cdots}_{\cdots}$ is an extensor of the type indicated by its free indices, then the same may be asserted of T.

(2) If $V^{\alpha c}T_{\alpha c\cdots}^{\cdots}$ is an extensor, then T is likewise an extensor assuming of course that V^c is arbitrary.

In a preceding note [3],[1] we have proved that if $T^{c}_{\alpha a}$ and $T^{\alpha a}_{c}$ are extensors of range 0 to $\mathbf{M}$ and of the type indicated by their indices, then their extensive derivatives $DT|^{c}_{\alpha a}$ and $D_1T|^{\alpha a}_{c}$ defined by

$$DT|^{c}_{\alpha a} = T^{c}_{\alpha-1\cdot a} + T^{c\cdot\cdot\prime}_{\alpha a} + T^{b}_{\alpha a}L^{c}_{b}, \tag{1.2}$$

$$\begin{aligned} D_1T|^{\alpha a}_{c} &= \frac{\mathbf{M}+1-\mathbf{A}}{\mathbf{M}+1}T^{\mathbf{A}a}_{c} \\ &\quad + \frac{\mathbf{A}}{\mathbf{M}+1}\left(T^{\mathbf{A}-1\cdot a\prime}_{c} - T^{\mathbf{A}-1\cdot a}_{b}L^{b}_{c}\right), \qquad \mathbf{A} = \alpha, \end{aligned} \tag{1.3}$$

are extensors of the extended range 0 to $\mathbf{M}+1$. The hair line is introduced to emphasize that the indices c and αa designate components of the derived extensor and not of the original. The subscript on D in (1.3) will serve to distinguish the two derivatives (upper and lower) in case $\mathbf{M}=0$ and consequently T is a tensor—a special case. The proper range of α on the left is 0 to $\mathbf{M}+1$, while the proper range of the first index on the right is 0 to $\mathbf{M}$. The special statements needed for the cases $\alpha=0$ and $\alpha=\mathbf{M}+1$ will be contained in (1.2) and (1.3) if we adopt, as we now do, the following convention:

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Any symbol bearing an indicial number outside of the proper range of the index in question is to be given the value zero. Thus, in particular, $T_{-1\cdot\bullet}=T_{\mathbf{M}+1\cdot\bullet}=0$.

The quantities obtained by repeatedly applying upper and lower extensive differentiation to the Kronecker delta $\delta^{\bullet}_{\bullet}$ will be called the extended components of connection. The result of contracting these components with the $V^{\alpha\bullet}$ and $U_{\alpha\bullet}$ defined in equation (1.1) is the $\mathbf{M}$th order intrinsic derivative [3] of V and U.

2. **The derived extensors.** Since the quantities $V^{\alpha\bullet}$ and $U_{\alpha\bullet}$ of (1.1) are extensors and are made up of the derivatives of V and U, it is quite obvious that one can develop extensors from higher order tensors by first contracting with $V^{\bullet}$, $U_{\bullet}$, and so on, differentiating the product, "factoring" out $V^{\alpha\bullet}$, $U_{\bullet\bullet}$, and so on, and then applying the quotient law. As a preliminary to this procedure, we introduce certain generalized binomial coefficients one of which at least—the multinomial coefficient—is well known. These quantities arise from applying the Leibnitz rule for differentiating a product and formula (1.1). To simplify the writing, we restrict ourselves to the specific definition of the typical special cases.

DEFINITION 2.1.

$$\binom{\mathrm{M}}{\mathrm{A},\ \mathrm{B},\ \Gamma}=\frac{\mathrm{M}!}{\mathrm{A}!\mathrm{B}!\Gamma!(\mathrm{M}-\mathrm{A}-\mathrm{B}-\Gamma)!},\qquad \mathrm{M}-\mathrm{A}-\mathrm{B}-\Gamma\geqq 0,$$

$$\binom{\mathrm{M}}{\mathrm{A},\ \mathrm{B},\ \Gamma}=0,\qquad \text{if } \mathrm{M}-\mathrm{A}-\mathrm{B}-\Gamma<0;$$

$$\begin{bmatrix}\mathrm{A},\ \mathrm{B},\ \Gamma\\ \mathrm{M}\end{bmatrix}=\binom{\mathrm{M}}{\mathrm{M}-\mathrm{A},\ \mathrm{M}-\mathrm{B},\ \mathrm{M}-\Gamma}\binom{\mathrm{M}}{\mathrm{A}}^{-1}\binom{\mathrm{M}}{\mathrm{B}}^{-1}\binom{\mathrm{M}}{\Gamma}^{-1},$$
$$\mathrm{A}+\mathrm{B}+\Gamma-2\mathrm{M}\geqq 0,$$

$$\begin{bmatrix}\mathrm{A},\ \mathrm{B},\ \Gamma\\ \mathrm{M}\end{bmatrix}=0,\qquad \text{if } \mathrm{A}+\mathrm{B}+\Gamma-2\mathrm{M}<0;$$

$$\left\{\begin{matrix} \mathrm{A},\ \mathrm{B} \\ \mathrm{M},\ \Gamma,\ \Delta \end{matrix}\right\} = \begin{bmatrix} \mathrm{A},\ \mathrm{B} \\ \mathrm{M} \end{bmatrix} \begin{pmatrix} \mathrm{A} + \mathrm{B} - \mathrm{M} \\ \Gamma,\ \Delta \end{pmatrix}$$

$$= \frac{\mathrm{A}!\mathrm{B}!}{\mathrm{M}!\Gamma!\Delta!(\mathrm{A} + \mathrm{B} - \mathrm{M} - \Gamma - \Delta)!} .$$

THEOREM 2.1. *If* T^{abc} *is a tensor of the type* (3, 0, 0) (*that is, three superscripts, no subscripts, and weight zero*) *and the necessary derivatives exist, then the quantities* E *defined by*

$$E^{\alpha a\cdot\beta b\cdot\gamma c\cdot} = \begin{bmatrix} \mathrm{A},\ \mathrm{B},\ \Gamma \\ \mathrm{M} \end{bmatrix} T^{abc(\mathrm{A}+\mathrm{B}+\Gamma-2\mathrm{M})}, \qquad \mathrm{A} = \alpha,\ \mathrm{B} = \beta,\ \Gamma = \gamma,$$

are the components of an extensor of the type (3, 0, 0, 0, 0).

PROOF. $(T^{abc}A_aB_bC_c)^{(\mathrm{M})}$ is an invariant for arbitrary choice of the covariant vectors A, B, C. Expanding this derivative by the Leibnitz rule for differentiating a multinomial and, in accordance with our convention, defining a symbol which bears a negative prime to be zero, we have, after dropping the indices a, b, c,

$$(TABC)^{(\mathrm{M})} = \sum \begin{pmatrix} \mathrm{M} \\ \alpha,\ \beta,\ \gamma \end{pmatrix} A^{(\alpha)}B^{(\beta)}C^{(\gamma)}T^{(\mathrm{M}-\alpha-\beta-\gamma)}; \qquad \alpha, \beta, \gamma : 0 \text{ to } \mathrm{M}.$$

We now replace the dummy indices α, β, γ with $\mathrm{M}-\bar{\alpha}$, $\mathrm{M}-\bar{\beta}$, $\mathrm{M}-\bar{\gamma}$, drop the bars, and express A sup $(\mathrm{M}-\alpha)$ in terms of $A_{\alpha a}$ by (1.1). The result after introducing definition (2.1) and the definition of E is

$$(T^{abc}A_aB_bC_c)^{(\mathrm{M})} = E^{\alpha a\cdot\beta b\cdot\gamma c}A_{\alpha a}B_{\beta b}C_{\gamma c}$$

and our theorem follows from the quotient law.

Remark. From the proof, we see that in the generalization of this theorem to the case in which T is of contravariant order q, the multiplier 2 in the expression $-2\mathrm{M}$ which occurs in the definition of E would be replaced with $q-1$, thus

$$\begin{bmatrix} \mathrm{A},\ \mathrm{B},\ \Gamma,\ \cdots \\ \mathrm{M} \end{bmatrix} T^{abc\cdots(\mathrm{A}+\mathrm{B}+\Gamma\cdots-(q-1)\mathrm{M})}.$$

This situation arises because we replaced α with $\mathrm{M}-\alpha$, and so on.

THEOREM 2.2. *If* T^{ab}_{cd} *is a tensor of the type* (2, 2, 0), *then the quantities* E *defined by*

$$E^{\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} = \left\{\begin{matrix} \mathrm{A},\ \mathrm{B} \\ \mathrm{M},\ \Gamma,\ \Delta \end{matrix}\right\} T^{ab(\mathrm{A}+\mathrm{B}-\mathrm{M}-\Gamma-\Delta}_{cd} \qquad \mathrm{A} = \alpha,\ \mathrm{B} = \beta,\ \Gamma = \gamma,\ \Delta = \delta,$$

are the components of an extensor of the type (2, 2, 0, 0, 0).

PROOF. Taking q to be two in the generalization of Theorem 2.1, it follows that for arbitrary U and V, the quantities

$$\begin{bmatrix} A, B \\ M \end{bmatrix} (T^{ab}_{cd} U^c V^d)^{(A+B-M)}$$

constitute the components of an extensor of the type (2, 0, 0, 0, 0). But

$$\begin{aligned} \begin{bmatrix} A, B \\ M \end{bmatrix} & (T^{ab}_{cd} U^c V^d)^{(A+B-M)} \\ & = \sum_{\gamma, \delta = 0} \begin{bmatrix} A, B \\ M \end{bmatrix} \binom{A + B - M}{\gamma, \delta} T^{ab(A+B-M-\gamma-\delta)}_{cd} U^{c(\gamma)} V^{d(\delta)} \\ & = E^{\alpha a \cdot \beta b}_{\gamma c \cdot \delta d} U^{c(\gamma)} V^{d(\delta)} \end{aligned}$$

and the theorem follows from the quotient law. With regard to the range of indices it should be borne in mind that the proper range for α, β, γ, δ is zero to M. Confining α and β to this range, we note in addition that $[{}^{A,B}_{M}]$ is zero if $A+B<M$, thus the effective range of $A+B-M$ is 0 to M inclusive. Consequently, the maximum range of γ and δ is 0 to M and this may be taken as the actual range for whenever $\gamma+\delta>A+B-M$, the corresponding terms of the sum vanish.

Remark. If T were contravariant of unspecified order q, E would be defined by

$$E^{\alpha a \cdot \beta b \cdots}_{\gamma c \cdot \delta d} = \begin{Bmatrix} A, B, \cdots \\ M, \Gamma, \Delta \end{Bmatrix} T^{ab \cdots [A+B \cdots -(q-1)M-\Gamma-\Delta]}_{cd}.$$

THEOREM 2.3. *If* T_{cd} *is a tensor of the type* (0, 2, 0), *then the quantities* E *defined by*

$$E_{\gamma c \cdot \delta d} = \binom{M}{\Gamma, \Delta} T_{cd}^{\cdot \cdot (M-\Gamma-\Delta)}$$

are the components of an extensor of the type (0, 2, 0, 0, 0).

PROOF. In the proof of the preceding theorem delete the indices a, b and the symbol $[{}^{A,B}_{M}]$ and replace $(A+B-M)$ with M. The theorem then follows from the quotient law.

Remark. The generalization of this theorem to higher order tensors is obvious.

3. **Recursion formulas.** To facilitate later developments, we list

here some recursion formulas for the E's and L's. Specifically, these relationships express the E's and L's and their derivatives for a given range M in terms of the corresponding quantities for range $\mathrm{M}+1$. The higher range $\mathrm{M}+1$ will be indicated by means of a star. To illustrate the notation and method, we develop the first formula in detail. In the right members of these formulas the capitals A, B, Γ, and Δ are equal to α, β, γ, δ, respectively, but do not generate sums. From the definition of E sup $\alpha a\cdot\beta b$, we have

$$
\begin{aligned}
E^{\alpha a\cdot\beta b} &= \begin{bmatrix}\mathrm{A},\ \mathrm{B}\\ \mathrm{M}\end{bmatrix} T^{ab(\mathrm{A}+\mathrm{B}-\mathrm{M})}\\
&= \begin{bmatrix}\mathrm{A},\ \mathrm{B}\\ \mathrm{M}\end{bmatrix}\begin{bmatrix}\mathrm{A}+1,\ \mathrm{B}\\ \mathrm{M}+1\end{bmatrix}^{-1}\begin{bmatrix}\mathrm{A}+1,\ \mathrm{B}\\ \mathrm{M}+1\end{bmatrix} T^{ab(\mathrm{A}+1+\mathrm{B}-\mathrm{M}-1)}\\
&= \begin{bmatrix}\mathrm{A},\ \mathrm{B}\\ \mathrm{M}\end{bmatrix}\begin{bmatrix}\mathrm{A}+1,\ \mathrm{B}\\ \mathrm{M}+1\end{bmatrix}^{-1} E^{*\mathrm{A}+1\cdot a\cdot\beta b}\\
&= \frac{\mathrm{M}+1}{\mathrm{A}+1}\, E^{*\mathrm{A}+1\cdot a\cdot\beta b}.
\end{aligned}
$$

Thus we have established

$$
E^{\alpha a\cdot\beta b} = \frac{\mathrm{M}+1}{\mathrm{A}+1}\, E^{*\mathrm{A}+1\cdot a\beta b} = \frac{\mathrm{M}+1}{\mathrm{B}+1}\, E^{*\alpha a\cdot\mathrm{B}+1\cdot b}. \tag{3.1}
$$

Similarly, it may be shown that

$$
E^{\alpha a\cdot\beta b'} = \frac{(\mathrm{M}+1)(\mathrm{A}+\mathrm{B}-\mathrm{M}+1)}{(\mathrm{A}+1)(\mathrm{B}+1)}\, E^{*\mathrm{A}+1\cdot a\cdot\mathrm{B}+1\cdot b}, \tag{3.2}
$$

$$
E'_{\gamma c\cdot\delta d} = \frac{\mathrm{M}+1-\Gamma-\Delta}{\mathrm{M}+1}\, \overset{*}{E}_{\Gamma c\cdot\Delta d}, \tag{3.3}
$$

$$
E_{\gamma-1\cdot c\cdot\delta d} = \frac{\Gamma}{\mathrm{M}+1}\, \overset{*}{E}_{\Gamma c\cdot\delta d}, \qquad E_{\gamma c\cdot\delta-1\cdot d} = \frac{\Delta}{\mathrm{M}+1}\, \overset{*}{E}_{\gamma c\cdot\Delta d}, \tag{3.4}
$$

$$
E^{\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} = \frac{\mathrm{M}+1}{\mathrm{A}+1}\, E^{*\mathrm{A}+1\cdot a\cdot\beta b}_{\gamma c\cdot\delta d} = \frac{\mathrm{M}+1}{\mathrm{B}+1}\, E^{*\alpha a\cdot\mathrm{B}+1\cdot b}_{\gamma c\cdot\delta d}, \tag{3.5}
$$

$$
E^{\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} = \frac{(\mathrm{M}+1)(\Gamma+1)}{(\mathrm{A}+1)(\mathrm{B}+1)}\, E^{*\mathrm{A}+1\cdot a\mathrm{B}+1\cdot b}_{\Gamma+1\cdot c\cdot\delta d}, \tag{3.6}
$$

$$
E^{\alpha a\cdot\beta b'}_{\gamma c\cdot\delta d} = \frac{(\mathrm{M}+1)(\mathrm{A}+\mathrm{B}-\mathrm{M}-\Gamma-\Delta+1)}{(\mathrm{A}+1)(\mathrm{B}+1)}\, E^{*\mathrm{A}+1\cdot a\cdot\mathrm{B}+1\cdot b}_{\Gamma c\cdot\Delta d}. \tag{3.7}
$$

The extended components of connection. As we have remarked before, the extended components of connection can be obtained from the Kronecker delta by applying repeatedly upper and lower extensive differentiation. In the case of upper extensive differentiation (that in which the superscript retains its tensor character) the **M**th extensive derivative may be obtained by applying the formula: $D^{\mathrm{M}}\delta|^{c}_{\alpha a} = \binom{\mathrm{M}}{\mathrm{A}}\Gamma^{c}_{\mathrm{A}a}$ and Kawaguchi's recursion formula [**4**, p. 105, (13.6) and **3**],

$$\Gamma^{c}_{\alpha-1\cdot a} = \Gamma^{c\prime}_{\alpha a} + \Gamma^{b}_{\alpha a}L^{c}_{b}.$$

Thus, if we let $L^{c}_{\alpha a}$ and $L^{*c}_{\alpha a}$ denote the extended components of connection for the cases **M** and **M**+1, then since $\Gamma^{c}_{\alpha a} = D^{\mathrm{M}}\delta|^{c}_{\alpha a}$ (by definition), we have the following table of values

$$L^{c}_{\mathrm{M}a} = \delta^{c}_{a}, \qquad L^{c}_{\mathrm{M}-1\cdot a} = \binom{\mathrm{M}}{\mathrm{M}-1}L^{c}_{a},$$

$$L^{c}_{\mathrm{M}-2\cdot a} = \binom{\mathrm{M}}{\mathrm{M}-2}(L^{c\prime}_{a} + L^{b}_{a}L^{c}_{b}),$$

and so on, and the obvious relationship,

$$L^{c}_{\mathrm{A}a}\binom{\mathrm{M}}{\mathrm{M}-\mathrm{A}}^{-1} = L^{*c}_{\mathrm{A}+1\cdot a}\binom{M+1}{\mathrm{M}-\mathrm{A}}^{-1}.$$

Thus we have

$$L^{c}_{\alpha a} = \frac{\mathrm{A}+1}{\mathrm{M}+1}\,L^{*c}_{\mathrm{A}+1\cdot a}. \tag{3.8}$$

Also, in geodesic coordinates Kawaguchi's formula may be written in the form $L^{c\prime}_{\alpha a} = \binom{\mathrm{M}}{\mathrm{A}}\binom{\mathrm{M}}{\mathrm{A}-1}^{-1}L^{c}_{\mathrm{A}-1}$ and this in conjunction with (3.8) yields

$$L^{c\prime}_{\alpha a} = \frac{\mathrm{M}+1-\mathrm{A}}{\mathrm{M}+1}\,L^{*c}_{\mathrm{A}a}. \tag{3.9}$$

Formula (3.9) is valid for substitution provided there is no subsequent differentiation.

Turning to the lower extensive derivative of the Kronecker delta (in this case the subscript retains its tensor character), the situation is somewhat simpler. The rule for the computation of this **M**th derivative is as follows: Component upper $0c$ lower d of the **M**th derivative of δ^{c}_{d} is given by $D_{\mathrm{M}}\delta|^{0c}_{d} = \delta^{c}_{d}$, the other components may be obtained from the recursion formula $D_{\mathrm{M}}\delta|^{\gamma c}_{d} = D_{\mathrm{M}}\delta|^{\gamma-1\cdot c\prime}_{d} - D_{\mathrm{M}}\delta|^{\gamma-1\cdot c}_{b}L^{b}_{d}$. Thus since we define $L^{\gamma c}_{d}$ to be $D_{\mathrm{M}}\delta|^{\gamma c}_{d}$, we have the following table for the extended components of connection, $L^{\gamma c}_{d}$:

$$L_d^{0e} = \delta_d^e, \qquad L_d^{1e} = -L_d^e, \qquad L_d^{2e} = -L_d^{e\prime} + L_b^e L_d^b,$$
$$L_d^{3e} = (-L_d^{e\prime} + L_b^e L_d^b)' - (-L_b^{e\prime} + L_c^e L_b^c) L_d^b,$$

and so on. Consequently, we have the equations,

$$L_d^{\gamma e} = L_d^{*\gamma e}, \qquad \text{for } \gamma < \mathrm{M} + 1; \tag{3.10}$$

$$L_d^{\gamma+1\cdot e} = L_d^{\gamma e\prime} = L_d^{*\gamma+1\cdot e}, \qquad \text{in geodesic coordinates.} \tag{3.11}$$

Equation (3.11) like equation (3.9) is valid for substitution only in case there is no subsequent differentiation.

4. The higher intrinsic derivatives as contractions. In case M is unity, we know from previous work [2, pp. 291–301] that a full contraction of any one of the E's with the L's yields the first order intrinsic derivative of the corresponding T. Accordingly, we turn our attention to the induction from M to $M+1$. Thus in the proofs that follow, we assume that a complete contraction of the unstarred quantities yields the Mth order intrinsic derivative and then attempt to establish the corresponding result for the starred quantities. Our first step is to compute the intrinsic derivative of the contraction of the unstarred quantities at the origin of a geodesic coordinate system and then, by means of formulas (3.1) through (3.11), express the result in terms of the starred quantities. With this in mind we are now ready to consider our final propositions.

THEOREM 4.1. *If $E_{\gamma e\cdot\delta d}$ denotes $\binom{\mathrm{M}}{\Gamma,\Delta}T_{ab}^{\cdot\cdot(\mathrm{M}-\Gamma-\Delta)}$, then the contraction $E_{\gamma e\cdot\delta d}L_a^{\gamma e}L_b^{\delta d}$ is the* M*th intrinsic derivative of the tensor T_{ab}.*

PROOF. If this contraction yields the Mth intrinsic derivative, then in geodesic coordinates the (M+1)th derivative is given by

$$E'_{\gamma e\cdot\delta d}L_a^{\gamma e}L_b^{\delta d} + E_{\gamma e\cdot\delta d}L_a^{\gamma e\prime}L_b^{\delta d} + E_{\gamma e\cdot\delta d}L_a^{\gamma e}L_b^{\delta d\prime}.$$

By applying formulas (3.3) and (3.10), the first term transforms into

$$\sum_{\gamma,\delta=0}^{\mathrm{M}+1} \frac{\mathrm{M}+1-\gamma-\delta}{\mathrm{M}+1} E^*_{\gamma e\cdot\delta d}L_a^{*\gamma e}L_b^{*\delta d}. \tag{4.1}$$

The increase in the range of summation from M to M+1 is legitimate because the additional terms vanish since either the fractional coefficient or E^* vanishes if $\gamma+\delta \geqq \mathrm{M}+1$.

Similarly, the second term may be made to assume the form

$$\sum_{\gamma=0}^{\mathrm{M}+1} \frac{\gamma}{\mathrm{M}+1} E^*_{\gamma e\cdot\delta d}L_a^{*\gamma e}L_b^{*\delta d} \tag{4.2}$$

by replacing γ with $\bar{\gamma}-1$, dropping the bar, and applying equations (3.4), (3.11), and (3.10). The lower range of γ should be one but the presence of the factor γ allows us to drop it to zero. The range of δ is 0 to $\mathbf{M}$ and of course this repeated index generates a sum. If we allow δ to take on the value $\mathbf{M}+1$, then the E^* would vanish unless γ has the value zero, but this value eliminates the term. Consequently, in (4.2) we may regard both of the indices γ and δ as generating sums over the range 0 to $\mathbf{M}+1$. Evidently, the third term may be treated similarly.

Obviously, the sum of the coefficients involved in the expressions (4.1), (4.2), and so on, is unity, and the theorem is established. Furthermore, it is quite apparent that this particular case is typical. The addition of more subscripts would not call for a change in procedure and the sum of the fractional coefficients would again be one.

Turning to the problem of mixed tensors, the question to be investigated may be formulated as follows:

THEOREM 4.2. *If* $E^{*\alpha a\cdot\beta b}_{\gamma c\cdot\delta d}$ *denotes* $\left\{ {\mathrm{A},\mathrm{B} \atop \mathbf{M},\Gamma,\Delta} \right\} T^{*ab(\mathrm{A}+\mathrm{B}-\mathbf{M}-\Gamma-\Delta)}_{cd}$ *then the contraction*

$$E^{\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} L^{\cdot e}_{\alpha a} L^{f}_{\beta b} L^{\gamma c}_{g} L^{\delta d}_{h}$$

is the Mth *intrinsic derivative of the mixed tensor* T^{ef}_{gh}.

The method of verifying this statement is essentially that of Theorem 4.1. Accordingly, we may omit some of the explanation.

PROOF. The derivative of the contraction in question consists of five terms the first of which has a prime on the first E, the second a prime on the first L, and so on. By means of the relationships (3.7), (3.8), and (3.10), the first term may be written,

$$\frac{(\mathbf{M}+1)(\alpha+\beta-\mathbf{M}-\gamma-\delta+1)}{(\alpha+1)(\beta+1)} E^{*\alpha+1\cdot a\cdot\beta+1\cdot b}_{\gamma c\cdot\delta d} \frac{(\alpha+1)}{\mathbf{M}+1} \cdot L^{*\cdot e}_{\alpha+1\cdot a} \frac{(\beta+1)}{\mathbf{M}+1} L^{*f}_{\beta+1\cdot b} L^{*\gamma c}_{g} L^{*\delta d}_{h}.$$

Here there is summation on α, β, γ, δ from 0 to $\mathbf{M}$. By replacing α and β with $\bar{\alpha}-1$ and $\bar{\beta}-1$, and subsequently dropping the bars, we get

$$\frac{(\alpha+\beta-\mathbf{M}-\gamma-\delta-1)}{\mathbf{M}+1} E^{*\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} L^{*\cdot e}_{\alpha a} L^{*f}_{\beta b} L^{*\gamma c}_{g} L^{*\delta d}_{h}. \tag{4.3}$$

It will be recalled that E^* vanishes whenever $\alpha+\beta<\mathbf{M}+1+\gamma+\delta$. Consequently, we may drop the range on α and β to zero and increase the range of γ and δ to $\mathbf{M}+1$. Thus the range of summation of α, β, γ, and δ in the expression (4.2) may be taken to be 0 to $\mathbf{M}+1$.

The second term of the derivative, that which contains the derivative of $L^{e}_{\alpha a}$, after application of the formulas (3.5), (3.9), (3.8), and (3.10) assumes the form

$$\frac{\mathbf{M}+1}{\beta+1} E^{*\alpha a\cdot\beta+1\cdot b}_{\gamma c\cdot\delta d} \frac{\mathbf{M}+1-\alpha}{\mathbf{M}+1} L^{*e}_{\alpha a} \frac{\beta+1}{\mathbf{M}+1} L^{*f}_{\beta+1\cdot b} L^{*\gamma c}_{g} L^{*\delta d}_{h}$$

or, replacing β with $\bar{\beta}-1$ and dropping the bar,

$$\frac{\mathbf{M}+1-\alpha}{\mathbf{M}+1} E^{*\alpha a\cdot\beta b}_{\gamma c\cdot\delta d} L^{*e}_{\alpha a} L^{*f}_{\beta b} L^{*\gamma c}_{g} L^{*\delta d}_{h}. \tag{4.4}$$

The summation on α, γ, and δ is 0 to $\mathbf{M}$ and on β is 1 to $\mathbf{M}+1$. We observe that if α is given the value $\mathbf{M}+1$, the coefficient vanishes and recall that if $\alpha+\beta$ is less than $\mathbf{M}+1+\gamma+\delta$ the E^* is zero. Consequently, if we assign β the value zero, α the value $\mathbf{M}$, then $\alpha+\beta<\mathbf{M}+1+\gamma+\delta$. Therefore, α and β may each be given the range 0 to $\mathbf{M}+1$. Furthermore, if γ is given the value $\mathbf{M}+1$, α the value $\mathbf{M}$, then again $\alpha+\beta<\mathbf{M}+1+\gamma+\delta$. Hence, we conclude that the range of α, β, γ, δ in (4.4) may be taken to be 0 to $\mathbf{M}+1$.

If we denote the product of the E with the L's that occurs in (4.4) by P, then the third term of the derivative in question (it contains $L^{f}_{\beta b}$) may be transformed into the expression

$$\sum \frac{\mathbf{M}+1-\beta}{\mathbf{M}+1} P. \tag{4.5}$$

To treat the fourth and fifth terms, we apply equations (3.6), (3.8), (3.10), (3.11). The result is

$$\frac{(\mathbf{M}+1)(\gamma+1)}{(\alpha+1)(\beta+1)} E^{*\alpha+1\cdot a\cdot\beta+1\cdot b}_{\gamma+1\cdot c\cdot\delta d} \frac{\alpha+1}{\mathbf{M}+1} \frac{\beta+1}{\mathbf{M}+1} L^{*e}_{\alpha+1\cdot a} L^{*f}_{\beta+1\cdot b} L^{*\gamma+1\cdot c}_{g} L^{*\delta d}_{h},$$

which upon following previous procedures may be written

$$\sum \frac{\gamma}{\mathbf{M}+1} P. \tag{4.6}$$

Similarly, from the last term of the derivative, we get

$$\sum \frac{\delta}{\mathbf{M}+1} P. \tag{4.7}$$

The sum of the expressions (4.3) through (4.7) is

$$\sum \frac{\alpha+\beta-\mathrm{M}-\gamma-\delta-1+\mathrm{M}+1-\alpha+\mathrm{M}+1-\beta+\gamma+\delta}{\mathrm{M}+1} P \quad \text{or} \quad \sum P$$

and the theorem is established.

Remark. If we delete the subscript letters throughout in formulas (3.5) and (3.7), they become (3.1) and (3.2), respectively. Consequently, equations (4.3) and (4.4) with the necessary deletions constitute the proof of the purely contravariant contraction theorem which stated briefly is:

THEOREM 4.3. *The* M*th intrinsic derivative of the tensor* T^{ab} *is given by* $E^{\alpha a\cdot\beta b}L^{a}_{\alpha a}L^{b}_{\beta b}$.

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TEXAS UNIVERSITY

A NOTE ON THE INTERRELATION OF SUBSETS OF INDEPENDENT VARIABLES OF A CONTINUOUS FUNCTION WITH CONTINUOUS FIRST DERIVATIVES[1]

WASSILY LEONTIEF

Let X be a set of n variables $x_1, x_2, \cdots, x_n$ and y a continuous function of these variables, $y=f(x_1, x_2, \cdots, x_n)$, also written as $f(X)$ with continuous partial derivatives of the first, second, and third orders. Throughout the following discussion all the first partial derivatives of $f(x_1, x_2, \cdots, x_n)$ are assumed to be different from zero.

Let S be a proper subset of X and S' the subset which is complementary to S in X.

DEFINITION I. A subset S of independent variables is locally functionally separable at a point $(a_1, a_2, \cdots, a_n)$ within the set X in $y=f(X)$ if there exist some function ϕ with continuous first derivatives and defined in some neighborhood of $(a_1, a_2, \cdots, a_\nu)$ and another function ψ also with continuous first derivatives and defined in some neighborhood of $(b, a_{\nu+1}, a_{\nu+2}, \cdots, a_n)$ where $b=\phi(a_1, a_2, \cdots, a_\nu)$ such that

$$y = f(X) = \psi(\phi(S), S'). \tag{1}$$

The function $\phi(S)$ is locally separable in $f(X)$.

From the definition of a locally separable subset it follows that each of the original variables, that is, each of the elements of X, is a locally separable subset of X in $f(X)$; so is also the set X itself.

Let ${}_{ij}R$ be defined by

$${}_{ij}R = f'_i / f'_j$$

where f'_i and f'_j are the partial derivatives of $f(x_1, x_2, \cdots, x_n)$ with respect to x_i and x_j. The necessary and sufficient condition for ${}_{ij}R$ to be independent of x_k is

$${}_{ij}R'_k = \frac{f''_{ik} f'_j - f''_{jk} f'_i}{[f'_j]^2} = 0$$

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[1] The subject of this note suggested itself by a study of "production functions" and "consumption functions" in mathematical economics.

The problem considered in this paper bears some relation to the XIIIth problem of Hilbert: D. Hilbert, Nach. Ges. Wiss. Göttingen (1900) p. 280; cf. also Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. I, problems 119 and 119a, pp. 61–62 and 220–222.

where f''_{ik} and f''_{jk} are the second partial derivatives of $f(x_1, x_2, \cdots, x_n)$. Thus if ${}_{ij}R'_k = 0$ then ${}_{ji}R'_k = 0$.

Since

$$(2) \qquad {}_{jl}R'_k f'_l f'_i = {}_{il}R'_k f'_l f'_j - {}_{ij}R'_k [f'_j]^2,$$

if ${}_{il}R$ and ${}_{ij}R$ are independent of x_k, ${}_{jl}R$ is also independent of x_k.

Similarly the relationship

$$(3) \qquad {}_{ik}R'_j - {}_{jk}R'_i = {}_{ij}R'_k [f'_j/f'_k]^2$$

implies that if ${}_{ik}R$ is independent of x_j and ${}_{jk}R$ is independent of x_i then ${}_{ij}R$ is independent of x_k.

THEOREM I. *A subset S is locally functionally separable from X in $f(X)$ if and only if the following set of equations is satisfied: ${}_{s_is_j}R'_{s'} = 0$. The first two subscripts to R refer to any two identical or different elements in the subset S, while the last subscript indicates any element of the subset S' which is complementary to S in X.*

Let S be a locally separable subset, that is, relationship (1) is known to be true. If x_i and x_j are elements in S and x_k is an element in S', then ${}_{ij}R = f'_i/f'_j = \phi'_i(S)/\phi'_j(S)$ and consequently ${}_{ij}R'_k = 0$.

In order to show that the local separability condition stated above is not only necessary but also sufficient let the function $\phi(S)$ be defined by

$$\phi(S) = f(S, S'^0),$$

where $f(S, S'^0)$ is obtained from $f(X)$ by assigning to each of the $n-v$ elements $x_{v+1}, x_{v+2}, \cdots, x_n$ of the subset S' a constant value $x^0_{v+1}, x^0_{v+2}, \cdots, x^0_n$.

The subset S comprising the elements $x_1, x_2, \cdots, x_v$ is locally functionally separable in X if the function $f(X)$ can be shown to be expressed in terms of $f(S, S'^0)$ and of the set of the $n-v$ variables $x_{v+1}, x_{v+2}, \cdots, x_n$. Let these be considered as parameters in $f(X)$. The derivatives of the functions $f(X)$ and $f(S, S'^0)$, with respect to the v variables $x_1, x_2, \cdots, x_v$, form a matrix

$$\begin{Vmatrix} f'_1(S, S') & f'_2(S, S') & \cdots f'_v(S, S') \\ f'_1(S, S'^0) & f'_2(S, S'^0) & \cdots f'_v(S, S'^0) \end{Vmatrix}.$$

All the terms of the first row of this matrix are proportional to the corresponding terms of the second row if the relationship

$$f'_i(S, S'^0)/f'_j(S, S'^0) = f'_i(S, S')/f'_j(S, S')$$

holds for all $i, j \leq v$. It obviously holds if $S' = S'^0$, that is, in the special case when the parameters $x_{v+1}, x_{v+2}, \cdots, x_n$, which are elements of S', are assigned the particular values $x^{\circ}_{v+1}, x^{\circ}_{v+2}, \cdots, x^{\circ}_{v}$. But if ${}_{ij}R'_k = 0$ for all $i, j \leq v$ and $k > v$, that is, if the last term of the equation above is independent of the variables comprised in the subset S', this equation holds also if $x_{v+1}, x_{v+2}, \cdots, x_n$ acquire values other than $x^{\circ}_{v+1}, x^{\circ}_{v+2}, \cdots, x^{\circ}_{n}$. In this case the two rows of the matrix are linearly dependent: the first row equals the second multiplied by $f'_1(S, S')/f'_1(S, S'^{\circ})$. Since, furthermore, the element $f'_v(S, S'^{\circ})$ located in the lower right-hand corner of the matrix is by assumption different from zero, $f(X)$ can, according to the general theorem on functional dependence, be expressed in the form $f(X) = \psi(f(S, S'^{\circ}), S')$, where ψ is a function having continuous first, second, and third partial derivatives, or if we substitute $\phi(S)$ for $f(S, S'^{\circ})$

$$f(X) = \psi(\phi(S), S')$$

which shows that the condition of local separability of subset S in X as stated in Theorem I is a sufficient one.

The first half of the foregoing argument holds also if the function $\phi(S)$ by which S is locally separable from X in $f(X)$ is more generally defined as $\phi(S) = \Phi(f(S, S'^{\circ}))$ where Φ is some continuous function with locally nonvanishing first derivatives.

The conditions of simultaneous local separability of mutually exclusive locally separable subsets of any given set of independent variables can be stated as follows:

LEMMA. *Given A, B, C, three mutually exclusive subsets of set X and G the rest of X, if each of the sets A, B, C is locally functionally separable from X in $f(X)$, these subsets can also be separated simultaneously from X in $f(X)$.*

Proceeding along the line of argument used in proving Theorem I, let the individually locally separable functions $\alpha(A)$, $\beta(B)$, $\gamma(C)$ be given the particular forms $f(A, B^{\circ}, C^{\circ}, G^{\circ})$, $f(A^{\circ}, B, C^{\circ}, G^{\circ})$ and $f(A^{\circ}, B^{\circ}, C, G^{\circ})$. Differentiating $f(X)$ and the three separable functions with respect to all the variables comprised in subsets A, B, and C, form the four-rowed matrix:

$$\left\| \begin{array}{cccccccc} f'_{a_1}(A,B,C,G) & \cdots & f'_{b_1}(A,B,C,G) & \cdots & f'_{c_1}(A,B,C,G) & \cdots & f'_{c_r}(A,B,C,G) \\ f'_{a_1}(A,B^{\circ},C^{\circ},G^{\circ}) & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & f'_{b_1}(A^{\circ},B,C^{\circ},G^{\circ}) & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & f'_{c_1}(A^{\circ},B^{\circ},C,G^{\circ}) & \cdots & f'_{c_r}(A^{\circ},B^{\circ},C,G^{\circ}) \end{array} \right\|.$$

The first m columns of this matrix contain the derivatives with respect to the m variables $a_1, a_2, \cdots, a_m$ contained in the subset A, the next r columns consist of derivatives with respect to the r variables $b_1, b_2, \cdots, b_r$ included in subset B and the last v columns are composed of the derivatives with respect to the v elements of subset C. All elements of subset G appearing under the function signs in row one are regarded as parameters. An argument similar to that presented in the proof of Theorem I shows that the four rows of the matrix are linearly dependent. Since all derivatives are assumed to be different from zero, the determinant formed by the second, third, and fourth elements of the first, second, and third columns (as written out above) respectively is also different from zero: the rank of the matrix is three. From these two properties of the matrix it follows that $f(X)$ is a function of $f(A, B^0, C^0, G^0)$, $f(A^0, B, C^0, G^0)$, $f(A^0, B^0, C, G^0)$ and of the variables comprized in the subset G. In general:

$$f(X) = \Phi(\alpha(A), \beta(B), \gamma(C), G).$$

THEOREM II. (a) *If two functionally locally separable subsets S and T of X in $f(X)$, of which neither is a subset of the other, intersect each other, then each of the three nonoverlapping sets—one of which comprises all elements of S which are not elements of T, the second consists of all the elements of T which are not elements of S, and the third contains all elements common to S and T—is also a functionally locally separable subset of X in $f(X)$.* (b) *Furthermore if $\phi(S \cup T)$ is a function of $S \cup T$ locally separable in $f(X)$ the three nonoverlapping sets mentioned above are additively locally separable subsets of $S \cup T$ in $\phi(S \cup T)$; that is, if A, B, and C are these three subsets, there exist functions α, β, and γ with continuous first derivatives such that*

$$\phi(S \cup T) = \alpha(A) + \beta(B) + \gamma(C).$$

Let X be partitioned in four nonoverlapping subsets A, B, C, and G,

$$X = A \cup B \cup C \cup G,$$

and the two overlapping subsets defined as

$$S = A \cup B, \qquad T = C \cup B.$$

S and T are locally separable from X in $f(X)$, that is,

$$f(X) = v(\phi(A \cup B), C \cup G), \tag{4}$$

$$f(X) = w(\theta(C \cup B), A \cup G). \tag{5}$$

First let it be shown that these two equations imply

$$f(X) = u(\alpha(A), \beta(B), \gamma(C), G). \tag{6}$$

From (4) and (5) it follows, according to Theorem I, that

$$\begin{array}{lll} {}_{ab}R'_c = 0, & {}_{aa}R'_c = 0, & {}_{bb}R'_c = 0, \\ {}_{ab}R'_g = 0, & {}_{aa}R'_g = 0, & {}_{bb}R'_g = 0, \\ {}_{cb}R'_a = 0, & {}_{cc}R'_a = 0, & {}_{bb}R'_a = 0, \\ {}_{cb}R'_g = 0, & {}_{cc}R'_g = 0, & {}_{bb}R'_g = 0, \end{array} \tag{7}$$

where a, b, c, and g represent any elements in A, B, C, and G respectively, while the combinations of two identical subscripts such as aa or bb refer to any two elements, ***identical or different***, both belonging to the same set, in this instance A or B respectively.

From ${}_{ab}R'_c = 0$ and ${}_{cb}R'_a = 0$ we have, by (3), ${}_{ac}R'_b = 0$. Let the subscript c_1 refer to one particular element in C and b_k to one particular element in B, then on the basis of the last relationship ${}_{a_1c_1}R'_{b_k} = {}_{a_2c_1}R'_{b_k} = \cdots = {}_{a_mc_1}R'_{b_k} = 0$, where $a_1, a_2, \cdots, a_m$ refer to all the m elements of A. Applying to these m equations the rule derived from (2) we have ${}_{a_ia_j}R'_{b_k} = 0$ where a_i and a_j refer to any elements in A, and b_k to any element in B; in short, ${}_{aa}R'_b = 0$. By an analogous argument we find that ${}_{cc}R'_b = 0$. These two equations in combination with the eight equations in the last two columns of (7) constitute according to Theorem I a sufficient condition for the local separability of each of the three subsets A, B, and C of X in $f(X)$. According to the lemma proved above the individual local separability of subsets A, B and C in set X implies their simultaneous local separability. Thus it is shown that (6) follows from (4) and (5).

Now we can proceed to demonstrate that A, B and C represent additively locally separable subsets of the set $A \cup B \cup C$.

Let ${}_{\alpha\beta}\mathfrak{R}$ be defined, in analogy to ${}_{ij}R$, by

$${}_{\alpha\beta}\mathfrak{R} = u'_\alpha / u'_\beta ;$$

then expressing the first two equations in the first column of (7) in terms of (6) we have

$${}_{ab}R'_c = \frac{\alpha'_a}{\beta'_b}\,{}_{\alpha\beta}\mathfrak{R}'_\gamma \gamma'_c = 0,$$

$${}_{ab}R'_g = \frac{\alpha'_a}{\beta'_b}\,{}_{\alpha\beta}\mathfrak{R}'_g = 0.$$

Since the assumed properties of function $f(X)$ exclude $(\alpha'_a/\beta'_b)\gamma'_c = 0$ and $\alpha'_a/\beta'_b = 0$, it follows that

(8) $${}_{\alpha\beta}\mathfrak{R}'_{\gamma} = 0 \quad \text{and} \quad {}_{\alpha\beta}\mathfrak{R}'_{g} = 0.$$

Being thus independent of γ, and of any element g of G, ${}_{\alpha\beta}\mathfrak{R}$ can be considered to be a function of α and β alone:

(8a) $${}_{\alpha\beta}\mathfrak{R} = {}_{\alpha\beta}\mathfrak{R}(\alpha, \beta).$$

Analogous arguments show that

(9) $${}_{\gamma\alpha}\mathfrak{R}'_{\beta} = 0 \quad \text{and} \quad {}_{\gamma\alpha}\mathfrak{R}'_{g} = 0,$$

(10) $${}_{\beta\gamma}\mathfrak{R}'_{\alpha} = 0 \quad \text{and} \quad {}_{\beta\gamma}\mathfrak{R}'_{g} = 0,$$

and consequently

(9a) $${}_{\gamma\alpha}\mathfrak{R} = {}_{\gamma\alpha}\mathfrak{R}(\gamma, \alpha),$$

(10a) $${}_{\beta\gamma}\mathfrak{R} = {}_{\beta\gamma}\mathfrak{R}(\beta, \gamma).$$

Multiplying all the left-hand and all the right-hand terms of the three equations (8a), (9a), and (10a) with each other and expressing the resulting equation in logarithms we have:

$$\begin{aligned} &\log {}_{\alpha\beta}\mathfrak{R}(\alpha, \beta) + \log {}_{\gamma\alpha}\mathfrak{R}(\gamma, \alpha) + \log {}_{\beta\gamma}\mathfrak{R}(\beta, \gamma) \\ &\qquad = \log u'_{\alpha}/u'_{\beta} + \log u'_{\gamma}/u'_{\alpha} + \log u'_{\beta}/u'_{\gamma} = \log 1 = 0. \end{aligned}$$

If ${}_{\alpha\beta}\mathfrak{R}(\alpha, \beta)$, ${}_{\gamma\alpha}\mathfrak{R}(\gamma, \alpha)$ or ${}_{\beta\gamma}\mathfrak{R}(\beta, \gamma)$ is negative, they can be made positive by temporarily replacing one of the variables α, β, or γ by its negative. An opposite substitution must then be made at the later stage of the argument after natural numbers have again been substituted for logarithms. Successive partial differentiation of both sides of the last equation with respect to α and β gives

$$\frac{\partial^2 \log {}_{\alpha\beta}\mathfrak{R}(\alpha, \beta)}{\partial\alpha\partial\beta} = 0,$$

which means that $\log {}_{\alpha\beta}\mathfrak{R}(\alpha, \beta)$ can be written as a sum of two terms, one of which is a function of α alone and the other a function of β alone:

$$\log {}_{\alpha\beta}\mathfrak{R}(\alpha, \beta) = \log \lambda(\alpha) + \log \mu(\beta)$$

or

$$u'_{\alpha}/u'_{\beta} = \lambda(\alpha)\mu(\beta).$$

The characteristic equations of this partial differential equation are

$$\frac{d\alpha}{1/\lambda(\alpha)} = \frac{d\beta}{-\mu(\beta)} = \frac{du}{0} = \frac{d\gamma}{0} = \frac{dx_k}{0} = \cdots = \frac{dx_n}{0},$$

where $x_k, \cdots, x_n$ are variables in G and its general integral is

$$r(\alpha) + t(\beta) + m(\gamma, G, y) = 0, \tag{11}$$

where

$$y = f(X) = u(\alpha, \beta, \gamma, G).$$

Expressing the two equations in (9) in terms of (11) and in accordance with the definition of ${}_{\alpha\gamma}\Re$ and using implicit differentiation to find the partial derivatives of u, we have,

$$
\begin{aligned}
{}_{\alpha\gamma}\Re'_\beta &= \frac{\partial}{\partial\beta}\left(\frac{r'(\alpha)}{m'_\gamma(\gamma, G, y)}\right) = -\frac{r'(\alpha)m''_{\gamma y}(\gamma, G, y)u'_\beta}{[m'_\gamma(\gamma, G, y)]^2} = 0, \\
{}_{\alpha\gamma}\Re'_g &= \frac{\partial}{\partial g}\left(\frac{r'(\alpha)}{m'_\gamma(\gamma, G, y)}\right) \\
&= -\frac{r'(\alpha)\left[m''_{\gamma y}(\gamma, G, y)u'_g + m''_{\gamma g}(\gamma, G, y)\right]}{[m'_\gamma(\gamma, G, y)]^2} = 0.
\end{aligned}
$$

In the operations of partial differentiation indicated by the middle terms of the two expressions, α and γ in accordance with Theorem I are held constant; in the first expression y and β, in the second y and g, are allowed to vary. The partial derivative $m'_\gamma(\gamma, G, y)$ is by assumption different from zero. Since no other functions in the third terms of these two expressions can vanish,

$$
\begin{aligned}
m''_{\gamma y}(\gamma, G, y) &= 0, \\
m''_{\gamma y}(\gamma, G, y)u'_g + m''_{\gamma g}(\gamma, G, y) &= 0,
\end{aligned}
$$

from which

$$m''_{\gamma g}(\gamma, G, y) = 0.$$

These equations imply that function m has the general form

$$m(\gamma, G, y) = K(\gamma) + l(G, y).$$

Substitution in (11) gives

$$r(\alpha) + t(\beta) + K(\gamma) + l(G, y) = 0 \tag{12}$$

and solving for y, we obtain

$$y = h(r(\alpha) + t(\beta) + K(\gamma), G).$$

But α is a function of A, β is a function of B, and γ a function of C; thus the same relationship can be written as

$$y = f(X) = \kappa(\alpha(A) + \beta(B) + \gamma(C), G),$$

which shows that:

i. The function $\alpha(A)+\beta(B)+\gamma(C)$ of the set $A \cup B \cup C$ is locally separable in $f(X)$ and this set itself is a locally functionally separable subset of X in $f(X)$.

ii. The sets A, B and C are additively locally separable subsets of $A \cup B \cup C$ in $\alpha(A)+\beta(B)+\gamma(C)$.

HARVARD UNIVERSITY

UNSYMMETRICAL APPROXIMATION OF IRRATIONAL NUMBERS

RAPHAEL M. ROBINSON

1. **Introduction.** In a recent paper, B. Segre[1] showed that for any $\tau \geqq 0$ an irrational number ξ can be approximated by infinitely many fractions A/B in such a way that

$$-\frac{1}{(1+4\tau)^{1/2}B^2} < \frac{A}{B} - \xi < \frac{\tau}{(1+4\tau)^{1/2}B^2}.$$

For $\tau = 0$, this places A/B to the left of ξ and within a distance $1/B^2$ from it. This type of approximation was known to be possible, since alternate convergents to the continued fraction representing ξ satisfy this condition. For $\tau = 1$, the inequality becomes

$$-\frac{1}{5^{1/2}B^2} < \frac{A}{B} - \xi < \frac{1}{5^{1/2}B^2},$$

so that we have the classical theorem of Hurwitz.[2] For other values of τ, approximations from both sides are permitted, but the errors allowed on the two sides are different; hence the term unsymmetrical approximation. The result here was new, and is so related to Hurwitz's inequality that one side is strengthened and the other weakened.

Notice that the result for $\tau > 1$ is weaker than the result for $\tau < 1$. For suppose that $\tau > 1$, and apply the theorem with τ replaced by $1/\tau$ to the irrational number $-\xi$. In this way, the permissible errors on the right and left are interchanged, and we see that ξ has infinitely many approximations A/B satisfying

$$-\frac{1}{(\tau^2+4\tau)^{1/2}B^2} < \frac{A}{B} - \xi < \frac{\tau}{(\tau^2+4\tau)^{1/2}B^2},$$

which is stronger than the original inequality. *It is therefore sufficient to prove Segre's theorem for* $0 \leqq \tau \leqq 1$.

Segre's proof depends on considering whether certain regions con-

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[1] B. Segre, *Lattice points in infinite domains, and asymmetric Diophantine approximations*, Duke Math. J. vol. 12 (1945) pp. 337–365.

[2] A. Hurwitz, *Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche*, Math. Ann. vol. 39 (1891) pp. 279–284.

tain lattice points. C. D. Olds[3] has given a proof for the case $\tau \geqq 1$, making use of Farey series. We shall consider the problem, making use of continued fractions, and shall prove Segre's theorem (in §6) and somewhat more. In particular, we shall show (in §7) that for any $\epsilon > 0$ the inequality

$$-\frac{1}{(5^{1/2}-\epsilon)B^2} < \frac{A}{B} - \xi < \frac{1}{(5^{1/2}+1)B^2}$$

has infinitely many solutions. This result is interesting since it shows that *one side of Hurwitz's inequality can be strengthened without essentially weakening the other.*

In §2 we state a few known results about continued fractions which we shall need to use, and in §§3–5 we develop the theory that is needed for the last two sections. It may be remarked that §3 is of some interest in itself, since we show there that *the best approximations to an irrational number ξ from either side are convergents to ξ.*

2. **Known results about continued fractions.** Besides the convergents $A_n/B_n = [q_0, q_1, \cdots, q_n]$ to the continued fraction $\xi = [q_0, q_1, q_2, \cdots]$, we shall consider as approximations to ξ the secondary convergents

$$\frac{A_n + A_{n-1}}{B_n + B_{n-1}} = [q_0, \cdots, q_{n-1}, q_n + 1],$$

$$\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = [q_0, \cdots, q_{n-1}, q_n - 1].$$

However, in case $q_{n+1} = 1$ the first reduces to the convergent A_{n+1}/B_{n+1}, and in case $q_n = 1$ the second reduces to the convergent A_{n-2}/B_{n-2}; in such cases they will not be called secondary convergents. Notice that the first is on the opposite side of ξ from A_n/B_n, while the second is on the same side as A_n/B_n.

If in general we put

$$\left| \frac{A}{B} - \xi \right| = \frac{1}{\lambda B^2},$$

then it is seen that for the convergent A_n/B_n the value of λ is

$$\lambda_n = \beta_n + \frac{1}{\alpha_n},$$

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[3] C. D. Olds, *Note on an asymmetric Diophantine approximation*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 261–263.

and that for the secondary convergents considered, the values of λ are

$$\lambda_n' = 1 + \frac{1}{\beta_n - 1} - \frac{1}{\alpha_n + 1}, \qquad \lambda_n'' = 1 + \frac{1}{\alpha_n - 1} - \frac{1}{\beta_n + 1},$$

respectively, where

$$\alpha_n = [q_n, q_{n-1}, \cdots, q_1], \qquad \beta_n = [q_{n+1}, q_{n+2}, \cdots].$$

We shall use the notation

$$[\cdots, q_{-2}, q_{-1}, [q_0], q_1, q_2, \cdots]$$

to denote the sum of two continued fractions

$$[q_0, q_1, q_2, \cdots] + [0, q_{-1}, q_{-2}, \cdots].$$

With this notation,

$$\begin{aligned} \lambda_n &= [q_1, \cdots, q_{n-1}, q_n, [q_{n+1}], q_{n+2}, \cdots], \\ \lambda_n' &= [q_1, \cdots, q_{n-1}, q_n, 1, [0], q_{n+1} - 1, q_{n+2}, \cdots], \\ \lambda_n'' &= [q_1, \cdots, q_{n-1}, q_n - 1, [0], 1, q_{n+1}, q_{n+2}, \cdots]. \end{aligned}$$

Besides the classical results that every convergent represents ξ with an error less than $1/B^2$, and that every approximation for which the error is less than $1/2B^2$ is a convergent, we shall also need the result that *every approximation with an error less than* $1/B^2$ *is either a convergent or a secondary convergent.*[4]

3. The best approximations are convergents.

A classical result about continued fractions is that the best approximations to an irrational number ξ are included among the convergents to the continued fraction representing ξ. This can be understood in the following sense: *The inequality*

$$\left| \frac{A}{B} - \xi \right| < \frac{1}{\mu B^2}$$

has infinitely many solutions, all of which are convergents to ξ, *if* μ *is a suitable number*; indeed, μ may be any constant such that $2 \leq \mu \leq 5^{1/2}$.

We shall show that in a somewhat similar sense the best approximations to ξ from the right are included among the convergents to ξ. More precisely, *the inequality*

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[4] This result was stated without proof by P. Fatou, *Sur l'approximation des incommensurables et les séries trigonométriques*, C. R. Acad. Sci. Paris vol. 139 (1904) pp. 1019–1021. See also R. M. Robinson, *The approximation of irrational numbers by fractions with odd or even terms*, Duke Math. J. vol. 7 (1940) pp. 354–359.

$$0 < \frac{A}{B} - \xi < \frac{1}{\mu B^2}$$

has infinitely many solutions, all but a finite number of which are convergents to ξ, if μ is a suitable number, depending on ξ. It is shown, in the theorem below, that μ may always be taken as one of the three numbers 1, 4/3, 2. A similar result of course holds also for approximations from the left.

LEMMA 1. *Suppose $q_{n+1} \geqq 2$, so that λ_n' actually corresponds to a secondary convergent.*

(a) *If $\lambda_n' > 1$, then $\lambda_{n-1} > 4/3$.*

(b) *If $\lambda_n' > 4/3$, then $\lambda_{n-1} > 2$ or $\lambda_{n+1} > 2$.*

PROOF. (a) From $\lambda_n' > 1$, we find $\beta_n < \alpha_n + 2$. Hence

$$\lambda_{n-1} = \beta_{n-1} + \frac{1}{\alpha_{n-1}} = \alpha_n + \frac{1}{\beta_n} > \alpha_n + \frac{1}{\alpha_n + 2}\cdot$$

Since this expression increases with α_n, which is at least 1, we find that $\lambda_{n-1} > 4/3$.

(b) Proof by contradiction. Suppose $\lambda_{n-1} \leqq 2$ and $\lambda_{n+1} \leqq 2$. From the first follows $\alpha_n < 2$, and the second gives $\beta_{n+1} < 2$, and hence

$$\beta_n = q_{n+1} + \frac{1}{\beta_{n+1}} > 2 + \frac{1}{2}\cdot$$

Therefore

$$\lambda_n' = 1 + \frac{1}{\beta_n - 1} - \frac{1}{\alpha_n + 1} < 1 + \frac{2}{3} - \frac{1}{3} = \frac{4}{3}\cdot$$

LEMMA 2. *Suppose $q_n \geqq 2$, so that λ_n'' actually corresponds to a secondary convergent.*

(a) *If $\lambda_n'' > 1$, then $\lambda_n > 4/3$.*

(b) *If $\lambda_n'' > 4/3$, then $\lambda_n > 2$ or $\lambda_{n-1} > 2$.*

PROOF. (a) From $\lambda_n'' > 1$, we find $\alpha_n < \beta_n + 2$, hence $\lambda_n > 4/3$.

(b) Proof by contradiction. Suppose $\lambda_n \leqq 2$ and $\lambda_{n-1} \leqq 2$. From the first follows $\beta_n < 2$, and the second gives $\alpha_{n-1} < 2$ and hence $\alpha_n > 5/2$. Therefore $\lambda_n'' < 4/3$.

Notice that both lemmas assert that if a secondary convergent gives a good approximation to ξ, then a convergent can be found (with a nearby subscript) giving an even better approximation to ξ from the same side.

THEOREM. *At least one of the three inequalities*

$$0 < \frac{A}{B} - \xi < \frac{1}{B^2}, \qquad 0 < \frac{A}{B} - \xi < \frac{3}{4B^2}, \qquad 0 < \frac{A}{B} - \xi < \frac{1}{2B^2}$$

has infinitely many solutions, of which all but a finite number are convergents to ξ.

PROOF. The first inequality has infinitely many solutions. If it has infinitely many solutions which are not convergents, then they must be secondary convergents. Thus by the lemmas, corresponding convergents can be found satisfying the second inequality, which thus has infinitely many solutions. If it has infinitely many solutions which are not convergents, then they are again secondary convergents. Corresponding convergents can be found satisfying the third inequality, which thus has infinitely many solutions, all of which must be convergents.

4. **The moduli of approximability.** The degree of approximation possible for the irrational number ξ is measured by the number

$$M(\xi) = \limsup \lambda_n,$$

the modulus of approximability of the number ξ. In fact, the inequality

$$\left| \frac{A}{B} - \xi \right| < \frac{1}{\mu B^2}$$

has infinitely many solutions if $M(\xi) > \mu$, but not if $M(\xi) < \mu$. The function $M(\xi)$ has been studied by various authors.[5] In a similar way, the degrees of approximation possible from the right and from the left are measured by

$$M^+(\xi) = \limsup \lambda_{2n+1}, \qquad M^-(\xi) = \limsup \lambda_{2n},$$

which may be called the moduli of approximability from the right and left, respectively. Evidently

$$M(\xi) = \max \{ M^+(\xi), M^-(\xi) \}.$$

Notice that

$$M^+(-\xi) = M^-(\xi), \qquad M^-(-\xi) = M^+(\xi),$$

so that the roles of the two moduli are interchangeable.

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[5] See J. F. Koksma, *Diophantische Approximationen*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 4, Berlin, 1936, chap. 3, §2.

The numbers

$$\xi_k = [k,\ 1,\ k,\ 1,\ k,\ 1,\ \cdots]$$

will be of particular interest for our problem. It is clear that

$$M^+(\xi_k) = [\cdots,\ 1,\ k,\ 1,\ [k],\ 1,\ k,\ 1,\ \cdots] = \rho_k,$$
$$M^-(\xi_k) = [\cdots,\ k,\ 1,\ k,\ [1],\ k,\ 1,\ k,\ \cdots] = \sigma_k.$$

Evaluating these continued fractions, we find that

$$\rho_k = (k^2 + 4k)^{1/2}, \qquad \sigma_k = (k^2 + 4k)^{1/2}/k.$$

We noticed in §1 that it is sufficient to prove Segre's theorem for $0 \leqq \tau \leqq 1$. For these values of τ, the error allowed on the right is less than or equal to the error allowed on the left. *Now to prove that an inequality of the form*

$$-\frac{1}{\sigma B^2} < \frac{A}{B} - \xi < \frac{1}{\rho B^2},$$

where $\rho \geqq \sigma$, has infinitely many solutions for each irrational number ξ, it is sufficient to consider those numbers for which $M^+(\xi) \geqq M^-(\xi)$. For if $M^-(\xi) > M^+(\xi)$, then $-\xi$ satisfies the condition imposed. Applying the inequality to $-\xi$, we see that we can satisfy

$$-\frac{1}{\rho B^2} < \frac{A}{B} - \xi < \frac{1}{\sigma B^2}$$

infinitely often. If $\rho = \sigma$, this is the required inequality. If $\rho > \sigma$, then $M^+(\xi) \geqq \sigma$ or $M^-(\xi) \geqq \rho$. Either of these conditions leads to $M^-(\xi) > \sigma$. Hence we can satisfy the inequality

$$-\frac{1}{\sigma B^2} < \frac{A}{B} - \xi < 0$$

infinitely often.

5. **Classification of irrational numbers.** In this section, we shall consider certain classes of irrational numbers, and show that together they include every irrational ξ such that $M^+(\xi) \geqq M^-(\xi)$. The classes are defined by the following conditions, where k denotes a positive integer.

(1_k) $q_{2n} \to k$, $q_{2n+1} \to 1$, and hence $M^+(\xi) = \rho_k$, $M^-(\xi) = \sigma_k$.

(1_∞) $q_{2n} \to \infty$, $q_{2n+1} \to 1$, and hence $M^+(\xi) = \infty$, $M^-(\xi) = 1$.

(2_k) $M^+(\xi) > \rho_k$, $M^-(\xi) > \sigma_k$.

$$(3_k) \qquad \limsup q_{2n} = k+1, \ \liminf q_{2n} = k, \ q_{2n+1} \to 1.$$

LEMMA. *If* $M^+(\xi) \geqq M^-(\xi)$, *but we do not have* $q_{2n+1} \to 1$, *then* ξ *belongs to class* (2_1) *or* (2_2).

PROOF. Let $a = \limsup q_{2n}$, $b = \limsup q_{2n+1}$. By hypothesis, $b > 1$. We have the following cases:

$$\begin{array}{ll} b \geqq 3. & M^+(\xi) \geqq M^-(\xi) \geqq 3. \\ b = 2,\ a \geqq 4. & M^+(\xi) \geqq 4,\ M^-(\xi) \geqq 2. \\ b = 2,\ a \leqq 3. & M^+(\xi) \geqq M^-(\xi) \geqq [4, [2], 4] = 5/2. \end{array}$$

Thus ξ belongs to the classes (2_1), (2_2), (2_1), respectively.

THEOREM. *An irrational number* ξ *with* $M^+(\xi) \geqq M^-(\xi)$ *belongs to at least one of the classes* (1_k), (1_∞), (2_k), (3_k).

PROOF. Because of the lemma, it will be sufficient to consider an irrational number ξ with $q_{2n+1} \to 1$, but not belonging to any class (1_k), (1_∞), or (3_k), and to show that it belongs to class (2_k) for some k.

If $\limsup q_{2n} = \infty$, we have $M^+(\xi) = \infty$. Since ξ does not belong to the class (1_∞), we do not have $q_{2n} \to \infty$, and hence $M^-(\xi) > 1$. Thus ξ belongs to class (2_k) for any sufficiently large k.

It remains to consider the case when $\limsup q_{2n}$ is finite. Since ξ does not belong to any class (1_k) or (3_k), we must have

$$\limsup q_{2n} - \liminf q_{2n} \geqq 2.$$

Hence there is an integer $k \geqq 2$, such that

$$\limsup q_{2n} = k+1, \qquad \liminf q_{2n} \leqq k-1.$$

From the first condition alone, it follows that

$$M^+(\xi) \geqq [1, 1, [k+1], 1, 1] = k+2 > \rho_k.$$

Making use of the fact that infinitely many partial quotients do not exceed $k-1$, and that ultimately none exceed $k+1$, we also have

$$M^-(\xi) \geqq [1, k+1, [1], k-1, 1] = [k, 1, k, [1], k, 1, k] > \sigma_k.$$

The inequalities here are clear, and the equality in the middle may be verified by actual expansion of the continued fractions. Thus ξ belongs to class (2_k).

6. **Segre's theorem.** We shall put

$$\rho_\kappa = (\kappa^2 + 4\kappa)^{1/2}, \qquad \sigma_\kappa = (\kappa^2 + 4\kappa)^{1/2}/\kappa,$$

even when κ is not a positive integer. In particular, $\rho_\infty = \infty$ and $\sigma_\infty = 1$.

If we replace τ by $1/\kappa$, Segre's inequality becomes

$$(\mathrm{I}_\kappa) \qquad -\frac{1}{\sigma_\kappa B^2} < \frac{A}{B} - \xi < \frac{1}{\rho_\kappa B^2}.$$

We are to show that this inequality has infinitely many solutions A/B for each irrational number ξ and each κ such that $1 \leqq \kappa \leqq \infty$. As noted in §4, for these values of κ we need consider only values of ξ such that $M^+(\xi) \geqq M^-(\xi)$. It is thus sufficient to prove the theorem for each of the classes defined in §5.

For numbers of class (1_∞), it is clear that (I_κ) is satisfied by infinitely many approximations from the right if $\kappa < \infty$, and by infinitely many approximations from the left if $\kappa = \infty$. For numbers of the other classes, the situation is made clearer by first formulating the following lemmas.

Lemma 1. *If for some* κ_0 *we have*

$$M^+(\xi) > \rho_{\kappa_0}, \qquad M^-(\xi) > \sigma_{\kappa_0},$$

then (I_κ) *has infinitely many solutions for each* κ.

Proof. Since ρ_κ increases and σ_κ decreases as κ increases, we have $M^+(\xi) > \rho_\kappa$ for $\kappa \leqq \kappa_0$, and $M^-(\xi) > \sigma_\kappa$ for $\kappa \geqq \kappa_0$. Hence (I_κ) is satisfied by infinitely many approximations from the right if $\kappa \leqq \kappa_0$, and by infinitely many approximations from the left if $\kappa \geqq \kappa_0$.

Lemma 2. *If for some* κ_0 *we have*

$$M^+(\xi) = \rho_{\kappa_0}, \qquad M^-(\xi) = \sigma_{\kappa_0},$$

then (I_κ) *has infinitely many solutions for each* κ, *except possibly for* $\kappa = \kappa_0$.

Proof. $M^+(\xi) > \rho_\kappa$ for $\kappa < \kappa_0$, and $M^-(\xi) > \sigma_\kappa$ for $\kappa > \kappa_0$.

For numbers of class (2_k), the hypothesis of Lemma 1 is satisfied with $\kappa_0 = k$. Hence the desired conclusion follows. For numbers of class (1_k), the hypothesis of Lemma 2 is satisfied with $\kappa_0 = k$. Hence the conclusion follows for $\kappa \neq k$. Now for numbers of class (1_k), we have

$$\lambda_{2n+1} \to \rho_k, \qquad \lambda_{2n} \to \sigma_k,$$

one of the limits being approached from above and the other from below. Hence for numbers of class (1_k), the inequality (I_k) can also be satisfied.

It remains to prove (I_κ) for numbers of class (3_k). We first notice that for numbers of this class

$$\rho_k \leqq M^+(\xi) \leqq \rho_{k+1}, \qquad \sigma_{k+1} \leqq M^-(\xi) \leqq \sigma_k.$$

The upper bounds here are sharp, but the lower bounds are not. It is easily seen that the sharp lower bounds are given by

$$M^+(\xi) \geqq [\cdots, k, 1, k, 1, [k+1], 1, k, 1, k, \cdots] = \rho_k',$$
$$M^-(\xi) \geqq [\cdots, k+1, 1, k+1, [1], k, 1, k+1, \cdots] = \sigma_{k+1}'.$$

It is clear that

$$\rho_k' = 1 + \rho_k = 1 + (k^2 + 4k)^{1/2}.$$

Replacing k by $k-1$ in the other expression and expanding, we find that for $k \geqq 2$,

$$\sigma_k' = \frac{k(k+1) + (3k-1)(k^2+4k)^{1/2}}{2k(2k-1)}.$$

We shall show below that

$$\rho_k' > \rho_{k+0.8}, \qquad \sigma_k' > \sigma_{k-0.5}.$$

From these inequalities, it follows that

$$M^+(\xi) \geqq \rho_k' > \rho_{k+0.8}, \qquad M^-(\xi) \geqq \sigma_{k+1}' > \sigma_{k+0.5},$$

for any number ξ of class (3_k). Hence the hypothesis of Lemma 1 is satisfied for numbers of class (3_k) with a suitable κ_0, and indeed with any κ_0 such that

$$k + 0.5 \leqq \kappa_0 \leqq k + 0.8.$$

The proof of the theorem will therefore be complete.

PROOF THAT $\rho_k' > \rho_{k+0.8}$. The inequality to be proved is

$$1 + (k^2 + 4k)^{1/2} > \left\{(k+0.8)^2 + 4(k+0.8)\right\}^{1/2}.$$

If we square and simplify, this reduces to

$$50(k^2 + 4k)^{1/2} > 40k + 71.$$

Squaring again, we have

$$900k^2 + 4320k > 5041,$$

which is true, since the left side is at least 5220.

PROOF THAT $\sigma_k' > \sigma_{k-0.5}$. If we multiply through by $2k(2k-1)$, the required inequality takes the form

$$k(k+1) + (3k-1)(k^2+4k)^{1/2} > 2k\left\{(2k-1)^2 + 8(2k-1)\right\}^{1/2}.$$

If we square and simplify, this reduces to

$$(3k-1)(k+1)(k^2+4k)^{1/2} > (3k-1)k(k+3) - 2,$$

which is certainly true if

$$(k+1)(k^2+4k)^{1/2} > k(k+3).$$

Squaring this, we obtain the inequality

$$k^4+6k^3+9k^2+4k > k^4+6k^3+9k^2,$$

which is true.

7. **Some stronger results.** The inequality

$$(\mathrm{I}_k) \qquad -\frac{1}{\sigma_k B^2} < \frac{A}{B} - \xi < \frac{1}{\rho_k B^2},$$

where k is a positive integer, is a special case of Segre's inequality. Now for numbers of class (1_k), we have

$$M^+(\xi) = \rho_k, \qquad M^-(\xi) = \sigma_k.$$

It follows, as noted by Segre, that we cannot increase both of the numbers ρ_k and σ_k. Actually, for numbers of the class (1_k), we have

$$\lambda_{2n+1} \to \rho_k, \qquad \lambda_{2n} \to \sigma_k,$$

one of the limits being approached from above and the other from below; which one is approached from which side depends on the beginning of the continued fraction expansion. Hence we cannot increase either of the numbers ρ_k or σ_k, leaving the other one fixed.

Nevertheless, *there are infinitely many solutions to the inequality*

$$(\mathrm{I}_k') \qquad -\frac{1}{(\sigma_k-\epsilon)B^2} < \frac{A}{B} - \xi < \frac{1}{\rho_k' B^2},$$

for any $\epsilon>0$, although $\rho_k' = \rho_k+1$. In a similar way, if $k \geq 2$ *the inequality*

$$(\mathrm{I}_k'') \qquad -\frac{1}{\sigma_k' B^2} < \frac{A}{B} - \xi < \frac{1}{(\rho_k-\epsilon)B^2}$$

has infinitely many solutions for each $\epsilon>0$, although $\sigma_k' > \sigma_k$. We shall confine our discussion to the inequality (I_k'), since the proof of (I_k'') is entirely similar.

To prove the inequality (I_k'), we need consider only numbers ξ for which $M^+(\xi) \geq M^-(\xi)$, as shown in §4. Now (I_k') is satisfied by approximations from the right for numbers of classes (1_j), (2_j), and (3_j), if $j>k$, since $\rho_j \geqq \rho_{k+1} > \rho_k'$. It is satisfied by approximations from the left for numbers of the classes (1_j) and (2_j) if $j \leqq k$, since $\sigma_j > \sigma_k - \epsilon$, and for numbers of class (3_j) if $j<k$, since then $\sigma_{j+1} > \sigma_k - \epsilon$.

It remains only to consider the numbers of class (3_k). For these numbers we have $M^+(\xi) \geqq \rho_k'$, and indeed $\lambda_{2n+1} > \rho_k'$ has infinitely many solutions, since it is true whenever n is sufficiently large and $q_{2n+2} = k+1$. Thus in this case, (I_k') can be satisfied by approximations from the right.

It is easily seen that there are numbers of class (3_k) for which $M^-(\xi) < \sigma_k$ and $M^+(\xi)$ is arbitrarily near to ρ_k'. It follows that we cannot replace ρ_k' by any larger constant on the right side of the inequality (I_k').

A particularly interesting case of the theorem just proved is obtained by putting $k=1$: *For any* $\epsilon > 0$, *every irrational number* ξ *has infinitely many approximations* A/B *such that*

$$-\frac{1}{(5^{1/2}-\epsilon)B^2} < \frac{A}{B} - \xi < \frac{1}{(5^{1/2}+1)B^2}.$$

The coefficient of B^2 *in the denominator on the right cannot be replaced by any larger constant.*

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A NEW PROOF OF HILBERT'S NULLSTELLENSATZ

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Introduction. A number of proofs of Hilbert's Nullstellensatz can be found in the literature. One, based on elimination theory and due to A. Rabinowitsch, is reproduced in van der Waerden's *Moderne Algebra*, vol. 2, p. 11. In a later chapter van der Waerden gives another proof which is based on the method of specialization in fields of algebraic functions (pp. 59–61). The finishing touches to this proof (p. 65) presuppose the decomposition theorem for polynomial ideals. In his Ergebnisse monograph *Idealtheorie*, p. 46, Krull gives an ideal-theoretic proof which, while it is based on the simple remark by Rabinowitsch, is of an advanced nature, since the proof makes use of the full dimension theory of algebraic varieties developed in §17, pp. 41–43. The main "Dimensionssatz" of p. 43 is based on a result which is proved only in §48, pp. 129–134. Moreover, the concept of integral dependence and the "Normalization theorem" of Emmy Noether are used in Krull's proof.

In the present note we give first of all a short proof of Hilbert's Nullstellensatz which makes use only of the rudiments of field theory and ideal theory. Actually we give two new proofs of the Nullstellensatz. A lemma used in the second proof enables us to establish a result on finite integral domains which we were not able to find in the literature. This result is as follows:

If $R=\mathrm{K}[\xi_1, \xi_2, \cdots, \xi_n]$ *is a finite integral domain over a field* K *and if* $\overline{\mathrm{K}}$ *is the algebraic closure of* K *in* R, *then* $\overline{\mathrm{K}}$ *contains all the fields which are contained in* R.

1. **First proof of the Nullstellensatz.** Let $\mathfrak{P}_n$ denote the polynomial ring $\mathrm{K}[x_1, x_2, \cdots, x_n]$ in n indeterminates x_i, over a given ground field K. By a point $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ we mean an ordered n-tuple of algebraic quantities α_i over K. Our convention will be that conjugate n-tuples over K represent the same point over K. Let S_n denote the (linear) space of all points. By a zero α of an ideal $\mathfrak{A}$ in $\mathfrak{P}_n$ we mean a point α such that $f(\alpha)=0$ for every polynomial $f(x)$ $[=f(x_1, x_2, \cdots, x_n)]$ in $\mathfrak{A}$. The totality of zeros of $\mathfrak{A}$ is the algebraic variety in S_n determined by the ideal $\mathfrak{A}$ and shall be denoted by $\mathcal{V}(\mathfrak{A})$.

If W is an algebraic variety in S_n, we shall denote by $\mathfrak{J}(W)$ the ideal in $\mathfrak{P}_n$ consisting of all polynomials which vanish on W (that is, at every point of W). The Hilbert Nullstellensatz asserts the following:

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H_1: *If* $\mathfrak{A}$ *is an ideal in* $\mathfrak{P}_n$ *then* $\mathfrak{J}(\mathcal{V}(\mathfrak{A}))=$ *radical of* $\mathfrak{A}$.

It has been shown by Rabinowitsch that the following special case of Hilbert's Nullstellensatz is equivalent to the full Nullstellensatz:

H_2: *If* $\mathcal{V}(\mathfrak{A})$ *is an empty set then* $\mathfrak{A}$ *is the unit ideal.*

For the convenience of the reader we reproduce here the original proof of Rabinowitsch. Let x_{n+1} be an extra indeterminate and let $\mathfrak{P}_{n+1}$ be the polynomial ring $K[x_1, x_2, \cdots, x_{n+1}]$. If $f(x)$ is any polynomial in $\mathfrak{P}_n$ which vanishes on $\mathcal{V}(\mathfrak{A})$ and if we set $g=1+x_{n+1}f$, then the ideal $\mathfrak{B}$ generated in $\mathfrak{P}_{n+1}$ by $\mathfrak{A}$ and g is such that $\mathcal{V}(\mathfrak{B})$ is empty. Hence by the special case H_2 of the Nullstellensatz it follows that $\mathfrak{B}=(1)$. There exists therefore an identity of the form

$$\sum_i A_i h_i + gh = 1, \qquad A_i \in \mathfrak{A};\ h_i,\ h \in \mathfrak{P}_{n+1}.$$

On replacing x_{n+1} in this identity by $-1/f$ and clearing the denominators we find a relation of the form $\sum A_i B_i = f^\rho$, where $B_i \in \mathfrak{P}_n$ and ρ is a non-negative integer. Hence f belongs to the radical of $\mathfrak{A}$, and since the inclusion: radical of $\mathfrak{A} \subseteq \mathfrak{J}(\mathcal{V}(\mathfrak{A}))$ is trivial, this establishes the Nullstellensatz.

Let R_n denote any finite integral domain over K which can be obtained from K by an n-fold ring extension: $R_n = K[\xi_1, \xi_2, \cdots, \xi_n]$. To prove H_2 we first show that H_2 is an immediate consequence of the following statement:

H_3^n: *If a finite integral domain* R_n *(over* K*) is a field then it is an algebraic extension of the field* K.

For if $\mathfrak{A}$ is an ideal in $\mathfrak{P}_n$, *different from the unit ideal*, then $\mathfrak{A}$ is contained in some maximal ideal of $\mathfrak{P}_n$. If $\mathfrak{p}$ is such a maximal ideal and if we set $R_n = \mathfrak{P}_n/\mathfrak{p} = K[\alpha_1, \alpha_2, \cdots, \alpha_n]$, then R_n is a field and, therefore, if we assume H_3^n, it follows that the α_i are algebraic over K, that is, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a point. This point is a zero of $\mathfrak{p}$ and hence a fortiori a zero of $\mathfrak{A}$, since $\mathfrak{A} \subseteq \mathfrak{p}$. We have thus shown that if $\mathfrak{A} \neq (1)$ then $\mathcal{V}(\mathfrak{A})$ is not empty, and this establishes H_2.

The proof of Hilbert's Nullstellensatz is thus reduced to proving H_3^n. We shall prove H_3^n by induction with respect to n, since H_3^1 is trivial (if $K[\xi_1]$ is a field, then $1/\xi_1 = f(\xi_1) \in K[\xi_1]$ and hence ξ_1 is a root of the polynomial $xf(x)-1$).

Granted that H_3^{n-1} is true, let R_n be a given finite integral domain $K[\xi_1, \xi_2, \cdots, \xi_n]$ and let it be assumed that R_n is a field. Under this assumption we shall have $R_n = K(\xi_1)[\xi_2, \xi_3, \cdots, \xi_n]$, where $K(\xi_1)$ is the field generated over K by ξ_1. Hence if we apply H_3^{n-1} to R_n *thought of as an* R_{n-1} *over the field* $K(\xi_1)$, we conclude that R_n *is an algebraic extension of* $K(\xi_1)$. To complete the proof of H_3^n it remains only to show that ξ_1 *is algebraic over* K.

Each ξ_i, $i=2, 3, \cdots, n$, is a root of a polynomial $f_i(X)$ with coefficients in $\mathrm{K}[\xi_1]$. Let b_i be the leading coefficient of $f_i(X)$, $b_i \neq 0$. If ω is any element of R_n, there will exist an integer ρ (depending on ω) such that the product $\omega\cdot(b_2b_3\cdots b_n)^\rho$ can be expressed as a linear combination, *with coefficients in* $\mathrm{K}[\xi_1]$, of the $m_2m_3\cdots m_n$ power products $\xi_2^{j_2}\xi_3^{j_3}\cdots\xi_n^{j_n}$, $0\leqq j_i\leqq m_i-1$, where m_i is the degree of $f_i(X)$.

Let ν be the relative degree of R_n over $\mathrm{K}(\xi_1)$ and let $\omega_1=1$, $\omega_2, \cdots, \omega_\nu$ be a linear basis of R_n over $\mathrm{K}(\xi_1)$. We can find an element b_1 in $\mathrm{K}[\xi_1]$, $b_1\neq 0$, such that for each of the above $m_2m_3\cdots m_n$ power products $\xi_2^{j_2}\xi_3^{j_3}\cdots\xi_n^{j_n}$ the following is true: in the expression of $b_1\cdot\xi_2^{j_2}\xi_3^{j_3}\cdots\xi_n^{j_n}$ as a linear combination of the elements $\omega_1, \omega_2, \cdots, \omega_\nu$, with coefficients in $\mathrm{K}(\xi_1)$, the coefficient of ω_1 is in $\mathrm{K}[\xi_1]$. It follows that if we set $b=b_1b_2\cdots b_n$ then b is an element of $\mathrm{K}[\xi_1]$, different from zero, having the following property: for any element ω in R_n there exists an integer ρ such that if $\omega b^\rho = a_1+a_2\omega_2+\cdots+a_\nu\omega_\nu$, $a_i\in\mathrm{K}(\xi_1)$, then $a_1\in\mathrm{K}[\xi_1]$. Now let ζ be an arbitrary element of $\mathrm{K}[\xi_1]$, $\zeta\neq 0$, and let us apply this result to $\omega=1/\zeta$. Since $1, \omega_2, \cdots, \omega_\nu$ are independent over $\mathrm{K}(\xi_1)$ and since $b^\rho/\zeta\in\mathrm{K}(\xi_1)$, it follows then that $b^\rho/\zeta=a_1$, that is, *any element* ζ *of* $\mathrm{K}[\xi_1]$, $\zeta\neq 0$, *divides some power of the fixed element* b of $\mathrm{K}[\xi_1]$. The existence of an element $b\neq 0$ with this property clearly implies that $\mathrm{K}[\xi_1]$ cannot be a ring of polynomials in one indeterminate over K. Hence ξ_1 is algebraic over K, and this completes the proof of H_3^n.

2. **Second proof.** In this section we shall give a second proof of the Nullstellensatz. If we take for $\mathfrak{A}$ a prime ideal then H_1 yields the following result:

H_4^n: *If* $\mathfrak{p}$ *is a prime ideal in* $\mathfrak{P}_n$ *then* $\mathfrak{J}(\mathcal{V}(\mathfrak{p}))=\mathfrak{p}$.

This is the most important consequence of the Nullstellensatz, since it shows that there exists a one-to-one correspondence between the irreducible varieties in S_n and the prime ideals in $\mathfrak{P}_n$. It is therefore desirable to prove H_4^n directly. On the other hand it is easy to deduce from H_4^n the full Nullstellensatz. In fact, H_4^n implies that if $\mathfrak{p}\neq(1)$ then $\mathcal{V}(\mathfrak{p})$ is not empty. Since any ideal $\mathfrak{A}$ in $\mathfrak{P}_n$ which is different from the unit ideal is contained in some prime ideal $\mathfrak{p}\neq(1)$ (for instance in a maximal ideal), we see that from H_4^n follows H_2, that is, the full Nullstellensatz.

We shall prove H_4^n by induction with respect to n, since H_4^1 is trivial. The proof will be based on the following lemma:

LEMMA 1. *Let* Ω *be an integral domain and let* R *be an integral domain contained in* Ω. *If* Ω *is a simple ring extension of* R: $\Omega=R[\omega]$, *then there* *exists in R an element $a \neq 0$ with the following property: if $\mathfrak{p}$ is a prime ideal in R such that $a \notin \mathfrak{p}$, then $\Omega \cdot \mathfrak{p} \neq (1)$.*

PROOF. We shall use small Latin and Greek letters to indicate respectively elements of R and Ω. In the proof we may assume that ω is algebraic over R, since otherwise the assertion of the lemma is trivial (if ω is a transcendental over R then $\Omega\mathfrak{A} \neq (1)$ for *any* ideal $\mathfrak{A}$ in R, $\mathfrak{A} \neq (1)$). Let

$$b_0\omega^{\nu} + b_1\omega^{\nu-1} + \cdots + b_{\nu} = 0, \qquad b_0 \neq 0,\ b_i \in R, \tag{1}$$

be an equation of least degree ν which ω satisfies over R. If $\mathfrak{A}$ is any ideal in R then any element ζ of $\Omega\mathfrak{A}$ is of the form: $\zeta = f(\omega)$, where f is a polynomial with coefficients in $\mathfrak{A}$. If we use (1) to reduce the degree of f we see that there exists an integer ρ (depending on ζ) such that

$$\zeta \cdot b_0^{\rho} = c_1\omega^{\nu-1} + c_2\omega^{\nu-2} + \cdots + c_{\nu}, \qquad c_i \in \mathfrak{A}. \tag{2}$$

If the element ζ also belongs to R, say $\zeta = z$, then (2) is a relation of algebraic dependence for ω over R, *of degree less than ν*. Hence necessarily $c_1 = c_2 = \cdots = c_{\nu-1} = 0$ and

$$zb_0^{\rho} = c_{\nu} \in \mathfrak{A}. \tag{3}$$

If $z = 1$ then (3) yields $b_0^{\rho} \in \mathfrak{A}$, that is, b_0 is in the radical of $\mathfrak{A}$. *Hence if the radical of $\mathfrak{A}$ does not contain b_0 then $\Omega\mathfrak{A} \neq (1)$.* It follows that the element b_0 is an element a the existence of which is asserted in the lemma.

We now come to the proof of H_4^n. Let $f(x)$ be an element of $\mathfrak{I}(\mathcal{V}(\mathfrak{p}))$. We have to show that

$$f(x) \in \mathfrak{p}. \tag{4}$$

We pass to the ring of residual classes $\Omega = \mathfrak{P}_n/\mathfrak{p} = K[\xi_1, \xi_2, \cdots, \xi_n]$, where ξ_i is the $\mathfrak{p}$-residue of x_i. Let $\zeta = f(\xi)$. We have to show that

$$\zeta = 0. \tag{5}$$

The case $\mathfrak{p} = (0)$ is trivial. Hence we may assume that $\mathfrak{p} \neq (0)$. In this case there exists a non-identical algebraic relation between the elements $\xi_1, \xi_2, \cdots, \xi_n$, with coefficients in K. We may therefore assume that ξ_n is algebraically dependent on $\xi_1, \xi_2, \cdots, \xi_{n-1}$, over K. We set $R = K[\xi_1, \xi_2, \cdots, \xi_{n-1}]$. To our present two rings R and Ω we apply Lemma 1. Let a be an element of R, $a \neq 0$, the existence of which is asserted in the lemma.

We shall now show that if H_4^{n-1} is granted, then the hypothesis that (5) is not satisfied leads to a contradiction. Every element of Ω

is algebraic over R. In particular, let

$$d_0\zeta^s + d_1\zeta^{s-1} + \cdots + d_s = 0, \qquad d_i \in R, \tag{6}$$

be a relation of algebraic dependence for ζ over R. Let us assume that (5) is false: $\zeta \neq 0$. Then we may also assume that $d_s \neq 0$. Since a and d_s are in R, these elements can be expressed as polynomials in $\xi_1, \xi_2, \cdots, \xi_{n-1}$, with coefficients in K:

$$a = g(\xi_1, \xi_2, \cdots, \xi_{n-1}), \qquad a \neq 0; \tag{7}$$

$$d_s = h(\xi_1, \xi_2, \cdots, \xi_{n-1}), \qquad d_s \neq 0. \tag{7a}$$

Let $f_1(x) = g(x)h(x) \in \mathfrak{P}_{n-1}$, and let $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{P}_{n-1}$. It is clear that $R \cong \mathfrak{P}_{n-1}/\mathfrak{p}_1$ and that $\xi_1, \xi_2, \cdots, \xi_{n-1}$ are the $\mathfrak{p}_1$-residues of $x_1, x_2, \cdots, x_{n-1}$. *We shall show now that*

$$f_1(x) \in \mathfrak{I}(\mathcal{V}(\mathfrak{p}_1)). \tag{8}$$

Let $P_1(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ be any point of $\mathcal{V}(\mathfrak{p}_1)$. If P_1 is a zero of the polynomial $g(x)$ there is nothing to prove. Assume then that P_1 is not a zero of $g(x)$. The ideal $\mathfrak{I}(P_1)$ is a maximal (prime) ideal in $\mathfrak{P}_{n-1}$ which contains $\mathfrak{p}_1$, since $P_1 \in \mathcal{V}(\mathfrak{p}_1)$. Hence if we set $\mathfrak{p}_0 = \mathfrak{I}(P_1)/\mathfrak{p}_1$, then $\mathfrak{p}_0$ is a maximal prime ideal in R, and the assumption that P_1 is not a zero of $g(x)$ signifies, in view of (7), that $a \notin \mathfrak{p}_0$. Hence, by Lemma 1, the ideal $\Omega \cdot \mathfrak{p}_0$ is not the unit ideal. Let $\mathfrak{p}_0'$ be a maximal prime ideal in Ω which contains the ideal $\Omega \cdot \mathfrak{p}_0$. Since $\mathfrak{p}_0 \subseteq \mathfrak{p}_0'$ and since $\mathfrak{p}_0$ is maximal in R, it follows that $\mathfrak{p}_0' \cap R = \mathfrak{p}_0$. Hence the $\mathfrak{p}_0'$-residues of $\xi_1, \xi_2, \cdots, \xi_{n-1}$ can be identified with their $\mathfrak{p}_0$-residues, that is, with the algebraic quantities $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ respectively. Let α_n be the $\mathfrak{p}_0'$-residue of ξ_n. Since $\text{K}[\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n] = \Omega/\mathfrak{p}_0'$ is a field, α_n is algebraically dependent on $\text{K}[\alpha_1, \alpha_2, \cdots, \alpha_{n-1}]$, whence also α_n is algebraic over K. Hence $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n)$ is a point P of S_n. Since the coördinates of this point are residues of $\xi_1, \xi_2, \cdots, \xi_n$ and since $\text{K}[\xi_1, \xi_2, \cdots, \xi_n] = \mathfrak{P}_n/\mathfrak{p}$, it follows that the point P belongs to $\mathcal{V}(\mathfrak{p})$. Hence P is also a zero of the polynomial $f(x)$, since, by hypothesis, $f(x)$ vanishes on $\mathcal{V}(\mathfrak{p})$. This implies that $f(\xi)$, that is, ζ, belongs to $\mathfrak{p}_0$. But then, in view of (6), also d_s belongs to $\mathfrak{p}_0$, and therefore, by (7a), $h(\alpha) = 0$, and this establishes (8).

3. **An application.** By the hypothesis H_4^{n-1} of our induction it follows from (8) that $f_1(x)$ belongs to $\mathfrak{p}_1$. Hence $f_1(\xi) = 0$, that is, $ad_s = 0$, in contradiction with $a \neq 0$ and $d_s \neq 0$. This completes the proof of H_4^n.

We shall now proceed to the proof of the theorem stated in the introduction. Let R be a finite integral domain over K: $R = \text{K}[\xi_1, \xi_2, \cdots, \xi_n]$, and let K$'$ be the set of elements of R which are algebraic over K. It is clear that K′ is not only a ring but also a field, since if $\alpha \in \mathrm{K}'$ then $\mathrm{K}(\alpha) = \mathrm{K}[\alpha]$. Since we have $R = \mathrm{K}'[\xi_1, \xi_2, \cdots, \xi_n]$, we shall replace K by K′. We assume therefore that K *is maximally algebraic* in R (that is, $\mathrm{K} = \mathrm{K}'$).

LEMMA 2. K *is a maximal subfield of* R.

PROOF. Let $\mathfrak{p}$ be a fixed maximal prime ideal in R and let x be any element of R which does not belong to K. By H_n^* the field $R/\mathfrak{p}$ is an algebraic extension of K. Hence x satisfies a congruence of the form: $f(x) \equiv 0 \pmod{\mathfrak{p}}$, where f is a polynomial with coefficients in K, not all zero. Since $\mathfrak{p} \neq (1)$, $f(x)$ is not a unit in R. On the other hand since K is maximally algebraic in R and since $x \notin \mathrm{K}$, x is a transcendental over K. Hence $f(x) \neq 0$. This shows that $\mathrm{K}(x)$ is not contained in R. Since this is true for any element x of R which is not in K, the lemma follows.

COROLLARY. *If F is any subfield of* K *such that R is a finite integral domain over F, then the algebraic closure of F in R coincides with* K.

For the algebraic closure of F in R is contained in K and on the other hand it must be a maximal subfield of R, in view of Lemma 2.

We shall need the following generalization of Lemma 1:

LEMMA 3. *If an integral domain Ω is a finite ring extension of an integral domain R, then for any element α in Ω, $\alpha \neq 0$, there exists a corresponding element a in R, $a \neq 0$, having the following property: if the radical of an ideal $\mathfrak{A}$ in R does not contain a, then the radical of $\Omega\mathfrak{A}$ does not contain α.*

PROOF. Let Ω be an n-fold ring extension of R. We shall prove the lemma by induction with respect to n.

Case $n = 1$. We use the notation of the proof of Lemma 1 and we consider separately two cases, according as ω is transcendental or algebraic with respect to R.

(a) ω *is a transcendental over* R. We have $\alpha = g(\omega)$, where g is a polynomial with coefficients in R. Let a be the leading coefficient of g, $a \neq 0$. If $\mathfrak{A}$ is an ideal in R and if some power of α belongs to $\Omega \cdot \mathfrak{A}$, then the same power of a must belong to $\mathfrak{A}$. Hence this element a has the desired property.

(b) ω *is algebraic over* R. In this case also α is algebraic over R. Let $d_0\alpha^s + d_1\alpha^{s-1} + \cdots + d_s = 0$, $d_i \in R$, $d_s \neq 0$, be an equation of algebraic dependence for α over R. If some power α^σ of α belongs to $\Omega \cdot \mathfrak{A}$, then also d_s^σ belongs to $\Omega \cdot \mathfrak{A}$, since α divides d_s in Ω. It follows from (3), for $z = d_s^\sigma$, that $d_s^\sigma b_0^\rho \in \mathfrak{A}$, where ρ is a suitable integer. Therefore if some power of α belongs to $\Omega\mathfrak{A}$ then some power of

$d_s b_0$ belongs to $\mathfrak{A}$. Consequently the element $a = d_s b_0$ has the desired property.

Case $n > 1$. Let $\Omega = R[\xi_1, \xi_2, \cdots, \xi_n]$, $\Omega_1 = R[\xi_1, \xi_2, \cdots, \xi_{n-1}]$, so that $\Omega = \Omega_1[\xi_n]$. By the case $n = 1$, the lemma is applicable if we replace R by Ω_1. Let α_1 be an element of Ω_1 which plays now the role of a. By the case $n - 1$, the lemma is applicable if we replace Ω by Ω_1 and α by α_1. This yields an element a of R, $a \neq 0$, which has the desired property in relation to the rings R, Ω and the given element α of Ω. The proof of Lemma 3 is now complete.

We now go back to our original finite integral domain $R = \mathrm{K}[\xi_1, \xi_2, \cdots, \xi_n]$, where we assume that K is maximally algebraic in R. We have to show that if F is a field contained in R then F is already contained in K. Let x be an arbitrary element of R which does not belong to K. We apply Lemma 3 to the two integral domains $\mathrm{K}[x]$ and R (these two rings now play respectively the role of the rings R and Ω of the lemma). We take for α the element 1 of R (nevertheless it is clear that our inductive proof of Lemma 3 could not have been carried out if we had restricted ourselves to the special case $\alpha = 1$). In this special case, Lemma 3, and the fact that x is a transcendental over K, yield the following result: there exists a polynomial $g(x)$ in $\mathrm{K}[x]$ such that any irreducible polynomial in $\mathrm{K}[x]$ which does not divide $g(x)$ is a nonunit in R. Now let Δ be the intersection of the two fields K and F (the two fields have in common at least the prime subfield of K). We can certainly find a polynomial in $\Delta[x]$ which is relatively prime to $g(x)$ in $\mathrm{K}[x]$. If $f(x)$ is such a polynomial, then by the preceding result $f(x)$ is a nonunit in R. Consequently $f(x)$ does not belong to the *field* F, and since the coefficients of $f(x)$ are in F *it follows that* x *is not in* F. Since x was an arbitrary element of R not belonging to K, we conclude that $F \subseteq \mathrm{K}$, q.e.d.

From the theorem just proved and from the corollary to Lemma 2 we draw immediately the following consequence:

If R is a finite integral domain over a field K *and if F is any subfield of R such that R is a finite integral domain also over F, then R has the same degree of transcendency over* K *as it has over F.*

This result is of course also a consequence of the fundamental theorem in the dimension theory of ideals in finite integral domains: *if r is the degree of transcendency of R over* K *then every minimal prime ideal in R is of dimension* $r - 1$. On the other hand, the above result gives an a priori reason for the fact that when the dimension of a prime ideal in R is defined *relative to* K *as ground field*, the resulting dimension theory is intrinsically related to R.

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ON THE STRUCTURE OF ALGEBRAS WITH NONZERO RADICAL

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Introduction. If one attempts to make a systematic study of algebras with nonzero radical one soon realizes that the main difficulty is that of singling out a set of structural characteristics which could constitute a suitable center of interest in a general theory. In the study of simple and semisimple algebras, the full matrix algebras over the groundfield serve as models for the "perfect" structure, and the greatest departure from this which one has to consider consists in the ground field being replaced by a division algebra. On the other hand, even among the nilpotent algebras, it would be difficult to decide what type is to be regarded as representing the "perfect" structure.

The first type that one might think of in this connection is the following: There is a single element, x, with $x^{n+1}=0$, for some positive integer n, whose powers $x, x^2, \cdots, x^n$ constitute a linear basis for the algebra. However, this is evidently far too special a type to serve as a standard. A natural generalization of this, which turns out to be more suitable, is obtained by replacing the one-dimensional subspace, (x), by a subspace of arbitrary dimension. With suitable additional requirements in the case of non-nilpotent algebras, we are led to the notion of a "quasicyclic" algebra whose radical has a considerably more transparent structure than a general nilpotent algebra.

We also introduce a more restrictive notion, that of a "maximal" algebra, and it will be shown that every algebra whose quotient by the radical is separable or $\{0\}$ is the homomorphic image of a "related" maximal (quasicyclic) algebra. Since the structure of maximal algebras is determined completely by separable algebras and their representations our structure theory decomposes into two parts: The study of two-sided ideals contained in the radical of a maximal algebra, and the study of separable algebras and their representations.

Throughout this paper we shall deal only with algebras whose quotients by the radical are either $\{0\}$ or separable. In particular, this will be the case for all algebras over a perfect field. It is to be noted that in this approach there is no need for a general representation theory of non-semisimple algebras.

1. **Related algebras.** We consider algebras B, with radical R, such that B/R is separable or $\{0\}$. Two such algebras, B_1 and B_2, are said

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to be related if:

(1) $B_1/R_1^2 \approx B_2/R_2^2$.

(2) The indices of nilpotency of R_1 and R_2 are the same.

By the index of nilpotency of a nilpotent algebra, R, we shall mean the least non-negative integer k for which $R^{k+1} = \{0\}$.

By Wedderburn's structure theorem,[1] we know that B contains a subalgebra, A, which is mapped isomorphically onto B/R by the natural homomorphism of B onto B/R. Then—as a vector space—B is the direct sum of A and R, and for any such decomposition the multiplication in B induces the structure of a two-sided B/R-module[2] in B. Moreover, R^2 is a submodule of R, and the induced B/R-module structure in R/R^2 is determined uniquely by B, or even by B/R^2. Related algebras determine isomorphic modules R/R^2. If we denote by T a model for the A-module R/R^2 and write the linear transformations in T which correspond to an element $a \in A$ as $t \to a \cdot t$ and $t \to t \cdot a$, we obtain a model for B/R^2 whose underlying vector space is the direct sum (A, T) of A and T, and where multiplication is defined by the formula

$$(a_1, t_1)(a_2, t_2) = (a_1a_2, a_1 \cdot t_2 + t_1 \cdot a_2).$$

For, if $\bar{\sigma}$ is an operator isomorphism of T onto R/R^2, the mapping $(a, t) \to \bar{a} + \bar{\sigma}\{t\}$, where $\bar{a}$ is the coset mod R^2 of a in B/R^2, is evidently an isomorphism of this algebra onto B/R^2.

Later, we shall construct a maximal algebra related to B of which B is a homomorphic image by suitably "enlarging" the radical $(0, T)$ of this algebra. For this construction we require a few auxiliary notions and a decomposition theorem for two-sided modules. These will be given in the next two sections.

2. A**-modules.** Let A be an algebra over the field F, T a vector space over F. Let σ be a homomorphism of A into the algebra of linear transformations of T, σ^* an antihomomorphism of A into this algebra. If, for all $a_i \in A$, we have $\sigma\{a_1\}\sigma^*\{a_2\} = \sigma^*\{a_2\}\sigma\{a_1\}$ we say that the set (A, T, σ, σ^*), or, simply, T is a (two-sided) A-module. If we write $(\sigma\{a\})\{t\} = a \cdot t$, $(\sigma^*\{a\})\{t\} = t \cdot a$, we may express our conditions by saying that the $\cdot$ operations satisfy all the formal requirements for a multiplication. We define submodules and quotient modules in the usual manner. A module M is said to be semisimple if to every submodule K we can find a complementary submodule L such that M is the direct sum of K and L. M is called simple if $\{0\}$

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[1] See, for instance, N. Jacobson, *The theory of rings*, Mathematical Surveys, vol. 2, chap. 5, §20.

[2] Cf. §2 of this paper.

and M are its only submodules. We shall require the following theorem:

THEOREM 2.1. *If A is separable then every two-sided A-module is semisimple.*

This is an easy generalization of the well known corresponding theorem concerning semisimple algebras and their representations, that is, their left modules.[3] In fact, denote by A^* the algebra obtained by adjoining a new identity element, 1, to A. Let e be the identity element of A. Then we have $(1-e)A=\{0\}=A(1-e)$, and $(1-e)^2=1-e$, whence we see that A^* is the direct sum of A and the ground field F. Hence A^* is separable. Now let $\tilde{A}^*$ be the algebra inversely isomorphic with A^*. Then $\tilde{A}^*$ is evidently separable also, and hence, by well known results,[4] the Kronecker product $A^*\times\tilde{A}^*$ is separable, or, in particular, semisimple. Hence every left $A^*\times\tilde{A}^*$-module is semisimple. Now if M is any A-module we can define in M the structure of a left $A^*\times\tilde{A}^*$ module by setting $1\cdot m=m=m\cdot 1$, for every $m\in M$, and $(a_1^*\times\tilde{a}_2^*)\cdot m=a_1^*\cdot m\cdot a_2^*$. The statement that this left $A^*\times\tilde{A}^*$-module is semisimple is then evidently equivalent to the statement that the given two sided A-module M is semisimple. (Because of the 1, there is a 1-1 correspondence between two-sided submodules and one-sided submodules of M.)

3. **Remarks on Kronecker products.** It will be convenient to review some basic facts concerning Kronecker products of vector spaces. Let $L_1, \cdots, L_n$ be a set of (finite-dimensional) vector spaces over a field F. We denote by $M(L_1, \cdots, L_n)$ the vector space formed by the n-linear functions on the direct sum of these spaces, that is, by the functions f mapping $(L_1, \cdots, L_n)$ into F in such a way that f is linear on each L_i when the coordinates in the remaining L_j are kept fixed. By the Kronecker product, $L_1\times \cdots \times L_n$, of the L_i is meant the vector space dual to $M(L_1, \cdots, L_n)$, that is, the vector space whose elements are the linear mappings of $M(L_1, \cdots, L_n)$ into F. If $(z_1, \cdots, z_n)\in(L_1, \cdots, L_n)$ we obtain an element $z_1\times\cdots\times z_n$ in $L_1\times\cdots\times L_n$ by setting $(z_1\times\cdots\times z_n)\{f\}=f\{z_1, \cdots, z_n\}$, for every $f\in M(L_1, \cdots, L_n)$. Evidently, the mapping $(z_1, \cdots, z_n)\rightarrow z_1\times\cdots\times z_n$ is n-linear. Moreover, the images $z_1\times\cdots\times z_n$ span $L_1\times\cdots\times L_n$ over F. It follows that the mapping $(z_1\times\cdots\times z_k, z_{k+1}\times\cdots\times z_l)\rightarrow z_1\times\cdots\times z_l$ induces a bilinear mapping of $(L_1\times\cdots\times L_k, L_{k+1}\times\cdots\times L_l)$ into $L_1\times\cdots\times L_l$.

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[3] N. Jacobson, loc. cit. chap. 4, §12.

[4] See van der Waerden, *Moderne Algebra*, 1st ed., vol. 2, §119.

Moreover, this mapping is distributive. If $u \in L_1 \times \cdots \times L_k$, and $v \in L_{k+1} \times \cdots \times L_l$, we shall denote the image of (u, v) under the above mapping by $u \times v$. Then it is easy to see that $(u \times v) \times w = u \times (v \times w)$. Thus the mapping $(u, v) \rightarrow u \times v$ behaves like a multiplication. Indeed, we shall use this operation later in order to define a multiplication in a vector space which is the direct sum of Kronecker products.

4. **Maximal related extensions and quasicyclic algebras.** Let B be an algebra, R the radical of B. We say that B is quasicyclic if the following conditions are satisfied:

(1) B contains a subalgebra A which is mapped isomorphically onto B/R by the natural homomorphism of B onto B/R, so that R becomes an A-module in the natural fashion.

(2) R, as an A-module (two-sided), admits of a direct decomposition into submodules $R_1, \cdots, R_n$, where R_k is the ordinary kth power of R_1, and n is the index of nilpotency of R.

If B is an algebra and β is a homomorphism of B onto an algebra A, the pair (B, β) is called an extension of A. (B, β) is called a related extension of A if B and A are related. A is called maximal if, in every related extension (B, β) of A, β is an isomorphism. A related extension (B, β) of A is called maximal if B is maximal.

Our main result is the following:

THEOREM 4.1. *Let B be an algebra with radical R such that B/R is separable or $\{0\}$. Then there exists a maximal related extension (C, γ) of B, such that C is quasicyclic. If (C_i, γ_i) is a maximal related extension of B_i, $i = 1, 2$, and if B_1 and B_2 are related, then C_1 and C_2 are isomorphic.*

PROOF. Let A be a subalgebra of B which is mapped isomorphically onto B/R by the natural homomorphism of B onto B/R. Now consider the A-module R, and the submodule R^2. By Theorem 2.1, there exists a complementary submodule T such that R, as an A-module, is the direct sum of T and R^2. Let n be the least integer k such that $R^{k+1} = \{0\}$. (We may assume $n \geqq 1$.) We form the Kronecker products $T^{(k)}$ of T by itself k times, $k = 1, \cdots, n$. We can make each $T^{(k)}$ into an A-module by setting $a \cdot (t_1 \times \cdots \times t_k) = at_1 \times \cdots \times t_k$, and $(t_1 \times \cdots \times t_k) \cdot a = t_1 \times \cdots \times t_k a$, where at_1 and $t_k a$ stand for the products in B, which lie in T, by our construction of T. We consider the direct sum, S, of these $T^{(k)}$. There is a unique linear mapping σ of S into R such that $\sigma\{t_1 \times \cdots \times t_k\} = t_1 \cdots t_k$. Evidently, σ is an operator homomorphism of the A-module S onto the A-module R. The fact that σ is onto follows from the fact that every $r \in R$ can be written as a sum of products of elements in T. Now we make S into a nilpotent algebra related to R by means of the Kronecker multiplication discussed in §3. More precisely, for $S_{k_1} \in T^{(k_1)}$ and $S_{k_2} \in T^{(k_2)}$, we set $S_{k_1} * S_{k_2} = S_{k_1} \times S_{k_2}$, when $k_1+k_2 \leqq n$, and 0 otherwise. Then σ is evidently a homomorphism of the algebra S onto the algebra R. This still requires modification:

Let U_2 be the subspace of $T^{(2)}$ which is spanned by the elements of the form $t_1 a \times t_2 - t_1 \times a t_2$, with $a \in A$, $t_1, t_2 \in T$. If U_k is already defined as a subspace of $T^{(k)}$ we define U_{k+1} as the subspace of $T^{(k+1)}$ which is spanned by $T \times U_k$ and $U_k \times T$. Evidently, each U_k is a submodule of $T^{(k)}$, and is contained in the kernel of σ. Now we can choose a submodule V_k of $T^{(k)}$ which is complementary to U_k. Denote the projection of $T^{(k)}$ onto V_k by π_k. Write $T = V_1$, and let $E = V_1 + \cdots + V_n$ (this sum is direct). We define a multiplication in E by setting, for $z_k \in V_k$, $z_i \circ z_j = \pi_{i+j}\{z_i \times z_j\}$, when $i+j \leqq n$, and 0 otherwise. This is evidently bilinear. Moreover, since $\pi_{i+j}\{z_i \times z_j\} \times z_k - z_i \times \pi_{j+k} \cdot \{z_j \times z_k\} \in U_{i+j+k}$ when $i+j+k \leqq n$, we have always $(z_i \circ z_j) \circ z_k = z_i \circ (z_j \circ z_k)$. Thus E has the structure of a nilpotent algebra. The mapping σ induces a homomorphism of the algebra E onto the algebra R; in fact, $\sigma\{z_i \circ z_j\} = \sigma\{z_i * z_j\}$. Since the index of nilpotency of E is not greater than n, by construction, and since $\sigma\{E^n\} = R^n$, it follows that the index of nilpotency of E is exactly n. Evidently, $E/E^2 \approx T \approx R/R^2$, whence E is related to R. Finally, σ induces an operator homomorphism of the A-module E onto the A-module R.

Now we construct an algebra, C, out of the direct sum (A, E) of the vector spaces of A and E by setting

$$(a_1, e_1)(a_2, e_2) = (a_1 a_2,\ a_1 \cdot e_2 + e_1 \cdot a_2 + e_1 \circ e_2).$$

Since E is an A-module and since, evidently, $a \cdot (e_1 \circ e_2) = (a \cdot e_1) \circ e_2$, and $(e_1 \circ e_2) \cdot a = e_1 \circ (e_2 \cdot a)$, the associativity condition reduces to $(e_1 \cdot a) \circ e_2 = e_1 \circ (a \cdot e_2)$. This is satisfied in virtue of our definitions for the U_k and π_k. If we define $\gamma\{(a, e)\} = a + \sigma\{e\}$, we see immediately that γ is a homomorphism of C onto B. Moreover, γ induces an isomorphism of $C/E^2 \approx (A, T)$ onto B/R^2. Hence C is related to B. Evidently, C is quasicyclic. For convenience of reference, let us call a related extension (C, γ) of B, which is obtained by a construction such as we have just described, a standard related extension of B.

Now let B_1 and B_2 be any two related algebras with radicals R_1 and R_2, respectively. Let ρ be an isomorphism of B_1/R_1^2 onto B_2/R_2^2. Let (C_1, γ_1) and (C_2, γ_2) be standard related extensions of B_1 and B_2, respectively. We wish to show that C_1 and C_2 are isomorphic:

We have, in a notation paralleling the above, $B_i = (A_i, R_i)$, and

$C_i=(A_i, E_i)$, where A_i is a subalgebra of B_i which is mapped isomorphically onto B_i/R_i by the natural homomorphism of B_i onto B_i/R_i. Since $A_i \approx B_i/R_i$, ρ induces an isomorphism λ of A_1 onto A_2 and an isomorphism μ of $T_1 \approx R_1/R_1^2$ onto $T_2 \approx R_2/R_2^2$. For $a \in A_1$, $t \in T_1$, we shall have $\mu(at)-\lambda(a)\mu(t) \in E_2^2$ and $\mu(ta)-\mu(t)\lambda(a) \in E_2^2$, whence $\mu(at)=\lambda(a)\mu(t)$ and $\mu(ta)=\mu(t)\lambda(a)$. Evidently, μ may be extended in a unique fashion to a linear isomorphism $\hat{\mu}$ of $(T_1, \cdots, T_1^{(n)})$ onto $(T_2, \cdots, T_2^{(n)})$, such that, for $t_j \in T_1$,

$$\hat{\mu}(t_1 \times \cdots \times t_k) = \mu(t_1) \times \cdots \times \mu(t_k).$$

Clearly, $\hat{\mu}$ maps each $U_{k,1}$ onto $U_{k,2}$, where the $U_{k,i}$ are the subspaces of the $T_i^{(k)}$, defined as were the U_k in our construction above. It follows that $\hat{\mu}$ induces an isomorphism $\bar{\mu}$ of E_1 onto E_2, such that $\bar{\mu}(z)-\hat{\mu}(z) \in U_{k,2}$, for $z \in T_1^{(k)}$. From this and the above it follows that $\bar{\mu}(a \cdot e)=\lambda(a) \cdot \bar{\mu}(e)$, and $\bar{\mu}(e \cdot a)=\bar{\mu}(e) \cdot \lambda(a)$, for all $e \in E_1$ and $a \in A_1$. Finally, the mapping $(a, e) \rightarrow (\lambda(a), \bar{\mu}(e))$ is evidently an isomorphism of C_1 onto C_2.

Now let (C, γ) be a standard extension of B. Let (D, δ) be any related extension of C, and construct a standard related extension (C^*, γ^*) of D. Then, since D and B are related, it follows from what we have just proved that C^* is isomorphic with C. Hence the homomorphism $\delta\gamma^*$ of C^* onto C must be an isomorphism. A fortiori, δ is an isomorphism. Hence C is maximal. This completes the proof of our theorem.

We obtain immediately the following corollary:

COROLLARY. *Every maximal algebra is quasicyclic.*

5. Some illustrative remarks.

It is interesting that some algebras whose structure one would regard as nondegenerate on purely intuitive grounds are, indeed, quasicyclic or even maximal. Thus, the algebra formed by all n by n matrices with zero coordinates above the main diagonal is maximal. The nilpotent algebra of matrices with zero coordinates on and above the main diagonal is quasicyclic, though evidently not maximal.

A simple example of an algebra which is not quasicyclic is the following: Let N be the maximal nilpotent algebra generated by an element x with $x^7=0$, such that the set of powers $(x, x^2, \cdots, x^6)$ forms a linear basis for N. Let Z be the subalgebra generated by x^2 and x^3. Then Z has a linear basis $(x^2, x^3, x^4, x^5, x^6)$, Z^2 has a basis (x^4, x^5, x^6), while Z^3 is the one-dimensional vector space spanned by x^6. If $Z=Z_1+Z_1^2+Z_1^3$ were a quasicyclic decomposition of Z, then Z_1 would have to contain an element of the form $x^3+\alpha x^4+\beta x^5+\gamma x^6$. Then Z_1^2

would contain x^6, that is, $Z_1^2 \supseteq Z^3$, which gives a contradiction. Hence Z is not quasicyclic.

6. **An application.** We shall apply the above methods to a particularly simple case in which a complete result can be obtained.

Let B be an algebra over the field F, and let R be the radical of B. We consider the case in which B/R is a central simple algebra over F, that is, B/R is a full matrix ring of degree d (say) with coordinates in a division algebra Φ over F, such that F is the center of Φ. Then there exists a subalgebra A of B which is mapped isomorphically onto B/R by the natural homomorphism of B onto B/R, and R is a two-sided A-module in the natural fashion. In addition to the above we shall assume that the quotient module R/R^2 is simple, and that it is not annihilated by A on either side. (This implies, in particular, that the identity element of A acts as the identity transformation in R/R^2, both on the right and on the left.) Let $\tilde{A}$ be the algebra anti-isomorphic with A. Then, by well known results,[5] $A \times \tilde{A}$ is simple (in fact, it is a full matrix algebra over F), and, as in the proof of Theorem 2.1, we may regard R/R^2 as a nontrivial simple left $A \times \tilde{A}$-module. Now the algebra A itself may be regarded in the natural way as such a simple left $A \times \tilde{A}$ module, and since any two nontrivial simple left modules for a simple algebra are isomorphic, we may conclude that there exists an isomorphism λ of R/R^2 onto A (as a vector space over F) such that $\lambda\{a \cdot r\} = a\lambda\{r\}$, and $\lambda\{r \cdot a\} = \lambda\{r\}a$, for every $a \in A$, and every $r \in R/R^2$. Therefore, in constructing a maximal related extension of B we may take for the space T of §4 a copy, $A^{(1)}$, of A, and we shall have $a_1 \cdot a_2^{(1)} = (a_1a_2)^{(1)}$; $a_2^{(1)} \cdot a_1 = (a_2a_1)^{(1)}$, where we indicate by $a \to a^{(1)}$ a fixed isomorphism of A onto $A^{(1)}$.

Next, we shall have to compute the subspace U_2 of $T \times T$. For this purpose, we may as well use the isomorphic space $A \times A$; U_2 will be the subspace of $A^{(1)} \times A^{(1)}$ which corresponds to the subspace U_2' of $A \times A$ spanned by all elements of the form $a_1a_2 \times a_3 - a_1 \times a_2a_3$. For a moment, it will be convenient to identify A with Φ_d, and to introduce the usual matrix units e_{ij}; $1 \leqq (i, j) \leqq d$. Thus, each element of A may be written $a = \sum_{i,j} \phi_{ij}e_{ij}$, with $\phi_{ij} \in \Phi$, and we have $(\phi e_{ij})(\phi' e_{pq}) = \delta_{jp}\phi\phi' e_{iq}$, where δ_{jp} stands for the usual Kronecker symbol. Now we have

$$e_{ij}e_{rs} \times e_{pq} - e_{ij} \times e_{rs}e_{pq} = \delta_{jr}e_{is} \times e_{pq} - \delta_{sp}e_{ij} \times e_{rq}.$$

For $j = r$ and $s \neq p$ this gives $e_{is} \times e_{pq}$, and for $j = r$ and $s = p$ we get $e_{is} \times e_{sq} - e_{ir} \times e_{rq}$. It follows that every element of $A \times A$ is congruent mod U_2' to a sum of elements of the form $\phi e_{i1} \times \phi' e_{1j}$. Furthermore, U_2' contains all elements of the form

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[5] N. Jacobson, loc. cit. chap. 5, §13.

$$\phi_1\phi_2 e_{i1} \times \phi_3 e_{1j} - \phi_1 e_{i1} \times \phi_2\phi_3 e_{1j}.$$

Take $\phi_1 = \phi\phi'$, $\phi_2 = (\phi')^{-1}$, $\phi_3 = \phi'$; then we see that U_2' contains $\phi e_{i1} \times \phi' e_{1j} - \phi\phi' e_{i1} \times e_{1j}$. Hence every element of $A \times A$ is congruent mod U_2' to a sum of elements of the form $\phi e_{i1} \times e_{1j}$.

Now consider the mapping $a_1 \times a_2 \rightarrow a_1 a_2$. This induces a bilinear mapping of $A \times A$ into A which evidently annihilates U_2'. Hence, if $\sum_{i,j} \phi_{ij} e_{i1} \times e_{1j} \in U_2'$, it follows that $\sum_{i,j} \phi_{ij} e_{ij} = 0$, that is, $\phi_{ij} = 0$. Therefore, the above mapping induces an isomorphism of $A \times A / U_2'$ onto A. Since $(\phi e_{i1} \times e_{1j}) \cdot \phi' e_{pq} = \phi e_{i1} \times \phi' e_{1j} e_{pq} \equiv \phi\phi' e_{i1} \times e_{1j} e_{pq} (U_2')$, and similar relations hold for the left operations, this isomorphism is actually an operator isomorphism. Therefore, we may identify V_2 (cf. §4) with a copy, $A^{(2)}$, of A, with the same relations as we had obtained above for T and $A^{(1)}$. Almost identical computations show that each V_k may be identified with a copy $A^{(k)}$ of A.

It follows that, if (C, γ) is a maximal related extension of B, C is isomorphic with the algebra obtained as follows: Let $A^{(0)}, A^{(1)}, \cdots, A^{(n)}$ be $(n+1)$ copies of A, where n is the index of nilpotency of R. Define $a_1^{(r)} \circ a_2^{(s)} = (a_1 a_2)^{(r+s)}$, for $r + s \leqq n$, and 0 otherwise, and extend this by linearity to give the structure of an algebra over the direct sum of the vector spaces, $A^{(0)}, \cdots, A^{(n)}$. (Evidently, the radical of C is $A^{(1)} + \cdots + A^{(n)}$, and we have $A^{(k)} = (A^{(1)})^k$.) A still simpler description of this algebra, C, is obtained by noting that C can be obtained from the polynomial ring $A[x]$ by dividing out the two-sided ideal generated by x^{n+1}.

Now let I be any nonzero two-sided ideal which is contained in the radical $A^{(1)} + \cdots + A^{(n)}$. Then the projections $I^{(k)}$ of I onto the $A^{(k)}$ are evidently submodules of the A-module $A^{(1)} + \cdots + A^{(n)}$. Let k_0 be the least k for which $I^{(k)} \neq \{0\}$. Since $A^{(k_0)}$ is a simple module we must have $I^{(k_0)} = A^{(k_0)}$. Now $A^{(n-k_0)} \circ I^{(k_0)} \subseteq I$. But $A^{(n-k_0)} \circ I^{(k_0)} = A^{(n)}$. Hence we have $I \supseteq A^{(n)}$. We may now consider $I/A^{(n)}$ in $C/A^{(n)}$ and conclude that $I \supseteq A^{(n-1)}$, unless $k_0 = n$, and so on. Finally, we conclude that $I = A^{(k_0)} + \cdots + A^{(n)}$. But this shows that the homomorphism γ of C onto B must be an isomorphism, for otherwise we could take for I the kernel of γ and conclude that $B \approx C/A^{(k_0)}$, which contradicts the assumption that the index of nilpotency of R is n. Hence every algebra such as B is maximal and has the same simple structure as the algebra C which we have constructed.

In the general case, it is not difficult to construct the maximal algebras, starting from a decomposition of the A-module R/R^2 into a direct sum of simple submodules. The real difficulties are met in the attempt to enumerate the two-sided ideals which are contained in the radical.

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ON A CONJECTURE ABOUT INFINITE CLASS FIELDS

GEORGE WHAPLES

If we are given any algebraic extension field, of finite degree, of a given ground field, then the p-adic completion of the extension field, under any one of its valuations[1] (prime spots) is an algebraic extension of the completion of the ground field under the same valuation. Our original extension field (in the large) thus determines a set of algebraic extensions of p-adic ground fields. We shall refer to these extensions as the *local components* of the original field. If our extension field (in the large) is normal, then any two valuations of the extension field which induce the same valuation in the ground field determine isomorphic local components; hence in case of a normal extension field we can think of a local component as determined by a valuation of the ground field.

When our extension field is not of finite degree we must modify this definition, since the p-adic closure of such an extension field will in general not be algebraic over the ground field.[2] For a normal extension of infinite degree we define the local component as follows: The original extension is the splitting field[3] of a certain set of polynomials with coefficients in the ground field. Define the local component to be the splitting field of this same set of polynomials over the p-adic extension of the ground field. It is easy to show that this field is independent of the set of polynomials used (indeed, one could use the set of all polynomials of the ground field which split in the extension

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[1] For theory of valuations, see E. Artin and G. Whaples, *Axiomatic characterization of fields by the product formula for valuations*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 469–492, and the literature cited there.

[2] See Ostrowski, *Über einige Fragen der allgemeine Körpertheorie*, Journal fur Mathematik vol. 143 (1914) pp. 225–284. I am indebted to the referee for a correction of the first version of this paper and for this reference.

[3] The *splitting field* is the smallest subfield of the algebraic closure in which all the given polynomials split into linear factors.

field) and that it is equal to the product[4] of the local components (in the original sense) of all subfields of finite degree of our original field.

We restrict ourselves here to the case where the ground field is of finite degree over the rational field and the extension field is normal. It is known that when the extension field is of finite degree, it is completely determined by its local components.[5] To prove this, take any two non-isomorphic extension fields, consider one of them as ground field and their product as extension field, and apply the theorem: In any extension field there are an infinite number of primes of ground field which do not split completely.

It has been conjectured[6] that the same thing is true for abelian extensions, even when they are of infinite degree. (One sees that the above proof does not work, for the field which would have to be considered as ground field would now be of infinite degree.) The purpose of this note is to disprove the conjecture. First, we observe that the conjecture is equivalent to the seemingly stronger statement: *If one abelian extension includes another everywhere locally, it includes it also in the large*; that is, if we are given two abelian extensions (of infinite or finite degree) of a fixed ground field, such that at every valuation of the ground field the local component of the first includes (in sense of isomorphism) the local component of the second, then the first field includes the second. To show that the second conjecture follows from the first, take two abelian extension fields which satisfy the assumption that the local component of first field everywhere includes that of second. Form the product of the two fields. The local component of the product field is everywhere the *same* as that of first field; hence the original conjecture would give: product field equals first field—hence second field is subfield of first.[7]

However, both conjectures are false; in fact, given any finite abelian extension field, an infinite abelian extension can be constructed which includes it everywhere locally, but not in the large. Namely, let a finite abelian extension field be given. Arrange the prime spots of the ground field in some definite order. Our infinite extension field will be the product of a set of constituent fields, where one constituent corresponds to each prime spot of the ground field. The constituent corresponding to a given prime spot can be any finite abelian extension satisfying the conditions:

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[4] The *product* of a set of fields is the smallest subfield of the algebraic closure which includes them all.

[5] This follows from a theorem of M. Bauer, *Über einen Satz von Kronecker*, Archiv der Mathematik und Physik vol. 6 (1904) pp. 218–219. For a modern, non-analytic proof of this theorem see Chevalley, *La théorie du corps de classes*, Ann. of Math. vol. 41 (1940) pp. 394–418.

[6] C. Chevalley, *Généralisation de la théorie de corps de classes pour les extensions infinies*, Journal de Mathématiques pures et appliquées (9) vol. 15 (1936) pp. 359–371.

[7] The author owes thanks to Professor Chevalley for some simplification of this part of the argument.

(a) Its local component includes the local component of the given field at the prime spot to which it corresponds.

(b) Each of its cyclic subfields is ramified at some place at which all the preceding fields (that is, all the fields corresponding to preceding primes) are unramified.

Grunwald's theorem[8] shows that such constituent fields can be constructed. (b) can be achieved by making a product of cyclic extension fields, each of which is totally ramified at at least one place where all the fields already constructed are unramified. Condition (a) implies that the product field includes the given field everywhere locally. But we shall show that (b) implies that every cyclic subfield of our product is ramified at some place where the given field is unramified; hence given field cannot be subfield.

The proof is best carried out by means of group characters. Clearly we can replace our infinite product field by a product of a finite number of the constituents, since every cyclic subfield of the field we have constructed is contained in such a product. (The introduction of this finite subfield could be avoided by using directly the characters of the whole field; yet it seems simpler this way.)

Consider, then, any extension of our ground field which is formed by taking the product of a finite number of fields satisfying (a), (b) above. Its Galois group is abelian and finite. Each character of the Galois group determines a cyclic subfield under the correspondence: character goes into subfield of elements left fixed by the subgroup on which the character takes value 1; and every cyclic subfield is determined by some character. Call a character ramified (at a certain prime) if the subfield which it determines is ramified there. Clearly a character is ramified (at any spot) exactly when it is not identically 1 on the ramification group (at that spot); so it follows that, at any fixed prime spot, a product of a ramified character and an unramified character is ramified.

The characters of the constituent fields, that is, the characters which determine the fields described in (b), are evidently a basis for the group of all characters of the finite abelian group of our field.

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[8] W. Grunwald, *Ein allgemeines Existenztheorem für algebraische Zahlkörper*, Journal für Mathematik vol. 169 (1932) pp. 103–107. For a nonanalytic proof see Whaples, *Nonanalytic class field theory and Grunwald's Theorem*, Duke Math. J. vol. 9 (1942) pp. 455–473.

(Indeed, condition (b) implies that this group is simply the direct product of the groups of the constituent fields.) Any character can be expressed as a product of characters determining cyclic subfields of the constituent fields; in this product, all characters belonging to the same constituent can be united into a single one, and that character which is not identically 1 and belongs to the latest of the constituents will be ramified at some spot where all the others are unramified, and where our given field is unramified.

UNIVERSITY OF WISCONSIN

SEQUENCES OF IDEAL SOLUTIONS IN THE TARRY-ESCOTT PROBLEM

H. L. DORWART

1. **Introduction.** In the Tarry-Escott problem (sometimes called the problem of multi-degree equalities, or of equal sums of like powers), one seeks integral solutions of the k equations

$$\sum_{i=1}^{s} x_i^l = \sum_{i=1}^{s} y_i^l, \qquad l = 1, 2, \cdots, k. \tag{1}$$

The usual notation is to represent a solution of (1) by

$$a_1, \cdots, a_s \overset{k}{=} b_1, \cdots, b_s. \tag{2}$$

Such a solution is called *trivial* if the a's form a permutation of the b's, and will be called *semi-trivial* if any $a_i = a_j$ or $b_i = b_j$. A solution is said to be in *reduced* form when $\sum a_i = 0$, $(a_i, b_i) = 1$, and solutions having the same reduced form are called *equivalent* solutions.

It is easily shown that for nontrivial solutions, $s > k$. The case $s = k+1$, called the *ideal* or *optimum* case [1, 2],[1] is of particular interest in many applications [5]. For a given k, $N(k)$ is defined as the least value of s for which (1) has nontrivial solutions. It is known in general [7] that $N(k) \leqq k(k+1)/2$, but numerical examples [6] give $N(k) = k+1$ for $k = 1, 2, \cdots, 9$.

In order to decrease the number of the equations (1), many writers have imposed the conditions

$$x_i = -y_i, \qquad i = 1, 2, \cdots, s, \text{ for } s \text{ odd}, \tag{3}$$

or

$$x_{s+1-i} = -x_i, \quad y_{s+1-i} = -y_i, \qquad i = 1, 2, \cdots, s/2, \text{ for } s \text{ even}. \tag{4}$$

Solutions of (1) subject to (3) or (4) will be called *symmetric* solutions. It is evident that the conditions for symmetry are sufficient to assure that symmetric solutions are reduced.

By use of the binomial theorem, one finds that (2) implies

$$Ma_1 + K, \cdots, Ma_s + K \overset{k}{=} Mb_1 + K, \cdots, Mb_s + K, \tag{5}$$

and

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$(6)\qquad a_1, \cdots, a_s, b_1 + h, \cdots, b_s + h \overset{k+1}{=} b_1, \cdots, b_s, a_1 + h, \cdots, a_s + h,$$

where M, K and h are any integers.

Equation (6) shows the existence of multi-degree equalities of arbitrarily high degree, but in general each time the degree is raised by one, the number of terms on each side of the equality is doubled. However, by taking for h a difference $|a_i - a_j|$ or $|b_i - b_j|$ which occurs t times, t terms on one side will cancel with the same number of terms on the other side. The set of all differences $|a_i - a_j|$ and $|b_i - b_j|$ will be called the *difference table* for a given equality.

If the process indicated in (6) is applied to a nontrivial ideal solution and leads to another nontrivial ideal solution, these ideal solutions will be said to be *in sequence*. Thus for a nontrivial ideal equality of degree k to be in sequence with one of degree $k+1$, the value used for h in (6) must occur exactly $2(k+1) - (k+2) = k$ times in the difference table for the ideal equality of degree k.

The following sequence of ideal solutions has been noted [5].

$$\begin{array}{lr} & 1, 9 \overset{1}{=} 4, 6, \\ h = 2, & 1, 8, 9 \overset{2}{=} 3, 4, 11, \\ h = 1, & 1, 5, 8, 12 \overset{3}{=} 2, 3, 10, 11, \\ h = 7, & 1, 5, 9, 17, 18 \overset{4}{=} 2, 3, 11, 15, 19, \\ h = 8, & 1, 5, 10, 18, 23, 27 \overset{5}{=} 2, 3, 13, 15, 25, 26. \end{array} \qquad (7)$$

It is the purpose of this paper to determine all sequences of ideal solutions which begin with degrees one, two and three (the nonsymmetric case for degree three in not completed). The appearance of certain substitution groups in the solutions, apparently not previously observed, will be noted. These groups are then used to simplify many of the computations which would otherwise be very tedious. Although the methods of the paper do not lead to new solutions, it is believed that the tabulation of complete information concerning sequences beginning with low degrees sheds new light on the structure of existing solutions.

2. Sequences beginning with the first degree. All reduced ideal solutions of first degree are of necessity in the symmetric form

$$a,\ -a \overset{1}{=} b,\ -b. \tag{8}$$

If we apply (6) with $h=2a$ ($h=2b$ gives similar results) and then (5) with $M=1$ and $K=-a$ to (8), there results

$$a+b,\ a-b,\ -2a \overset{2}{=} -a-b,\ -a+b,\ 2a. \tag{9}$$

The difference table for this equality contains $3a+b$, $3a-b$ and $2b$ each repeated two times. Applying (6) to (9) with $h=3a+b$, followed by (5) with $M=2$ and $K=-3a-b$, we have

$$\pm(7a+b,\ a+3b) \overset{3}{=} \pm(5a+3b,\ 5a-b). \tag{10}$$

The relationship of (10) and other solutions found in this and succeeding sections to results given in [2] will be pointed out later.

When the difference table for (10) is formed, it is found by equating all a_i-a_j with each other and with all b_i-b_j that for the same difference to occur exactly three times (and thus lead to another nontrivial ideal solution), a/b must have one of the following sixteen values: $3, 1/2, -5, 1/4, -1/4, -3/7, -1/6, -5/9, -3/11, -2/5, 1/7, -2, 2, 3/5, -2/3, -1/9$. Fourteen distinct ideal equalities of degree 1 are obtained from these, leading to only six distinct ideal equalities of degree 2, which in turn give rise to four ideal equalities of degree 3. These imply two ideal equalities of degree 4 and finally three ideal equalities of degree 5. Here the sequences end. The relationship of these qualities to each other is shown in the accompanying chart (p. 390).

The ideal equality

$$\pm(1,\ 11,\ 12) \overset{5}{=} \pm(4,\ 9,\ 13)$$

is the reduced form of the fifth degree equality in (7) and is of particular interest. Two applications of (6) to the last equality of (7) give

$$h=13,\quad 1, 5, 10, 16, 27, 28, 38, 39 \overset{6}{=} 2, 3, 13, 14, 25, 31, 36, 40,$$

$$h=11,\quad 1, 5, 10, 24, 28, 42, 47, 51 \overset{7}{=} 2, 3, 12, 21, 31, 40, 49, 50.$$

Thus we have an example of an equality of degree 6 which, although not itself ideal, leads to an ideal equality of degree 7 (see [2, p. 632 (C)]). This particular ideal equality of degree 7 was first found by Tarry and has been used as the basis of extensive calculations [8]. Until recently [6], it was not known if ideal equalities of degree greater than 7 existed. The best results for $N(k)$, $k>9$, are still given by numerical examples rather than by theoretical methods.

Application of (6) to (9) with $h=3a-b$ followed by (5) with $M=2$ and $K=-3a+b$ leads to

$$\pm\,(7a-b,\ a-3b) \overset{3}{=} \pm\,(5a-3b,\ 5a+b), \tag{11}$$

and with $h=2b$ followed by (5) with $M=1$ and $K=-b$ leads to

$$\pm\,(2a+b,\ a-2b) \overset{3}{=} \pm\,(2a-b,\ a+2b). \tag{12}$$

However, neither (11) nor (12) leads to further ideal solutions differing from those in the chart.

We now summarize the results of this section in the following theorem.

THEOREM 1. *Every reduced ideal equality of degree* 1 *is symmetric and implies an ideal equality of degree* 2 *of the form* (9), *which in turn implies three ideal equalities of degree* 3 *of the form* (10), (11) *and* (12). *The only sequences beginning with degree* 1 *which extend to degrees* 4 *and* 5 *are equivalent to those of the chart; none extends beyond degree* 5.

3. **Sequences beginning with the second degree.** The general ideal solution of second degree [3, p. 52] is

$$AG+BD,\quad AD,\quad BG \overset{2}{=} AD+BG,\quad AG,\quad BD, \tag{13}$$

where A, B, D, G are any integers. Let $A=\lambda B$, $G=\mu D$, and (13) becomes

$$\lambda\mu+1,\quad \lambda,\quad \mu \overset{2}{=} \lambda+\mu,\quad \lambda\mu,\quad 1, \tag{14}$$

where λ and μ are rational numbers. When (14) is put in reduced form by the application of (5) with $M=3$ and $K=-\lambda\mu-\lambda-\mu-1$, it is found that the conditions for trivial solutions are λ or $\mu=0,\ 1$, and for symmetric solutions are λ or $\mu=-1,\ 2,\ 1/2$.

The symmetric ideal solutions of degree two are all equivalent to

$$\lambda,\quad 1-\lambda,\quad -1 \overset{2}{=} -\lambda,\quad \lambda-1,\quad 1, \tag{15}$$

given directly by (14) with $\mu=-1$, with $\mu=2$ followed by (5) with $M=1$ and $K=-\lambda-1$, or with $\mu=1/2$ followed by (5) with $M=2$ and $K=-\lambda-1$. If $\lambda=(a+b)/2a$, (15) reduces to (9), which is therefore equivalent to the general symmetric solution of degree two. Also, we now see that there are no symmetric sequences which *begin* with degree two.

The fact that special cases of (14) are given by λ or $\mu=0$, 1 and -1, 2, 1/2 suggests a possible connection with the cross-ratio group G_6 of projective geometry. It is easy to show that (15) is invariant under the substitutions λ, $1-\lambda$, $(\lambda-1)/\lambda$, $1/\lambda$, $1/(1-\lambda)$, $\lambda/(\lambda-1)$ which form this group.

The difference table for (15) contains $\lambda+1$, $\lambda-2$, $2\lambda+1$ each twice, and the equality in λ which corresponds to (10) is

$$(10') \qquad \pm(3\lambda+1,\ \lambda-3) \stackrel{3}{=} \pm(3\lambda-1,\ \lambda+3).$$

It is evident that (10′) is invariant under the substitutions of the group λ, $-\lambda$, $1/\lambda$, $-1/\lambda$. The values of λ for which (10′) leads to an ideal equality of degree 4 (corresponding to the sixteen values of a/b given in the preceding section) are $\pm 2/3$, $\pm 3/2$, $\pm 2/5$, $\pm 5/2$, $\pm 3/4$, $\pm 4/3$, ± 4, $\pm 1/4$. Hence we now see why these sixteen values give only four distinct equalities of degree 3.

Corresponding to (11) and (12) are

$$(11') \qquad \pm(3\lambda-4,\ \lambda+2) \stackrel{3}{=} \pm(3\lambda-2,\ \lambda-4)$$

which is invariant under λ, $2-\lambda$, $\lambda/(\lambda-1)$, $(\lambda-2)/(\lambda-1)$, and

$$(12') \qquad \pm(4\lambda-1,\ 2\lambda-3) \stackrel{3}{=} \pm(4\lambda-3,\ 2\lambda+1),$$

invariant under λ, $1-\lambda$, $\lambda/(2\lambda-1)$, $(\lambda-1)/(2\lambda-1)$. The groups for (10′), (11′) and (12′) are abstractly the same G_4.

If neither λ nor $\mu=0$, ± 1, 1/2, 2 in (14) we have a nontrivial, nonsymmetric solution. Forming the difference table, we find that for (14) to give rise to an ideal equality of degree 3,

$$\begin{aligned} \mu = \lambda,\quad & 1-\lambda,\quad \lambda/(\lambda-1),\quad 1/\lambda,\quad 1/(1-\lambda),\quad (\lambda-1)/\lambda,\\ & (\lambda+1)/(2-\lambda),\quad (\lambda-2)/(2\lambda-1),\quad (2\lambda-1)/(\lambda+1),\\ & (\lambda+1)/(2\lambda-1),\quad (1-2\lambda)/(2-\lambda),\quad (2-\lambda)/(1+\lambda). \end{aligned}$$

These substitutions form a G_{12}. The first six are, of course, the cross ratio subgroup, and lead to semi-trivial solutions. The remaining six all give

$$(16) \qquad \lambda^2-2\lambda,\ -\lambda^2+1,\ 2\lambda-1 \stackrel{2}{=} 0,\ \lambda^2-\lambda+1,\ -\lambda^2+\lambda-1,$$

which is invariant under the G_{12}. An examination of the difference table for (16) shows that a nontrivial equality of degree 3 can be formed from (16) by the application of (6) only if $h=\lambda^2-\lambda+1$. Making this application followed by (5) with $M=2$ and $K=-\lambda^2+\lambda-1$, we have

$$(17)\qquad \begin{aligned} &3\lambda^2-3\lambda+3,\ -\lambda^2+5\lambda-3,\ -3\lambda^2+\lambda+1,\ \lambda^2-3\lambda-1\\ &\overset{3}{=}\lambda^2+3\lambda-1,\ -\lambda^2-\lambda+3,\ 3\lambda^2-5\lambda+1,\ -3\lambda^2+3\lambda-3,\end{aligned}$$

which is also invariant under the G_{12}.

The difference table for (17) is found to contain the same value three times (and one which will lead to a nontrivial equality of degree 4) for twelve numerical values of λ. One of these is 3 and the others can be obtained by substituting $\lambda=3$ in the last eleven elements of the G_{12}. The only sequence given is the following.

$$\begin{aligned} &7,\ 0,\ -7\overset{2}{=}5,\ 3,\ -8,\\ (18)\qquad k=7,\qquad &21,\ 3,\ -1,\ -23\overset{3}{=}17,\ 13,\ -9,\ -21,\\ k=22,\qquad &14,\ 12,\ -4,\ -5,\ -17\overset{4}{=}16,\ 7,\ 3,\ -10,\ -16.\end{aligned}$$

Replacing λ by c/d in (16) and (17), we have the following theorem.

THEOREM 2. *The only sequences of ideal solutions beginning with degree two are nonsymmetric and are equivalent to*

$$c^2-2cd,\ -c^2+d^2,\ 2cd-d^2\overset{2}{=}0,\ c^2-cd+d^2,\ -c^2+cd-d^2,$$
$$3c^2-3cd+3d^2,\ -c^2+5cd-3d^2,\ -3c^2+cd+d^2,$$
$$c^2-3cd-d^2\overset{3}{=}c^2+3cd-d^2,$$
$$-c^2-cd+3d^2,\ 3c^2-5cd+d^2,\ -3c^2+3cd-3d^2,$$

or to (18).

4. **Sequences beginning with the third degree.** A. *Symmetric case.* The general symmetric solution of third degree is [2, (4)]

$$(19)\qquad a_1,\ a_2,\ -a_1,\ -a_2\overset{3}{=}b_1,\ b_2,\ -b_1,\ -b_2$$

where $a_1=p_1p_2+p_3p_4$, $b_1=p_1p_2-p_3p_4$, $a_2=p_1p_3-p_2p_4$, $b_2=p_1p_3+p_2p_4$. Solution (10′) is the special case $p_1=\lambda$, $p_2=3$, $p_3=p_4=1$, (11′) the case $p_1=2$, $p_2=\lambda$, $p_3=\lambda-2$, $p_4=-1$, and (12′) the case $p_1=2\lambda-1$, $p_2=1$, $p_3=2$, $p_4=-1$.

Let $p_2=\alpha p_3$, $p_4=\beta p_1$. Then (19) becomes

$$(20)\qquad \pm(\alpha+\beta,\ 1-\alpha\beta)\overset{3}{=}\pm(\alpha-\beta,\ 1+\alpha\beta)$$

where α and β are rational, and where α and β play an interchangeable role. Equality (20) is trivial if α or $\beta = 0, \pm 1$, and semi-trivial if

$$\beta = \alpha,\ -\alpha,\ 1/\alpha,\ -1/\alpha,\ (1+\alpha)/(1-\alpha),\ -(1+\alpha)/(1-\alpha),$$
$$(1-\alpha)/(1+\alpha),\ -(1-\alpha)/(1+\alpha).$$

These substitutions form a G_8. Three equal differences leading to a nontrivial equality of degree 4 occur in the difference table for (20) if $\beta, -\beta, 1/\beta, -1/\beta = (3\alpha+1)/(\alpha+1), (3\alpha-1)/(\alpha-1), (\alpha+3)/(\alpha+1), (\alpha-3)/(\alpha-1), (\alpha+3)/(3\alpha-1), (3\alpha+1)/(3-\alpha)$. But since (20) is invariant under $\beta, -\beta, 1/\beta, -1/\beta$ we need consider only the six expressions in α. Finally, since the first four expressions are invariant under $\alpha, -\alpha, 1/\alpha, -1/\alpha$, and the last two under α and $1/\alpha$, we need consider only $\beta = (3\alpha+1)/(\alpha+1)$ and $\beta = (\alpha+3)/(3\alpha-1)$.

(i) Let $\beta = (3\alpha+1)/(\alpha+1)$. Then (20) becomes

$$(21)\quad \pm(3\alpha^2+2\alpha+1, \alpha^2-2\alpha-1) \overset{3}{=} \pm(3\alpha^2-1, \alpha^2+4\alpha+1),$$

which is trivial if $\alpha = 0, \pm 1, -1/2, -1/3$ and which is given by [2, (11)] with $a = \alpha^2+4\alpha+1$, $b = 3\alpha^2+2\alpha+1$, and $k = \alpha^2-2\alpha-1$. The only value for k leading to a nontrivial ideal equality of degree 4 is $2\alpha^2-4\alpha-2$, giving

$$(22)\quad \begin{aligned} &2\alpha^2-\alpha-1,\ \alpha^2+2\alpha+1,\ -3\alpha-1,\ -\alpha^2+2\alpha+1,\ -2\alpha^2 \\ &\overset{4}{=} 2\alpha^2,\ \alpha^2-2\alpha-1,\ 3\alpha+1,\ -\alpha^2-2\alpha-1,\ -2\alpha^2+\alpha+1. \end{aligned}$$

There are sixteen values of α for which the sequence continues, twelve of them (in three sets of four each) when substituted in (21) leading to

$$\pm(1, 17) \overset{3}{=} \pm(11, 13), \qquad \pm(3, 11) \overset{3}{=} \pm(7, 9),$$
$$\pm(1, 13) \overset{3}{=} \pm(7, 11),$$

all of which appear in the chart. The remaining four values, $-3/2, 1/5, -1/7, -3/4$ all give

$$\pm(11, 23) \overset{3}{=} \pm(17, 19),$$

$$(23)\quad k = 34,\quad 18, 17, -1, -14, -20 \overset{4}{=} 20, 14, 1, -17, -18,$$

$$k = 19, \qquad \pm(15, 47, 59) \overset{5}{=} \pm(9, 53, 55).$$

(ii) Let $\beta = (\alpha+3)/(3\alpha-1)$. Then (20) becomes

$$(24)\quad \pm(3\alpha^2+3, \alpha^2+1) \overset{3}{=} \pm(3\alpha^2-2\alpha-3, \alpha^2+6\alpha-1),$$

which is trivial if $\alpha=0,\ \pm 1,\ 2,\ -1/2,\ -3,\ 1/3$, and which is given by [2, (13)] with $u=3\alpha^2-2\alpha-3,\ v=\alpha^2+6\alpha-1,\ k=\alpha^2+1$. For $\alpha=-2$, (24) gives the fourth equality of degree 3 of the chart, namely

$$\pm(5,15)\overset{3}{=}\pm(9,13).$$

The sequence beginning with (24) continues for $k=\alpha^2+1$ with

$$\begin{aligned}(25)\qquad & 2\alpha^2-\alpha-1,\ \alpha^2+3\alpha,\ -3\alpha+1,\ -\alpha^2+\alpha+2,\ -2\alpha^2-2\\ & \overset{4}{=}2\alpha^2+2,\ \alpha^2-\alpha-2,\ 3\alpha-1,\ -\alpha^2-3\alpha,\ -2\alpha^2+\alpha+1,\end{aligned}$$

and then for $k=\alpha^2-4\alpha-1$ with

$$\begin{aligned}(26)\qquad & \pm(5\alpha^2-4\alpha+3,\ 3\alpha^2-6\alpha-5,\ \alpha^2+10\alpha+1)\\ & \overset{5}{=}\pm(5\alpha^2-6\alpha-3,\ 3\alpha^2+4\alpha+5,\ \alpha^2-10\alpha+1),\end{aligned}$$

or for $k=2\alpha^2+2\alpha-2$ with

$$\begin{aligned}(27)\qquad & \pm(3\alpha^2+\alpha+1,\ 2\alpha^2-3,\ \alpha^2+4\alpha-2)\\ & \overset{5}{=}\pm(3\alpha^2-2,\ 2\alpha^2+4\alpha-1,\ \alpha^2-\alpha+3).\end{aligned}$$

However, there are no real values of α giving an ideal equality of degree 6.

B. *Nonsymmetric case*. The general nonsymmetric solution of third degree is [2, (10)]

$$(28)\qquad \begin{aligned} a_1 &= -mr+ns+t, & b_1 &= -r+s+mnt,\\ a_2 &= mr-ns+t, & b_2 &= r-s+mnt,\\ a_3 &= mr+ns-t, & b_3 &= r+s-mnt,\\ a_4 &= -mr-ns-t, & b_4 &= -r-s-mnt,\end{aligned}$$

where m and n are rational numbers not equal to $0,\ \pm 1$ and where

$$\begin{aligned} r &= -(m^2-1)nU^2-2(n^2-1)UV+(n^2-1)nV^2,\\ s &= (m^2-1)U^2-2(m^2-1)nUV-(n^2-1)V^2,\\ t &= (m^2-1)U^2+(n^2-1)V^2,\end{aligned}$$

with U and V relatively prime integers.

Equality (17) is given by (28) with $m=\lambda,\ n=-(\lambda-1)/\lambda$, $t=-(\lambda-2),\ r=\lambda+1,\ s=\lambda(2\lambda-1)$. The equality of degree 3 of (18) is given by $m=5,\ n=3,\ U=1,\ V=-3$.

To determine sequences starting with (28), it is best to form the difference table for the transformation used in [2, (6)]. This leads to a series of linear conditions which are then incorporated in [2, (8)] and finally in the parametric solution. The following example shows the existence of such sequences. With $m=n=2$, $U=3$, $V=1$, we have

$$31,\ 13,\ -21,\ -23 \overset{3}{=} 29,\ 11,\ -7,\ -33,$$

$$k = 18,\quad 47,\ 13,\ -15,\ -21,\ -23 \overset{4}{=} 49,\ -3,\ -5,\ -7,\ -33.$$

THEOREM 3. *All symmetric sequences beginning with degree three are equivalent to* (21)(22), *to* (24)(25)(26) *or* (27) (*in all of which* α *is rational*), *or to* (23). *Nonsymmetric sequences beginning with degree three exist and can be determined by the method of this paper.*

5. **Sequences beginning with higher than the third degree.** General parametric ideal solutions have not been found beyond the third degree, but many special solutions are known [1, 2, 4, 6]. Although probably only sequences beginning with low degrees extend for more than 2 or 3 steps, sequences beginning with high degrees do exist. This paper will be concluded with two numerical examples of sequences which were obtained from known ideal equalities of degrees 6 and 8 by the use of a theorem of Escott [5, page 617] which gives the *predecessor* of a given equality. In neither case can the sequence be extended in either direction.

The first example is

$$1,\ 18,\ 35,\ 59,\ 72 \overset{4}{=} 2,\ 15,\ 39,\ 56,\ 73,$$

$$k = 17,\quad 1,\ 19,\ 32,\ 59,\ 72,\ 90 \overset{5}{=} 2,\ 15,\ 39,\ 52,\ 76,\ 89,$$

$$k = 13,\quad 1,\ 19,\ 28,\ 59,\ 65,\ 90,\ 102 \overset{6}{=} 2,\ 14,\ 39,\ 45,\ 76,\ 85,\ 103,$$

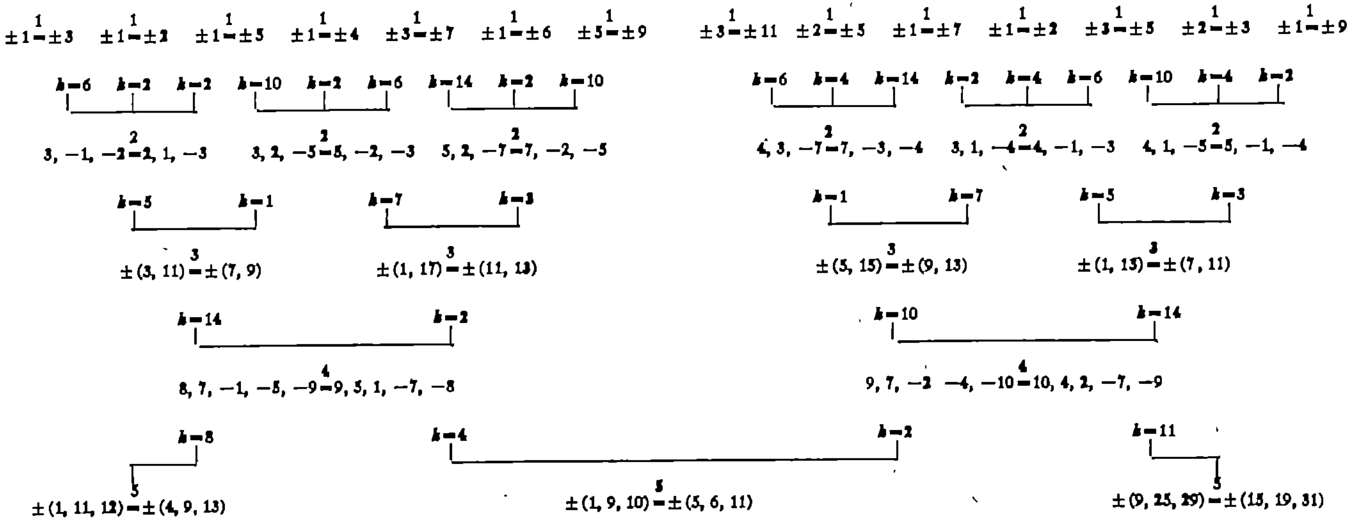
where the equality of degree 6 is due to Escott [5, p. 616].

The second example is

$$1,\ 25,\ 31,\ 72,\ 87,\ 128,\ 134,\ 158 \overset{7}{=} 2,\ 18,\ 43,\ 59,\ 100,\ 116,\ 141,\ 157,$$

$$k = 41,\quad 1,\ 25,\ 31,\ 84,\ 87,\ 134,\ 158,\ 182,\ 198 \overset{8}{=} 2,\ 18,\ 42,\ 66,\ 113,\ 116,\ 169,\ 175,\ 199,$$

where the equality of degree 8 is due to Letac [6, p. 48].


±1 =¹ ±3
±1 =¹ ±2
±1 =¹ ±5
±1 =¹ ±4
±3 =¹ ±7
±1 =¹ ±6
±5 =¹ ±9
±3 =¹ ±11
±2 =¹ ±5
±1 =¹ ±7
±1 =¹ ±2
±3 =¹ ±5
±2 =¹ ±3
±1 =¹ ±9
k=6 k=2 k=2
k=10 k=2 k=6
k=14 k=2 k=10
k=6 k=4 k=14
k=2 k=4 k=6
k=10 k=4 k=2
3, −1, −2 =² 2, 1, −3
3, 2, −5 =² 5, −2, −3
5, 2, −7 =² 7, −2, −5
4, 3, −7 =² 7, −3, −4
3, 1, −4 =² 4, −1, −3
4, 1, −5 =² 5, −1, −4
k=5 k=1
k=7 k=3
k=1 k=7
k=5 k=3
±(3, 11) =³ ±(7, 9)
±(1, 17) =³ ±(11, 13)
±(5, 15) =³ ±(9, 13)
±(1, 13) =³ ±(7, 11)
k=14 k=2
k=10 k=14
8, 7, −1, −5, −9 =⁴ 9, 5, 1, −7, −8
9, 7, −2 −4, −10 =⁴ 10, 4, 2, −7, −9
k=8
k=4 k=2
k=11
±(1, 11, 12) =⁵ ±(4, 9, 13)
±(1, 9, 10) =⁵ ±(5, 6, 11)
±(9, 25, 29) =⁵ ±(15, 19, 31)

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THE NON-EXISTENCE OF A CERTAIN TYPE OF ODD PERFECT NUMBER

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For any perfect number[1] expressed in the form $n = a_0 a_1 \cdots a_t$, where

$$a_0 = p_0^{\alpha_0}, \quad a_1 = p_1^{\alpha_1}, \cdots, a_t = p_t^{\alpha_t}$$

and $p_0, p_1, \cdots, p_t$ are the distinct prime factors of n, it can be shown that a unique one of the prime powers a_i has an even divisor sum $\sigma(a_i)$. Throughout we shall suppose that the primes p_i and hence the prime powers a_i to be so numbered that

$$(1) \qquad \sigma(a_0) \equiv 0; \quad \sigma(a_i) \equiv 1 \qquad i = 1, 2, \cdots, t, \pmod 2.$$

Then with the abbreviations

$$(2) \qquad \sigma_0 = \sigma(a_0)/2; \quad \sigma_i = \sigma(a_i), \qquad i = 1, 2, \cdots, t,$$

the condition for n to be perfect may be written in the form

$$(3) \qquad \sigma(n)/2 = \sigma_0 \sigma_1 \cdots \sigma_t = a_0 a_1 \cdots a_t = n.$$

For the even perfect numbers, which are the only kind known, it is well known that $p_0 = 2^q - 1$, $\alpha_0 = 1$, $p_1 = 2$, $\alpha_1 = q - 1$, $t = 1$, where q is any prime such that $2^q - 1$ is also prime. Then $\sigma_1 = 2^q - 1 = a_0$ and $\sigma_0 = 2^{q-1} = a_1$ so that σ_0 and σ_1 are the prime powers a_0 and a_1 in rereverse order. It is natural to inquire whether there may exist odd perfect numbers such that analogously $\sigma_0, \sigma_1, \cdots, \sigma_t$ are the prime powers $a_0, a_1, \cdots, a_t$ in a different order. In the following it will be proved that no odd perfect numbers of this form can exist.

We first establish an algebraic identity. Throughout this paper the product notation $\prod_{i=a}^{b} x_i$ is used with the convention that $\prod_{i=a}^{b} x_i = 1$ if $a > b$.

LEMMA 1. *Let $c_1, c_2, \cdots, c_t$ be any $t \geqq 2$ integers (more generally, elements of a commutative ring with a unit element). Then,*

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[1] For a summary of results concerning perfect numbers (including those cited above) with references see L. Dickson, *History of the theory of numbers*, vol. 1, 1919, pp. 1–33. For a more recent paper with references to other recent literature on the subject, see A. Brauer, *On the non-existence of odd perfect numbers of form* $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 712–718.

$$\sum_{j=1}^{t}\left[\prod_{i=1}^{j-1}(c_i-1)\prod_{i=j+1}^{t}c_i\right]=\prod_{i=1}^{t}c_i-\prod_{i=1}^{t}(c_i-1). \tag{4}$$

PROOF. The identity holds for $t=2$, both members reducing to c_1+c_2-1. Proceeding by induction, assume the identity holds for $t=m$. Multiplying both members by c_{m+1} and adding $\prod_{i=1}^{m}(c_i-1)$, we have

$$\begin{aligned}\sum_{j=1}^{m+1}\left[\prod_{i=1}^{j-1}(c_i-1)\prod_{i=j+1}^{m+1}c_i\right] &= c_{m+1}\sum_{j=1}^{m}\left[\prod_{i=1}^{j-1}(c_i-1)\prod_{i=j+1}^{m}c_i\right]+\prod_{i=1}^{m}(c_i-1)\\ &= c_{m+1}\left[\prod_{i=1}^{m}c_i-\prod_{i=1}^{m}(c_i-1)\right]+\prod_{i=1}^{m}(c_i-1)\\ &= \prod_{i=1}^{m+1}c_i-\prod_{i=1}^{m+1}(c_i-1),\end{aligned}$$

the first and last members of which are the members of the required identity for $t=m+1$, thus completing the induction.

For an odd integer $n=a_0a_1\cdots a_t$ to be perfect a well known necessary condition is that with the p's numbered according to (1)

$$\alpha_0\equiv p_0\equiv 1 \pmod 4. \tag{5}$$

For such prime powers we have the following:

LEMMA 2. *Let $\alpha>1$ be an integer and p a prime such that $\alpha\equiv p\equiv 1$ (mod 4). Then $\sigma(p^\alpha)$ is divisible by at least two distinct odd primes.*

PROOF. It is sufficient to exhibit two odd nontrivial divisors of $\sigma(p^\alpha)$ which are relatively prime. We have

$$\sigma(p^\alpha)=\frac{p^{\alpha+1}-1}{p-1}=2\cdot\frac{p^{(\alpha+1)/2}+1}{2}\cdot\frac{p^{(\alpha+1)/2}-1}{p-1}. \tag{6}$$

Then the required divisors are

$$d_1=\frac{p^{(\alpha+1)/2}+1}{2}\quad\text{and}\quad d_2=\frac{p^{(\alpha+1)/2}-1}{p-1}. \tag{7}$$

They are both odd since $\sigma(p^\alpha)=1+p+p^2+\cdots+p^\alpha\equiv\alpha+1\equiv 2$ (mod 4). They are coprime, since $2d_1-(p-1)d_2=2$ so that if there were a common divisor of d_1 and d_2, it would have to divide 2. Finally, they are nontrivial divisors since $d_1>d_2>1$ for $\alpha>1$.

We are now able to prove our theorem.

THEOREM. *Let* $n = a_0 a_1 \cdots a_t$, *where*

$$a_0 = p_0^{\alpha_0}, \quad a_1 = p_1^{\alpha_1}, \cdots, a_t = p_t^{\alpha_t}$$

and $p_0, p_1, \cdots, p_t$ *are distinct odd primes. Then, if each of the quantities* σ_i, $i = 0, 1, \cdots, t$, *defined in* (2) *is a power of a prime,* n *is not a perfect number.*

PROOF. Assume that n is perfect. Then (3) holds; and since the σ_i are prime powers, by the fundamental theorem of arithmetic, they must each equal one of the a_j, $i, j = 0, \cdots, t$ with $i \neq j$, because $\sigma_i \equiv 1 \not\equiv 0 \pmod{p_i}$. That is, $\sigma_0, \cdots, \sigma_t$ are the prime powers $a_0, \cdots, a_t$ in a different order.

Without loss of generality we may suppose that the p's are numbered recursively in the following manner: p_0 has already been chosen in accord with (1) (or (5), which amounts to the same thing). Choose as p_1 that prime p_i for which $a_i = \sigma_0$, as p_2 that prime p_j for which $a_j = \sigma_1$, and in general choose as p_m that prime p_r for which $a_r = \sigma_{m-1}$. This process can be continued until a prime p_k is reached such that $\sigma_k = a_l$ with $l < k$ so that we cannot set $a_{k+1} = \sigma_k$. We shall now show that this cannot occur until the primes have been completely numbered; that is, when $k = t$, and then $l = 0$. First suppose $0 < l < k \leq t$. Then we have both $\sigma_k = a_l$ and $\sigma_{l-1} = a_l$ so that in the product, $\sigma_0 \cdots \sigma_l \cdots \sigma_k \cdots \sigma_t$, p_l occurs to at least the power $2\alpha_l$ contrary to (3). Next suppose $l = 0$ but $k < t$. Then $\sigma_k = a_0$, and

$$a_1 a_2 \cdots a_k a_0 = \sigma_0 \sigma_1 \cdots \sigma_{k-1} \sigma_k. \tag{8}$$

Hence from (3), numbering the p_m, $m = k+1, k+2, \cdots, t$ in any order,

$$a_{k+1} a_{k+2} a_t = \sigma_{k+1} \sigma_{k+2} \cdots \sigma_t. \tag{9}$$

But this is impossible, since

$$\begin{aligned} &\sigma_{k+1} \sigma_{k+2} \cdots \sigma_t \\ &\quad = (1 + p_{k+1} + p_{k+1}^2 + \cdots + p_{k+1}^{\alpha_{k+1}})(1 + \cdots + p_{k+2}^{\alpha_{k+2}}) \\ &\qquad \cdots (1 + \cdots p_t^{\alpha_t}) > p_{k+1}^{\alpha_{k+1}} p_{k+2}^{\alpha_{k+2}} \cdots p_t^{\alpha_t} \\ &\quad = a_{k+1} a_{k+2} \cdots a_t. \end{aligned} \tag{10}$$

The only remaining possibility is, then, $k = t$, $l = 0$. Thus the p's, and hence the a's, have been completely numbered as follows:

$$a_m = \sigma_{m-1}, \quad m = 1, 2, \cdots, t; \qquad a_0 = \sigma_t. \tag{11}$$

In view of (5) and Lemma 2, we must have $\alpha_0 = 1$, since otherwise

σ_0 would not be a power of a prime.

Now, evaluating the σ's, equations (11) become

$$(12)\qquad a_1 = \frac{p_0+1}{2}; \quad a_m = \frac{p_{m-1}^{\alpha_{m-1}+1}-1}{p_{m-1}-1}, \quad m = 2, 3, \cdots, t; \qquad p_0 = \frac{p_t^{\alpha_t+1}-1}{p_t-1}.$$

With the definitions,

$$(13)\qquad a_m = p_m^{\alpha_m}, \qquad b_m = \frac{1}{p_m-1}, \qquad m = 1, 2, \cdots, t,$$

equations (12) become

$$(14)\qquad a_m = b_{m-1}p_{m-1}a_{m-1} - b_{m-1} \qquad \text{for } m = 2, 3, \cdots, t,$$

$$(15)\qquad a_1 = \frac{p_0+1}{2}, \qquad p_0 = b_t p_t a_t - b_t.$$

Eliminating p_0 from (15) gives

$$(16)\qquad 2a_1 - 1 = b_t p_t a_t - b_t.$$

By repeated application of the recursion formula (14) we find that

$$(17)\qquad a_m = \left(\prod_{i=1}^{m-1} b_i p_i\right)a_1 - \sum_{j=1}^{m-1} b_j \prod_{i=j+1}^{m-1} b_i p_i, \qquad m = 2, 3, \cdots, t,$$

which is readily verified by induction. From (16) and (17) with $m = t$

$$(18)\qquad 2a_1 - 1 = \left(\prod_{i=1}^{t} b_i p_i\right)a_1 - \sum_{j=1}^{t} b_j \prod_{i=j+1}^{t} b_i p_i$$

or

$$(19)\qquad \left(\prod_{i=1}^{t} b_i p_i - 2\right)a_1 - \sum_{j=1}^{t} b_j \prod_{i=j+1}^{t} b_i p_i + 1 = 0.$$

Multiplying by $\prod_{i=1}^{t}(p_i - 1)$ and using (13), (19) becomes

$$(20)\qquad \left[\prod_{i=1}^{t} p_i - 2\prod_{i=1}^{t}(p_i-1)\right]a_1 - \sum_{j=1}^{t}\left[\prod_{i=1}^{j-1}(p_i-1)\prod_{i=j+1}^{t} p_i\right] + \prod_{i=1}^{t}(p_i-1) = 0.$$

Utilizing the identity (4), (20) becomes

$$\left[\prod_{i=1}^{t} p_i - 2\prod_{i=1}^{t} (p_i - 1)\right] a_1 - \left[\prod_{i=1}^{t} p_i - 2\prod_{i=1}^{t} (p_i - 1)\right] = 0 \tag{21}$$

so that

$$(a_1 - 1)\left[\prod_{i=1}^{t} p_i - 2\prod_{i=1}^{t} (p_i - 1)\right] = 0. \tag{22}$$

Hence, either

$$p_1^{\alpha_1} = a_1 = 1 \tag{23}$$

or

$$\prod_{i=1}^{t} p_i = 2\prod_{i=1}^{t} (p_i - 1). \tag{24}$$

(23) is impossible since $p_1 \geqq 3$. (24) is also impossible, since the right member is even while the left member, being the product of odd primes, is odd. Thus the assumption that n is perfect leads to a contradiction, and the theorem is proved.

Our results may evidently be restated in the following form:

COROLLARY. *If $n = a_0 a_1 \cdots a_t$ is an odd perfect number, at least two of the divisor sums $\sigma(a_i)$ must have a common factor greater than* 1.

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EXTENSIONS OF DIFFERENTIAL FIELDS. III

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The purpose of the present note is to show how the point of view of a preceding paper[1] can be used in developing the concepts of resolvent, dimension, and order introduced by J. F. Ritt in his theory of algebraic differential equations.[2] The present development, in addition to being simpler in some instances, has the advantage of being valid for abstract differential fields as opposed to fields of meromorphic functions of a complex variable, as used by Ritt. I shall also take the opportunity to correct mistakes in a related paper.[3] The notation and definitions used will be as in Extensions I and II.

1. **Resolvents, dimension, and order.** Let $\mathcal{F}$ be a differential field (ordinary or partial) of characteristic 0, and let $y_1, \cdots, y_n$ be unknowns. If Π is a prime differential ideal in $\mathcal{F}\{y_1, \cdots, y_n\}$ other than $\mathcal{F}\{y_1, \cdots, y_n\}$ itself then Π has a generic solution $\eta_1, \cdots, \eta_n$. If the degree of differential transcendency of $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ over $\mathcal{F}$ is q then $0 \leq q < n$, and precisely q of the elements $\eta_1, \cdots, \eta_n$ are differentially algebraically independent over $\mathcal{F}$. Suppose, say, that $\eta_1 \cdots, \eta_q$ are independent in this way, that is, that Π does not contain a nonzero differential polynomial in $y_1, \cdots, y_q$, but does in $y_1, \cdots, y_q, y_j$ for each $j > q$. In Ritt's terminology $y_1 \cdots, y_q$ is a complete set of arbitrary unknowns for Π. It is natural to call q the *dimension* of Π (in symbols, dim Π).

Suppose henceforth that $\mathcal{F}$ is ordinary. It is easy to see that the degree of transcendency of $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ over $\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle$ (both these differential fields being considered as fields) is finite. We denote the degree of transcendency of any field $\mathcal{K}$ over a subfield $\mathcal{G}$ by $\partial^\circ\mathcal{K}/\mathcal{G}$. It will be seen that it is natural to call the integer $\partial^\circ\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle/\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle$ the *order* of Π with respect to $y_1, \cdots, y_q$ (when the set of arbitrary unknowns is understood, for example when $q = 0$, we use the notation: ord Π).

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[1] Kolchin, *Extensions of differential fields*, I, Ann. of Math. vol. 43 (1942) pp. 724–729. We shall refer to this paper as *Extensions* I.

[2] The subject matter treated here, together with some of the material from *Extensions* I, is roughly parallel to the contents of §§24–31, 75 of Ritt, *Differential equations from the algebraic standpoint*, Amer. Math. Soc. Colloquium Publications, vol. 14, New York, 1932.

[3] Kolchin, *Extensions of differential fields*, II, Ann. of Math. vol. 45 (1944) pp. 358–361. We shall refer to this paper as *Extensions* II.

If $\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle$ contains a nonconstant (which is the case either when $\mathcal{F}$ does or when $q>0$) then by Extensions I there is an ω such that $\mathcal{F}\langle\eta_1, \cdots, \eta_q, \omega\rangle = \mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$. Let $A = A(\eta_1, \cdots, \eta_q, w)$ be an irreducible differential polynomial in $\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle\{w\}$, with solution $w = \omega$, of lowest possible order. Since ω and its first ord Π derivatives must be algebraically dependent over $\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle$, the order of A is not greater than ord Π. On the other hand, if the order of A is p then the pth derivative (and consequently all the derivatives) of ω is algebraically dependent over $\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle$ on ω and its first $p-1$ derivatives, so that ord $\Pi = \partial^\circ\mathcal{F}\langle\eta_1 \cdots, \eta_n\rangle/\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle = \partial^\circ\mathcal{F}\langle\eta_1, \cdots, \eta_q, \omega\rangle/\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle \leqq p$. Therefore the order of A in w is $p =$ ord Π. $A(y_1, \cdots, y_q, w)$ is called a ***resolvent*** of Π. (Actually, this is a slight generalization of Ritt's resolvent, which must be in $\mathcal{F}\{y_1, \cdots, y_q, w\}$ instead of merely in $\mathcal{F}\langle y_1, \cdots, y_q\rangle\{w\}$.)

Let $\mathcal{G}$ be a differential extension field of $\mathcal{F}$, let $\{\Pi\} = \Pi_1 \cap \cdots \cap \Pi_r$ be the decomposition into prime components (that is, prime differential ideals none of which contains another) of the perfect differential ideal generated by Π in $\mathcal{G}\{y_1, \cdots, y_n\}$, and let $A_1(y_1, \cdots, y_q, w) \cdots A_s(y_1, \cdots, y_q, w)$ be the complete factorization of $A(y_1, \cdots, y_q, w)$ in $\mathcal{G}\langle y_1, \cdots, y_q\rangle\{w\}$. Each $A_i(y_1, \cdots, y_q, w)$ is of order p in w, for a factor of $A(y_1, \cdots, y_q, w)$ of order less than p would be a common factor of the coefficients in $A(y_1, \cdots, y_q, w)$ when $A(y_1, \cdots, y_q, w)$ is considered as a polynomial in w_p, the pth derivative of w. We shall now establish Ritt's result that $r = s$ and each $A_i(y_1, \cdots, y_q, w)$ is a resolvent of one Π_j. This result implies that Π decomposes if and only if $A(y_1, \cdots, y_q, w)$ factors, and that each prime component in the decomposition has the same order as Π has.

Let $\eta_1', \cdots, \eta_n'$ be a generic solution of Π_1. Then (by *Extensions* I, §1) $\eta_1', \cdots, \eta_n'$ is a generic solution of Π, so that $\eta_1' \to \eta_1, \cdots, \eta_n' \to \eta_n$ generates an isomorphism of $\mathcal{F}\langle\eta_1', \cdots, \eta_n'\rangle$ onto $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$. Therefore if we let ω' be the same differential rational function over $\mathcal{F}$ of $\eta_1', \cdots, \eta_n'$ that ω is of $\eta_1, \cdots, \eta_n$, we shall have

$$\mathcal{F}\langle\eta_1', \cdots, \eta_q', \omega'\rangle = \mathcal{F}\langle\eta_1', \cdots, \eta_n'\rangle.$$

Now ω' is a solution of $A' = A(\eta_1', \cdots, \eta_q', w)$, and therefore of some $A_i' = A_i(\eta_1', \cdots, \eta_q', w)$, say of A_1'. Furthermore, ω' is not a solution of two different A_i''s, for ω' does not annul the separant $\partial A'/\partial w_p = \partial(A_1' \cdots A_s')/\partial w_p$. Let ω'' be a generic solution of the prime component of $\{A_1'\}$ in $\mathcal{G}\langle\eta_1', \cdots, \eta_q'\rangle\{w\}$ not containing the separant $\partial A_1'/\partial w_p$. Then ω'' is a generic solution of the prime component of $\{A'\}$ in $\mathcal{F}\langle\eta_1', \cdots, \eta_q'\rangle\{w\}$ not containing the separant $\partial A'/\partial w_p$, so that $\omega''\rightarrow\omega'$ generates an isomorphism of $\mathcal{J}\langle\eta_1', \cdots, \eta_q', \omega''\rangle$ onto $\mathcal{J}\langle\eta_1', \cdots, \eta_q', \omega'\rangle$, and a homomorphism of $\mathcal{G}\langle\eta_1', \cdots, \eta_q'\rangle\{\omega''\}$ onto $\mathcal{G}\langle\eta_1', \cdots, \eta_q'\rangle\{\omega'\}$. Therefore, if for each $i>q$ we let η_i'' be the same differential rational function over $\mathcal{J}\langle\eta_1', \cdots, \eta_q'\rangle$ of ω'' as η_i' is of ω', then $\eta_1', \cdots, \eta_q'$, $\eta_{q+1}'', \cdots, \eta_n''$ is a generic solution of Π and a solution of some Π_i. Since $\eta_1', \cdots, \eta_n'$ must be a solution of the same Π_i, and since one Π_i does not contain another, $\eta_1', \cdots, \eta_q', \eta_{q+1}'', \cdots, \eta_n''$ is a solution of Π_1, and indeed a generic one.

Therefore $\eta_{q+1}''\rightarrow\eta_{q+1}', \cdots, \eta_n''\rightarrow\eta_n'$ generates an isomorphism of $\mathcal{G}\langle\eta_1', \cdots, \eta_q', \eta_{q+1}'', \cdots, \eta_n''\rangle$ onto $\mathcal{G}\langle\eta_1', \cdots, \eta_n'\rangle$, A_1' is an irreducible differential polynomial in $\mathcal{G}\langle\eta_1', \cdots, \eta_q'\rangle\{w\}$, with solution $w=\omega'$, of minimal degree, and $A_1(y_1, \cdots, y_q, w)$ is a resolvent of Π_1. In the same way, every Π_i has an $A_j(y_1, \cdots, y_q, w)$ as a resolvent, so that $r\leqq s$. To show that there is no $A_j(y_1, \cdots, y_q, w)$ left over, for any j let ω_j be a generic solution of the prime component of $\{A_j'\}$ in $\mathcal{G}\langle\eta_1', \cdots, \eta_n'\rangle\{w\}$ not containing $\partial A_j'/\partial w_p$. For each $i>q$ let η_{ji} be the same differential rational function over $\mathcal{J}\langle\eta_1', \cdots, \eta_q'\rangle$ of ω_j as η_i' is of ω'. Then $\eta_1', \cdots, \eta_q', \eta_{j,q+1}, \cdots, \eta_{jn}$ is a generic solution of Π and therefore a solution of some Π_i, say Π_{i_0}. Therefore ω_j is a solution of the A_k' for which $A_k(y_1, \cdots, y_q, w)$ is a resolvent of Π_{i_0}. This implies that $A_k(y_1, \cdots, y_q, w)$ is divisible by $A_j(y_1, \cdots, y_q, w)$, so that $k=j$ and $A_j(y_1, \cdots, y_q, w)$ is a resolvent of a Π_i.

If $q=0$ and $\mathcal{J}$ consists solely of constants it is still true that each prime component of $\{\Pi\}$ has the same order as Π. To see this introduce a new unknown u and let $\mathcal{J}'=\mathcal{J}\langle u\rangle$, $\mathcal{G}'=\mathcal{G}\langle u\rangle$. The perfect differential ideal generated by Π in $\mathcal{J}'\{y_1, \cdots, y_n\}$ is clearly prime and has the same order as Π has. The prime components of the perfect differential ideal generated by Π in $\mathcal{G}'\{y_1, \cdots, y_n\}$ are the perfect differential ideals generated by $\Pi_1, \cdots, \Pi_r$, and have the same order. Therefore ord $\Pi_i=$ ord Π for each i.

2. **Corrections to Extensions** II. We refer now to the proof on page 359 of *Extensions* II. The derivation of the equation $\omega K(z)-H(z)=\alpha A(z)$ is incorrect, for it rests on the unjustified assumption (see lines 18 and 17 from the bottom) that $\partial A(z)/\partial y_p\in\mathcal{G}\{z\}$. To save the proof we delete in toto lines 22–4 from the bottom ("Denote the $\cdots A(z)$:"), and replace them by the following considerations.

Let $\omega=H(y)/K(y)$ be any coefficient in $A(z)$ not merely an element of $\mathcal{J}$, with $H(y)$, $K(y)$ free of common divisor. Clearly $\omega K(z)-H(z)\in\Sigma$.

Denote the lowest common denominator of the coefficients in $A(z)$ by $D(y)$, and let $B(y, z)=D(y)A(z)$. Then $B(y, z)\in\mathcal{F}\{y, z\}$, and $B(y, y)=0$. Since $A(z)$ is irreducible and one of the coefficients in $A(z)$ is unity, the irreducible factors of $B(y, z)$ are distinct and all have the same order in z as $A(z)$ has.

Denoting the order of $B(y, z)$ in y by p, let $B_1(y, z)$ be an irreducible factor of $B(y, z)$ of order p in y. Let Λ_1 be the prime component of $\{B_1(y, z)\}$ which contains neither of the separants of $B_1(y, z)$. No other irreducible factor of $B(y, z)$ is in Λ_1, for such a factor would have the same order in z as $B_1(y, z)$ and would be divisible by $B_1(y, z)$. Let y, ζ_1 be a generic solution of Λ_1. $B(y, z)\in\Lambda_1$ but the separant of $B(y, z)$ with respect to z is not in Λ_1 (for otherwise the separant of $B_1(y, z)$ would be in Λ_1). Therefore ζ_1 is a nonsingular solution of $A(z)$, a solution of Σ, and a solution of $\omega K(z)-H(z)$. Thus $H(y)K(z)-K(y)H(z)$ vanishes for the generic solution y, ζ_1 of Λ_1, and is in Λ_1. With order in y clearly not greater than p, $H(y)K(z)-K(y)H(z)$ must be divisible by $B_1(y, z)$.

Similarly, $H(y)K(z)-K(y)H(z)$ is divisible by all the irreducible factors $B_1(y, z), \cdots, B_s(y, z)$ of $B(y, z)$ which have order p in y. Since all these $B_i(y, z)$'s are distinct we may write

$$H(y)K(z) - K(y)H(z) = L(y, z)B_1(y, z) \cdots B_s(y, z),$$

where $L(y, z)\in\mathcal{F}\{y, z\}$. Moreover, if we denote the degree of $B(y, z)$ in y_p (the pth derivative of y) by d, we see that the degree of $H(y)K(z)-K(y)H(z)$ in y_p is not greater than d, that of $B_1(y, z) \cdots B_s(y, z)$ is d, so that $L(y, z)$ is of degree 0 in y_p, that is, of order not greater than $p-1$ in y.

Let $B_{s+1}(y, z)$ be an irreducible factor of $B(y, z)$ of order $p-1$ in y, let Λ_{s+1} be the prime component of $\{B_{s+1}(y, z)\}$ not containing the separants of $B_{s+1}(y, z)$, and let y, ζ_{s+1} be a generic solution of Λ_{s+1}. As with y, ζ_1 before, we see that y, ζ_{s+1} is a solution of $H(y)K(z)-K(y)H(z)$. But y, ζ_{s+1} is not a solution of any $B_i(y, z)$ with $i\leq s$, for no such $B_i(y, z)$ is in Λ_{s+1}. Hence y, ζ_{s+1} is a solution of $L(y, z)$, and $L(y, z)\in\Lambda_{s+1}$. This implies, since the order of $L(y, z)$ in y is not greater than $p-1$, that $L(y, z)$ is divisible by $B_{s+1}(y, z)$.

Similarly, $L(y, z)$ is divisible by all the irreducible factors $B_{s+1}(y, z), \cdots, B_t(y, z)$ of order $p-1$ in y, so that

$$H(y)K(z) - K(y)H(z) = M(y, z)B_1(y, z) \cdots B_t(y, z),$$

where $M(y, z)\in\mathcal{F}\{y, z\}$. Moreover, if we denote the degree of $B(y, z)$ in y_p, y_{p-1} by e, we see that the degree of $H(y)K(z)-K(y)H(z)$ in y_p, y_{p-1} is not greater than e, that of $B_1(y, z) \cdots B_t(y, z)$ is e, so that $M(y, z)$ is of degree 0 in y_p, y_{p-1}, that is, of order not greater than $p-2$ in y.

Continuing in this way we finally arrive at an equation

$$H(y)K(z) - K(y)H(z) = P(z)B_1(y, z) \cdots B_v(y, z),$$

where $B_1(y, z), \cdots, B_v(y, z)$ are all the irreducible factors of $B(y, z)$. Since $H(z)$, $K(z)$ have no common divisor, $H(y)K(z)-K(y)H(z)$ has no factor free of y that is not also free of z. Therefore $P(z) \in \mathcal{J}$, and $H(y)K(z)-K(y)H(z)=aB(y, z)$, where $a \in \mathcal{J}$. The desired equation $\omega K(z)-H(z)=\alpha A(z)$ immediately follows.

The rest of the proof of the theorem as given in *Extensions* II is apparently correct.

Of the two examples given in *Extensions* II, the proof for Example 2 is incorrect, and I do not yet know whether that example is valid.

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TWO-DIMENSIONAL GEOMETRIES WITH ELEMENTARY AREAS

HERBERT BUSEMANN

Whereas area in spaces with a smooth Riemann metric has been widely studied, very little is known regarding area in spaces with general metrics. It is natural to ask, first, in which general spaces the most familiar types of formulas for area hold.

The present note answers this question in two cases for two-dimensional spaces in which the geodesic connection is locally unique.[1] It shows under very weak assumptions regarding the nature of area:

I. *If (and only if) locally an area exists for which triangles with equal sides have equal area, then the space is a locally isometric map of either the euclidean plane, or a hyperbolic plane, or a sphere.*

Consequently, Hero's and the corresponding non-euclidean formulas[2] are (up to constant factors) the only possible formulas for area in terms of the sides, and each formula is characteristic for its respective geometry.

II. *If (and only if) locally an area* α *exists such that the area of the triangle* pab *depends only on* p, *the local branch of the geodesic* $\mathfrak{g}$ *that contains the segment* $\mathfrak{s}(a, b)$, *and the distance* ab (the euclidean geometry is, of course, the special case $\alpha = p\mathfrak{g}\cdot ab/2$), *then the space is a locally isometric map of a Minkowski plane.*

The exact hypotheses regarding the space R are these: (1) *R is a metric space.* (2) *R is finitely compact.* (3) *R is two-dimensional.* (4) *R is convex.* If xy denotes the distance of x and y, let (xyz) denote the statement that the three points x, y, z are different and that $xy+yz=xz$. (5) *Every point p has a neighborhood $U(p)$ such that for any two different points x, y in $U(p)$ a point z with (xyz) exists.* (6) *If (xyz_1), (xyz_2) and $yz_1 = yz_2$, then $z_1 = z_2$.*[3]

The following facts are known to hold in R: If $S(p, \rho)$ denotes the set of points x with $px < \rho$, then a $\rho(p) > 0$ exists such that: $S(p, 3\rho(p))$ is homeomorphic to a circular disk [1, p. 29]. The segment $\mathfrak{s}(a, b)$

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[1] The assumptions are formulated further on. They are equivalent to the conditions A, B, C, D in [1, pp. 11, 12]. Numbers in brackets refer to the references cited at the end of the paper.

[2] In spherical geometry L'Huilier's formula for the defect can be used; for a proof see [4, p. 134]. The analogous hyperbolic formula may be obtained in the same manner. A slightly different form is found in [5, p. 129].

[3] The requirements (5) and (6) are equivalent to D in [1, p. 12] or [2, p. 215], see [2, Theorem (4.1)].

is unique for any two points of $S(p, 3\rho(p))$ [2, (4.5)]. If a, b are different points of $S(p, \rho(p))$, then $\mathfrak{s}(a, b) \subset S(p, 2\rho(p))$ [2, (1.15)] and $\mathfrak{s}(a, b)$ is subsegment of a (unique) segment $\mathfrak{h}(a, b) = \mathfrak{s}(a', b')$ with $pa' = pb' = 3\rho(p)$ and $xp < 3\rho(p)$ for $(a'xb')$. Three points c_1, c_2, c_3 of $S(p, 2\rho(p))$ will be called collinear when $c_i \in \mathfrak{h}(a, b)$ for suitable $a \neq b$.

If a, b, c are three noncollinear points of $S(p, \rho(p))$, then the union of the three segments $\mathfrak{s}(a, b)$, $\mathfrak{s}(b, c)$, $\mathfrak{s}(c, a)$ is a closed Jordan curve in $S(p, 2\rho(p))$ and bounds exactly one convex subset $T(abc)$ of $S(p, 2\rho(p))$. (Convexity of a set E means that $a, b \in E$ implies $\mathfrak{s}(a, b) \subset E$.) Therefore a segment connecting two points of $T(abc)$ is again in $S(p, 2\rho(p))$.

A symmetric function $\alpha_p(a_1a_2a_3)$ defined for any triple a_1, a_2, a_3 in $S(p, \rho(p))$ is called an area if:

A: $0 \leq \alpha_p(a_1a_2a_3) < \infty$, *and* $\alpha_p(a_1a_2a_3) = 0$ *if and only if* a_1, a_2, a_3 *are collinear.*

B: *If* (a_2ba_3) *then* $\alpha_p(a_1a_2b) + \alpha_p(a_1ba_3) = \alpha_p(a_1a_2a_3)$.

Statement I will be proved by showing:

(1) If in $S(p, 2\delta(p))$, $0 < \delta(p) \leq \rho(p)/2$, an area $\alpha(a_1a_2a_3)$ exists such that $a_ia_k = a_i'a_k'$ implies $\alpha(a_1a_2a_3) = \alpha(a_1'a_2'a_3')$, then $S(p, \delta(p))$ is isometric to the interior of a circle of the euclidean plane, a hyperbolic plane or a sphere.

Consider two different points a, b in $S(p, \delta(p))$ and two points c_1, c_2 (not on $\mathfrak{s}(a, b)$) in $S(p, \delta(p))$ with $c_ia = c_ib$ such that $\mathfrak{s}(a, b)$ and $\mathfrak{s}(c_1, c_2)$ intersect at a point q. Such c_i exist, compare [1, Theorem III, 2.5′]. Then

$$\alpha(c_1c_2a) = \alpha(c_1c_2b). \tag{2}$$

If $am = mb = ab/2$, then

$$\alpha(c_iam) = \alpha(c_ibm), \qquad i = 1, 2. \tag{3}$$

These relations and properties A, B of α imply that q coincides with m. For if this were not the case it may be assumed that (amq), because the triangle inequality implies that q cannot coincide with a or b. Then $\alpha(c_imq) > 0$, but $\sum_i [\alpha(c_iam) + \alpha(c_imq)] = \sum_i \alpha(c_iaq) = \alpha(c_1c_2a)$ $= \alpha(c_1c_2b) = \sum_i \alpha(c_iqb) = \sum_i [\alpha(c_imb) - \alpha(c_imq)]$, which contradicts (3). Hence the points c_1, c_2, m are collinear.

Let c be any other point in $S(p, \delta(p))$ which is on the bisector $B(a, b)$ of a and b, that is the locus $xa = xb$. Then either $\mathfrak{s}(c_1, c)$ or $\mathfrak{s}(c_2, c)$ intersects $\mathfrak{s}(a, b)$, compare [1, Theorem III, 2.5′]. Therefore c is collinear either with c_1 and m or with c_2 and m. Hence $c \in \mathfrak{h}(c_1, c_2)$ so that $B(a, b)$ is linear. (1) follows then from [2, (15.2)].

Theorem I would become meaningless if the passage: "an area exists" were replaced by "every area that satisfies A and B." For instance, if $\phi(\xi, \eta)$ is any nonconstant positive continuous function in the euclidean plane, then $\alpha(abc) = \int_{T(abc)}\phi(\xi, \eta)d\xi d\eta$ will satisfy A and B but not the relation $\alpha(a_1' a_2' a_3') = \alpha(a_1a_2a_3)$ for any two triangles with equal (euclidean) sides. This remark applies also to Theorem II.

This theorem will be proved in the following form:

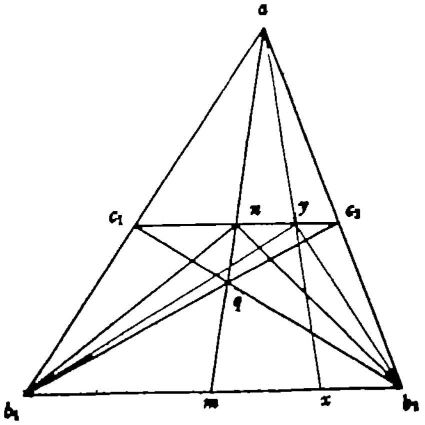
(4) If every point p of R has a neighborhood $S(p, 2\delta(p))$, $0 < \delta(p) \leq \rho(p)/2$, such that in $S(p, 2\delta(p))$ an area $\alpha_p(xab)$ exists which depends only on x, $\mathfrak{h}(a, b)$,[4] and ab, then the universal covering space of R is a Minkowski plane.

The *proof* for (4) is not as easy as for (1). If (abc), then $\mathfrak{h}(a, b) = \mathfrak{h}(b, c)$. Property B of area and standard arguments[5] show that for any point x not on $\mathfrak{h}(a, b)$

$$\alpha(xab):\alpha(xbc) = ab:bc, \tag{5}$$

where α stands for α_p.

Consider now three noncollinear points a, b_1, b_2 in $S(p, \delta(p))$. Let m be the center of $\mathfrak{s}(b_1, b_2)$, (mqa) and put $c_1 = \mathfrak{h}(b_2, q) \cap \mathfrak{s}(b_1, a)$, $c_2 = \mathfrak{h}(b_1, q) \cap \mathfrak{s}(b_2, a)$, $n = \mathfrak{s}(a, m) \cap \mathfrak{s}(c_1, c_2)$. Then (anq).



The relation (5) yields $\alpha(ab_1m) = \alpha(ab_2m)$ and $\alpha(qb_1m) = \alpha(qb_2m)$, hence

$$\alpha(qb_1a) = \alpha(qb_2a) \tag{6}$$

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[4] $\mathfrak{h}(a, b)$ is the exact form of the term "local branch of the geodesic that contains $\mathfrak{s}(a, b)$" employed in II. It is necessary to use that branch instead of the geodesic itself because the example of a torus with a euclidean metric shows that a geodesic may cross an arbitrary neighborhood of a given point infinitely often.

[5] See for instance [6, pp. 1, 2].

so that by (5)

$$\frac{\alpha(qb_1c_1)}{\alpha(qc_1a)} = \frac{b_1c_1}{c_1a} = \frac{\alpha(b_2b_1c_1)}{\alpha(b_2c_1a)} = \frac{\alpha(qb_1b_2) + \alpha(qb_1c_1)}{\alpha(qb_2a) + \alpha(qc_1a)} = \frac{\alpha(qb_1b_2)}{\alpha(qb_2a)}$$

and in the same way

$$\frac{\alpha(qb_2c_2)}{\alpha(qc_2a)} = \frac{b_2c_2}{c_2a} = \frac{\alpha(qb_1b_2)}{\alpha(qb_1a)}.$$

Therefore it follows from (6) that

$$b_1c_1 : c_1a = b_2c_2 : c_2a, \tag{7}$$

$$\alpha(qb_1c_1) = \alpha(qb_2c_2), \quad \text{and} \quad \alpha(qc_1a) = \alpha(qc_2a). \tag{8}$$

Now (5) yields

$$\frac{\alpha(ac_1n)}{\alpha(qc_1n)} = \frac{an}{qn} = \frac{\alpha(ac_2n)}{\alpha(qc_2n)},$$

hence, by (8), $\alpha(ac_1n) = \alpha(ac_2n)$ and then by (5)

$$c_1n = c_2n. \tag{9}$$

Applying (9) and (5)

$$\frac{\alpha(ac_1b_2)}{\alpha(anb_2)} = \frac{2\alpha(anc_2) + 2\alpha(b_2nc_2)}{\alpha(anc_2) + \alpha(b_2nc_2)} = 2$$

so that

$$\frac{an}{am} = \frac{2\alpha(anb_2)}{2\alpha(amb_2)} = \frac{\alpha(ac_1b_2)}{\alpha(ab_1b_2)} = \frac{ac_1}{ab_1}. \tag{10}$$

Successive dyadic subdivisions of $\mathfrak{s}(b_1, b_2)$ and applications of (9) and (10) yield:

(11) If (b_1xb_2) and $\mathfrak{s}(a, x) \cap \mathfrak{s}(c_1, c_2) = y$ then $ay : ax = ac_i : ab_i$ and $c_1y : c_2y = b_1x : b_2x$.

This leads to:

(12) If z_1, z_2 are points of $\mathfrak{h}(a, b) \cap S(p, \delta(p))$ then $\alpha(z_1z_2y)$ is constant for fixed z_i and $y \in \mathfrak{h}(c_1, c_2) \cap S(p, 2\delta(p))$.

For

$$\frac{\alpha(ab_1b_2)}{\alpha(c_1b_1b_2)} = \frac{ab_1}{c_1b_1} = \frac{ax}{yx} = \frac{\alpha(ab_ix)}{\alpha(yb_ix)} = \frac{\alpha(ab_1b_2)}{\alpha(yb_1b_2)}$$

so that $\alpha(b_1b_2y)$ is constant. The general statement follows easily from (5).

We call $\mathfrak{h}_2=\mathfrak{h}(c_1, c_2)$ parallel to $\mathfrak{h}_1=\mathfrak{h}(b_1, b_2)$. (8) and (9) show $\alpha(c_1c_2b_1)=\alpha(c_1c_2b_2)$ so that also $\mathfrak{h}_1$ is parallel to $\mathfrak{h}_2$. The relation (5) yields:

(13) Let $\mathfrak{h}_3$ be a parallel to $\mathfrak{h}_1$ and between $\mathfrak{h}_1$ and $\mathfrak{h}_2$, and $s_i, t_i\in\mathfrak{h}_i$, $(t_1t_3t_2)$, $(s_1s_3s_2)$, then $s_1s_3:s_2s_1=t_1t_3:t_2t_1$. Hence $\mathfrak{h}_3$ is also parallel to $\mathfrak{h}_2$.

Every point x of $S(p, \delta(p))$, in particular p itself, is an interior point of a parallelogram with vertices in $S(p, \delta(p))$, where parallelogram is defined as a quadrangle in which opposite sides lie on parallels in the present sense. For, if $\mathfrak{s}(a, c)$ and $\mathfrak{s}(b, d)$ are noncollinear segments with the same center, then (13) shows that $abcd$ is a parallelogram.

Call P the closed convex set in $S(p, 2\delta(p))$ bounded by the parallelogram $ouwv$, where the vertices are in $S(p, \delta(p))$ and w is opposite to o. Coordinates ξ, η can be introduced in P as follows: The parallels to $\mathfrak{h}(o, v)$ and $\mathfrak{h}(o, u)$ through a given point $q\in P$ may intersect $s(o, u)$ and $s(o, v)$ in q_1 and q_2 respectively. Then $oq_1=\xi$ and $oq_2=\eta$ are the coordinates of q. A line $L=\mathfrak{h}(a, b)\cap P\neq 0$ intersects the boundary of P. Let, for instance, $\mathfrak{s}(o, u)\cap L=a=(\beta, 0)$. Choose any two different points $q_0=(\xi_0, \eta_0)$ and $q=(\xi, \eta)$ on L which are different from a. If $\xi_0=\beta$, then $\xi=\beta$ is the equation of L. If $\xi_0\neq\beta$ let the parallel to $\mathfrak{h}(o, v)$ through a intersect the parallels to $\mathfrak{h}(o, u)$ through q_0 and q_1 at b_0 and b_1 respectively. Then by (11) and (13)

$$\frac{\xi-\beta}{\xi_0-\beta}=\frac{aq}{aq_0}=\frac{ab}{ab_0}=\frac{\eta}{\eta_0}$$

so that ξ and η satisfy a linear equation.

Besides the given distance q_1q_2 introduce in P a euclidean metric $e(q_1, q_2)=[(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2]^{1/2}$ where $q_i=(\xi_i, \eta_i)$. Then the lines L coincide with the euclidean straight line pieces in P; moreover because of (13), the distances $e(q_1, q_2)$ and q_1q_2 are proportional when q_1, q_2 vary on a fixed line L.

This and (13) imply that the universal covering space $\bar{R}$ of R is the whole euclidean plane with the euclidean straight lines as geodesics, and that on each straight line the distance $\bar{q}_1\bar{q}_2$ in $\bar{R}$ is proportional to the euclidean distance $e(\bar{q}_1, \bar{q}_2)$. Therefore (13) holds for R, the parallels now meaning ordinary euclidean lines. It also follows that the circles $\bar{p}\bar{x}=\rho$ about a fixed point p are homothetic. To see that the metric in $\bar{R}$ is Minkowskian it must be shown that circles with different centers $\bar{p}_1$, $\bar{p}_2$ are also homothetic.

Consider two parallel lines $\bar{L}_1$ and $\bar{L}_2$ and a point $\bar{p}$ between them. Let $\bar{f}_1$ be a foot of $\bar{p}$ on $\bar{L}_1$ and let the line through $\bar{p}$ and $\bar{f}_1$ intersect $\bar{L}_2$ at $\bar{f}_2$. Let another line through $\bar{p}$ intersect $\bar{L}_i$ at $\bar{a}_i$. Then by (13)

$1 \geq p\bar{f}_1/pa_1 = p\bar{f}_2/pa_2$, hence $\bar{f}_2$ is a foot of p on $\bar{L}_2$. If $(pq\bar{f}_2)$, then $\bar{f}_2$ is the only foot of p on $\bar{L}_2$ (see [2, (7.9)]), and by the preceding argument $\bar{f}_1$ is a foot of q on $\bar{L}_1$, hence $\bar{f}_1$ is the only foot of p on $\bar{L}_1$. This means that the circles in R are convex [1, Theorems IV, 3.1, 3]. If the lines through p_1 and $\bar{f}_1$ and through p_2 and $\bar{f}_2$ are parallel, and $\bar{L}_1$ is a supporting line to the circle $p_1x = p_1\bar{f}_1$ at $\bar{f}_1$, then the parallel to $\bar{L}_1$ through $\bar{f}_2$ is a supporting line of $p_2x = p_2\bar{f}_2$ [1, Theorem IV, 7.1]. Therefore the two circles are homothetic and the metric of R is Minkowskian.

The "only if" part in II follows from the fact that in a Minkowskian plane the ordinary area of a euclidean plane with the same geodesics will satisfy A and B.

The plane, cylinder, torus, Moebius strip, and the one-sided torus are the only two-dimensional manifolds that can carry locally euclidean metrics (compare [3, Chapter III, §VII]). The arguments used to prove this fact show also that these *are the only manifolds that can carry locally Minkowskian metrics.* If a Minkowski metric is given, then cylinders and tori exist which have locally this metric. But a *Moebius strip and a one-sided torus with this metric exist, if and only if the Minkowski metric admits a reflection in some straight line,* see [3, pp. 81–85].

The preceding discussion suggests the question: in which geometries is the locus of the vertex x of a triangle xab with the fixed base $\mathfrak{s}(a, b)$, $a \neq b$, a fixed area, and on a given side of $\mathfrak{h}(a, b)$ a straight line? However, as long as area only satisfies A and B, this question is not reasonable, as the following example shows: Represent the hyperbolic plane by a euclidean model, in which the straight line pieces in the interior K of the euclidean unit circle are the hyperbolic geodesics. For three points a, b, c in K define $\alpha(abc)$ as the content of the triangle in the sense of the imbedding euclidean metric. Then α satisfies A and B and the above condition.

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ON PRINCIPAL LINES AND PRINCIPAL POINTS

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Recently Chang[1] has enriched the contact theory of two plane curves by virtue of many projective correspondences. The object of this paper is to establish some analogous results regarding the contact theory of two space curves.

In an appendix to Fubini and Čech's book[2] Bompiani remarked that the contact theory of two space curves can be treated by considering a double ratio analogous to that of C. Segre.[3] Accordingly Palozzi[4] has obtained principal lines and principal points associated with two space curves which are tangent but have distinct osculating planes or possess a contact of higher order at an ordinary point. But in the latter case Stouffer[5] has shown that neither principal lines nor principal points exist when the principal plane becomes the common osculating plane of the two curves. Naturally in Stouffer's case we should require new covariant figures. We solve this problem by obtaining a new covariant line.

Between the points P and $\bar{P}$ of the two space curves C and $\bar{C}$, which are tangent or intersect at an ordinary point, we define a correspondence G, such that the tangents to C and $\bar{C}$ at corresponding points of G intersect. This correspondence plays an important role in this paper because the joins of the corresponding points of G constitute developables. We can deduce from G new definitions of Bompiani's[6] principal lines and principal points for two intersecting space curves.

Two space curves C and $\bar{C}$ intersecting at an ordinary point $O(0, 0, 0)$ with distinct osculating planes may be represented by the

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expansions

$$y = ax^3 + bx^4 + \cdots, \quad z = rx^2 + sx^3 + \cdots, \tag{1}$$

$$x = \bar{a}y^3 + \bar{b}y^4 + \cdots, \quad z = \bar{r}y^2 + \bar{s}y^3 + \cdots, \tag{2}$$

respectively, where x, y, z are nonhomogeneous coordinates of a point in space. After denoting the coordinates of the corresponding points P and $\bar{P}$ by (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \bar{a}\bar{y}^3 + \bar{b}\bar{y}^4 + \cdots$$

and

$$\bar{z} = \bar{r}\bar{y}^2 + \bar{s}\bar{y}^3 + \cdots,$$

we easily see from the vanishing of the determinant

$$\begin{vmatrix} x & y & z & 1 \\ 1 & dy/dx & dz/dx & 0 \\ \bar{x} & \bar{y} & \bar{z} & 1 \\ d\bar{x}/d\bar{y} & 1 & d\bar{z}/d\bar{y} & 0 \end{vmatrix} \tag{3}$$

that the correspondence G is given by

$$x = \tau_1\bar{y} + \tau_2\bar{y}^2 + \cdots, \tag{4}$$

where

$$\tau_1 = \pm\left(\frac{\bar{r}}{r}\right)^{1/2}, \qquad \tau_2 = \frac{s\tau_1^3 - \bar{s}}{-r\tau_1}. \tag{5}$$

The line $P\bar{P}$ has the equation

$$\frac{X - x}{\bar{x} - x} = \frac{Y - y}{\bar{y} - y} = \frac{Z - z}{\bar{z} - z}, \tag{6}$$

which evidently approaches the principal line[7]

$$X = -\tau_1 Y, \qquad Z = 0 \tag{7}$$

as $x \to 0$, X, Y, Z being the current coordinates of a point. In addition, the edges of regression of the developables constituted by the lines (6) pass through

$$\left(\frac{\bar{r}\tau_1}{-\bar{s} + s\tau_1^3}, \frac{-\bar{r}}{-\bar{s} + s\tau_1^3}, 0, 1\right), \qquad \left(\tau_1 = \pm\left(\frac{\bar{r}}{r}\right)^{1/2}\right), \tag{8}$$

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[7] Bompiani, loc. cit.

respectively. We have, therefore, reached the following theorem.

THEOREM 1. *When C and $\bar{C}$ intersect at an ordinary point O with distinct tangents, there are two and only two developables containing both C and $\bar{C}$. The edges of regression of these two developables are tangent to the two principal lines at the two principal points.*

Let M be a moving point on C. We denote the two edges of regression in the above theorem by E_{11} and E_{12}. If the tangents of E_{11} and E_{12} at their points M_{11} and M_{12} pass through M, then the line $M_{11}M_{12}$ constitutes a developable with edge of regression E_{21} as M varies on C. Similarly we obtain E_{22} by substituting $\bar{C}$ for C. Clearly on E_{2i} $(i=1, 2)$ there is a point M_{2i} at which the tangent to E_{2i} passes through M_{11} and the line $M_{21}M_{22}$ constitutes a developable with edge of regression E_{31} when M_{11} varies on E_{11}. Thus we obtain E_{ki} $(k=1, 2, 3, \cdots; i=1, 2)$. For brevity we denote E_{k1} and E_{k2} by (E_k) and call (E_k) the G-transform of (E_{k-1}).

Since G is one-to-two, a point P on C corresponds to two points $\bar{P}_1$ and $\bar{P}_2$ on $\bar{C}$. We denote the locus of the points of intersection N_{11} and N_{12} between the tangent of C at P and those of $\bar{C}$ at $\bar{P}_1$ and $\bar{P}_2$ by F_{11} and F_{12}. When P moves along C, the point of intersection of the tangents of F_{11} and F_{12} at N_{11} and N_{12} is a new curve F_{21}. The interchange between $\bar{C}$ and C gives F_{22}. In this way we obtain the curves F_{ki} $(k=1, 2, \cdots; i=1, 2)$, where (F_k), which denotes the two curves F_{k1} and F_{k2}, is evidently the G-transform of (F_{k+1}).

THEOREM 2. *With regard to the curves* (1) *and* (2) *there is an infinite sequence of pairs of curves*

$$\cdots, (F_k), (F_{k-1}), \cdots, (F_1), (C, \bar{C}), (E_1), (E_2), \cdots, (E_k), \cdots$$

including the pair C and $\bar{C}$ such that every pair is the G-transform of the preceding pair.

In case the osculating planes of C and $\bar{C}$ at O coincide, the expansions (1) and (2) should be replaced by

$$y = ax^2 + bx^3 + \cdots, \qquad z = rx^3 + sx^4 + \cdots, \tag{9}$$

and

$$x = \bar{a}y^2 + \bar{b}y^3 + \cdots, \qquad z = \bar{r}y^3 + \bar{s}y^4 + \cdots, \tag{10}$$

respectively. Now (4) becomes

$$x = \mu_1\bar{y} + \mu_2\bar{y}^2 + \cdots \tag{11}$$

where

$$(12)\qquad \mu_1^3 = \frac{\bar r}{r}, \qquad \mu_2 = \frac{1}{2r\mu_1^2}\left(- s\mu_1^4 + a\bar r\mu_1^2 + \bar s - \bar a r\mu_1^2\right).$$

Here the correspondence G is one-to-three. Clearly the joins of the corresponding points of G approach principal lines when P approaches O and constitute three developables whose edges of regression E_{1k} $(k=1, 2, 3)$ pass through the three principal points respectively.

With respect to two roots of $\mu_1^3=\bar r/r$ we obtain through (11) two correspondences between points of C and $\bar C$. In virtue of them we may determine a sequence of pairs of curves

$$(13)\qquad \cdots, (F_k^{(l)}), \cdots, (C, \bar C), \cdots, (E_k^{(l)}), \cdots, \qquad (k = 1, 2, \cdots; l = 1, 2, 3),$$

analogous to that of Theorem 2. Consequently we obtain a total of three sequences of transforms.

We call the point of contact of the join $P\bar P$ and E_{1l} the corresponding point of P or $\bar P$ on E_{1l}. We have arrived at the following theorem.

THEOREM 3. *When C and $\bar C$ intersect at an ordinary point O with distinct tangents but common osculating plane, there are three developables passing through C and $\bar C$ and three sequences of G-transforms* (13). *In particular the points corresponding to O on the G-transforms of C and $\bar C$ are the three principal points.*

If C and $\bar C$ have a common tangent at O, we take it for the x-axis. The expansions of C and $\bar C$ now become

$$(14)\qquad y = x^2\sum_0^\infty a_i x^i, \qquad z = x^2\sum_0^\infty b_i x^i$$

and

$$(15)\qquad y = x^2\sum_0^\infty \bar a_i x^i, \qquad z = x^2\sum_0^\infty \bar b_i x^i.$$

If the osculating planes at O of C and $\bar C$ are distinct, the correspondence G is analytically given by

$$(16)\qquad x = \bar x + m\bar x^2 + n\bar x^3 + \cdots,$$

where

$$m = \frac{1}{2(a_0\bar b_0 - \bar a_0 b_0)}\left[(b_0 - \bar b_0)(a_1 - \bar a_1) - (a_0 - \bar a_0)(b_1 - \bar b_1)\right],$$

$$n = -m^2 - \frac{m}{(a_0\bar{b}_0 - \bar{a}_0 b_0)}\left[2(a_0 b_1 - a_1 b_0) + 3(a_1\bar{b}_0 - \bar{a}_0 b_1) + (a_0\bar{b}_1 - \bar{a}_1 b_0)\right] + \frac{1}{a_0\bar{b}_0 - \bar{a}_0 b_0}\left[(a_2 - \bar{a}_2)(b_0 - \bar{a}_0) - (a_0 - \bar{a}_0)(b_2 - \bar{b}_2)\right].$$

Clearly the limit of the join of the corresponding points P and $\bar{P}$ when $P \to O$ is the principal line[8]

$$Y = \frac{a_0 - \bar{a}_0}{m} X, \qquad Z = \frac{b_0 - \bar{b}_0}{m} X. \tag{17}$$

Since the Halphen plane[9] osculates the edge of regression of the developable consisting of the joins $P\bar{P}$ at the principal point,[10] we have proved the following theorem.

THEOREM 4. *There is a unique developable passing through two space curves which are tangent at an ordinary point with distinct osculating planes. The edge of regression of this developable passes through the principal point, is tangent to the principal line, and osculates the Halphen plane.*

Now the sequence of G-transforms of C and $\bar{C}$ are terminated on both sides. Both E_1 and F_1 contain one curve only. In particular, F_1 has at O a point of inflexion[11] with the hyperosculating plane

$$\frac{b_0 Y - a_0 Z}{\bar{b}_0 Y - \bar{a}_0 Z} = \frac{a_0 b_1 - a_1 b_0}{\bar{a}_0 \bar{b}_1 - \bar{a}_1 \bar{b}_0} \tag{18}$$

which is a *new covariant plane*.

If C and $\bar{C}$ have at O contact of order $k+2$, then (14) and (15) become

$$y = x^2 \sum_0^\infty a_i x^i, \qquad z = x^3 \sum_0^\infty b_i x^i, \tag{19}$$

and

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[8] Palozzi, loc. cit.

[9] Palozzi, loc. cit.

[10] Palozzi, loc. cit.

[11] Chang, *Contributions to projective theory of singular points of space curves*, in the hands of The International Cultural Service of China.

$$y = x^2\sum_0^\infty \bar{a}_i x^i, \qquad z = x^2\sum_0^\infty \bar{b}_i x^i, \tag{20}$$

where

$$a_l = \bar{a}_l \quad (l = 0, 1, \cdots, k); \qquad b_m = \bar{b}_m \quad (m = 0, 1, \cdots, k-1).$$

Here either $a_{k+1} \neq \bar{a}_{k+1}$ or $b_k \neq \bar{b}_k$ must be true. By a little consideration of the transformation

$$x = \xi, \qquad y = \eta, \qquad \zeta = (\bar{b}_k - b_k)y + (a_{k+1} - \bar{a}_{k+1})z,$$

we find two cases are essential to consider: firstly, $a_{k+1} = \bar{a}_{k+1}$, $b_k \neq \bar{b}_k$ and secondly, $a_{k+1} \neq \bar{a}_{k+1}$, $b_k = \bar{b}_k$. In the latter case the Halphen plane and $Z = 0$ coincide but in the former they do not.

For the case $b_k \neq \bar{b}_k$, $a_{k+1} = \bar{a}_{k+1}$ the correspondence G may be analytically given by

$$x = \bar{x} + \sum_{\nu=k+2}^{\infty} C_\nu \bar{x}^{\nu+1}, \tag{21}$$

where

$$\begin{aligned} C_{k+2} &= -\frac{1}{2a_0}(a_{k+2} - \bar{a}_{k+2}), \\ C_{k+3} &= \frac{1}{2a_0(b_k - \bar{b}_k)}(a_{k+2} - \bar{a}_{k+2})(b_{k+1} - \bar{b}_{k+1}) \\ &\quad + \frac{3a_1(k+2)}{4a_0^2}(a_{k+2} - \bar{a}_{k+2}) - \frac{a_{k+3} - \bar{a}_{k+3}}{a_0}. \end{aligned} \tag{22}$$

After a little consideration we can prove the following theorem.

THEOREM 5. *If C and $\bar{C}$ have at O contact of order $k+2$ $(k \geqq 1)$ and if $a_{k+1} = \bar{a}_{k+1}$, $b_k \neq \bar{b}_k$, then the edge of regression E of the unique developable containing C and $\bar{C}$ passes through the principal point T. Furthermore, the tangent of E at T is the principal line and the osculating plane of E at T coincides with the Halphen plane of C and $\bar{C}$.*

If in the last two paragraphs we put $k = 0$, then the correspondence G is one-to-four and (21) should be replaced by

$$x = \alpha_i \bar{x} + C_2(\alpha_i)\bar{x}^2 + C_3(\alpha_i)\bar{x}^3 + \cdots \qquad (i = 1, 2, 3, 4), \tag{23}$$

where α_i $(i = 1, 2, 3, 4)$ are the four roots of the biquadratic equation

$$b_0 t^4 - 4b_0 t^3 + (3b_0 + 3\bar{b}_0)t^2 - 4\bar{b}_0 t + \bar{b}_0 = 0.$$

It is very easy to prove the following theorem.

THEOREM 6. *If C and $\bar C$ have at O contact of order two where $b_0 \neq \bar b_0$, then there are four developables passing through C and $\bar C$. Only one of these four developables is not tangent to the plane $Z=0$ but tangent to the Halphen plane of C and $\bar C$ along the principal line. The edge of regression of this particular developable passes through the principal point.*

By virtue of every two correspondences of (23) we can derive a sequence of G-transforms associated with C and $\bar C$. Therefore we have six sequences.

Now we consider the special case that C and $\bar C$ are tangent and that their osculating planes at the point of contact are coincident. The expansions of C and $\bar C$ are (19) and (20) where $a_0 \neq \bar a_0$. The correspondence G is also one-to-four and may be expressed by

$$x = \beta_i \bar x + D_2(\beta_i)\bar x^2 + D_3(\beta_i)\bar x^3 + \cdots \qquad (i = 1, 2, 3, 4), \tag{24}$$

where β_i $(i=1, 2, 3, 4)$ are the roots of

$$a_0 b_0 t^4 - 4\bar a_0 b_0 t^3 + 3(\bar a_0 b_0 + a_0 \bar b_0)t^2 - 4a_0\bar b_0 t + \bar a_0 \bar b_0 = 0. \tag{25}$$

In general $t \neq 1$ and we arrive at the following theorem.

THEOREM 7. *There are four developables passing through the two space curves* (19) *and* (20) *which are tangent at an ordinary point with common osculating plane.*

Naturally we can derive six sequences of curves by G-transforms. In general, all of these curves pass through O and possess common tangent $Y=0$, $Z=0$ and common osculating plane $Z=0$.

In the last three paragraphs $b_0 \neq \bar b_0$. If the projecting cones of C and $\bar C$ with common vertex on $Z=0$ have contact of third order, we have

$$b_0 = \bar b_0.$$

Since $a_0 \neq \bar a_0$, a single root of (25) is 1, and one of the expressions (24) becomes

$$x = \bar x - \frac{b_1 - \bar b_1}{3\bar b_0}\bar x^2 + \cdots . \tag{26}$$

The developable consisting of the joins of the points of C and $\bar C$ corresponding under (26) *is tangent to $Z=0$ along the generator*

$$Z = 0, \qquad \frac{X}{b_1 - \bar b_1} + \frac{Y}{3b_0(a_0 - \bar a_0)} = 0. \tag{27}$$

Furthermore, *the edge of regression of this developable intersects* $Z=0$ *at a new covariant point.*

The plane $Z=0$ intersects the developables of C and $\bar{C}$ in two plane curves

$$y = \frac{3}{4}a_0x^2 + \frac{3}{8}\frac{a_0b_1}{b_0}x^3 + \cdots, \qquad z = 0 \tag{28}$$

and

$$y = \frac{3}{4}\bar{a}_0x^2 + \frac{3}{8}\frac{\bar{a}_0\bar{b}_1}{\bar{b}_0}x^3 + \cdots, \qquad z = 0, \tag{29}$$

which have at O contact of order one and, therefore, possess a covariant line r_0[12] which is coincident with (27). Hence we have reached the following theorem.

THEOREM 8. *If on any plane through* $Y=0$, $Z=0$ *the projections of the curves* (19) *and* (20) *from any point on* $Z=0$ *have at* O *contact of order three, one of the four developables mentioned in Theorem* 7 *does not pass through* $Y=0$, $Z=0$ *but through the covariant line* r_0 *of the curves* (28) *and* (29).

If $b_0 \neq \bar{b}_0$, *the* r_0 *of* (28) *and* (29) *is found to be*

$$Z = 0, \qquad 3(\bar{a}_0 - a_0)X + \left(\frac{\bar{b}_1}{\bar{b}_0} - \frac{b_1}{b_0}\right)Y = 0, \tag{30}$$

which is an important covariant line associated with the curves (19) *and* (20).

Now we deal with the remaining case that the coefficients in (19) and (20) are connected by the relations

$$a_m = \bar{a}_m, \qquad b_m = \bar{b}_m \qquad (m = 0, 1, \cdots, k), \tag{31}$$

where $a_{k+1} \neq \bar{a}_{k+1}$ $(k \geq 0)$. (21) should be replaced by

$$x = \bar{x} + \sum_{r=k+1}^{\infty} D_r\bar{x}^{r+1}, \tag{32}$$

where $D_{k+1+\tau} = D_{k+1+\tau}(D_{k+1})$ $(\tau > 0)$ and

$$\begin{aligned} D_{k+1}[6b_0(a_{k+1} - \bar{a}_{k+1}) - 2a_0(b_{k+1} - \bar{b}_{k+1})]& \\ + (a_{k+1} - \bar{a}_{k+1})(b_{k+1} - \bar{b}_{k+1}) &= 0. \end{aligned} \tag{33}$$

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[12] Bompiani, loc. cit. Chang, loc. cit. footnote 1.

Consequently we have proved the following theorem.

THEOREM 9. *If C and $\bar{C}$ have at O contact of order $k+2$ $(k \geqq 0)$ where $a_{k+1} \neq \bar{a}_{k+1}$, $b_k = \bar{b}_k$, then there is a unique developable passing through these two curves.*

The edge of regression E^* of this developable passes through O. Since E^* and C (or $\bar{C}$) have at O a common tangent $Y=0$, $Z=0$ and common osculating plane $Z=0$, the application of our results of Theorems 7 and 8 to E^* and C (or $\bar{C}$) leads to a covariant line. But we particularly remark that *the developable of Theorem* 9 *is tangent to* $Z=0$ *along*

$$Z = 0, \qquad \frac{X}{b_{k+2} - \bar{b}_{k+2}} + \frac{Y}{3b_0(a_{k+1} - \bar{a}_{k+1})} = 0$$

when $b_{k+1} = \bar{b}_{k+1}$, $b_{k+2} \neq \bar{b}_{k+2}$.

A direct analogy of Chang's correspondence[13] F is that the osculating planes of two space curves at their corresponding points pass through one and the same point on the common tangent of these two curves. In general, there are two ways to generalize the correspondence F. Firstly, let the points P and $\bar{P}$ of the curves C and $\bar{C}$ in N-dimensional space so correspond that a prescribed $(N-1-2\mu)$-dimensional space $(2\mu+1<N)$ and the osculating linear spaces $S_\mu(P)$ and $S_\mu(\bar{P})$ of C and $\bar{C}$ at P and $\bar{P}$ belong to one and the same hyperplane, and secondly, let P and $\bar{P}$ so correspond that the common part of the osculating linear spaces $S_\nu(P)$ and $S_\nu(\bar{P})$ of ν dimensions, where $2\nu \geqq N$, are linearly dependent with a given $(2N-2\nu)$-dimensional space. These will be, however, established on another occasion.

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[13] Chang, loc. cit. footnote 1.

SOME PROBLEMS IN CONFORMAL MAPPING

D. C. SPENCER

1. Introduction. Attention will be confined to a group of problems centering around so-called schlicht functions—that is, functions regular in a given domain and assuming no value there more than once. The type of problem we consider involves determination of precise bounds for certain quantities depending on the function f, as f ranges over the schlicht functions in question. Since, for suitable normalization of the functions at some fixed point of the domain, the resulting family of functions is compact or normal, the extremal schlicht functions always exist and the problem is to characterize them.

Interest was focused on this category of questions by the work of Koebe in the years 1907–1909, who established for the family of functions f of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$, schlicht and regular in $|z|<1$, a series of properties, among them the theorem of distortion bearing Koebe's name. This theorem asserts the existence of bounds for the absolute value of the derivative $f'(z)$, these bounds depending only on $|z|$. Further efforts were directed toward finding the precise values of the bounds asserted by Koebe's theorem, but success was not attained until 1916 when Bieberbach, Faber, Pick and others gave a final form to the theorem of distortion. At the same time the precise bound for $|a_2|$ was given, namely 2, and the now famous conjecture was made that $|a_n|\leqq n$ for every n. Since 1916 this group of problems has attracted the attention of many, and there is now a considerable literature.

The present state of this sphere of questions will be described briefly in a general sort of way, and a few outstanding problems will be indicated, but no attempt at completeness has been made.

2. The coefficient problem. Let S be the family of functions

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{2.1}$$

which are regular and schlicht in $|z|<1$. The most famous problem concerning these functions is whether $|a_n|\leqq n$ $(n=2, 3, \cdots)$, with equality for any n only in the case when

$$f(z) = \frac{z}{(1-\eta z)^2} = z + 2\eta z^2 + 3\eta z^3 + \cdots, \qquad |\eta| = 1. \tag{2.2}$$

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It is this problem which has stimulated much of the research leading to the various methods, in particular the method of parametric representation given by Löwner [7][1] in 1923 and the recent methods (see [11, 12]) in which the extremal function is compared with infinitesimal variations more general than those provided by Löwner's method.

There are several short proofs that $|a_n| \leqq n$ ($n=2, 3, \cdots$) when all the coefficients are real, and a simple proof, due to Rogosinski [10], will be sketched here. Without loss of generality we may assume that f is regular and schlicht in $|z| \leqq 1$, for every schlicht function is the limit of such functions. Since f has real coefficients, it takes conjugate values for conjugate values of z and so the map of $|z|<1$ by f is symmetrical with respect to the real axis. Since f is schlicht, Im f and Im z have the same sign in $|z| \leqq 1$. The function

$$\frac{1-z^2}{z} f(z) = p(z) \tag{2.3}$$

therefore has a positive real part in $|z|<1$ since on the boundary we have, writing $z=e^{i\theta}$, $\operatorname{Re} p(e^{i\theta}) = 2 \sin\theta \cdot \operatorname{Im} f(e^{i\theta}) \geqq 0$. It follows that the coefficients of p are majorized by those of the function

$$\frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots .$$

Since the coefficients of this function as well as the coefficients of the function

$$\frac{z}{1-z^2} = z + z^3 + z^5 + \cdots \tag{2.4}$$

are positive, we see at once, multiplying both sides of (2.3) by (2.4), that the coefficients of f are majorized by the coefficients of the function

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots . \tag{2.5}$$

Similarly, $|a_n| \leqq n$ when f maps $|z|<1$ on a star-like domain. In both cases equality is attained only when f is of the form (2.2). We see here a connection between schlicht functions and functions with positive real part, a connection that will be further emphasized in §3 below.

In the general case $|a_n| < e \cdot n$ (see [6(a)]), but the only known precise inequalities are: $|a_2| \leqq 2$ and $|a_3| \leqq 3$, with equality in either case only if f has the form (2.2). It is comparatively easy to prove that $|a_2| \leqq 2$, and many proofs of this result are known. The inequality $|a_3| \leqq 3$ lies much deeper, only two essentially different proofs being known. The first proof was given by Löwner [7] in 1923, using his method of parametric representation; the second, recently given, makes use of methods quite different from Löwner's (see [11(a)]).

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Although the problem of obtaining precise bounds for the coefficients has attracted the most attention, the central problem of the theory is really the so-called coefficient problem. Given the point $(a_2, a_3, \cdots, a_n)$, we say that

$$f(z) = z + b_2 z^2 + b_3 z^3 + \cdots$$

belongs to the point $(a_2, a_3, \cdots, a_n)$ if f is regular and schlicht in $|z| < 1$ and if

$$b_2 = a_2, \; b_3 = a_3, \cdots, \; b_n = a_n.$$

Conversely we say that the point belongs to f. The nth region of variability S_n is the set of points $(a_2, a_3, \cdots, a_n)$ in a Euclidean space of $2n-2$ real dimensions each of which belongs to some $f(z)$. The interior of S_n is a bounded, simply-connected domain containing the origin $a_2=0$, $a_3=0$, $\cdots$, $a_n=0$, and S_n is the closure of its interior. The only one of these regions which is trivial is S_2, and it is the circle $|a_2| \leqq 2$.

The coefficient problem is the problem of finding S_n for $n=2, 3, \cdots$; this problem was introduced by Bieberbach.

A method has recently been developed which characterizes S_n for general n—explicitly for $n=3$ and implicitly for $n>3$.[2] The boundary of S_3 may be expressed by equations involving only elementary functions, but for $n>3$ the boundary is complicated and cannot be so expressed. We therefore state the result only for $n=3$.

We begin with the trivial observation that, given any schlicht function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, the function

$$e^{-i\theta} f(e^{i\theta} z) = z + a_2 e^{i\theta} z^2 + a_3 e^{2i\theta} z^3 + \cdots$$

is also normalized and schlicht. Hence if $S_3^{(0)}$ is the subset of S_3 for which a_2 is real, the whole of S_3 is the set of points $(a_2 e^{i\theta}, a_3 e^{2i\theta})$, $0 \leqq \theta < \pi$, where (a_2, a_3) belongs to $S_3^{(0)}$. In other words, S_3 is obtained from $S_3^{(0)}$ by rotations. Since the region $S_3^{(0)}$ is symmetrical with respect to the plane $a_2=0$, it is sufficient to consider the part of the boundary of $S_3^{(0)}$ which lies in the half-space $a_2\geqq 0$. This portion of the boundary of $S_3^{(0)}$ is made up of the following two analytic surfaces plus their intersection:

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[2] See [11(c)]. The region of variability of (a_2, a_3) when a_2 and a_3 are both real had been given previously by Peschl [8].

(1) Suppose that $0\leqq\alpha\leqq\pi/2$. Let $-2(\sin\alpha-\alpha\cos\alpha)\leqq\mu\leqq 2(\sin\alpha-\alpha\cos\alpha)$ and define $\lambda=2\cos\alpha\{\log(\cos\alpha)-1\}$. Then one of the two surfaces is defined by

$$\begin{aligned} a_2 &= (\lambda^2+\mu^2)^{1/2} \\ a_3 &= \lambda^2+\mu^2+2\cos\alpha(\lambda-i\mu)+(2\cos^2\alpha+1)\frac{\lambda-i\mu}{\lambda+i\mu}\cdot \end{aligned} \tag{2.6}$$

Any function $w=f(z)$ belonging to a point of this surface generally maps $|z|<1$ on the w-plane minus a forked slit composed of a ray $\arg(w)=$ constant extending from $w=\infty$ to some finite point where there is a fork composed of two prongs which form angles $2\pi/3$ with the ray. On the edge of this surface one or both prongs degenerate to a point. If $\alpha=\pi/2$, $w=f(z)$ maps $|z|<1$ on the w-plane minus two rays $\arg(w)=$ constant which make an angle π at $w=\infty$; in this case a_2, a_3 lie on the parabola $a_2=\mu$, $a_3=\mu^2-1$, $0\leqq\mu\leqq 2$.

(2) Suppose that $0<r<1$, $-\pi/2\leqq\phi\leqq\pi/2$. Let

$$\rho^2 = 1+6r^2+r^4+4r(1+r^2)\cos 2\phi,$$

$$\tan\alpha = \frac{2r\sin 2\phi}{1+2r\cos 2\phi+r^2} \qquad \left(-\frac{\pi}{2}<\alpha<\frac{\pi}{2}\right),$$

$$C_1 = \frac{(1+r)^2}{2r}\cos\phi, \qquad C_2 = \frac{(1-r)^2}{2r}\sin\phi,$$

$$\lambda = C_1\log\frac{\rho}{(1+r)^2}+C_2\alpha-2\cos\phi,$$

$$\mu = C_2\log\frac{\rho}{(1-r)^2}-C_1\alpha+2\sin\phi.$$

The second analytic surface is defined by

$$\begin{aligned} a_2 &= (\lambda^2+\mu^2)^{1/2} \\ a_3 &= \lambda^2+\mu^2+(C_1+iC_2)(\lambda-i\mu) \\ &\quad +\left(r+\frac{1}{r}+\cos 2\phi\right)\frac{\lambda-i\mu}{\lambda+i\mu}\cdot \end{aligned} \tag{2.7}$$

If $f(z)$ belongs to a point on this boundary surface of $S_3^{(0)}$, then $w=f(z)$ maps $|z|<1$ on the w-plane minus a single curved analytic slit extending from $w=\infty$ to some finite point. As $r\rightarrow 0$, the slit tends to a straight line arg $(w)=$constant and the corresponding f tends to $z/(1+e^{-i\phi}z)^2$. As $r\rightarrow 1$,

$$\lambda \rightarrow 2\cos\phi\{\log(\cos\phi)-1\},\qquad \mu\rightarrow 2(\sin\phi-\phi\cos\phi).$$

That is, $r=1$ corresponds to the edge of intersection of the two surfaces. For functions $w=f(z)$ belonging to this edge, one of the two prongs of the fork is absent.

Even in the case $n=3$ we observe that the equations defining the boundary of S_3 are sufficiently complicated that it is difficult to infer directly from them that $|a_3|\leqq 3$. Since for $n>3$ the situation is much more complicated, the precise bounds for the coefficients, a specific question about these regions, remain undetermined. Other questions concerning these regions may be asked; for example, what are the maximum and minimum distances of the boundary of S_n from the origin (distance from the origin being equal to $(\sum_{\nu=2}^{n}|a_\nu|^2)^{1/2}$). In the case of S_3, the maximum distance of the boundary from the origin is $(2^2+3^2)^{1/2}=13^{1/2}$, attained only for functions of the form (2.2), and the minimum distance of the boundary from the origin is $3^{1/2}/2$, attained only for functions of the form

$$\frac{z}{1-\eta z/2^{1/2}+\eta^2z^2}=z+\frac{1}{2^{1/2}}\eta z^2-\frac{1}{2}\eta^2z^3+\cdots,\quad |\eta|=1. \tag{2.8}$$

To each boundary point $(a_2, a_3, \cdots, a_n)$ of S_n there belongs one and only one function $w=f(z)$, and this function maps the unit circle on the w-plane minus piecewise analytic slits. Given any complex numbers $p_2, p_3, \cdots, p_n, \sum_{\nu=2}^{n}|p_\nu|^2=1$, let

$$\begin{aligned} L &= \operatorname{Re}(p_2a_2+p_3a_3+\cdots+p_na_n)\\ &= (p_2a_2+\bar{p}_2\bar{a}_2+\cdots+p_na_n+\bar{p}_n\bar{a}_n)/2. \end{aligned} \tag{2.9}$$

If the maximum value of L in S_n is M, then S_n lies entirely on one side of the $(2n-3)$-dimensional hyperplane $L=M$. Since S_n cannot be convex for $n>2$, such a supporting hyperplane can touch the boundary only at a well-defined subset of boundary points of S_n. Any function $w=f$ belonging to a point of this subset maps $|z|<1$ on the w-plane minus one or more analytic slits meeting at infinity and each of these slits has no finite critical points and is therefore unforked.

If for two different sets of numbers $p_2, p_3, \cdots, p_n$ the corresponding L's given by (2.9) both have a local extremum at the same boundary point $(a_2, a_3, \cdots, a_n)$ of S_n, then the function f belonging to this point is algebraic. These algebraic boundary functions map $|z|<1$

onto the plane minus slits. It would be interesting to study the character of algebraic schlicht functions which map onto slit regions.

The method used to characterize S_n has been sketched in [11(c)] and a detailed exposition is in course of preparation by A. C. Schaeffer and the author. Therefore, no attempt will be made to describe the method here. We remark only that the starting point of the method is to maximize a certain function $F(a_2, \bar{a}_2, a_3, \bar{a}_3, \cdots, a_n, \bar{a}_n)$ in S_n, the maximum being attained at a boundary point $(a_2, a_3, \cdots, a_n)$. If f belongs to this boundary point, a differential equation for f is obtained by making infinitesimal variations of this extremal function. The method then consists of a study of the resulting differential equation.

In the next section (§3), Löwner's method of parametric representation will be briefly discussed. This method provides an ϵ-variation of any schlicht function, but the variation is one-sided in that ϵ can have only one sign. This defect does not occur in the more powerful variational methods recently developed (see [11] and [12]).

3. Relations between coefficients of schlicht functions and coefficients of functions with positive real part. In the case of functions with positive real part, the regions of variability of the coefficients have been obtained by Carathéodory [1], Toeplitz [13], and others. If numbers $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ are given, then there is a function

$$p(z) = 1 + 2\sum_{\nu=1}^{\infty} c_\nu z^\nu \tag{3.1}$$

which is regular and has positive real part in $|z| < 1$ with

$$c_1 = \gamma_1, \ c_2 = \gamma_2, \cdots, c_{n-1} = \gamma_{n-1}$$

if and only if the Hermitian form

$$H = \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} \gamma_{\mu-\nu} x_\mu \bar{x}_\nu \tag{3.2}$$

is positive semi-definite, where $\gamma_0 = 1$ and $\gamma_{-k} = \bar{\gamma}_k$ is the complex conjugate of γ_k. In geometrical language, there is a function $p(z)$ with $\operatorname{Re} p(z) > 0$ in $|z| < 1$ if and only if $(\gamma_1, \gamma_2, \cdots, \gamma_{n-1})$ lies inside or on the boundary of the smallest convex region containing the curve $e^{i\theta}, e^{2i\theta}, \cdots, e^{i(n-1)\theta}$ (θ real), and the interior of this region is characterized by the property that H is positive definite there. The nth region of variability P_n is the set of points $(c_1, c_2, \cdots, c_{n-1})$ each of which belongs to some $p(z)$. The points of P_n have the one-to-one parametric representation

$$(3.3)\qquad \begin{aligned} c_k &= \sum_{j=1}^{n-1} \mu_j e^{ik\theta_j}, &\qquad 1 \leq k \leq n-1,\ \mu_j \geq 0,\\ \sum_{j=1}^{n-1} \mu_j &\leq 1, &\qquad 0 \leq \theta_j < 2\pi,\end{aligned}$$

and the boundary of P_n is characterized by the condition that $\sum_{j=1}^{n-1}\mu_j = 1$. To each boundary point defined by (3.3) there belongs only one function, namely

$$(3.4)\qquad p(z) = \sum_{j=1}^{n-1} \mu_j \frac{1 + e^{i\theta_j}z}{1 - e^{i\theta_j}z}, \qquad \sum_{j=1}^{n-1} \mu_j = 1.$$

Rogosinski [10] has studied the class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots$$

which are regular for $|z| < 1$ and have the property that $\operatorname{Im} f$ and $\operatorname{Im} z$ have the same sign in $|z| < 1$, a property which implies that all the coefficients are real. We have seen that schlicht functions with real coefficients have this property, and the argument used above to prove that $|a_n| \leq n$ for schlicht functions with real coefficients extends at once to this wider class of power series, a class of functions which Rogosinski has called "typically-real." In fact, if f is typically-real then

$$(3.5)\qquad \frac{1 - z^2}{z} f(z) = p(z)$$

is a function of positive real part, and conversely. From this simple relation the variability regions T_n for the coefficients of typically-real functions are easily obtained from the corresponding regions P_n for functions with positive real part. It is readily seen that T_n is the smallest convex region containing the nth variability region for schlicht functions all of whose coefficients are real. This raises an interesting question: what is the smallest convex region containing S_n itself?

A domain containing $w = 0$ is said to be star-like (with respect to $w = 0$) if any point of it can be joined to the origin by a straight-line segment which lies in the domain. The variability regions S_n^* for the subfamily of schlicht functions which map $|z| < 1$ onto star-like domains are also related to the regions P_n in a simple way. This follows from the fact that if f maps $|z| < 1$ onto a star-like domain, then

$$(3.6)\qquad \frac{f(z)}{zf'(z)} = p(z),$$

where $p(z)$ is regular and has positive real part in $|z|<1$. Conversely, if $p(z)$ is regular and has positive real part in $|z|<1$, then

$$f(z) = z \exp \int_0^z \frac{1-p(\zeta)}{\zeta p(\zeta)}\, d\zeta \tag{3.7}$$

is regular in $|z|<1$ and maps the unit circle on a star-like domain. Writing

$$p(z) = 1 + 2\sum_{\nu=1}^{\infty} c_\nu z^\nu,$$

we have

$$\begin{aligned} a_2 &= -2c_1, \qquad a_3 = -c_2 + 4c_1^2, \\ a_4 &= (-2c_3 + 14c_1c_2 - 24c_1^3)/3, \cdots . \end{aligned} \tag{3.8}$$

The parametric representation of P_n given by (3.3) may be used to define a parametric representation of S_n^*. Let

$$a_k = a_k(\theta_1, \theta_2, \cdots, \theta_{n-1}; \mu_1, \mu_2, \cdots, \mu_{n-1}) \tag{3.9}$$

be the kth coefficient of the function

$$f(z) = \frac{z}{\prod_{\nu=1}^{n-1} (1 - e^{i\theta_\nu} z)^{2\mu_\nu}}, \qquad \mu_\nu \geq 0, \quad \sum_{\nu=1}^{n-1} \mu_\nu \leq 1, \tag{3.10}$$

which maps $|z|<1$ on a star-like domain. As the parameters vary over the parameter space, the point $(a_2, a_3, \cdots, a_n)$ sweeps out S_n^*. The boundary of S_n^* is characterized by the condition

$$\sum_{\nu=1}^{n-1} \mu_\nu = 1, \tag{3.11}$$

and to each boundary point defined in this way there belongs the unique function given by (3.10). In this case the function (3.10) maps $|z|<1$ on the plane minus q $(1 \leqq q \leqq n-1)$ straight-line slits pointing toward the origin, and adjacent slits form at ∞ an angle equal to $2\mu_\nu\pi$.

The intersection of the boundary of S_n with S_n^* is the subset of the boundary of S_n^* for which

$$\mu_\nu = \frac{m_\nu}{k-1}, \qquad \sum_{\nu=1}^{n-1} m_\nu = k-1, \tag{3.12}$$

where m_ν and k are integers, $m_\nu \geqq 0$, $2 \leqq k \leqq n$.

The method of Löwner [7] provides a deeper connection between schlicht functions and functions with positive real part. Heuristically, the method may be briefly described as follows.

For each τ, $0\leqq\tau\leqq T$ $(T>0)$, let $S(z, \tau)$, $S(0, \tau)=0$, be a schlicht function mapping $|z|<1$ onto a star-like domain D^*. Then for any $t>0$, the values of the function $e^{-t}S(z, \tau)$ lie inside D^* and so the function $\phi=\phi(z, t, \tau)=S^{-1}(e^{-t}S(z, \tau), \tau)$, where S^{-1} denotes the function inverse to S, is regular, schlicht and bounded by 1 in $|z|<1$. We have

$$\left.\frac{d\phi}{dt}\right|_{t=0} = -\frac{S(z, \tau)}{S'(z, \tau)},$$

and so

$$S^{-1}(e^{-\Delta t}S(z, t), t) = z - \frac{S(z, t)}{S'(z, t)}\Delta t + o(\Delta t). \tag{3.13}$$

Now let us define a one-parameter family of functions

$$g(z, t) = \gamma(t)(z + a_2(t)z^2 + a_3(t)z^3 + \cdots), \qquad \gamma(t) > 0, \tag{3.14}$$

which are regular and schlicht in $|z|<1$ for $0\leqq t\leqq T$ and such that

$$g(z, t + \Delta t) = g(S^{-1}(e^{-\Delta t}S(z, t), t), t). \tag{3.15}$$

By (3.13)

$$g(z, t + \Delta t) = g(z, t) - z\frac{\partial g(z, t)}{\partial z}p(z, t)\Delta t + o(\Delta t) \tag{3.16}$$

where $p(z, t)=S(z, t)/(zS'(z, t))$ has positive real part in $|z|<1$. Dividing by Δt and letting Δt approach zero, we obtain the differential equation

$$\frac{\partial g(z, t)}{\partial t} = -z\frac{\partial g(z, t)}{\partial z}p(z, t). \tag{3.17}$$

If we divide both sides of (3.17) by z and then take $z=0$, we have

$$\frac{d\gamma(t)}{dt} = -\gamma(t), \qquad \gamma(t) = \gamma(0)e^{-t}. \tag{3.18}$$

The equation (3.17) may also be given a direct geometrical interpretation. Suppose that $g(z, t)$ and $p(z, t)$ are both regular and $\operatorname{Re} p>0$ in $|z|\leqq 1$ for $0\leqq t\leqq T$. Taking $z=e^{i\theta}$ in (3.16) we have

$$(3.19)\qquad g(e^{i\theta}, t + \Delta t) = g(e^{i\theta}, t) + i\frac{\partial g(e^{i\theta}, t)}{\partial \theta} p(e^{i\theta}, t)\Delta t + o(\Delta t).$$

The map D_t of $|z| < 1$ by $g(z, t)$ is a domain bounded by an analytic curve; the quantity $i\partial g(e^{i\theta}, t)/\partial\theta$ has the direction of the inner normal to the boundary of D_t at the point $w = g(e^{i\theta}, t)$ and $i\partial g(e^{i\theta}, t)/\partial\theta \cdot p(e^{i\theta}, t)$ is a vector which makes an angle less than $\pi/2$ with this inner normal. Equation (3.19) states that the boundary point $g(e^{i\theta}, t+\Delta t)$ of the domain $D_{t+\Delta t}$ lies inside D_t. Thus, as t increases from 0 to T, the domain D_t shrinks and, if $t' < t''$, $D_{t''} \subset D_{t'}$. Writing

$$(3.20)\qquad p(z, t) = 1 + 2\sum_{\nu=1}^{\infty} c_\nu(t) z^\nu,$$

and equating coefficients of $z^n (n \geqq 2)$ in (3.17), we obtain

$$(3.21)\qquad \frac{da_n(t)}{dt} + (n - 1)a_n(t) = -2\sum_{\nu=1}^{n-1} \nu a_\nu(t) c_{n-\nu}(t).$$

Integrating this equation between the limits t' and t'', $0 \leqq t' < t'' \leqq T$,

$$(3.22)\qquad \begin{aligned} &e^{(n-1)t''}a_n(t'') - e^{(n-1)t'}a_n(t') \\ &\qquad = -2\int_{t'}^{t''} e^{(n-1)\tau} \sum_{\nu=1}^{n-1} \nu a_\nu(\tau) c_{n-\nu}(\tau) d\tau. \end{aligned}$$

In particular,

$$(3.23)\qquad \begin{aligned} &e^{t''}a_2(t'') - e^{t'}a_2(t') = -2\int_{t'}^{t''} e^{\tau} c_1(\tau) d\tau, \\ &e^{2t''}a_3(t'') - e^{2t'}a_3(t') \\ &\qquad = -2\int_{t'}^{t''} e^{2\tau} c_2(\tau) d\tau + e^{2t''}a_2(t'')^2 - e^{2t'}a_2(t')^2. \end{aligned}$$

If the sense of the parameter t is reversed by replacing t by $T - t$, and if (for simplicity) we again write $c_\nu(t)$ for $c_\nu(T - t)$, $a_\nu(t)$ for $a_\nu(T - t)$, the formula (3.22) becomes

$$(3.24)\qquad \begin{aligned} &e^{-(n-1)t''}a_n(t'') - e^{-(n-1)t'}a_n(t') \\ &\qquad = 2\int_{t'}^{t''} e^{-(n-1)\tau} \sum_{\nu=1}^{n-1} \nu a_\nu(\tau) c_{n-\nu}(\tau) d\tau. \end{aligned}$$

Sometimes one, sometimes the other, of these two formulas is more convenient.

The above method establishes a curve $(a_2(t), a_3(t), \cdots, a_n(t))$ connecting two points of S_n, and this curve is the transform of a corresponding curve $(c_1(\tau), c_2(\tau), \cdots, c_{n-1}(\tau))$ in the coefficient region P_n of functions having positive real part in $|z| < 1$.

In Löwner's formulation (see [7]) the function $p(z, t)$ is restricted to be of the form

$$(3.25) \qquad p(z, t) = (1 + \kappa(t)z)/(1 - \kappa(t)z),$$

where $\kappa(t)$ is a continuous function of t, $|\kappa(t)| = 1$. In this case the iteration defined by (3.15) introduces a continually lengthening Jordan-arc slit; that is, if $t'' > t'$, then $D_{t''}$ (the map of $|z| < 1$ by $g(z, t'')$) is obtained by removal of a slit from $D_{t'}$ (the map of $|z| < 1$ by $g(z, t')$). Actually, Löwner's parameter t moves in the opposite sense (corresponding to (3.24)), in which case the inclusion relation between $D_{t'}$ and $D_{t''}$ is reversed, and he showed that any schlicht function of an everywhere dense set can be connected to the function $f(z) = z$ by a curve corresponding to a function $p(z, t)$ of the form (3.25) which depends continuously on t. If the functions $p(z, t)$ are not restricted to be of the form (3.25) and if the continuity in t is dropped, then[3] it can be shown that any schlicht function can be connected to $f(z) = z$, but it is not known that this can be done using only functions of the form (3.25), even without continuity.

Take $t'' = T$, $t' = 0$ in (3.24), and let the value T of the parameter correspond to $f(z) = z$, in which case $a_k(T) = 0$ $(k = 2, 3, \cdots)$. Writing $a_k(0) = a_k$ $(k = 2, 3, \cdots)$, we then obtain, by recursion, the formulas:

$$(3.26) \qquad a_2 = -2\int_0^T e^{-\tau}c_1(\tau)d\tau,$$

$$(3.27) \qquad a_3 = -2\int_0^T e^{-2\tau}c_2(\tau)d\tau + 4\left(\int_0^T e^{-\tau}c_1(\tau)d\tau\right)^2,$$

$$\begin{aligned} a_4 = &-2\int_0^T e^{-3\tau}c_3(\tau)d\tau + 12\int_0^T e^{-2\tau}c_2(\tau)d\tau\cdot\int_0^T e^{-\tau}c_1(\tau)d\tau \\ (3.28) \qquad &-4\int_0^T\int_{\tau_1=\tau}^T e^{-2\tau-\tau_1}c_2(\tau)c_1(\tau_1)d\tau d\tau_1 \\ &-8\left(\int_0^T e^{-\tau}c_1(\tau)d\tau\right)^3, \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

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[3] An unpublished result of A. C. Schaeffer and the author.

$$a_n = \sum (-1)^k \Gamma_{\alpha_1\alpha_2\cdots\alpha_k} \cdot \int_{\tau_1=0}^{T}\int_{\tau_2=\tau_1}^{T}\cdots\int_{\tau_k=\tau_{k-1}}^{T} \exp\left[-\sum_{\nu=1}^{k}\alpha_\nu\tau_\nu\right] \cdot \prod_{\nu=1}^{k} c_{\alpha_\nu}(\tau_\nu)\,d\tau_1 d\tau_2\cdots d\tau_k, \tag{3.29}$$

where

$$\Gamma_{\alpha_1\alpha_2\cdots\alpha_k} = 2^k(n-\alpha_1)(n-\alpha_1-\alpha_2)\cdots(n-\alpha_1-\alpha_2-\cdots-\alpha_k),$$

the $\alpha_1, \alpha_2, \cdots, \alpha_k$ being positive integers with sum $n-1$.

Now the interior points of S_n are characterized by the property that bounded functions belong to them. Given an interior point $(a_2, a_3, \cdots, a_n)$ of S_n, let the minimum maximum modulus of all functions f belonging to this point be $e^t (t \geqq 0)$. The set of points of S_n which are representable by the formulas (3.26) to (3.29), in which the $c_\nu(\tau)$ $(\nu = 1, 2, \cdots, n-1)$ are measurable functions of τ, is exactly the set of interior points of S_n corresponding to the values $t \leqq T$. If, therefore, we take $T = \infty$ in these formulas, any point of S_n may be given this integral representation for suitable choice of the curve $(c_1(\tau), c_2(\tau), \cdots, c_{n-1}(\tau))$. More generally, given any schlicht function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, there is a function $p(z, \tau) = 1 + 2\sum_{\nu=1}^{\infty} c_\nu(\tau)z^\nu$ such that the coefficients a_n $(n = 2, 3, \cdots)$ are given by (3.29). Conversely, given any function $p(z, \tau)$, the coefficients defined by (3.29) belong to a schlicht function. The correspondence between functions $f(z)$ and $p(z, \tau)$ is not one-to-one; in general infinitely many $p(z, \tau)$ correspond to a schlicht function f.

To any f which has real coefficients, there is a corresponding $p(z, \tau)$ which has real coefficients. Now if $c_1, c_2, c_3, \cdots$ belong to a function with positive real part, so do the numbers

$$\text{Re } c_1, \text{ Re } c_2, \text{ Re } c_3, \cdots$$

(this follows from the convexity of the family of functions p). Thus if in (3.29) we replace the $c_\nu(\tau)$ by their real parts, the resulting a_n belongs to a schlicht function with real coefficients. From this remark, it follows at once that $|a_2| \leqq 2$, $|a_3| \leqq 3$, for complex coefficients. For example, to show that $|a_3| \leqq 3$, we may suppose without loss of generality that $a_3 > 0$. For, given any schlicht function f, we have only to consider the function

$$e^{-i\theta}f(e^{i\theta}z) = z + a_2e^{i\theta}z^2 + a_3e^{2i\theta}z^3 + \cdots$$

which, for suitable choice of θ, will have a real non-negative third coefficient. But if a_3 is real we have from (3.27) (with $T=\infty$)

$$\begin{aligned} a_3 = &-2\int_0^\infty e^{-2\tau}\operatorname{Re} c_2(\tau)d\tau + 4\left(\int_0^\infty e^{-\tau}\operatorname{Re} c_1(\tau)d\tau\right)^2 \\ &-4\left(\int_0^\infty e^{-\tau}\operatorname{Im} c_1(\tau)d\tau\right)^2 \\ \leqq &-2\int_0^\infty e^{-2\tau}\operatorname{Re} c_2(\tau)d\tau + 4\left(\int_0^\infty e^{-\tau}\operatorname{Re} c_1(\tau)d\tau\right)^2. \end{aligned}$$

The expression on the right of this inequality is the third coefficient of some function with all coefficients real, and therefore as we have seen bounded by 3. This argument fails for a_4.

When the $c_\nu(\tau)$ $(\nu=1, 2, \cdots)$ are independent of τ, the coefficients given by (3.26)–(3.29) belong to the function

$$f(z)=e^{T}S^{-1}(e^{-T}S(z))$$

where $S(z)/(zS'(z))=p(z)=1+\sum_{\nu=1}^{\infty}c_\nu z^\nu$. In particular, for $T=\infty$, $f(z)=S(z)$ and the formulas reduce to (3.8).

Finally, let $(a_1(t), a_3(t), \cdots, a_n(t))$, $0\leqq t\leqq T$, be a curve lying on the boundary of S_n. Then it is obvious that the curve $(c_1(\tau), c_2(\tau), \cdots, c_{n-1}(\tau))$, of which $(a_2(t), a_3(t), \cdots, a_n(t))$ is the transform, must lie on the boundary of P_n, at least for almost all τ. On the other hand, if $(a_2(0), a_3(0), \cdots, a_n(0))$ is a boundary point of S_n and if $(c_1(\tau), c_2(\tau), \cdots, c_{n-1}(\tau))$ lies on the boundary of P_n for every τ, it does not necessarily follow that $(a_2(t), a_3(t), \cdots, a_n(t))$ is on the boundary of S_n for $0<t\leqq T$. The curve $(a_2(t), a_3(t), \cdots, a_n(t))$ lies on the boundary only if $(c_1(\tau), c_2(\tau), \cdots, c_{n-1}(\tau))$ is a curve of special form.

For example, in the case $n=3$, the region S_3 is bounded by two hypersurfaces H_1 and H_2 plus their intersection, where H_1 and H_2 are the hypersurfaces generated by rotating the surfaces (2.6) and (2.7) respectively. Let $(a_2(0), a_3(0))$, the initial point of the curve, be an interior point of H_2. Then

$$\begin{aligned} e^t a_2(t) &= a_2(0) - 2\int_0^t e^{\tau}e^{i\phi(\tau)}d\tau, \\ e^{2t}a_3(t) &= a_3(0) - a_2(0)^2 + e^{2t}a_2(t)^2 - 2\int_0^t e^{2\tau}e^{2i\phi(\tau)}d\tau \end{aligned} \tag{3.30}$$

is a curve lying on H_2, only if $e^{i\phi(\tau)}$ satisfies the equation

$$\operatorname{Im}\left\{A_0 e^{-\tau} e^{i\phi(\tau)} + B_0 e^{2i\phi(\tau)}\right\} = 0, \tag{3.31}$$

where A_0 and B_0 are numbers which can be given explicitly in terms of the point $(a_2(0), a_3(0))$. As t tends to infinity, $(a_2(t), a_3(t))$ tends to $(2\eta, 3\eta^2)$ for some η, $|\eta| = 1$, a point on the intersection of H_1 and H_2 belonging to the function $f(z) = z/(1-\eta z)^2$. No interior point of H_2 can be connected to a point on the intersection of H_1 and H_2 by a curve corresponding to a finite range of the parameter t.

Next, set

$$\begin{aligned} e^t a_2(t) &= -2 - 2\int_0^t e^{\tau} c_1(\tau)d\tau, \\ e^{2t} a_3(t) &= -1 + e^{2t} a_2(t)^2 - 2\int_0^t e^{2\tau} c_2(\tau)d\tau \end{aligned} \tag{3.32}$$

and take

$$c_1(\tau) = \begin{cases} e^{i\alpha(\tau)}, & 0 \leqq \tau \leqq t_0, \\ \cos\alpha(\tau) + iC, & t_0 < \tau < \infty, \end{cases} \tag{3.33}$$

$$c_2(\tau) = \begin{cases} e^{2i\alpha(\tau)}, & 0 \leqq \tau \leqq t_0, \\ \cos 2\alpha(\tau) + 2iC\cos\alpha(\tau), & t_0 < \tau < \infty, \end{cases} \tag{3.34}$$

where $\cos\alpha(\tau) = e^{-\tau}$, $0 \leqq \alpha(\tau) < \pi/2$, C is a constant, $0 \leqq C \leqq 1$, and t_0 is uniquely defined by $\sin\alpha(t_0) - \alpha(t_0)\cos\alpha(t_0) = C$. The function $p(z, \tau)$ corresponding to this curve is

$$p(z,\tau) = \begin{cases} \dfrac{1 + e^{i\alpha(\tau)}z}{1 - e^{i\alpha(\tau)}z} = \dfrac{1 + 2i\sin\alpha(\tau)z - z^2}{1 - 2\cos\alpha(\tau)z + z^2}, & 0 \leqq t \leqq t_0, \\ \dfrac{1 + 2iCz - z^2}{1 - 2\cos\alpha(\tau)z + z^2}, & t_0 < \tau < \infty. \end{cases} \tag{3.35}$$

For $t = 0$, $(a_2(0), a_3(0)) = (-2, 3)$; this point belongs to the function $f(z) = z/(1+z)^2$. As t increases from 0 to t_0, the point $(a_2(t), a_3(t))$ moves on the intersection of H_1 and H_2; as t increases from t_0 to ∞, $(a_2(t), a_3(t))$ moves on H_1 and as t tends to infinity approaches the point $(-2iC, 1-4C^2)$ belonging to $f(z) = z/(1+2iCz-z^2)$. Any point on H_1, or on the intersection of H_1 and H_2, is representable in one of the two forms $(a_2(t)e^{i\theta}, a_3(t)e^{2i\theta})$, $(\bar{a}_2(t)e^{-i\theta}, \bar{a}_3(t)e^{-2i\theta})$, for suitable choice of θ, C and t, $0 \leqq t \leqq \infty$. That is, any point on H_1, or any point on the intersection of H_1 and H_2, may be obtained from (3.32) by a rotation, or by a reflection on the axis-plane Im $a_2 =$ Im $a_3 = 0$, or by both.

We observe that the points $(-2iC, 1-4C^2)$, $0 \leqq C \leqq 1$, which lie on the boundary of S_n^*, correspond to $t = \infty$. Rotating through the angles $\pm\pi/2$, these points become $(2C, 4C^2-1)$, $-1 \leqq C \leqq 1$, and they lie on the parabola $a_3 = a_2^2 - 1$, $-2 \leqq a_2 \leqq 2$. This parabola plus its rotations is the intersection of the boundary of S_3 with S_3^*. The parabola is obtained from (3.32) by taking $c_1(\tau) = -1$, $c_2(\tau) = 1$. As t tends to infinity, $(a_2(t), a_3(t))$ approaches the point (2,3) belonging to $f(z) = z/(1-z)^2$, a rotation of the function $z/(1+z)^2$ corresponding to $t = 0$. Hence every point of H_1 can be connected to this function, or to a rotation of it, by a curve corresponding to a finite range of the parameter. This is a consequence of the fact that for functions $w = f$ belong to H_1, the portion of the boundary in the w-plane near $w = \infty$ is a straight line arg $(w) =$ constant (see §2).

The form of these curves lying on the boundary of S_3 has been calculated from the formulas (2.6) and (2.7). A derivation based on Löwner's method alone would be interesting.

4. Maps of the coefficient regions S_n.

Let

$$U_n = T(S_n) \tag{4.1}$$

be a continuous mapping of S_n on a region U_n. When U_n has the same dimensionality as S_n, a common type of map has the special form in which, if $(b_2, b_3, \cdots, b_n)$ is the image of the point $(a_2, a_3, \cdots, a_n)$ of S_n, then

$$b_\nu = b_\nu(a_2, a_3, \cdots, a_\nu); \tag{4.2}$$

that is, b_ν depends only on the coefficients $a_2, a_3, \cdots, a_\nu$.

This type of map frequently arises when we consider the coefficients of the powers of z of some functional F of f,

$$g = F(f), \tag{4.3}$$

where g depends on z in an analytic way. Thus we may take

$$g = (f(z))^k = z^{k-1}(z + b_2 z^2 + b_3 z^3 + \cdots). \tag{4.4}$$

In the case $k = -1$,

$$b_2 = -a_2, \quad b_3 = a_2^2 - a_3, \quad b_4 = 2a_2a_3 - a_2^3 - a_4, \cdots \tag{4.5}$$

or, generally,

$$b_{\nu+1} = \sum_{k_1\geqq 0, k_2\geqq 0, \cdots, k_\nu \geqq 0, k_1+2k_2+\cdots+\nu k_\nu = \nu} (-1)^{k_1+k_2+\cdots+k_\nu} \cdot \frac{(k_1+k_2+\cdots+k_\nu)!}{k_1!k_2!\cdots k_\nu!} a_2^{k_1}\cdots a_{\nu+1}^{k_\nu}.$$

We also obtain a mapping (4.2) if we set

$$g = \frac{f(z)}{f'(z)} = z + b_2 z^2 + b_3 z^3 + \cdots \tag{4.6}$$

which gives

$$b_2 = -a_2,\quad b_3 = 2(a_2^2 - a_3),\quad b_4 = 7a_2a_3 - 4a_2^3 - 3a_4, \cdots . \tag{4.7}$$

Another important case of (4.2) is defined by

$$f_k(z) = \{f(z^k)\}^{1/k} = z + b_2 z^{k+1} + b_3 z^{2k+1} + \cdots , \tag{4.8}$$

k being a positive integer. Here

$$\begin{aligned} b_2 &= \frac{a_2}{k}, \qquad b_3 = \frac{1}{k}\left(a_3 - \frac{k-1}{2k}a_2^2\right), \\ b_4 &= \frac{1}{k}\left(a_4 - \frac{k-1}{k}a_2a_3 + \frac{(k-1)(2k-1)}{6k^2}a_2^3\right), \cdots . \end{aligned} \tag{4.9}$$

The function f_k is schlicht (conversely if a function f_k of the form (4.8) is schlicht, so is the function $\{f_k(z^{1/k})\}^k$) and hence (4.9) is a mapping of S_n on a subset of itself. Finally let

$$z = f^{-1}(w) = w + b_2 w^2 + b_3 w^3 + \cdots \tag{4.10}$$

where

$$b_2 = -a_2,\quad b_3 = 2a_2^2 - a_3,\quad b_4 = 5a_2a_3 - 5a_2^3 - a_4, \cdots . \tag{4.11}$$

The maps defined by (4.5), (4.7), (4.9) and (4.11) have been studied by various authors, mainly from the point of view of obtaining precise upper bounds for the coefficients b_ν. In the case (4.5), precise bounds are known only for $\nu = 2$, 3 and 4, namely, $|b_2| \leqq 2$, $|b_3| \leqq 1$, $|b_4| \leqq 2/3$ (see [12(a)] and [3(c)]). In this case the conjecture has been suggested that $|b_n| \leqq 2/(n-1)$.

In the case (4.7), it is obvious that $|b_\nu| \leqq 2$ ($\nu = 2, 3, \cdots$) when f maps $|z| < 1$ on a star-like domain. However, there are points (a_2, a_3, a_4) of S_4 outside S_4^* for which $|b_4| > 2$ (see [11(b)]) and this in fact occurs even for points (a_2, a_3, a_4) belonging to schlicht functions with all coefficients real. The region B_n of points $(b_2, b_3, \cdots, b_n)$ defined by (4.7) was studied by Peschl [8] in preference to S_n itself, for the reason that B_n has a slightly simpler relationship to P_n than S_n has.

In (4.9) precise bounds of b_ν are known only for $\nu=2, 3$. It is clear that $|b_2| \leqq 2/k$, and the precise inequality for b_3 is (see [2])

$$|b_3| \leqq \frac{2}{k} \exp\left\{-\frac{2(k-1)}{k+1}\right\} + \frac{1}{k}.$$

The precise upper bounds for the b_ν defined by (4.11) were obtained by Löwner [7] using his method of parametric representation (these bounds may also be obtained in a different way (see [11(b)])). The inequalities are:

$$|b_\nu| \leqq \frac{1\ 3\cdot 5\cdots(2\nu-1)}{1\ 2\cdot 3\cdots(\nu+1)} 2^\nu \qquad (\nu = 2, 3, \cdots).$$

Equality is attained for any ν only if $f(z) = z/(1-\eta z)^2$, $|\eta| = 1$.

Of particular importance is the case in which the mapping (4.2) is linear in $a_2, a_3, \cdots, a_\nu$, that is when

$$b_\nu = c_1 + c_2 a_2 + \cdots + c_\nu a_\nu. \tag{4.12}$$

A linear mapping of this type arises whenever we consider the coefficients of the powers of ζ in the expansion

$$g(\zeta) = f(\phi(\zeta)) = b_1\zeta + b_2\zeta^2 + b_3\zeta^3 + \cdots \tag{4.13}$$

where

$$z = \phi(\zeta) = \beta_1\zeta + \beta_2\zeta^2 + \beta_3\zeta^3 + \cdots \tag{4.14}$$

is regular in a neighborhood of $\zeta = 0$. Writing

$$\{\phi(\zeta)\}^k = \sum_{\nu=k}^{\infty} \beta_\nu^{(k)} \zeta^\nu,$$

we see that

$$b_\nu = \beta_\nu^{(1)} + \beta_\nu^{(2)} a_2 + \beta_\nu^{(3)} a_3 + \cdots + \beta_\nu^{(\nu)} a_\nu \qquad (\nu = 1, 2, \cdots) \tag{4.15}$$

where $\beta_\nu^{(1)} = \beta_\nu$, $\beta_\nu^{(\nu)} = \beta_1^\nu$.

Suppose first that $\beta_1 = 1$ in (4.14) and let $z = \phi(\zeta)$ map some domain

G on $|z|<1$.[4] Then the variability region of $(b_2, b_3, \cdots, b_n)$ is the nth coefficient region $S_n(G)$ of functions regular and schlicht in G.

Roughly speaking, the coefficient problem is easier the larger G is.[5] For example, let λ be a small number, $0<\lambda<1$, and let $G=G(\lambda)$ be the exterior of the circle orthogonal to the real axis and cutting it at $\lambda(1+\lambda)^{-1}$ and $\lambda(1-\lambda)^{-1}$. Then if $\lambda=\lambda_n$ is small enough, the precise upper bound for b_n can be determined (see [11(b)]). However, these regions depend on n.

In particular, consider the class C of domains G which are generated from the unit circle by the family of normalized schlicht functions f. The interior of the unit circle itself belongs to C, and we denote it by E. If $S(G)$ is the family of functions (4.13) which are regular and schlicht in G, we define

$$\alpha_n = \alpha_n(G) = \sup_{g\subset S(G)} |b_n|$$

and write

$$\gamma_n = \inf_{G\subset C} \alpha_n(G), \qquad \Gamma_n = \sup_{G\subset C} \alpha_n(G).$$

Then (see [11(a)])

$$\alpha_n(E) = \gamma_n.$$

But, given n, there are other domains G (depending on n) for which $\alpha_n=\gamma_n$. Moreover, if $\alpha_n(E)=n$ $(n=2, 3, \cdots)$, we have $\Gamma_n=4^{n-1}$ $(n=2, 3, \cdots)$. Hence $\alpha_n(E)=n$ $(n=2, 3, \cdots)$ implies that the inequality

$$n \leqq \alpha_n(G) \leqq 4^{n-1}$$

is valid throughout C.

Let us drop the restriction that $\beta_1=1$, and suppose instead that $\phi(\zeta)$ is regular and bounded by 1 in $|\zeta|<1$. The family of functions $g(\zeta)$ given by (4.13) as f and ϕ vary is then the family of functions which are subordinate to schlicht functions, and the region of variability of the point $(b_2, b_3, \cdots, b_n)$ contains S_n. Concerning precise bounds for the b_ν, it is known only that $|b_2|\leqq 2$. However, if f (in

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[4] The domain may contain ∞. We suppose that G is schlicht although a more general problem is obtained by allowing G to be a multiply-connected Riemann domain. It is also clear that we may formulate the coefficient problem for points of G other than the origin.

[5] If G is the whole plane minus a single point (in which case no function $\phi(\zeta)$ exists), there is only one normalized function $g(\zeta)$ which is regular and schlicht in G; in this case the problem is completely trivial.

(4.13)) has real coefficients or maps $|z| < 1$ on a star-like domain, then $|b_\nu| \leq \nu$ ($\nu = 2, 3, \cdots$) (see [6(b)], where an excellent exposition of this family is given).

5. **Variability regions in general.** Let γ be a closed set in $|z| < 1$, and let

$$F = F(f(z), \overline{f(z)}, f'(z), \overline{f'(z)}, \cdots, f^{(n)}(z), \overline{f^{(n)}}(z) \mid \gamma)$$

be a functional depending on the values taken by $f(z), f'(z), \cdots, f^{(n)}(z)$ in γ. The value of F is a number which may be complex, and we suppose that F is defined and continuous whenever f belongs to the family S of schlicht functions. Moreover, writing $f^{(\mu)}(z) = x_\mu + iy_\mu$, let F be "differentiable" (for example, in the sense of Volterra) with respect to x_μ and y_μ ($\mu = 0, 1, \cdots, n$) when the functions are regarded as independent and only one of them is given an increment. However, we do not need to be precise since we shall be concerned here only with specific and rather trivial examples of functionals F. Now let $U = (U_1, U_2, \cdots, U_m)$ be the point in euclidean space defined by m functionals of this type:

$$U_k = F_k(f(z), \overline{f(z)}, \cdots, f^{(n)}(z), \overline{f^{(n)}(z)} \mid \gamma_k). \tag{5.1}$$

The region of variability V of U is the closed set of points U as f ranges over the family S of schlicht functions.

Thus, the nth region S_n of the coefficients is obtained by taking $U_k = f^{(k)}(0)/k!$ ($k = 2, 3, \cdots, n$). On the other hand the region of variability of the point

$$\left(\frac{f''(z)}{f'(z)}, \frac{f'''(z)}{f'(z)}, \cdots, \frac{f^{(n)}(z)}{f'(z)}\right) \tag{5.2}$$

for fixed z, $0 \leq |z| < 1$, is a linear map of S_n of the type (4.12). For let

$$g(\zeta) = f\left(\frac{\zeta + z}{1 + \bar{z}\zeta}\right) = \sum_{\nu=1}^{\infty} a_\nu \left(\frac{\zeta + z}{1 + \bar{z}\zeta}\right)^\nu = \sum_{\nu=0}^{\infty} \alpha_\nu \zeta^\nu,$$

where $\alpha_0 = g(0) = f(z)$, $\alpha_1 = g'(0) = f'(z)(1 - |z|^2)$. Then the function

$$\frac{g(\zeta) - \alpha_0}{\alpha_1} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{\alpha_1} \zeta^\nu$$

is a normalized schlicht function, and so the point $(\alpha_2/\alpha_1, \alpha_3/\alpha_1, \cdots, \alpha_n/\alpha_1)$ belongs to S_n. We have

$$f(\zeta) = g\left(\frac{\zeta - z}{1 - \bar{z}\zeta}\right) = \sum_{\nu=0}^{\infty} \alpha_\nu \left(\frac{\zeta - z}{1 - \bar{z}\zeta}\right)^\nu, \quad f^{(\nu)}(z) = \sum_{k=1}^{\nu} \alpha_k K(k, \nu, z)$$

where $K(k, \nu, z)$ depends only on k, ν and z; hence

$$\frac{f^{(\nu)}(z)}{f'(z)} = (1 - |z|^2) \sum_{k=1}^{\nu} \frac{\alpha_k}{\alpha_1} K(k, \nu, z) \qquad (\nu = 2, 3, \cdots).$$

If $U = f(z)$, z fixed, the region of variability is defined by

$$(5.3) \qquad \left| \log \frac{f(z)}{z} + \log (1 - |z|^2) \right| \leqq \log \frac{1 + |z|}{1 - |z|}$$

(see [4, 3(d)]), where $\log (f(z)/z)$ is the branch which vanishes for $z = 0$. Taking $U = f(z)/f'(z)$, the corresponding region V is given by

$$(5.4) \qquad \left| \log \left(\frac{f(z)}{zf'(z)} \right) \right| \leqq \log \frac{1 + |z|}{1 - |z|}$$

(see [5, 3(b)]), where $\log (f(z)/zf'(z))$ is the branch vanishing for $z = 0$.

From (5.3) and (5.4) we obtain at once the two classical inequalities

$$(5.5) \qquad \frac{|z|}{(1 + |z|)^2} \leqq |f(z)| \leqq \frac{|z|}{(1 - |z|)^2},$$

$$(5.6) \qquad \frac{1 - |z|}{(1 + |z|)^3} \leqq |f'(z)| \leqq \frac{1 + |z|}{(1 - |z|)^3}.$$

If we choose $U = \arg f'(z)$ $(\arg f'(0) = 0)$ then

$$(5.7) \qquad |\arg f'(z)| \leqq \begin{cases} 4 \arcsin |z|, & |z| \leqq 1/2^{1/2}, \\ \pi + \log \dfrac{|z|^2}{1 - |z|^2}, & 1/2^{1/2} < |z| < 1. \end{cases}$$

It is worth noting that although (5.6) was the starting point of these investigations, (5.7) was not proved until twenty years later when Golusin [3(a)] established it by using Löwner's method. Inequalities (5.6) and (5.7) constitute what are called the distortion and rotation theorems.

An interesting problem arises if we set $U = (|f|, |f'|)$. This problem has been solved by Robinson [9].

We may also take $U = (f(z), f(-z))$, or $U = (f'(z), f'(-z))$, where z is a fixed point of $|z| < 1$. Aspects of this problem have recently been studied by Golusin (see, for example, [3(e), (f), (g)]).

Another type of problem is obtained by setting

$$(5.8) \qquad U = \int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta, \qquad \lambda > 0, 0 < r < 1.$$

In this case the precise upper bound is unknown. Or take

$$(5.9)\qquad U = \int_0^r \int_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^2 \rho d\rho d\theta = \pi \sum_{n=1}^{\infty} n | a_n |^2 r^{2n}.$$

Here U is the area of the map of $|z| < r$ by f. It is obvious that this area is not less than πr^2 (with equality only if $f = z$), but the precise upper bound is not known.

A rough classification of the above types of problems is provided by the order n of the highest derivative involved. The existing literature (apart from papers on the coefficient problem) is mainly concerned with problems involving first and second derivatives only.

6. **Generalizations.** Two natural generalizations are obtained by: (i) relaxing the restrictions that the domain G_z of the variable z is schlicht and simply-connected; (ii) relaxing the restriction that the domain G_w of values $w = f(z)$ is schlicht. Both have been considered. We shall comment briefly on these two generalizations, and we discuss (i) first.

For simplicity, let us suppose that G_z is schlicht but possibly multiply-connected. We may then consider extremal properties of the family of functions $f(z)$ which are regular, single-valued and schlicht in G_z and which are properly normalized at some interior point z_0 of the region. Problems of this type have been investigated in numerous papers of H. Grötzsch.[6] The method used by Grötzsch was simplified and improved by H. Grunsky [5], and was later simplified further by G. Golusin ([3(b)]) who applied it to a series of problems in the schlicht conformal mapping of multiply-connected regions. In some instances these investigations have led to new and more complete results for simply-connected regions.

The method used is one of comparison with functions which map G_z onto the w-plane slit along finite arcs of logorithmic spirals which are obtained from the equation Im $(e^{-i\theta} \log w) = c$ by holding θ fixed and varying c. Similar comparisons can be made with functions which map G_z on the plane minus parallel straight-line slits.

More recently, problems of this type have been successfully attacked by variational methods in which the comparison functions are ϵ-variations of the extremals (see [12(b)]).

A generalization of type (ii) is obtained by considering the family of p-valent functions f which are regular in $|z| < 1$ and normalized in some way at $z = 0$ and which take no value in $|z| < 1$ more than p

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[6] See Berichte der Akademie der Wissenschaften, Leipzig, 1928–1932.

times, where p is some fixed positive integer. Although these functions have been intensively studied, few precise bounds for the family are known when $p>1$; most results obtained so far are statements about order of magnitude only.

The concept of p-valency may be generalized to fractional values of p as follows. Let $w=f(z)$ be regular in $|z|<1$, and let $A(R)$ denote the area (regions covered multiply being counted multiply) of that portion of the map of the unit circle by $w=f(z)$ which lies in the circle $|w|\leqq R$. If

$$A(R) \leqq p\pi R^2$$

for all $R>0$, where p is a positive (though not necessarily integral) number, we say that f is mean p-valent. With suitable normalization at $z=0$, the family of mean p-valent functions (p fixed) is compact, and it can be shown that

$$|f(z)| \leqq A(1-|z|)^{-2p} \qquad (p>0), \tag{6.1}$$

$$|a_n| \leqq Bn^{2p-1} \qquad (p>1/4), \tag{6.2}$$

where A and B are constants. (Here a_n is the coefficient of z^n in the expansion about $z=0$.)

The most interesting class of mean p-valent functions is the class of mean 1-valent functions f of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots.$$

For this family $|a_2|\leqq 2$ with equality only if $f=z/(1-\eta z)^2$, $|\eta|=1$, but there are mean 1-valent functions f for which $|a_3|>3$.[7] However, these functions resemble schlicht functions so far as orders of magnitude are concerned.

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JACOB DAVID TAMARKIN—HIS LIFE AND WORK

EINAR HILLE

Few members of our Society enjoyed such a firmly established and well deserved popularity as Jacob David Tamarkin during the twenty years that he stood in our midst. J. D., as his friends called him, was a personality and a colorful one; those who knew him well held him in fond affection and everybody respected his integrity and selfless service. The interest which he took in his fellowmen was most helpful and he stimulated mathematical activity wherever he went. His erudition and wise counsel were at the service of young and old; many of us are grateful to him for a start or a helping hand at a critical moment. He lived intensely and enjoyed it; he kept open house to his friends, and all mathematicians and musicians were his friends. He was the life of every mathematical gathering where his contagious laughter and roaring voice carried far and wide. It is a sad task to write his life; it is also a difficult one because so much of his colorful past is either hidden behind the famous curtain or cannot be told.[1]

J. D. was born in Chernigov (northern Ukraine; Russia) on June 28 (old style = July 11), 1888. His father was a physician, his mother belonged to the landed gentry, and he was their only child. His parents later moved to St. Petersburg where J. D. graduated from the Second Gymnasium in 1906. Among his classmates was Alexander Alexandrovich Friedmann, the future physicist.[2] They were bosom friends; they worked together in school and were destined to remain closely associated until the final separation in 1925. Their early interest went to number theory and resulted in a joint paper on quadratic congruences and the numbers of Bernoulli ([1] in the appended bibliography) which was dated October 5, 1905 and appeared in the Mathematische Annalen the following year. This was J. D.'s first paper and Friedmann became the first of his many collaborators of which at least twenty have been identified.[3] The authorities of the school thought that the achievement deserved special recognition and awarded them gold medals.

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[1] I am grateful to Professors C. R. Adams, R. C. Archibald, L. L. Silverman, S. Timoshenko, J. V. Uspensky (since deceased), and G. Vernadsky for valuable information bearing on various phases of J. D.'s life and work.

[2] Born June 17, 1888, died September 16, 1925. The facts concerning Friedmann's life, used in the following, are taken from his curriculum vitae in the Receuil de Géophysique, Leningrad, vol. 5 (1927) pp. 11–15.

[3] See the list before the Bibliography proper. Cf. footnote 6 according to which the names of N. M. Günther and J. V. Uspensky should be added to the list.

The two friends entered the St. Petersburg University in 1906 where they continued their work on number theory: the physico-mathematical faculty also responded with gold medals. In January 1908 they wrote a letter to K. Hensel, an extract of which appeared in Crelle's Journal in 1909 [2]. This paper brought their work on number theory to a finish. They had now come under the inspiring influence of V. A. Steklov who made them interested in mathematical physics: J. D. turned to boundary value problems, Friedmann to fluid mechanics. J. D. applied the method of Steklov to a study of the transversal vibrations of an elastic homogeneous rod in [3] which appeared in 1910; his joint paper with Steklov [4] is dated the same year. It was followed by a short note [5] on the heterogeneous case in 1912, but J. D. does not seem to have carried out his intention of publishing a detailed paper. By this time he was stalking bigger game.

He graduated[4] from the University in 1910 and got his first academic positions there and at the School of Communications[5] the same year and, incidentally, so did Friedmann. They continued working for the Magister degree in Applied Mathematics for a few years; Friedmann passed the examination in 1913, J. D. probably in 1912. Their dissertations were delayed by the war, J. D.'s 322 page book [I] on boundary value problems appeared in 1917, Friedmann's 516 page book on hydromechanics in 1922. In 1913 J. D. also became instructor (privatdocent?) at the Electro-Technical School and in 1917 (extraordinary?) professor at all three institutions in Petrograd. He married Helen Weichart in 1919; their son Paul was born in 1922.

J. D.'s research during this period was concentrated on the theory of boundary value problems for linear differential equations, leading up to his dissertation. Later he became interested in questions of approximation (methods of W. Ritz and of C. Störmer, interpolation and mechanical quadratures) possibly through N. Krylov with whom he wrote three papers.

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[4] Professors Timoshenko and Vernadsky have furnished the following information concerning Russian university conditions before 1918 which will help the reader to get the proper perspective. The student took a four year course and received a diploma of the first or the second class upon graduation. The requirements would correspond roughly to an American M.A. degree. The best students were encouraged to stay on and became candidates for degrees of which there were two, Magister and Doctor. The requirements for the latter were so severe that very few bothered to take it. The Magister degree was hard enough; it involved an examination and the writing and public defense of a dissertation. All academic degrees were abolished in 1918. A candidate who had passed the Magister examination could obtain the venia legendi and the title of privatdocent by giving a habilitation lecture on a preassigned topic. The number of ordinary professorships (cathedrae) was small, one started out with an extraordinary position.

[5] Instituta Inzhenerov Pooteĭ Soŏbshcheniya. J. D. used to refer to this institute as the School of Railroads; he may have had students from the railroad section. In 1912 he was appointed member of a Russian Government committee to study stresses in rails.

The very bad eyesight of J. D. presumably kept him out of the first world war. Friedmann on the other hand, who had written about airplane theory in 1911 and also had studied dynamical meteorology, saw active service with the Russian Air Force in 1914–1915 and later had a hand in research connected with aerial warfare. In 1918 Friedmann was called to the new university of Perm as professor of mechanics where he stayed for two years. J. D. accepted a call to Perm in 1920 and also stayed for two years as professor and dean. At this time starvation was widespread in Russia; living in Perm was easier than in Petrograd, but the climate was severe and J. D. nearly succumbed to an attack of pneumonia.

In 1922 we find J. D. back in Petrograd; to his three old professorships he has now added a fourth one at the Naval Academy. He also held consultantships with the Central Weather Bureau, the Optical and the Physico-Technical Institutes. The salaries were probably a minor consideration, but with each position went a set of ration cards which counted heavily in the household economy. The drawback of the many positions was that he had to spend his time on overcrowded trolley cars going from one institution to another all day long. During the three years that he remained in Petrograd, later called Leningrad, he produced an enormous amount of work. Comparatively little was pure mathematics, though the papers [14–16, 20, 21] were written during this time, [14] in collaboration with A. Besicovich and [15, 16] with Krylov. More effort must have gone into applied mathematics in collaboration with Friedmann [17–19] and still more into the three books [II, III, IV] with the co-authors J. S. Besicovich, G. Fichtenholz, A. Friedmann, and V. Smirnov.[4]

J. D. had been very well-to-do before the revolution razed all values. That his mathematical library was first rate, is not surprising, but he was also intensely fond of music and he possessed a musical library of some 6000 items including a complete collection of Russian music. He used to engage a string quartet to play at his home weekly; the concert lasted from eight in the evening until five or six in the morning with an intermission for a grand supper at midnight and a good breakfast in the morning. Shostakovich, at the age of sixteen, played part of his first symphony at one of these concerts in 1924. Life was precarious, however, and for a person with J. D.'s background Leningrad was becoming unhealthy: there was always the haunting fear of starvation—he lost ninety pounds during the revolution—and there was also the secret police. J. D. was a menshevik (social democrat) and as such he was fair game both to the okhrana of the czar and the cheka after the October revolution. In 1924 he decided to leave Russia; he arrived in this country in March 1925 and was joined by his wife and son a year later. One of J. D.'s best stories told how he tried to convince the American consul in Riga of his identity: the consul attempted to examine him in analytic geometry, but ran out of questions and gave up when J. D. threatened to take over the examination.

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[4] According to the Jahrbuch über die Fortschritte der Mathematik vol. 50 (1924) p. 149, the department of mathematics of the School of Communications published a collection of problems in higher mathematics to be used in the school. The authors are listed as A. A. Friedmann, N. M. Günther, J. D. Tamarkin, and J. V. Uspensky. This would add two more names to J. D.'s list of collaborators. J. D. apparently did not get much credit for his books. [II] has been issued in several editions; Smirnov added two more volumes to the book and J. D.'s name was omitted in the later editions. Neither [III] nor [IV] appeared with his name on the title page. According to J. D., [III] was awarded a prize by the Russian State Research Council and he got his share of the prize.

J. D. found asylum at Dartmouth College when he came over and stayed there for two years as visiting lecturer. He used these two years well as may be seen from the Bibliography [22–32]. It is still, in the main, boundary value problems and integral equations which interested him and in these fields he found new collaborators in Ch. E. Wilder and R. E. Langer who also urged him to write an English version of his dissertation [25]. His association with L. L. Silverman resulted in a paper [32] on Abel's theorem for Nörlund and Hausdorff means which was the point of departure of much of his later work.

He was called to Brown University in 1927 and here his scientific activities really flourished. His first seven years at Brown were probably the happiest in his life. He had security and his pent-up energies overflowed in all directions. He threw himself wholeheartedly into his new surroundings where he soon became a driving force; as Brown was pushed into the mathematical front rank, good students were attracted and were well trained (J. D. alone accounted for some twenty-two Ph.D.'s during his eighteen years of active service). His stimulating lectures were highly polished and represented up-to-date research in the field in question; mimeographed copies of his lecture notes were widely circulated and quoted. He read passionately and his students as well as his friends profited much from his constructive criticism.

His critical ability and great erudition were soon put to use. He became an associate editor of the Transactions in 1927, of the Annals in 1928, and of the Zentralblatt in 1932. He became one of the editors of the Transactions in 1932, but transferred to the Colloquium Committee in 1937. He was one of the editors of the Mathematical Reviews when it started in 1940 and of the Mathematical Surveys in 1945. J. D. was also one of the moving forces of the Organizing Committee for the International Congress which was to have been held in 1940. He was an influential and much trusted member of the Council of the Society since 1931 and Vice President in 1942–1943.

J. D. wrote his first letter to me in June 1925; we met in person at the Cornell meeting of the same year and remained in steady contact for the rest of his life. Our active collaboration started in the late fall of 1927, the first research project being the frequency of the characteristic values of an analytical kernel. This program was soon expanded to more general problems in the theory of linear integral equations, but our ambitious plans of writing the book in the field never materialized. In April 1928 we added Nörlund summability of Fourier series and in May Hausdorff summability. The latter also included questions of relative inclusion and turned out to be particularly fruitful since it led us into the theory of Fourier transforms and Laplace-Stieltjes integrals. A number of results never got beyond the stage of preliminary communications, however. A memoir on R. Schmidt summability which should have concluded the applications to Fourier series was put off indefinitely. Likewise planned memoirs on Hausdorff summability and on Laplace-Stieltjes integrals were postponed for various reasons. During the seven years that we worked together constantly we produced 24 notes and papers; to this should be added a paper with A. C. Offord [66] and one with G. Szegö [69]. Though J. D. devoted most of his energies to our joint work during this period, he wrote six short but good papers of his own and also found time to collaborate with H. L. Krall, S. Saks, and M. H. Stone. This was the time when the new functional analysis broke through in this country and J. D. in his writings and, perhaps still more, in his teaching became a strong and influential protagonist of the new ideas.

The happiness of J. D. came to an end in June 1934 with the sudden death of his wife and from this blow he never recovered. The effect on his scientific life was immediate and destructive. He published only four papers after 1935, but a great effort still went into his lecture notes and some of his best students belong to this later period. After the outbreak of the war, most of his energy was consumed by the course in partial differential equations [E] which he and W. Feller gave in the School of Applied Mechanics at Brown. He found time, however, to finish an excellent book on the moment problem [V], the first volume in the Mathematical Surveys, which he wrote with his old friend and university mate J. Shohat. His other co-authors during this period, in addition to those previously mentioned, are N. Dunford and A. Zygmund.

J. D.'s constitution had a number of weak spots which gradually became more serious; in February 1945 his heart threatened to give in completely, but he recovered and took part in the April meeting of the Council in his old vigorous manner. A trip to his beloved California broke him down again and he died in Washington, D. C. on November 18, 1945. He is survived by his son.

After this sketch of J. D.'s life, I shall proceed to a brief account of his scientific production which will be grouped according to subjects. If I should be found guilty of writing more fully about the investigations in which I have participated, this is due to my having a much better understanding of the ideas involved in these papers. J. D. published 71 research papers and 3 of his books are also of research character. He wrote 36 reviews in this BULLETIN as well as many reviews in the Zentralblatt and the Mathematical Reviews. These frequently contain original contributions and the same is true of his lecture notes.

1. **Number theory** [1, 2]. These papers were mentioned above. It is enough to add that the main problem in [1] is to give an explicit solution of the congruence $x^2 \equiv q \pmod{p}$ for the case in which $p \equiv 1 \pmod{4}$.

2. **Calculus; functions of real or complex variables** [9, 22–24, 29, 31, 39, 40, 47, 51]. Here [51] is an important contribution to the theory of conjugate functions in the case when $p = 1$ and extends work by M. Riesz, Hardy, Kolmogoroff, and Zygmund. The basic result is an inequality for $\int_0^{2\pi} |v(re^{i\theta})| \, d\theta$, $r < 1$, where $v(z)$ is the imaginary part of a function holomorphic in the unit circle. The inequality has various important consequences and is, in a certain sense, the best of its kind. [29] is connected with [I] and [25] and is concerned with the distribution of the zeros of certain entire functions of exponential type which arise in the theory of boundary value problems. Among the papers on functions of a real variable it is enough to mention [24], which contains a new proof for the closure of the trigonometric orthogonal system, and [39], which gives simple proofs of the basic properties of Lebesgue's monotone function associated with the set of Cantor.

3. **General theory of summability** [32, 56, 65]. J. D.'s lively interest in questions of summability dates from his association with Silverman. In [32] they proved that a series which is summable Hausdorff $[H, q(u)]$ is not summable Abel if the corresponding moment function $\int_0^1 u^s dq(u)$ has a zero in the right half-plane, while every Nörlund summable series is also summable Abel. The Hausdorff→Abel problem was later solved and generalized by O. Henrikson in Math. Zeit. vol. 39 (1935) pp. 501–510. J. D. found that Nörlund's definition of summability had been considered by G. F. Voronoi in a rare Russian publication from 1901, so he published an annotated translation of Voronoi's note in Ann. of Math. (2) vol. 33 (1932) pp. 422–428. Some questions raised by J. D. in this connection were solved later by R. E. Gilman and G. Szegö. In [56] J. D. and I considered the question of relative inclusion of Hausdorff means; we reformulated Hausdorff's conditions in terms of moment functions and mass functions and applied the results to various special questions. Our conditions for $[H, q(u)] \supset (C, n)$ are not correctly formulated (see Garabedian, Hille, and Wall, Duke Math. J. vol. 8 (1941) pp. 193–213). The fascinating problem of the arithmetic of Hausdorff matrices and of moment functions, raised in [56], is still unsolved. In [65] J. D. showed, as a consequence of the uniform boundedness theorem, that a certain plausible extension of the notion of regularity in the theory of summability actually reduces to the classical concept.

4. **Summability of Fourier series** [34–38, 50, 52–55]. In [36] and later papers we introduced various notions of effectiveness of a method of summability with respect to Fourier series, conjugate series, derived series, and so on. In [34] and [50] we determined sufficient conditions for different types of effectiveness of a Nörlund method of summation; in the most important case of positive monotone decreasing weight factors the conditions are also necessary. The corresponding problem for Hausdorff means was tackled in [34–37, 55]. A necessary condition for (F)-effectiveness was found to be that $C(s) \in L(-\infty, \infty)$ where $C(s)$ is the cosine transform of the function which equals $q(u) - u$ in $(0, 1)$ and zero elsewhere. That this condition is also sufficient is indicated in a footnote in [55], but the proof is still unpublished, and the sufficiency proof in [55] involves an additional restrictive end point condition on $q(u)$. If the latter holds and if $|C(s)|$ is dominated by a monotone function in $L(0, \infty)$, then the method is also (L)-effective, that is, sums the Fourier series of $f(x)$ to the sum $f(x)$ in the Lebesgue set of the function. For the conjugate series the corresponding sine transform plays a similar role.

During 1931–1933 we investigated the (F)-effectiveness of many methods of summation and were frequently led to a condition involving the integrability over $(-\infty, \infty)$ of the Fourier transform of a kernel associated with the method. A number of such results were announced in my symposium address on the summation of Fourier series in Bull. Amer. Math. Soc. vol. 38 (1932) pp. 505–528, but the detailed discussion which we had planned was not published when it was found that the main result of [38] was erroneous in the generality there stated.

In [54] we discussed a method of summability by convergence factors defined by a regular matrix in which each row is a factor sequence of type (L_p, L_p) with respect to a given orthogonal system. If the method is (L_p)-effective, it was shown to be (L_q)-effective for $p \leqq q \leqq p'$, and all (L_1)-effective methods were determined for trigonometric Fourier series. The main tool was the convexity theorem of M. Riesz for bilinear forms.

5. **Moment problems; Fourier and Laplace transforms** [V, 48, 58–63, 66, 67]. There are many moment problems in the literature; they may look artificial to the outsider, but they have led to the introduction of many new fruitful ideas in analysis and they have a bearing on a great variety of problems (continued fractions, functions of positive real part, positive functionals, quasi-analytic functions, and so on). The Hausdorff means are based on a sequence of moments for the interval $[0, 1]$ and lead to the Hausdorff moment problem: determine a function $q(u)$ never decreasing and bounded in $[0, 1]$ or of bounded variation in $[0, 1]$ such that the moments $\int_0^1 u^n dq(u)$, $n = 0, 1, 2, \cdots$, have given values. [48] is a contribution to this problem; J. D. gives a new proof of the Hausdorff-Bernstein-Widder theorem according to which a function completely monotone in $[0, \infty)$ is the Laplace-Stieltjes transform of a monotone bounded function. In [V] he and J. Shohat gave a survey of the theory of moment problems including history, general problems and methods, the Hamburger moment problem for $(-\infty, \infty)$, various modified problems, and the related theory of mechanical quadratures with an extensive bibliography.

The theories of Fourier and Laplace transforms are closely related to each other and to the theory of functions holomorphic in a half-plane, having boundary values in $L_p(-\infty, \infty)$ in terms of which the functions are representable by their proper Cauchy and Poisson integrals. One may ask when a given function $f(x)$ is such a boundary function; for $p = 2$ the answer has been given by Paley and Wiener and we extended their work to the range $1 \leqq p \leqq 2$ in [58]. We solved a similar question in [62] where $f(x)$ is supposed to be a Fourier transform. Related questions also occur in [67], but here the main problem is to give sufficient conditions under which the Fourier transform of the boundary function belongs to $L(-\infty, \infty)$. It was remarked above that the question of when a Fourier transform belongs to L is basic for the summation theory of Fourier series; a number of sufficient conditions were also proved in [55].

If $f(u) \in L_p$, $1<p<2$, Titchmarsh proved that the Fourier transform $F(t)$ of $f(u)$ exists as a limit in the mean of order p'; in [61] we proved conversely that $f(u)$ is the limit in the mean of order p, when $a \rightarrow \infty$, of the Fourier transform of the function which equals $F(-t)$ in $(-a, a)$ and zero outside. This is no longer true when $p=1$. The note also gave basic properties of the Dirichlet transform of $f(u)$ and served as the point of departure for later work on this transform by H. Kober and by myself. Note [66] contains the essence of a correspondence with A. C. Offord; it was observed, among other things, that for a fixed $p>2$ the elements of L_p having a Fourier transform in any L_q form a set of the first category in the metric space L_p.

The remaining papers under this heading deal with Laplace-Stieltjes integrals and are closely connected with [56]. The initial question is when an analytic function is a moment function, that is, the Laplace-Stieltjes transform of a function of bounded variation in $[0, \infty)$. Some sufficient conditions are stated in [59], but the main result here is that every function $f(s) \in H_p$, $1 \leqq p \leqq 2$, that is, such that $\int_{-\infty}^{\infty} |f(x+iy)|^p dy \leqq M$ for $x>0$, is the Laplace transform of a function $F(t)$ in $L_{p'}(0, \infty)$ (for $p=1$ proved by Bochner, for $p=2$ by Paley-Wiener, the general case was later discussed in detail by G. Doetsch along the lines indicated by us). When $2<p$, the situation is entirely different. In this case $f(s)$ is the Laplace-Stieltjes transform of a continuous function $A(t)$ and the functions $f(s)$ for which $A(t)$ is of bounded variation in any interval form a set of the first category in the metric space H_p. In [63] we introduced a new idea in the theory of summability of Laplace integrals based upon the trivial identity $f(s) = [g(s)]^{-1} g(s) f(s)$, where the multiplier $g(s)$ is supposed to be holomorphic, different from zero, and representable by an absolutely convergent Laplace-Stieltjes integral in a given half-plane $\Re(s) > \beta$. If $f(s)$ admits of a representation by an absolutely convergent Laplace-Stieltjes integral for $\Re(s) > \sigma$, so does $g(s)f(s)$, provided $\sigma \geqq \beta$, so that the transformation by multipliers is convergence preserving. If, in particular, $g(s) = s^{-\alpha}$, then the resulting representation for $f(s)$ is equivalent to that furnished by (C, α)-summability as long as $\Re(s) > 0$. These ideas were developed somewhat further in my address on Laplace integrals in the Proceedings of the Eighth Scandinavian Mathematical Congress in Stockholm 1934, but detailed proofs are still missing.

6. **Differential equations** [12, 14, 30, 41]. Here [12] contains a uniqueness theorem, [14] uses results of Besicovich to extend the asymptotic form of the solution of a linear differential system from real to complex values of a parameter, and [30] is mainly of didactic interest. Finally [41] contains a very useful estimate of the variation of the solution of a differential system with the coefficients of the system.

7. **Boundary value problems; Green's function** [I, 3–8, 25, 27, 43, 46]. This is the field which formed the center of J. D.'s interest during a period of almost twenty years. In [3–5] he is concerned with the problem of the transversal vibrations of a clamped rod which leads to a fourth order problem

$$[r(x)V'']'' = \lambda p(x)V, \qquad V(a) = V(b) = V'(a) = V'(b) = 0.$$

If the rod is homogeneous, $p(x)$ and $q(x)$ are positive constants and the problem is much simplified. J. D. proved that any twice differentiable function satisfying the boundary conditions can be represented as a uniformly convergent series in terms of characteristic functions. In these investigations he used the methods of Steklov which are restricted to self-adjoint problems.

In 1911–1912 he got the idea of attacking the boundary value problem for a general nth order linear differential equation by the method of Cauchy-Poincaré. This method requires a fairly accurate knowledge of the asymptotic character of the solutions for large values of the parameter as well as of the distribution of the characteristic values. J. D. proceeded to derive the required information and to apply it to the boundary value and expansion problem. During this process he found, much to his chagrin, that the problem had already been attacked by G. D. Birkhoff, using the same method, in Trans. Amer. Math. Soc. vol. 9 (1908) pp. 219–231, 373–395. He decided nevertheless to publish his results for the expansion problem in [6], replacing his own discussion of the asymptotic formulas by Birkhoff's. J. D. thought that Birkhoff's treatment of the expansion problem was insufficient and said so. Birkhoff supplied the omitted details in a paper which appeared a year later in the same journal (Rend. Circ. Mat. Palermo vol. 36 (1913) pp. 115–126) and J. D. acknowledged the sufficiency of the supplementary argument in [8]. For further details in this matter, I refer to Marston Morse, *George David Birkhoff and his* *mathematical work*, Bull. Amer. Math. Soc. vol. 52 (1946), especially pp. 360–362. One observation should be made, however. Birkhoff restricts himself to a function $f(x)$ which is made up of a finite number of pieces, each real, continuous, and with continuous derivative, and proves that the expansion converges to $[f(x-0)+f(x+0)]/2$ at all interior points and to certain linear combinations of $f(a+0)$ and $f(b-0)$ at the end points. Tamarkin, on the other hand, proves convergence to $[f(x-0)+f(x+0)]/2$ at interior points if $f(x)$ is merely of bounded variation and he also shows that the expansion is equi-convergent and equi-summable $(C, 1)$ with the trigonometric Fourier series of $f(x)$ in (a, b) when $f(x)$ is merely integrable. He does not say in what sense $f(x)$ is to be integrable, but even if he means Riemann integrable, his results contain those of Birkhoff for interior points as a very special case.

He continued these investigations in his dissertation [I] which is dated April, 1917. This is a practically self-contained, very detailed discussion of boundary value problems for linear differential systems. Starting with an outline of the theory of matrices, Lebesgue integration, and existence theorems, he proceeds to a discussion of the character of the solutions for large values of the complex parameter ρ, assuming the coefficient matrix to be analytic in ρ outside a circle, except for a pole at infinity. The boundary conditions for the interval (a, b) are of a very general nature, $U_k[y]=c_k$, $k=1, 2, \cdots, n$. In the revised version [25] of 1927 his $U_k[y]$ are what in modern language would be called linear bounded functionals on the space $C^{(n-1)}[a, b]$, that is,

$$U_k[y] = \sum_{m=0}^{n-1} \int_a^b y^{(m)}(t)\,da_{km}(t),$$

but in [I] he restricts himself to the case in which each $a_{km}(t)$ is the sum of a step function with a finite number of steps and an absolutely continuous function. The function of Green, the distribution of the characteristic values, and the expansion problem are studied in great detail. He proves the same type of equi-convergence and equi-summability theorems as in [6], but now explicitly for functions integrable in Lebesgue's sense. J. D.'s results in [I] represented the last word in the field for a number of years. These results, however, remained unknown outside of Russia practically until J. D. issued the English condensed and revised edition [25].

Papers [27] and [28] together with [43] mark different stages in the transition of J. D.'s interest from boundary value problems to the theory of integral equations. In [27] he shows that to a given integro-differential system you can construct a Green's function and the characteristic values of the problem are asymptotic to the characteristic values of a related differential system in such a manner that expansions in terms of fundamental functions of the two problems are equi-convergent and equi-summable. In [28] he and R. E. Langer show that certain integral equations are equivalent to integro-differential problems and hence soluble.

8. **Integral equations** [20, 21, 28, 33, 42, 44, 45, 64]. The papers [20, 21] were written in Petrograd in 1924 and published in this country two years later. The first is concerned with the inversion of Laplace's integral, the second with a generalization of Volterra's equation. In [26] J. D. is concerned with an integral equation whose kernel $K(x, t, \lambda)$ is a holomorphic function of λ in a domain Δ for almost all x, t. If the kernel is quadratically integrable for each λ in Δ, uniformly with respect to λ in any fixed closed subset of Δ, then either the resolvent kernel exists as a meromorphic function of λ in Δ or the resolvent does not exist for any λ in Δ. The basic tool is the theory of infinite determinants. [28] was discussed above. In [42] J. D. gave a careful modern treatment of Abel's integral equation.

The remaining papers are chiefly concerned with the frequency of the characteristic values. T. Carleman, generalizing a result of I. Schur, had proved the inequality $\sum|\lambda_n|^{-2} \leqq \iint |K(x, t)|^2 dxdt$ and had shown that the exponent of convergence could not be replaced by any smaller number than two even for continuous kernels. If, however, the kernel satisfies addition restrictions, it was known in some cases that better estimates hold. In [33, 45] we made a systematic investigation of the frequency problem under a large number of different assumptions of continuity, differentiability or analyticity. We replaced the integral equation by an equivalent system of linear equations for the Fourier coefficients of the solution in a suitable orthogonal system and discussed the equations with the aid of the theory of infinite determinants. We seem to have found this mode of attack independently of each other as well as the first crude estimates, but for the refined analysis the estimates given in [26] turned out to be very useful. J. D. also contributed the decisive artifice of modifying an optimum number of equations. This is probably our best joint paper, largely due to the ingenuity, knowledge, and technical skill displayed by J. D. In this investigation the kernel is supposed to be at least quadratically integrable. If merely the pth power is integrable for some p with $1 \leqq p < 2$, the situation is strikingly different. In [44] we exhibited a symmetric kernel in L_p whose spectrum is dense on the real axis as well as an unsymmetric kernel for which every $\lambda \neq 0$ is in the point spectrum. In [64] we showed that the transformation $g(x) = \int_{-\infty}^{\infty} K(x, t) f(t) dt$ is completely continuous (= compact) on L_p to itself when $\int [\int | K(x, t) |^{p'} dt]^{p/p'} dx < \infty$, provided $p > 1$. An example of J. von Neumann shows that this is no longer true for $p = 1$. Both [44] and [64] contain a number of additional results concerning special integral equations.

9. **Mathematical physics** [17–19]. These three papers arose from his collaboration with Friedmann in 1922–1924. I regret that I cannot read [17] which is concerned with the propagation of discontinuities in a compressible fluid. The results of this paper seem to have an important bearing on the problem of the vertical motion of the atmosphere, a problem to which Friedmann had given a number of contributions. This connection is stated in [18] which also contains some numerical observations bearing out the theoretical conclusions. [19] is not available to me and does not seem to have been reviewed in either mathematical or physical abstract journals.

10. **Approximations** [III, 10, 11, 13, 15, 16]. Of these [III] and [15] are not available to me. [10] is an elaboration of the method of W. Ritz written together with N. Krylov; the extensive corrections [11] are signed by J. D. alone. [13] deals with C. Störmer's method of approximate integration of ordinary differential equations; J. D. shows that the procedure is not merely suitable for numerical computations but is actually convergent. In [16] he and Krylov considered an interesting interpolation formula based upon the partial fraction expansions of the odd order derivatives of the cotangent; they establish convergence of the procedure at points of continuity and determine degree of approximation, and so on.

11. **Abstract spaces** [49, 57, 68–71]. These six notes are entirely unrelated. In [49] J. D. gave necessary and sufficient conditions in order that a subset of L_p be compact. He generalized results of Kolmogoroff; other conditions were later given by M. Riesz. In [57] he and S. Saks studied operators of the form $\int_0^1 f(t) d_t H(x, t) = U[f]$, where $H(x, t)$ is normalized and of bounded variation in t for almost all x. They show that if $U[f]$ is essentially bounded in x for each continuous $f(t)$, then the total variation of $H(x, t)$ is an essentially bounded function of x. [68] is written with M. H. Stone. The complex Euclidean spaces contain Hilbert space as a special case; the theorems refer to weak and strong forms of boundedness, completeness, and compactness and to conditions under which an operator or a matrix is bounded. In 1935–1936 J. D. lectured on polynomials of approxima-

tion [C], and [69] is an outgrowth of this course. The problem is that of estimating the ratio of the L_p-norms of $f'(x)$ and $f(x)$ when $f(x)$ is a polynomial of degree n. J. D. had found that the ratio is $O(n^2)$ and G. Szegö showed that this is the right order, but the best constant could not be found except when $p=2$ in which case the problem had already been solved by E. Schmidt. [70] is written with Nelson Dunford and its purpose is to establish theorems of the Fubini-Jessen type for abstract-valued functions. [71], written with A. Zygmund, is J. D.'s last publication. It contains a simplified proof of Thorin's extension of M. Riesz's convexity theorem from bilinear forms to entire functions of several complex variables.

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LIST OF J. D. TAMARKIN'S COLLABORATORS

[AB]	A. S. Besicovich
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[NK]	N. Krylov
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[AO]	A. C. Offord
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[JS]	J. A. Shohat
[LS]	L. L. Silverman
[VSm]	V. I. Smirnov
[VSt]	V. A. Steklov
[MS]	M. H. Stone
[GS]	Gabor Szegö
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[AZ]	Antoni Zygmund

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55. *On the summability of Fourier series.* III, Math. Ann. vol. 108 (1933) pp.
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58. *On a theorem of Paley and Wiener*, Ann. of Math. (2) vol. 34 (1933) pp.
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61. *On the theory of Fourier transforms*, Bull. Amer. Math. Soc. vol. 39 (1933)
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67. *On the absolute integrability of Fourier transforms*, Fund. Math. vol. 25
[EH] (1935) pp. 329–352.

68. *Elementary proofs of some known theorems of the theory of complex Euclidean
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Mimeographed Lecture Notes

A. *Integral equations*, Lectures delivered at Brown University during 1931–1932, Providence, 1932, iii+368 pp.

B. *Theory of Fourier series*, Lectures delivered at Brown University during 1932–1933, Providence, 1933, vi+319 pp.

C. *On the theory of polynomials of approximation*, Lectures delivered at Brown University during 1935–1936, Providence, 1936, 186 pp.

D. *Theory of abstract spaces*, Lectures delivered at Brown University during 1936–1937, Providence, 1937, 215 pp.

E. [WF] *Partial differential equations*, Brown University, Summer Session for Advanced Instruction and Research in Mechanics, 1941, x+268 pp. [Chapters I–III by J.D.T., chapters IV–VII by W.F.]

THE FEBRUARY MEETING IN NEW YORK

The four hundred twenty-third meeting of the American Mathematical Society was held at Columbia University on Saturday, February 22, 1947. The attendance was approximately two hundred including the following one hundred seventy-six members of the Society:

Milton Abramowitz, C. R. Adams, Leonidas Alaoglu, E. B. Allen, Warren Ambrose, T. W. Anderson, R. G. Archibald, R. F. Arens, L. A. Aroian, R. N. Ascher, W. A. Asprey, P. T. Bateman, M. F. Becker, Stefan Bergman, Arthur Bernstein, Lipman Bers, J. H. Bigelow, Gertrude Blanch, J. H. Blau, R. P. Boas, H. W. Bode, G. L. Bolton, W. H. Bradford, A. B. Brown, Bailey Brown, Hobart Bushey, J. H. Bushey, H. H. Campaigne, J. D. Campbell, Herman Chernoff, M. D. Clement, I. S. Cohen, J. E. Crawford, Franco Croci, H. M. Dadourian, John DeCicco, J. B. Díaz, S. P. Diliberto, J. L. Doob, C. H. Dowker, Y. N. Dowker, T. L. Downs, Arnold Dresden, Jacques Dutka, Bernard Epstein, Paul Erdos, Luis Esteban-Carrasco, W. H. Fagerstrom, J. M. Feld, A. D. Fialkow, Emanuel Fischer, D. A. Flanders, R. M. Foster, F. H. Fowler, G. A. Foyle, Gerald Freilich, L. J. Gårding, Abe Gelbart, B. P. Gill, Joseph Gillis, A. W. Goodman, H. J. Greenberg, E. J. Gumbel, F. C. Hall, Philip Hartman, K. E. Hazard, C. M. Hebbert, Alex Heller, A. D. Hestenes, Edwin Hewitt, Einar Hille, P. G. Hoel, A. J. Hoffman, T. R. Hollcroft, E. M. Hull, L. C. Hutchinson, Robert Jastrow, A. W. Jones, Bjarni Jónsson, Aida Kalish, Samuel Kaplan, Edward Kasner, M. E. Kellar, L. S. Kennison, D. E. Kibbey, H. S. Kieval, J. R. Kline, Morris Kline, E. R. Kolchin, B. O. Koopman, M. Z. Krzywoblocki, Jack Laderman, Joseph Lehner, R. A. Leibler, Benjamin Lepson, R. J. Levit, J. H. Lewis, Marie Litzinger, Charles Loewner, E. R. Lorch, Lee Lorch, A. N. Lowan, N. H. McCoy, Janet McDonald, A. W. McMillan, Brockway McMillan, L. A. MacColl, H. F. MacNeish, W. T. Martin, Imanuel Marx, F. I. Mautner, A. N. Milgram, Don Mittleman, G. F. Morecroft, I. R. Moses, Z. I. Mosesson, D. S. Nathan, C. A. Nelson, A. V. Newton, P. B. Norman, C. O. Oakley, L. G. Peck, A. M. Peiser, Anna Pell-Wheeler, R. S. Phillips, Everett Pitcher, E. L. Post, Walter Prenowitz, M. H. Protter, G. N. Raney, H. W. Reddick, R. G. D. Richardson, J. F. Ritt, H. E. Robbins, J. H. Roberts, H. E. Salzer, Pierre Samuel, Arthur Sard, A. T. Schafer, R. D. Schafer, I. E. Segal, C. E. Shannon, R. H. Shaw, Max Shiffman, D. N. Silver, David Singer, James Singer, P. A. Smith, Fritz Steinhardt, E. G. Straus, L. M. Straus, M. M. Sullivan, Fred Supnick, J. H. Taylor, Feodor Theilheimer, W. R. Transue, C. A. Truesdell, J. W. Tukey, Cengiz Uluçay, Philon Vassiliou, A. E. Ventriglia, A. D. Wallace, R. M. Walter, W. G. Warnock, Alan Wayne, Louis Weisner, F. P. Welch, G. W. Whitehead, Albert Wilansky, J. E. Wilkins, John Williamson, Jacob Wolfowitz, Bertram Yood, H. J. Zimmerberg, Leo Zippin, Antoni Zygmund.

The meeting Saturday morning was divided into two sections, one for papers in Analysis, Geometry and Topology, at which Professor Anna Pell-Wheeler presided, and one for papers in Algebra, Applied Mathematics, and Statistics, at which Professors W. T. Martin and E. R. Lorch presided.

The afternoon session, at which President Einar Hille presided, consisted of a business meeting and an invited address.

At the business meeting, Secretary J. R. Kline presented the following two changes in by-laws of the Society which had been voted by the Council and approved by the Board of Trustees.

ARTICLE V. ELECTIONS AND TERMS OF OFFICERS

Section 1. The term of office shall be two years in the case of the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. In every case, however, the officials specified in Articles I, II, III, and IV shall continue to serve until their successors have been duly elected and qualified.

The term of office in the case of the trustees to be elected at the annual meeting to be held between the fifteenth of December, 1948 and the fifteenth of January, 1949, shall be three years in the case of two of the trustees to be elected and shall be two years in the case of three of the trustees to be elected. Upon the expiration of the terms of office of the trustees elected at the annual meeting to be held between the fifteenth of December, 1948 and the fifteenth of January, 1949, the term of office of all trustees shall be two years.

On motion, this change in the by-laws was approved by the members of the Society.

ARTICLE VII. DUES

Section 2. The annual dues of persons elected by the Council to ordinary membership under the provisions of Article VI, Section 5, shall be ten dollars, with the following exceptions: (1) during the first two years of membership, the annual dues shall be six dollars; during the third and fourth years of membership, the annual dues shall be eight dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. Five dollars and fifty cents of the annual dues of those members who receive the Bulletin under the provisions of Article VI shall be allocated in payment therefore.

Section 7. Persons elected to ordinary membership as nominees of sustaining members or institutional contributing members under the provisions of Article VI, Sections 3–4, shall not be required to pay an initiation fee. They shall not be required to pay dues, so long as it is agreed that they are designated by the sustaining member or institutional contributing member under these provisions. If a nominee of a sustaining member or an institutional contributing member later becomes a dues-paying member, he shall pay dues at the appropriate rate per year (as provided in Section 2 of this Article) for the remainder (if any) of his first four years of membership, and ten dollars a year thereafter.

On motion, this change in the by-laws was approved by the members of the Society.

Professor N. H. McCoy of Smith College gave an address entitled *Subdirect sums of rings.*

Titles and cross references to the papers read follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Papers numbered 1–6 were presented in the section for Analysis, Geometry, and Topology, papers 7–12 in the section for Algebra, Applied Mathematics, and Statistics, papers 13–30 were presented by title. Paper 1 was read by Mr. Bradford, paper 3 by Professor Kasner, paper 8 by Professor Robbins, and paper 11 by Professor Díaz. Dr. Barsotti was introduced by Professor A. A. Albert, Professor Chang by Professor Oscar Zariski, Mr. Gerst by Professor J. F. Ritt, and Professor Samuelson by Dr. Frederick Mosteller.

1. R. N. Haskell and W. H. Bradford: *Sub-biharmonic functions.* Preliminary report. (Abstract 53-3-130.)

2. John DeCicco: *New proofs of the theorems of Kasner concerning the infinitesimal contact transformations of mechanics.* (Abstract 53-3-151.)

3. Edward Kasner and John DeCicco: *New theorems in the polar theory of a general algebraic curve.* (Abstract 53-3-154.)

4. Everett Pitcher: *Čech homology invariants of continuous maps.* (Abstract 53-3-159.)

5. Edwin Hewitt: *A characterization of rings of continuous real-valued functions.* (Abstract 53-5-171.)

6. J. H. Roberts: *A problem in dimension theory.* (Abstract 53-3-160.)

7. R. J. Levit: *The non-existence of a certain type of odd perfect number.* (Abstract 53-3-117.)

8. P. L. Hsu and H. E. Robbins: *Complete convergence and the law of large numbers.* (Abstract 53-3-156.)

9. Fred Supnick: *Cooperative phenomena.* II. *Structure of the two-dimensional Ising model.* (Abstract 53-3-149.)

10. H. J. Greenberg: *The determination of upper and lower bounds for the solution of the Dirichlet problem.* (Abstract 53-3-144.)

11. J. B. Díaz and H. J. Greenberg: *Upper and lower bounds for the solution of the general biharmonic boundary value problem.* (Abstract 53-3-126.)

12. M. Z. Krzywoblocki: *On certain cases of simple solutions of flow equations in compressible fluid with particular conditions.* (Abstract 53-3-145.)

13. Iacobo Barsotti: *Structure theorems for algebraic algebras without a finite basis.* (Abstract 53-3-107-*t*.)

14. Iacobo Barsotti: *Valuations in division algebras without a finite basis.* (Abstract 53-3-108-*t*.)

15. S. C. Chang: *Contributions to projective theory of singular points of space curves.* (Abstract 53-1-81-*t*.)

16. V. F. Cowling: *Some results for Dirichlet series.* (Abstract 53-3-125-*t*.)

17. J. J. Gergen and F. G. Dressel: *A minimal problem for harmonic functions.* (Abstract 53-3-128-*t*.)

18. Irving Gerst: *Meromorphic functions with simultaneous multiplication and addition theorems.* (Abstract 53-3-129-*t*.)

19. Edward Kasner and John DeCicco: *A partial differential equation of fourth order connected with rational functions of a complex variable.* (Abstract 53-1-83-*t*.)

20. Edward Kasner and John DeCicco: *Harmonic transformations and velocity systems.* (Abstract 53-3-153-*t*.)

21. L. B. Robinson: *Generation of an infinite system of functions with the unit circle as singular line from the Garvin series.* (Abstract 53-3-132-*t*.)

22. H. E. Salzer: *The approximation of numbers as sums of reciprocals.* (Abstract 53-3-133-*t*.)

23. H. E. Salzer: *Table of coefficients for interpolating in functions of two variables.* (Abstract 53-3-146-*t*.)

24. P. A. Samuelson: *A generalized Newton iteration.* (Abstract 53-3-147-*t*.)

25. P. A. Samuelson: *Generalization of the Laplace transform for any operator.* (Abstract 53-3-148-*t*.)

26. I. E. Segal: *Postulates for general quantum mechanics.* (Abstract 53-3-134-*t*.)

27. E. G. Straus: *On the existence of square-free numbers of the form* ax^2+b. (Abstract 53-3-118-*t*.)

28. D. V. Widder: *Inversion formulas for convolution transforms.* (Abstract 53-5-239-*t*.)

29. Bertram Yood: *On ideals in operator rings over Banach spaces.* Preliminary report. (Abstract 53-3-138-*t*.)

30. H. J. Zimmerberg: *Definite integral systems.* (Abstract 53-3-140-*t*.)

T. R. HOLLCROFT,
Associate Secretary

BOOK REVIEWS

Lectures on the calculus of variations. By G. A. Bliss. University of Chicago Press, 1946. 9+296 pp. $5.00.

Calculus of variations is in the main the study of properties of a real-valued function on a class $\mathfrak{M}$ with particular attention to maximizing and minimizing properties of certain elements in $\mathfrak{M}$. Normally the elements of $\mathfrak{M}$ are curves or surfaces embedded in an euclidean space. There are two main problems in the calculus of variations, problems in the large and problems in the small. The first of these is concerned with existence theorems of minimizing elements of various types, the classification and the counting of these elements and the connections between these minimizing elements and the topology of $\mathfrak{M}$. The calculus of variations in the small is concerned primarily with the properties of a particular minimizing element and the determination of those properties that will insure a minimum. Besides these two aspects, there is the fascinating study of the relations between the calculus of variations and differential equations, geometry, physical applications, and the like.

The book under review is concerned with the problem in the small. It contains the best introduction to the calculus of variations known to the reviewer, no matter what phase of the subject one wishes to pursue. The book undoubtedly will become the standard text for the beginner and will be used by the specialist as a source of material and ideas.

Professor Bliss restricts himself to the case in which the class $\mathfrak{M}$ referred to above is a class of continuous arcs C: $y_i(x)$ $(x_1 \leqq x \leqq x_2)$ $(i=1, \cdots, n)$ in euclidean $(x, y_1, \cdots, y_n)$-space. These arcs are assumed to have a piecewise continuously turning tangent with elements (x, y, y') in a prescribed region R of (x, y, y')-space. Professor Bliss begins first with the case when the end values $[x_1, y(x_1), x_2, y(x_2)]$ are fixed and develops the theory in a simple and direct manner. Later he permits these end values to lie on a prescribed end manifold. Finally he requires the arcs to satisfy in addition a given set of differential equations $\phi_\beta(x, y, y')=0$ $(\beta=1, \cdots, m<n)$. The function to be minimized is of the form

$$I(C) = g[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f(x, y, y')dx,$$

with $g=0$ in the fixed end point case. By this procedure the author begins with the simplest problem in the calculus of variations and skillfully leads the reader step by step to the development of the theory of the most comprehensive problem, known as the problem of Bolza, for which there exists a satisfactory theory of maxima and minima. In fact the book under review contains the first published account of this theory that is relatively complete. The problem of Bolza contains as special cases all but a very few of the hitherto formulated problems involving simple integrals. The development of its theory is largely due to Professor Bliss and his students, inspired by him.

The book is divided into two parts. Part I deals with the simpler problems and Part II with the problem of Bolza. Part I is divided into six chapters. The author has chosen to begin with the fixed end point problem in three dimensions. This is the simplest problem that is general in the sense that the extension to higher-dimensional spaces represents little more than formal notational changes.

Chapter I is concerned with necessary conditions for a minimum together with properties of the extremal family. Sufficient conditions for a minimum are established in Chapter II. In Chapter III a detailed discussion of the Hamiltonian theory is given, together with a discussion of Hamilton's principle in mechanics. These results are extended in Chapter IV to the plane and to spaces of higher dimensions. A similar extension is made in Chapter V to parametric problems. The case when the end points of the arc are allowed to vary is treated in Chapter VI.

The last three chapters, which constitute Part II, are concerned with the general problem of Bolza. In this part the author develops a theory of minima that is analogous to the theory developed for the simpler problem. It is the only published account of this theory that is relatively complete. The presence of differential equations introduces a concept of normality that is absent in the simpler case. The discussion of this concept, given in this book, is the clearest published exposition of this important topic. The author restricts himself for the most part to the normal case. This is the most interesting case since it includes most of the important problems of the Bolza type.

At the end of the book is found a bibliography for the problem of Bolza. In the papers here listed one can find additional material on the problem of Bolza. There is also found in the appendix a simple and elegant treatment of implicit function theorems and existence theorems for differential equations. These theorems are formulated so as to be conveniently applicable in the calculus of variations.

The book is exceptionally well written and easy to read. The author has succeeded in setting forth the theory of the calculus of variations in a simple and appealing manner. The reviewer's only regret is that Professor Bliss did not have occasion to include various other topics in the calculus of variations in which he has been interested and to which he has made numerous contributions. The book is a valuable addition to a mathematician's library.

M. R. HESTENES

The theory of potential and spherical harmonics. By W. J. Sternberg and T. L. Smith. (Mathematical Expositions, No. 3.) The University of Toronto Press, 1944. 312 pp. $3.35.

This is a book on a classical field of analysis treated in the classical way and restricted to classical theorems. Potential theory in the 19th century sense is no longer a familiar subject in a mathematical curriculum, and so the value of such a publication is to be considered with respect to a rather special audience. Its needs might equally well be served by a compact presentation in English covering the fundamental notions similar, for instance, to that in Courant-Hilbert, volume 2 (which actually presents a more complete picture as well) or the third volume of Goursat's *Cours d'analyse* with its illuminating problems. The preface suggests the research worker may find some ideas here but this is not borne out by the contents.

The chapter headings indicate the topics taken up: The Newtonian law of gravity, concept of the potential, the integral theorems of potential theory, analytic character of the potential, spherical harmonics, behavior of the potential at points of the mass, relation of potential to theory of functions, the boundary value problems of potential theory, the Poisson integral in the plane, the Poisson integral in space, the Fredholm theory of integral equations, general solution of the boundary value problem. The last chapters deserve commendation, especially for the excellent presentation of the Fredholm theory. The Riemann integral is used exclusively, and only an elementary acquaintance with real variable theory and the rudiments of functions of a complex variable is needed as a prerequisite. The book can be read by the first year graduate student in an American university.

The exposition is notably lucid, and in fact almost conversational in its naturalness, though there is little emphasis. Perhaps the principal results could have been singled out and more said about the power and generality of these theorems. The preface remarks on a "consistent" use of vector analysis. Without implying any criticism, it should be noted, however, that the treatment is not vectorial in spirit. The vectors are usually brought in as shorthand notations for the more complicated Cartesian expressions actually manipulated.

A carping pedant could find details to criticize. We list a few drawn at random. The writers never define *portion* of a region, and presumably intend subregion; they refer to a one-sided surface where one side of a two-sided surface is meant. There is a familiar lacuna on page 175; a line integral is first shown to vanish on all simple closed curves of class C', but this result is later quoted as having been established for *all* simple closed curves. In the reviewer's mind, such defects are of minor importance at this level of maturity.

The criticisms mentioned below are not too serious and can be met by the addition of a few pages of material in a later edition. They may be interpreted in the spirit of the authors' invitation for "suggestions for improving the text," though it will be clear that they derive in part from a somewhat different conception of values in this field. Selection of material is, of course, the author's prerogative and one cannot quarrel with a decision to leave out such topics as tesseral and toroidal harmonics. However, it seems to the reviewer that the book is not complete with an exposition of the classical viewpoint alone, and it is eminently desirable that the reader be given an understanding of what modern work in potential theory deals with and what some of the underlying difficulties are. (The last two chapters constitute a step in this direction, but they require considerably more sophistication than the rest of the book.) In this connection one may perhaps go too far in stressing pathological aspects, but surely a student cannot appreciate potential theory if no pathology is mentioned. Thus, for instance, the simple example of zero Dirichlet data, except at one point where the value is not defined, is omitted, nor is the Zaremba example cited. There is no mention of the Lebesgue spine, or on a higher level, the necessity for analogues of primends. Similarly, the reviewer feels Hadamard's classic example of the failure of the Dirichlet principle should have been given. Nothing is said about the possibility of definitions of assumption of boundary values other than the obvious one, and so the reader remains unaware of the spirit of so many of the contributions of the last 40 years. Here a single result such as that of A. J. Maria's, 1932, Bulletin note, would clarify the subject. The writers do not call attention to the role of convexity which appears already in such early methods as those of Schwarz. In this connection an example of the sort of region for which Poincaré's inequality is invalid would bring home the meaning of the restrictions made throughout the book. Even though Stieltjes or Radon integrals do not appear, perhaps something might have been said about singular types of mass distribution, and some mention made of sub- and superharmonic functions and their implications.

Some small natural additions in connection with applications would have increased the utility of the book considerably for the reader interested in physics. Thus for instance the method of images requires more complete treatment. Practically all the theory is available in the book for consideration of charge distribution on conductors and perhaps even polarization. Considering the details introduced in the treatment of conformal mapping, the Schwarz-Christoffel formula might well have been introduced in view of the applications in hydrodynamics and in the determination of the field in modern physical apparatus such as the cyclotron.

The problems are, in the main, formal exercises. At least a few interesting ones should have been included, say comparable to the reviewer's favorite: the proof of Maxwell's assertion that the inverse nth power law of attraction is singled out by the requirement that "similar bodies be similarly charged."

The index could stand amplification. Thus, for example, so important a term as "region" does not occur in the index and indeed is defined just once in the book in a footnote. On the whole, however, the book is meritorious and can be recommended to a student.

D. G. BOURGIN

Elementary matrices and some applications to dynamics and differential equations. By R. A. Frazer, W. J. Duncan, and A. R. Collar. Cambridge University Press, 1946. 14+416 pp. $4.00.

This is a reprint of the book which was first published in 1938 and reviewed in Bull. Amer. Math. Soc. vol. 45 (1939) p. 825. During the past eight years the use of matrices in problems of applied mathematics has become more widespread and part of the credit for this is due the book under review. The theory is well presented but it is regrettable that the authors have felt it necessary, in order to maintain the elementary level mentioned in the title, to refrain from a discussion of the canonical form of a matrix under change of basis and from the proof of the theorem that the number of linearly independent solutions of a system of linear differential equations with constant coefficients is the degree of the determinant of the system. The applications to dynamics (particularly those dealing with flutter problems in aerodynamics) are well chosen and realistic and the book may be recommended to anyone who wishes to know how matrices may be used to advantage in applied mathematics. The printing maintains the high standard set long ago by the Cambridge University Press.

F. D. MURNAGHAN

NOTES

For members of the Society who would like to have the photograph of the late George David Birkhoff and the tribute written by Rudolph Ernest Langer which appeared at the beginning of volume 60 of the TRANSACTIONS of the American Mathematical Society, the Society has had prepared some reprints which contain the photograph, dedication and eulogy between covers. Members may obtain a copy by sending a request to the Society accompanied by fifteen cents in stamps.

A mathematical institute, Mathematisch Centrum, was organized in Amsterdam in February, 1946, as a private foundation with support from the government of the Netherlands and of the city of Amsterdam. The general aim is to contribute to the development of pure and applied mathematics by arranging lectures and conferences, by supporting mathematical publications, by initiating and organizing mathematical research, and by providing a library and research center.

Professors Garrett Birkhoff and J. H. Van Vleck of Harvard University are members of the Advisory Committee of the recently organized American Institute of France, Incorporated. The Institute is designed to provide opportunities to American students and younger scholars to study in French institutions of learning, and to French scholars to study in American institutions. It also aims to provide specialized assistance to individual students and scholars, and to evaluate systematically work done abroad so that students may be properly accredited upon returning to their respective countries.

The University of Michigan announces a Conference on Algebra to be held Friday to Monday, July 25–28, 1947. Those who wish to present papers at the Conference, or who wish further information about it, are invited to communicate with R. M. Thrall, Department of Mathematics, University of Michigan.

The Institute of Mathematical Statistics has announced the election of these officers: President, Professor William Feller of Cornell University; Vice Presidents, Dr. J. H. Curtiss, Assistant to the Director of the National Bureau of Standards, and Mr. M. H. Hansen of the Bureau of the Census; Secretary-Treasurer, Professor P. S. Dwyer of the University of Michigan.

Professor Harald Cramér of the University of Stockholm has been awarded an honorary doctorate of science by Princeton University. During the spring term he has been visiting professor at Yale University.

Professor Heinz Hopf of the University of Zurich has been awarded an honorary doctorate of science by Princeton University. He is now visiting professor at Princeton University.

Professor J. W. Alexander of the Institute for Advanced Study has been awarded an honorary doctorate of science by Princeton University.

Professor J. R. Kline of the University of Pennsylvania has been elected to membership in the Society of Science and Letters at Warsaw.

Mr. W. D. Lambert of the United States Coast and Geodetic Survey has been elected correspondent of the French Academy of Sciences.

Mr. Milton Abramowitz of the Mathematical Tables Project, New York City, has received a Certificate of Award from the Navy Department.

Dr. A. S. Householder of the Monsanto Chemical Company has received a Bureau of Ordnance development award.

Mr. Jack Laderman of the Mathematical Tables Project, New York City, has received an award from the War Department.

Professor E. B. Wilson of the School of Public Health, Harvard University, has retired with the title emeritus.

Dr. A. Erdélyi of the University of Edinburgh has been appointed to a visiting professorship at the California Institute of Technology for the year 1947–1948. He will complete and edit certain works of the late Professor Harry Bateman.

Professor Cristobal de Losada y Puga of the Catholic University of Peru has been nominated by the President of the Republic of Peru as Minister of Public Education.

Dr. A. G. Walker of the University of Liverpool has been appointed to a professorship at the University of Sheffield.

Dr. P. H. Anderson, formerly with the War Assets Administration, has been appointed economic analyst with the Marketing Division, Office of Domestic Commerce, Commerce Department.

Dr. R. P. Bailey has been appointed to an associate professorship at the United States Naval Academy. This appointment was incorrectly announced in the January 1947 issue of this Bulletin.

Assistant Professor Herbert Busemann of Smith College has been appointed to a professorship at the University of Southern California.

Mr. Albert Cahn has accepted a position as mathematician with the Machinery Development Section of the National Bureau of Standards.

Mr. F. M. Carpenter of the State University of Iowa has been appointed to an assistant professorship at Missouri School of Mines, Rolla, Missouri.

Assistant Professor Lola M. Christy of Westminister College, New Wilmington, Pennsylvania, has retired.

Associate Professor H. S. M. Coxeter of the University of Toronto has been appointed visiting professor at the University of Notre Dame.

Mr. Paul Cramer of Huron College, Huron, South Dakota, has been appointed to an assistant professorship at Monmouth College, Monmouth, Illinois.

Dr. R. H. Downing of Fleetwings, Incorporated, Bristol, Pennsylvania, has been appointed to a professorship at the Army Air Forces Institute of Technology, Wright Field, Dayton, Ohio.

Assistant Professor B. H. Gere of the United States Naval Academy has been appointed to an associate professorship at Hamilton College.

Assistant Professor M. J. Gottlieb of Washington University is on leave of absence this term to work at the Institute for Advanced Study.

Professor L. E. Gurney of the University of Southern California has retired.

Assistant Professor J. F. Heyda of Franklin and Marshall College has accepted a position as a research mathematician at the Naval Ordnance Plant, Indianapolis, Indiana.

Dr. Herbert Jehle of the Franklin Institute has been appointed to an assistant professorship in the physics department of the University of Pennsylvania.

Professor Theodore Lindquist of Michigan State Normal College, Ypsilanti, Michigan, has retired.

Dr. J. K. L. MacDonald has been appointed to a professorship of mathematical physics at New York University.

Dr. Z. I. Mosesson of the Prudential Insurance Company of America, Newark, New Jersey, has been appointed assistant mathematician.

Dr. P. F. Neményi has accepted a position as physicist with the Naval Ordnance Laboratory at White Oak, Silver Spring, Maryland.

Professor Oystein Ore of Yale University has been on leave of absence this term.

Professor H. E. Spencer of Presbyterian College, Clinton, South Carolina, has been appointed to an assistant professorship at Virginia Polytechnic Institute.

The following promotions are announced:

O. K. Bower, University of Illinois, to an assistant professorship.

K. A. Bush, Mohawk College, to an associate professorship.

E. A. Cameron, University of North Carolina, to a professorship.

C. M. Erikson, Michigan State Normal College, Ypsilanti, Michigan, to a professorship.

V. A. Hoyle, University of North Carolina, to a professorship.

J. C. C. McKinsey, Oklahoma Agricultural and Mechanical College, to a professorship.

E. N. Oberg, State University of Iowa, to an associate professorship.

E. L. Welker, University of Illinois, to an associate professorship.

The following appointments to instructorships are announced: University of Arkansas: Mr. E. L. Eagle; University of Buffalo: Miss Helen M. Mazzuca, Mr. L. O. Ramer; University of Illinois: Miss Vivian R. Nuess, Dr. Jewell E. Schubert; Iowa State College of Agriculture and Mechanical Arts: Mrs. R. S. Banton, Mrs. J. V. Carr, Miss Bette L. Flatland, Mr. J. W. Markey, Mr. J. H. Watson; Ohio State University: Dr. Marjorie Alden; Queens College: Mr. Jack Moshman; United States Naval Academy: Mr. B. H. Buikstra; Westminster College, New Wilmington, Pennsylvania: Mr. W. A. Gibson; University of Wisconsin at Rhinelander: Miss Lillian Zarling.

Word has been received of the death of Professor Ettore Bortolotti of the University of Bologna on February 17, 1947, at the age of eighty years.

Professor H. M. Ackley of Western Michigan College of Education, Kalamazoo, Michigan, died February 8, 1947.

Mr. H. G. Avers of the United States Coast and Geodetic Survey died January 19, 1947.

Associate Professor J. F. Butler of the University of Detroit died February 18, 1947, at the age of fifty-one years. He had been a member of the Society since 1932.

Mr. R. V. A. Carpenter died on March 11, 1947. He had been a member of the Society since 1921.

Professor Emeritus B. F. Finkel of Drury College died on February 5, 1947. He had been a member of the Society since 1891.

Professor Emeritus W. A. Moody of Bowdoin College died on February 3, 1947. He joined the Society in 1891.

Mr. William Orange of Los Angeles City College died December 9, 1946.

Professor Emeritus B. L. Remick of Kansas State College died on March 18, 1947, at the age of seventy-nine years. He had been a member of the Society since 1913.

Professor J. V. Uspensky of Stanford University died on January 27, 1947. He had been a member of the Society since 1929.

Reverend G. L. Winkelmann of St. John's University, Collegeville, Minnesota, died January 23, 1947.

The following fifty-seven doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1946 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which degree was conferred, minor subject (other than mathematics) and the title of the dissertation are given in each case if available.

E. W. Barankin, California, June, *The characteristic values of linear transformations.*

P. T. Bateman, Pennsylvania, September, *On the representations of a number as the sum of three squares.*

Grace E. Bates, Illinois, June, *Free loops and nets and their generalizations.*

Richard Bellman, Princeton, October, *On the boundedness of solutions of non-linear difference equations.*

Kathleen E. Butcher, Michigan, October, *Topology of contiguous point spaces.*

J. D. Campbell, Illinois, June, *The parametric theory of singular parabolic partial differential equations.*

R. E. Carr, Iowa State, December, minor in chemistry and mathematical physics, *Enantiomorphism in mathematical models of organic molecules.*

S. H. Crandall, Massachusetts Institute of Technology, February, minor in electrical engineering, *On the Fourier series of the square of a function.*

D. R. Crosby, Princeton, October, *Tensor analysis in Finsler space and the problem of a unified theory of gravitation and electromagnetism.*

G. B. Dantzig, California, June, I. *Complete form Neyman-Pearson fundamental lemma.* II. *On the non-existence of tests of "student's" hypothesis having power functions independent of sigma.*

M. H. M. Esser, Northwestern, May, minor in physics, *Spectral expansion for self-adjoint differential transformations and for arbitrary self-adjoint transformations.*

Sister M. Celine Fasenmeyer, Michigan, March, *Some generalized hypergeometric polynomials.*

W. A. Ferguson, Illinois, June, minor in physics, *On the classification of finite metabelian groups with six generators.*

N. J. Fine, Pennsylvania, June, *On the Walsh functions.*

E. G. Fischer, Pittsburgh, June, minor in engineering, *Shock-excited lateral vibrations of a uniform bar with hinged ends.*

H. J. Greenberg, Brown, October, *The determination of upper and lower bounds for the solution of the Dirichlet problem.*

Edison Greer, Kansas, November, *A study of analytic surfaces by means of a projective theory of envelopes.*

G. H. Handelman, Brown, June, I. *Some problems in the study of thin-walled plastic sections.* II. *Aerodynamic pursuit curves for overhead attacks.*

E. L. Harder, Pittsburgh, June, *Solution of the general voltage regulator problem by electrical analogy.*

F. F. Helton, Illinois, October, minor in physics, *Quasi-analyticity related to sets of positive measure of functions of a complex variable.*

Erik Hemmingsen, Pennsylvania, June, *Some theorems in dimension-theory for normal Hausdorff spaces.*

Alfred Horn, California, June, *The asymptotic behavior of solutions of integral equations.*

A. R. Jacoby, Chicago, December, *Iterated integrals in the calculus of variations.*

R. C. James, California Institute of Technology, September, minor in physics, *Orthogonality in normed linear spaces.*

Bjarni Jónsson, California, September, *Direct decompositions of finite algebraic systems.*

Madeline Johnsen, Stanford, June, *Approximate evaluation of double probability integrals.*

L. G. Jones, Massachusetts Institute of Technology, June, minor in physics, *Differential equations with discontinuous right member.*

Leo Katz, Michigan, June, *Characteristics of frequency functions defined by first order difference equations.*

W. M. Kincaid, Brown, October, I. *On non-cut sets of locally connected continua.* II. *An application of orthogonal moments to problems in statically indeterminate structures.* III. *Numerical methods for finding characteristic roots and vectors of matrices.*

E. A. Knobelauch, Pennsylvania, June, *Extensions of homeomorphisms.*

H. G. Landau, Pittsburgh, June, *On the relations between certain criteria for the estimation of statistical parameters.*

Joseph Landin, Notre Dame, June, *Axiomatic theory of a singular non-Euclidean plane.*

E. L. Lehman, California, June, *Optimum tests of a certain class of hypotheses specifying the value of a correlation coefficient.*

W. S. Loud, Massachusetts Institute of Technology, September, minor in physics, *On the stationary solutions of van der Pol's equation with a forcing term.*

R. C. Lyndon, Harvard, October, *The cohomology theory of group extensions.*

G. R. MacLane, Rice, March, *Concerning the uniformization of certain Riemann surfaces allied to the inverse cosine and inverse gamma surfaces.*

P. R. Masani, Harvard, February, *Multiplicative Riemann integration in normed rings.*

Margaret S. Matchett, Indiana, June, *On the zeta function for idèles.*

Kenneth May, California, September, *On the mathematical theory of employment.*

H. J. Miser, Ohio State, August, *Generalized conformal representations of Fréchet surfaces.*

Kathryn A. Morgan, Stanford, June, *Representation of a positive binary form by a positive quaternary form.*

C. F. Mosteller, Princeton, October, *On some useful "inefficient" statistics.*

N. D. Nelson, Wisconsin, May, major in logic, *Recursive functions and intuitionistic number theory.*

M. H. Protter, Brown, June, *Generalized spherical harmonics.*

R. B. Saunders, Minnesota, June, minor in physics, *Polynomial approximation over certain infinite intervals.*

A. E. Schild, Toronto, June, *A new approach to kinematic cosmology.*

Jewell E. Schubert, Illinois, June, *Groups of order* 3^m *and class* 3.

E. J. Scott, Cornell, September, *Summability of the geometric series.*

R. H. Stark, Northwestern, August, minor in physics, *Some classes of monotone functions.*

J. C. Stewart, Illinois, October, *Between surfaces of revolution.*

E. B. Tolsted, Brown, June, *Limiting values of subharmonic functions.*

R. D. Wagner, Wisconsin, September, *The generalized Laplace equations in a function theory for commutative algebras.*

D. W. Western, Brown, June, *Inequalities of the Markoff and Bernstein type for integral norms.*

Ernest Williams, Michigan, June, *An invariant decomposition of a symmetric tensor of rank two in pseudo-Euclidean space.*

C. L. Woods, Ohio State, June, minor in physics, *A restricted class of convex functions.*

Margaret Y. Woodbridge, New York, February, *Variational problems with Euler equations of reduced order.*

Marie A. Wurster, Chicago, June, *A calculus of variations problem with required corners.*

Daniel Zelinsky, Chicago, August, *Integral sets in quasiquaternion algebras.*

The following doctorates were conferred in 1945, but were not included in the list in the preceding volume of this Bulletin (vol. 52 pp. 412–413).

C. L. Carroll, North Carolina, February, *Normal simple Lie algebras of Type* D *and order* 28 *over a field of characteristic zero.*

Fei Tsao, Minnesota, December, *General solution of the analysis of variants and convariants in the case of unequal or disproportionate numbers in the subclasses.*

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

Announcement. Beginning with the report of the 1947 Summer Meeting, this Bulletin will publish the abstracts of papers offered for presentation at a meeting of the Society as part of the report of the meeting. This arrangement will save considerable space in the Bulletin due to the fact that it will no longer be necessary to print the title of a paper and the name of the author as part of the abstract and also as part of the report of a meeting. The present plan of publishing abstracts was inaugurated in 1930 in the expectation that abstracts would appear in the Bulletin before the papers were read at meetings of the Society. Unfortunately a large proportion of the abstracts are not received in time to make such advance printing possible.

The Editors are pleased to announce in this connection that the Secretary of the Society plans to have available for distribution at as many meetings of the Society as possible mimeographed copies of the abstracts of papers to be presented. It is believed that such a distribution of the abstracts will be of considerable assistance to easier understanding of the papers presented.

Algebra and Theory of Numbers

164. Grace E. Bates: *Decompositions of a loop into characteristic free summands.*

If the additive loop L is the free sum of proper subloops L_1 and L_2, then neither L_1 nor L_2 is normal in L. Nevertheless, it is possible, with relatively few restrictions, to obtain decompositions of loops into characteristic free summands. The principal theorem of this paper is the following: If the loop L is the free sum of proper subloops A and B, then A is characteristic in L if and only if the following two conditions are satisfied: (i) no proper free summand of A is isomorphic to a free summand of B; (ii) no proper free summand of A is a free loop. There is also a strong refinement theorem for decompositions of loops into characteristic free summands. The concepts of free sums of loops and free loops, as well as existence theorems and subloop theorems, are taken from a previous paper by the author although their statements are here repeated for the convenience of the reader. (Received March 19, 1947.)

165. R. A. Beaumont: *Rings over a group.*

If G is an additive abelian group, G admits I_n, the domain of integers modulo n, as an operator domain. This n is chosen to be a positive integer or 0 according as G contains an element of maximum order n or not. Necessary and sufficient conditions that G be a vector space (finite or infinite-dimensional) over I_n are found. For groups G, which are the direct sum of cyclic groups, necessary and sufficient conditions that G be a ring are found in terms of multiplication constants in I_n. This general theorem is refined for certain classes of groups. Conditions that G be a ring are also obtained for certain groups which are not the direct sum of cyclic groups. (Received February 28, 1947.)

166. Arthur Bernhart: *Quantitative analysis of rings in minimal maps.*

Homogeneous equations are developed for the number of ways of coloring each side of an n-ring according to specified color schemes on the bounding ring. The solution is facilitated by arranging the schemes in magic squares, and by exploiting symmetry. This method provides a program for finding both reducible and irreducible configurations in the 4-color problem. (Received March 20, 1947.)

167. B. A. Bernstein: *Field in terms of multiplication and a unary operation.*

The author gives two sets of mutually independent postulates for fields in terms of multiplication and a unary operation. In one of these sets the result of the unary operation on an element a is a^+ $(=1+a)$, the "successor" of a. In the other set the result of the unary operation is a^* $(=1-a)$, the "complement" of a. The postulates are well adapted to bring out simply and directly properties of certain special elements of a field, properties hidden in the usual treatment of a field as a system of two binary operations. (Received March 17, 1947.)

168. William H. Durfee: *Quadratic forms over fields with a valuation.*

Let K be a field with a non-archimedean valuation which is complete with respect to the valuation and such that the residue class field has characteristic not two. The author considers the problem of the equivalence of quadratic forms over K and the representation of a given number of K by a quadratic form. Both problems are shown to be reducible to the case in which the forms have unit coefficients. Conditions are found under which the Hasse function $c(f)$ will be invariant and with its aid the necessary and sufficient conditions of Hasse for the representation of zero and the equivalence of two forms over a p-adic field are shown to apply to a larger class of fields. (Received March 10, 1947.)

169. Paul Erdős and Ivan Niven: *On the roots of a polynomial and its derivative.*

N. G. De Brujn has proved (*On the zeros of a polynomial and of its derivative*, K. Akademie van wetenschappen, Proceedings vol. 49 (1946) pp. 1037–1044) that the average of the absolute values of the imaginary parts of the roots of a polynomial with real coefficients is not less than the corresponding average for the derivative polynomial. The authors prove this result for polynomials with complex coefficients. More generally, it is proved that the average distance of the roots of a polynomial from a given straight line in the complex plane is not less than the corresponding average for the roots of the derivative; the averages are equal only if the roots of the polynomial are not distributed on both sides of the line. The same inequality holds for the average distances of the roots of any polynomial and its derivative from a fixed point in the plane; in this case the averages are equal only if the roots of a polynomial lie on a half line emanating from the fixed point. (Received March 19, 1947.)

170. C. J. Everett and George Whaples: *Representation of classes of finite sets.*

An arbitrary class of finite sets T_α possesses a choice function $f(T_\alpha)=t_\alpha \in T_\alpha$, $f(T_\alpha)\neq f(T_\beta)$ for $\alpha\neq\beta$, if and only if the union of every k of the T_α, $k=1, 2, \cdots$, contains at least k distinct elements. This generalizes the well known result of P. Hall for a finite class of finite sets. Necessary and sufficient conditions are given for the existence of a common representation of two partitions of a set when all components are finite. The existence of a common representation of left and right cosets of a group modulo a finite subgroup is obtained as a special case. This generalizes all results of Bull. Amer. Math. Soc. Abstract 53-1-98 from countable to arbitrary classes of finite sets. (Received March 19, 1947.)

171. Edwin Hewitt: *A characterization of rings of continuous real-valued functions.*

Let A be any commutative, associative algebra over the real number system, with unit e, and with the following properties: (1) for every $x\neq 0$ in A, there is an ideal M in A whose quotient ring is the real number field (such ideals are called *real*) such that $M(x)\neq 0$; (2) if $M(x)\geqq 0$ for all real ideals M in A, then $x=y^2$ for some y in A; (3) (x^2+e) always has an inverse, and there exists a real positive number α such that $M(\alpha e-(x^2+e)^{-1})\geqq 0$ for all real ideals M in A; (4) a topology is introduced into A by the following definition: if α is any positive real number, then $U_\alpha(x)$ is the set of all $y\in A$ such that $|M(y-x)|<\alpha$ for all real ideals M in A; the space A is to be complete under this topology; (5) if x has no inverse, then x is contained in some real ideal; (6) if I_0 and I_1 are intersections of real ideals such that the ideal I_0+I_1 is contained in no real ideal, then $I_0+I_1=A$. Under these hypotheses, it is proved that the algebra A is isomorphic to the ring of all real-valued continuous functions defined on some normal T_1-space. (Received March 3, 1947.)

172. R. E. Johnson: *The modules of a ring.* Preliminary report.

If I is a right ideal of the ring R, the additive groups of I and $R-I$ are well known examples of right R-modules. The author studies the converse problem of relating right, left and two-sided R-modules M to the ideal structure of R. If $M=xR$, then M is R-isomorphic to $R-I$, I a right ideal of R: in case M is irreducible, I is maximal. Any R-module is isomorphic to a subdirect sum of subdirectly irreducible R-modules. A study is made of the subdirectly irreducible R-modules. The Baer-Jacobson and Brown-McCoy radicals of a ring can be defined in terms of the annihilators of irreducible R-modules. These radicals are studied from this standpoint. The anti-radical of Baer is a left annihilator of the Baer-Jacobson radical. (Received March 20, 1947.)

173. G. K. Kalisch: *On extensions of topological fields.*

In the first part transcendental extensions are discussed and it is shown by means of a simple criterion of embeddability of topological integral domains in topological fields that, if F is a topological field, $F(x)$ can be topologized so as to preserve the topology of F. In the second part finite algebraic extensions are considered. If the ground field is complete, only one way of topologizing the extension is possible. If the ground field is not complete but is completable (that is, its completion is a field) then there are finitely many ways in which the extension can be topologized; these are determined, and their completions which are direct sums of finitely many complete fields are found in the case of separable extensions. In connection with part I of the paper it is pointed out that there exist non-locally compact connected fields, which, incidentally, are not complete, but are completable. (Received March 21, 1947.)

174. Irving Kaplansky and R. F. Arens: *Topological representation of algebras.*

The results obtained can be illustrated by the case of a ring A with unit satisfying $a^4 = a$. If A is countable, it is the set of all continuous functions from a compact totally disconnected space X to $GF(4)$, restricted on a closed subset of X to the values 0, 1. An example shows that this theorem fails if A is uncountable. Another kind of representation holds universally: here X has an involutory homeomorphism T and A consists of all functions satisfying $f(xT) = f(x)^2$. Results of this kind are proved for any commutative semi-simple algebraic algebra, and for a certain class of Banach algebras. Some results previously announced by the authors (Bull. Amer. Math. Soc. Abstracts 53-1-2 and 53-1-18) are subsumed. (Received March 15, 1947.)

175. D. H. Lehmer: *On the Tarry-Escott problem.*

In this paper the author considers the problem of finding b sets $S_0, S_1, \cdots, S_{b-1}$ of integers having equal sums of like powers. More explicitly, if $\sigma_k(r)$ denotes the sum of the kth powers of the members of S_r, it is required that $\sigma_k(0) = \sigma_k(1) = \cdots = \sigma_k(b-1)$ hold for $k = 0, 1, \cdots, n-1$. The case of $b = 2$ is well known (cf. Dickson's *History of the theory of numbers*, vol. 2, chap. 24). The above problem is said to be of order b and degree n. A family of solutions is obtained as follows. Let $m_1, m_2, \cdots, m_n$ be any n integers and let S_r be the set of all integers of the form $a_1m_1 + \cdots + a_nm_n$ where each a ranges over all non-negative integers less than b subject only to the condition that their sum be congruent to r modulo b. Then the sets S_r $(r = 0, 1, \cdots, b-1)$ form a solution of the general Tarry-Escott problem. Moreover, the degree of this solution is precisely n, except for the trivial case in which the sets S are merely permutations of one another. (Received March 21, 1947.)

176. D. H. Lehmer: *On the vanishing of Ramanujan's function.*

Ramanujan's function $\tau(n)$, defined as the coefficient of x^{n-1} in the expansion of the 24th power of the product $(1-x)(1-x^2)(1-x^3)\cdots$, is a well known numerical function. Nevertheless, the question of the possible vanishing of $\tau(n)$ has never been answered. Tables of $\tau(n)$ for $n \leqq 300$ show no case of $\tau(n) = 0$. In this paper it is shown that $\tau(n) \neq 0$ for $n < 3316799$ and that at most 8 of the first ten million values of $\tau(n)$ are zero. These results follow from a combination of several congruence properties of Ramanujan's function. (Received March 19, 1947.)

177. M. J. Norris: *Indeterminates over a universal algebra.*

A universal algebra S is considered as a set with an associated family of operations. An element of the algebra is called a polynomial in the elements of S_2 over S_1 if the element is in the least subset of S containing S_1 and S_2 closed with respect to the operations of S. By means of the notion of polynomial, the notions of algebraic elements and indeterminates are introduced. It is first shown that in the algebra of functions associated with S the identity function on S is an indeterminate over the class of functions constant on S. Other theorems of the above nature that would be expected are proved. The principle method used is that of algebraic induction. (Received March 21, 1947.)

178. L. J. Paige: *Neofields*. Preliminary report.

An additive loop L whose nonzero elements form a multiplicative group G such that multiplication is both right and left distributive with respect to addition is called a *neofield* N. Defining a group G to be admissible if and only if there exists a neofield N having G as its multiplication system, the author determines a necessary and sufficient condition that G be admissible and proves that the following groups are admissible: (i) abelian groups of rank $\rho=0$, (ii) abelian groups of rank $\rho\geqq 1$ and not possessing a unique element of order 2, (iii) groups G, all of whose elements have finite odd order. First proving that the center Z of an admissible group G is the multiplicative system of a sub-neofield $N'\subseteq N$, the author considers an extension problem of abelian groups to noncommutative admissible groups. In conclusion, an application of neofields to the coordinatization of projective planes is presented. (Received March 1, 1947.)

179. G. de B. Robinson: *On a theorem of Nakayama concerning the prime factors of the degrees of the irreducible representations of the symmetric group.*

R. Brauer has expressed the exponent of a prime p dividing the degree of an irreducible representation ϕ of a finite group G in the form $a-d+\epsilon$, where the order of G is $g=p^a g'$, $(g', p)=1$; p^d is the order of the *defect group* corresponding to ϕ, and ϵ is an unknown constant not less than 0 (Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 215–219). By refining Nakayama's theorem it is possible to define ϵ explicitly in the case of the symmetric group. Nakayama's conjecture (Jap. J. Math. vol. 17 (1940) pp. 165–184, 411–424) concerning the characterization of the blocks of characters follows. (Received March 14, 1947.)

180. R. D. Schafer: *The exceptional simple Jordan algebras.*

It is known that the only simple Jordan algebras over a nonmodular field $\mathfrak{F}$ which are not Jordan algebras of linear transformations are those of degree 3 and order 27 over their centers, having $\mathfrak{M}_3^8$ as split algebra. We show that (over their centers) all such exceptional algebras are reduced algebras, and that they consist of the J-symmetric elements of $\mathfrak{M}_3\times\mathfrak{C}$, where $\mathfrak{M}_3$ is the total matric algebra of degree 3, $\mathfrak{C}$ is any Cayley-Dickson algebra over $\mathfrak{F}$, and J is the involution $a\rightarrow p\bar{a}'p^{-1}$ of $\mathfrak{M}_3\times\mathfrak{C}$ where p is a nonsingular diagonal matrix. If $\mathfrak{F}$ is such that there exist inequivalent Cayley-Dickson division algebras over $\mathfrak{F}$, then there exist inequivalent Jordan algebras over $\mathfrak{F}$ with $\mathfrak{M}_3^8$ as split algebra. (Received March 4, 1947.)

181. M. F. Smiley: *Alternative regular rings without nilpotent elements.*

Forsythe and McCoy (*On the commutativity of certain rings*, Bull. Amer. Math.

Soc. vol. 52 (1946) pp. 523–526) showed, using a general theorem of G. Birkhoff (*Subdirect unions in universal algebra*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 764–768), that an associative regular ring R is a subdirect sum of division rings if and only if R has no nilpotent elements. This result is extended to alternative regular rings. It is also shown that every alternative ring which satisfies the condition $a^{n(a)}=a$ ($n(a)$ *an integer greater than one*) of Jacobson (*Structure theory for algebraic algebras of bounded degree*, Ann. of Math. (2) vol. 46 (1945) pp. 645–707) is associative. The proofs involve essentially new ideas only in case the additive order of some element of the ring is finite and divisible by three. (Received February 18, 1947.)

182. M. F. Smiley: *Binary systems which are almost loops.*

The investigation of H. B. Mann (*On certain systems which are almost groups*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 879–881) on the effect of postulating a right unit and left inverse for associative binary systems is extended to certain non-associative cases. The systems considered are found to be slight generalizations of loops with the inverse property (R. H. Bruck, *Some results in the theory of quasigroups*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 19–52). (Received February 18, 1947.)

183. B. M. Stewart: *Left associated matrices with elements in an algebraic domain.* II.

In the problem described in the author's Bull. Amer. Math. Soc. Abstract 47-5-202 the sufficiency proof was not completed for all cases. This omission is now remedied by showing that if the enlarged matrices A' and B' are left associates, then the matrices A and B are mutually left divisible, whence by a theorem of Steinitz it follows that the matrices A and B are left associates. This completes the proof, begun in the earlier abstract, that a necessary and sufficient condition that the matrices A and B be left associates over an algebraic domain is that the corresponding enlarged matrices A' and B' be left associates over the rational domain. Some study is made of the construction of the unimodular matrix P such that $PA=B$. (Received March 17, 1947.)

184. L. V. Toralballa: *A generalization of the finite integral.*

Given f, a function of $x_1, x_2, \cdots, x_n$, and A, a positive integer, consider the sum of the values of f over the set of all the ordered sets $(x_1, x_2, \cdots, x_s)$, $1 \leq s \leq n$, where for all i, $x_i \geq 1$, and $\sum_1^s x_i = A$. Three cases are treated: (1) f is a polynomial of the mth degree in the variables; (2) f is a factorial polynomial, that is, a sum of terms of the form $x_1^{(m_1)} x_2^{(m_2)} \cdots x_s^{(m_s)}$ where $x^{(m)} = x(x-1) \cdots (x-m+1)$; and (3) f is the quotient of two linear polynomials. General formulas are given for cases (1) and (2) and an algorithm for case (3). New formulas are also given for the so-called differences of zero. The methods used throughout are those standard in the calculus of finite differences. (Received March 20, 1947.)

185. J. A. Ward: *Analytic functions in linear algebras.*

This paper extends the results of a previous paper (*Theory of analytic functions in linear associative algebras*, Duke Math. J. vol. 7 (1940) pp. 233–248) by proving additional theorems on differentiation and integration and by giving a method for solving certain types of differential equations by use of linear associative algebras. (Received February 3, 1947.)

186. Morgan Ward: *Elliptic divisibility sequences.*

The author obtains all solutions of the difference equation $\omega_{m+n}\omega_{m-n} = \omega_{m+1}\omega_{m-1}\omega_n^2 - \omega_{n+1}\omega_{n-1}\omega_m^2$ over the field of rationals, and studies the arithmetical properties of integral solutions. These solutions include the well known Lucas function $u_n = (a^n - b^n)/(a-b)$ which satisfies a linear difference equation of order two; the more general numerical functions here investigated have very similar arithmetical properties. The conjecture of Lucas that there exists a connection between solutions of a linear difference equation of order three or four and elliptic functions is shown to be false so far as the difference equation studied is concerned despite Lucas' explicit assertion to the contrary (Amer. J. Math. vol. 1 (1878) p. 203.) (Received March 8, 1947.)

187. N. A. Wiegmann: *Some theorems on normal matrices with analogs of the generalized principal axis transformation.*

The following are among a number of theorems obtained on normal matrices: A necessary and sufficient condition that a matrix be normal is that its polar matrices be expressible as polynomials in the matrix; if A, B, and AB are normal, then BA is a normal matrix; a necessary and sufficient condition that the product AB of two normal matrices be normal is that each commute with the hermitian polar matrix of the other; the product AB of any two square matrices is normal if and only if there exist a unitary U and a nonsingular P such that $UAP = D$ and $P^{-1}BU^{CT} = T$ where D and T are certain diagonal and triangle matrices, respectively. Several analogs of the generalized principal axis transformation for a set of matrices A_α are shown to hold. (Received March 21, 1947.)

188. R. L. Wilson: *A finite method for the determination of the Galois group of an equation with an application to the problem of reducibility.*

This paper gives a sieve process for the determination of the Galois group, which depends only upon the ability to obtain the rational roots of a system of induced equations. If the given equation is of degree n, the Galois group G must be contained in the symmetric group of degree n. Hence, it is only necessary to ascertain which, if any, of the subgroups of the symmetric group contain G. If Γ is any subgroup of the symmetric group, the author has shown how to construct a function, ϕ, of the roots of the given equation which is invariant under precisely the permutations of Γ. The induced equation is then formed, having for its roots the function ϕ together with its conjugate functions. The theorem is then established that $G \subseteq \Gamma$ if and only if the induced equation has at least one rational root. By a different choice of the function ϕ, this same method is applied to the problem of the reducibility of a polynomial. (Received March 15, 1947.)

ANALYSIS

189. J. E. Bearman: *Rotations in the product of two Wiener spaces.* Preliminary report.

Let $X(t)$ and $Y(t)$ be elements of C, the space of all continuous functions on $0 \leq t \leq 1$ which vanish at $t=0$. Then if $F[X, Y]$ is a Wiener summable functional over $C \otimes C$, it is shown that the "rotation" $X(t) = x(t)\cos\theta - y(t)\sin\theta$, $Y(t) = x(t)\sin\theta + y(t)\cos\theta$

preserves measurability and measure, and that $\int_C^w\int_C^w F[X, Y]d_wXd_wY=\int_C^w\int_C^w F[x\cos\theta - y\sin\theta, x\sin\theta+y\cos\theta]d_wxd_wy$. (Received March 19, 1947.)

190. Richard Bellman: *On the solutions of the heat equation.* Preliminary report.

It is proved that if u is a solution of the heat equation, $\Delta u - u_t = 0$, $x, y, z \in R$, $t \geq 0$, $u = 0$ on B, where B is the boundary of the three-dimensional region R, then if one assumes that Green's transformations are permissible, $\int u^{2n}dV$ is a decreasing function of t. In a well known manner, it follows that $\max_R |u|$ is a decreasing function of t. Various consequences of this result are drawn concerning the bounds on particular solutions, and the stability of general solutions. (Received February 5, 1947.)

191. Richard Bellman: *On the stability of solutions of differential-difference equations.* Preliminary report.

It is shown that results concerning stability of solutions can be obtained for differential-difference equations of the form $x'(t) = ax(t) + bx(t+1) + f(x(t), x(t+1))$, where f is a nonlinear function, similar to those obtained for nonlinear differential equations. In particular, if e^{at}, $R(a) < 0$, is a particular solution of the equation with $f = 0$, then there exists a solution of the above equation with prescribed $x(0)$, provided that $|x(0)|$ is sufficiently small, and f is a power series in $x(t)$ and $x(t+1)$ lacking constant and first degree terms. (Received February 5, 1947.)

192. Stefan Bergman and Menahem Schiffer: *On Green's and Neumann's functions in the theory of partial differential equations.*

Let B be a finite plane domain with smooth boundary C and let $P(x, y)$ be continuous and positive in $B+C$. Define the scalar product $D\{\phi, \psi\} = \iint_B[(\partial\phi/\partial x)(\partial\psi/\partial x) + (\partial\phi/\partial y)(\partial\psi/\partial y) + P\phi\psi]dxdy$ of two functions ϕ, ψ in B. A function f will belong to class Ω if it has continuous derivatives in B and satisfies $D\{f, f\} < \infty$. It will belong to class Ω_0 if it belongs to Ω and vanishes on C. It will belong to class Γ if it belongs to Ω and satisfies the differential equation L: $\Delta\phi = P\phi$. A system of functions $\{f_\nu\}$ is called orthonormal if $D\{f_\nu, f_\mu\} = \delta_{\nu\mu}$; it is called complete with respect to a class if every element of this class may be developed into a series of the f_ν. The actual construction of Neumann's function of L in terms of a complete orthonormal system in Ω and Green's function of Laplace's equation is given. An analogous formula for Green's function of L is given (with Ω replaced by Ω_0). The difference between Neumann's and Green's function is the kernal function of a complete orthonormal system in Γ. The formulas contain as a particular case the representation of Green's function in terms of the eigenfunctions of $\Delta\phi - P\phi + \lambda\phi = 0$ with respect to B. (Received March 12, 1947.)

193. R. C. Buck: *Integral valued entire functions.*

Further characterizations of the class of entire functions of exponential type which take integral values at the positive integers are obtained in terms of the growth function $h(\theta, f)$, extending results of Pólya, Selberg, and Pisot. In particular, there are integral valued entire functions of relatively small growth which are not expressible as linear combinations of powers of algebraic integers. (Received March 22, 1946.)

194. R. P. Cesco: *On strong summability.*

Let $A = (a_{n\nu})$ be a regular matrix of real and non-negative numbers $a_{n\nu}$ $(n, \nu = 0, 1, 2, \cdots)$ and consequently strongly regular in the Hamilton-Hill sense (Amer. J. Math. vol. 60 (1938) p. 588). For such matrices the strong limit $[A, p]-\lim s_\nu = s$ of a series $\sum u_\nu$ with partial sums s_ν can be defined uniquely and has the property of finite additivity. These results are obtained: (A) *If* $[A, p]-\lim s_\nu = s$ *and* $s_\nu = O(1)$, *and if* $[A, p]-\lim t_\nu = t$, *then* $[A, p]-\lim s_\nu t_\nu = st$. (B) *If* $[A, p]-\lim s_\nu = s$ *and if* $q \leq p$, *then* $[A, q]-\lim s_\nu = s$. (C) *If* $[A, p]-\lim s_\nu = s$ *and if* $p \geq 1$, *then* $A-\lim s_\nu = s$. A matrix A is said to be ***strongly permanent*** if the series $\sum u_\nu$ strongly summable A remain strongly summable with same order and sum whenever a finite number of zeros is prefixed. It is proved: (D) *If the matrix A is strongly permanent, then the condition* $\liminf |u_\nu| = 0$ *is necessary for* $[A, p]-\lim s_\nu = s$. A matrix A is said to be stronger than convergence (or $A \succ E$) if at least one divergent sequence exists which is strongly summable A. It is proved: (E) *In order that $A \succ E$ it is necessary and sufficient that an increasing sequence $\{\nu_k\}$ of positive integers exist for which* $\lim \sum_{k=0}^{\infty} a_{n\nu_k} = 0$. (Received March 18, 1947.)

195. Komaravolu Chandrasekharan and Subbaramiah Minakshisundaram: *Some results on double Fourier series.*

Using Bochner's method of spherical summation of multiple Fourier series (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207) the two-dimensional analogues of (i) the Hardy-Littlewood convergence test, (ii) Rogosinski's theorem, and (iii) Bernstein's theorem on the approximation of a function belonging to Lipschitz class by the Cesàro means of its Fourier series (Zygmund, *Trigonometrical series*, pp. 34, 62, 181) are proved, in addition to other results on convergence and summability. (Received March 21, 1947.)

196. Komaravolu Chandrasekharan and Otto Szász: *On Bessel summation.*

$J_\mu(t)$ denotes the Bessel function of order μ: let $\alpha_\mu(t) = 2^\mu \Gamma(\mu+1) t^{-\mu} J_\mu(t)$, so that $\alpha_\mu(0) = 1$. The sequence $\alpha_\mu(nt)$, $n = 0, 1, 2, \cdots$, taken as convergence factors, defines a summability method, considered first by S. Minakshisundaram and K. Chandrasekharan. Thus a series $\sum a_n$ is said to be summable J_μ to the sum s if the series $\sum_{n=0}^{\infty} a_n \alpha_\mu(nt)$ converges in some interval $0 < t < t_0$ to a function $\phi_\mu(t)$, and if $\phi_\mu(t) \rightarrow s$ as $t \rightarrow 0$. The special case $\mu = 1/2$ gives the well known Lebesgue summability. The authors give some new results concerning the scale of J_μ summability, in particular in comparison to the scale of Cesàro summability, and including known results on Lebesgue summability. The method employs some partially new properties of Bessel functions. (Received March 22, 1947.)

197. Yael N. Dowker: *Invariant measure and the ergodic theorems.*

The ergodic theorems of G. D. Birkhoff and J. von Neumann presuppose an invariant measure. However, one can easily see that if the given measure is potentially invariant, that is, if there exists an invariant measure with no more null sets than the given measure, the ergodic theorems still hold for a broad class of functions (including all bounded measurable functions). It is shown that conversely the ergodic theorems imply that the measure is potentially invariant. Specifically, let S be an abstract space, let m be a measure defined for a Borel family F of sets of S with $m(S) < \infty$ and let T be a 1-1 measurable point transformation of S into itself such that $m(T^{-1}A) = 0$ whenever $m(A)=0$. Then m is potentially invariant if any one of the following conditions hold: 1 (Birkhoff). For every $f(x)\in L(S)$, the averages $F_n(x)(=1/n\sum_{i=0}^{n-1}f(T^ix))$ converge for almost all x. 2 (von Neumann). For every $f(x)\in L^2(S)$, $f(Tx)\in L^2(S)$ and the averages F_n converge in $L^2(S)$. 3. For every descending sequence of sets $A_r\in F$ whose intersection is empty, $\lim\sup_{n\to\infty}1/n\sum_{i=0}^{n-1}m(T^{-i}A_r)$ converges to zero. (Received February 17, 1947.)

198. W. F. Eberlein: *A nonlinear differential equation.*

The nonlinear differential equation $u''+8\beta\cos t\sin u=0$, which arises in a vibration problem, is related to both the pendulum equation and the Mathieu equation. Solutions of the type $u=t+a+v$, where $v(t+2\pi)=v(t)$ and $v=O(\beta)$ for small β, are discussed. It is shown that essentially only two exist, one $(a=0)$ being stable, and the other $(a=\pi)$ being unstable. (Received March 21, 1947.)

199. W. F. Eberlein: *A note on ergodic theory.*

Some applications to ergodic theory of a recent theorem of the author on weak compactness (cf. Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947)) are discussed. For example, what appears to be the most general mean ergodic theorem—in the sense of G. Birkhoff and Alaoglu—attainable for Abelian semi-groups of transformations becomes an immediate consequence of a fixed point theorem of A. Markov, an almost elementary proof of which has been sketched by Kakutani (Proc. Imp. Acad. Tokyo vol. 14 (1938)). (Received March 21, 1947.)

200. Bernard Epstein: *Some inequalities relating to conformal mapping upon canonical slit-domains.*

Let a domain D of the complex z-plane containing the point at infinity and bounded by any finite number of smooth curves be mapped conformally and biuniformly upon a domain D_θ of the ζ-plane bounded by rectilinear slits each of which makes the angle θ with the positive direction of the real axis, the mapping function having in the neighborhood of $z=\infty$ a Laurent expansion of the form $\zeta=z+(a_\theta/z)+\cdots$. (The coefficient of each inverse power of z will depend, of course, upon the angle θ.) By setting up a certain positive-definite integral taken over the domain D, and transforming it into an integral taken over the boundary curves, the following inequality is obtained: $\text{Re}(a_\theta e^{-2i\theta})\geq A/2\pi$, where A is the total area enclosed by the boundary curves. By employing certain results in the theory of systems of orthonormal functions, as developed by S. Bergman and M. Schiffer, it is then shown that the above inequality can be replaced by the following stronger one: $\text{Re}(a_\theta e^{-2i\theta})-|a_\theta|^2/(a_0-a_{\pi/2})\geq A/2\pi$. It is shown that each of these inequalities is the best possible in the sense that the factor $1/2\pi$ on the right-hand sides cannot be replaced by any larger constant. (Received March 5, 1947.)

201. Paul Erdős and George Piranian: *Over-convergence on the circle of convergence.*

Let the function $f(z)=\sum a_nz^n$ be regular in the unit circle and on the (closed) arc C of the unit circle; let $\phi(x)$ be a function such that $\phi(n)\geq\log|a_n|$ $(n=0, 1, \cdots)$ and $\phi'(x)\searrow 0$ as $x\to\infty$; and let m_i and n_i be two monotonic sequences such that $a_n=0$ when $m_i\leq n\leq n_i$ $(i=0, 1, \cdots)$. If the differences n_i-m_i are sufficiently large in relation to the quantities $\phi(n_i)$, the sequence $s_{m_i}(z)=\sum_{n=0}^{m_i}a_nz^n$ converges to $f(z)$ uniformly on C.

The simplest of the particular results in the paper is the following: If $|a_n| < 1$ $(n = 0, 1, \cdots)$, $\lim (n_i - m_i) = \infty$, and $k < 1$, there exists an integer i_0 such that $|s_{m_i} - f(z)| < \exp \{-(n_i - m_i)^k\}$ on C when $i > i_0$. (Received February 15, 1947.)

202. G. M. Ewing: *Variation problems formulated in terms of the Weierstrass integral.*

An integral of the form $\lim \sum f(x, y; \Delta x, \Delta y)$, introduced by Weierstrass, allows the admission of general rectifiable curves. The present paper is largely expository, bringing together an introduction to existence theorems of the Tonelli type and to related questions for the single-integral parametric problem in ordinary n-space. Results overlap with those of M. Aronszajn, C. Pauc, and L. Tonelli to whose papers the writer is indebted for his interest in the topic. Among the novel features of the present paper is the use of the W-integral in obtaining a simplified proof of upper-reducibility. (Received February 21, 1947.)

203. F. G. Gravalos: *A note on dynamical systems with two degrees of freedom.*

Painlevé stated without proof (*Leçons sur la théorie analytic des équations différentielles*, Paris, 1897, p. 543) that dynamical systems do not admit of integrals of the form $\phi(x_1, \cdots, x_n) = \text{const.}$ where the x's are the coordinates. Using this property, the proof of which is immediate, a rather descriptive theorem is obtained for the case of two degrees of freedom: If a dynamical system with two degrees of freedom admits of a fundamental set of conservative integrals (this terminology is taken from A. Wintner), one at least may be rewritten as a function of the angular momentum $xy' - yx'$, of x' and y', and containing x and y in no other form. (Received March 5, 1947.)

204. Leonard Greenstone: *Mapping by analytic functions*. Part II. *Pseudo-conformal distortion theorems*. Preliminary report.

The author extends results obtained in Part I. *Conformal mapping of multiply connected domains*, Trans. Amer. Math. Soc., to the case of mappings by analytic functions of several variables. In particular the following results are obtained: Let B^4 be a four-dimensional nondegenerate bounded domain, and suppose that b^3, its three-dimensional boundary, contains at least one limit point of third order. (Bergman, *Über die Kernfunktion eines Bereiches und ihr Verhalten am Rand*, I, J. Reine Angew. Math. vol. 169 (1933) pp. 1–42.) If B^4 maps into B_1^4 under pseudo-conformal transformation, then there exists a non-null subdomain $C^4 \subset B^4$ whose image under the transformation is C_1^4, and such that the Euclidean volume of C_1^4 is less than c, where c depends only on B^4 and C^4. An inequality is also given by means of which the distortion of Euclidean length of a Jordan curve under pseudo-conformal mapping may be computed. (Received March 24, 1947.)

205. William Gustin: *A bilinear integral identity for harmonic functions.*

It is shown that any two functions harmonic in open subsets of a euclidean space satisfy a certain bilinear integral identity. The associated quadratic integral identity is used to give a new proof of the theorem that a function harmonic in a connected open set D and vanishing over some non-null open subset of D must vanish throughout D. (Received March 24, 1947.)

206. H. J. Hamilton: *Mertens' theorem and sequence transformations.*

Mertens' theorem on the Cauchy-product for two convergent series is rephrased so as to admit of a valid converse. The resulting theorem is extended to multiple series in two forms. One of these is a dual of a recent theorem of I. M. Sheffer (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1036–1041). The type of convergence used is Pringsheim's and the proofs are effected by means of linear sequence transformation theory. (Received February 21, 1947.)

207. R. G. Helsel and Tibor Rado: *On the Cauchy area of a Fréchet surface.*

Rado (Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 102–106) has stated that $L(S) < \infty$ implies $C(S) = L(S)$, where $L(S)$ is the Lebesgue area of the Frechet surface S and $C(S)$ is a lower semi-continuous functional of S which is based on a construction given by Cauchy. The paper contains a proof of this assertion and makes use of recent results of L. Cesari to show that $C(S) = L(S)$ always. (Received March 15, 1947.)

208. M. R. Hestenes: *Sufficient conditions for isoperimetric multiple integral problem in the calculus of variations.*

In the present paper sufficient conditions are established for the problem of minimizing an integral $I(y) = \int_S f(x_1, \cdots, x_m, y, \partial y/\partial x_1, \cdots, \partial y/\partial x_m) dx_1 \cdots dx_m$ in the class of functions $y(x)$ (x on S) having the same boundary values and satisfying a set of isoperimetric conditions $I_k(y) = \text{constant}$ ($k = 1, \cdots, p$), where $I_L(y)$ is of the same form as $I(y)$. The sufficient conditions are the direct analogies of the corresponding conditions for the simple integral case and have the same generality (see Hestenes, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 93–118). Conditions for strong and weak relative minima are established. The method used is an extension of the indirect method first introduced by McShane (Trans. Amer. Math. Soc. vol. 52 (1942)) for the problem of Bolza. (Received March 20, 1947.)

209. Rufus Isaacs: *Inverse iterates.*

Let g be a function mapping an arbitrary space E into itself. An inverse iterate of g is a function f of the same type such that always $f(f(x)) = g(x)$. The author has found n.a.s. conditions for the existence of f. By a *linkage* (for g) is meant a subset of E minimally closed under the operation g and its complete inverse. Each linkage for f is the union of two (possibly identical) linkages for g. The existence problem is solved by giving criteria for two distinct linkages for g to be "matable" in this way and for a single linkage to be "self-mating." (Received March 15, 1947.)

210. Mark Kac, Raphael Salem, and Antoni Zygmund: *A gap theorem.*

The main result of the paper is as follows. Let $f(x)$ have period 2π and belong to L^2 and let its mean value be zero. Suppose, in addition, that $\int_0^{2\pi} (f - s_n)^2 dx = O(\log n)^{-\sigma}$, $\sigma > 0$, where s_n denotes the nth partial sum of the Fourier series of f. Let $\{\lambda_k\}$ be any positive sequence such that $\lambda_{k+1}/\lambda_k \geqq q > 1$. Then (1) if $\sigma > 2$, the series $\sum c_k f(\lambda_k x)$ converges almost everywhere provided $\sum c_k^2 \log^2 k < \infty$; (2) if $\sigma \leqq 2$, the series $\sum f(\lambda_k x)/k^{1-\delta}$ converges almost everywhere for every $\delta < \sigma/4$. The relation $1/(m+1)[f(x) + f(\lambda_1 x) + \cdots + f(\lambda_m x)] \to 0$ almost everywhere is a weak consequence of both results. Other theorems of the same type are proved by using properties of quasi-orthogonal functions. (Received February 12, 1947.)

211. L. H. Kanter: *On the roots of orthogonal polynomials and the related Christoffel numbers.*

Under appropriate conditions on the weight function $w(x, \tau)$ the Gauss-Jacobi quadrature formula may be differentiated with respect to the parameter τ, thus giving (1): $\int_a^b \rho(x) w_\tau(x, \tau)dx = \sum_{\nu=1}^n \rho(x_\nu)x_\nu'(\tau)\lambda_\nu(\tau) + \sum_{\nu=1}^n \rho(x_\nu)\lambda_\nu'(\tau)$ where $\rho(x, \tau)$ is a polynomial of degree $2n-1$ (Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23). If $\rho(x, \tau)$ is $h_\nu(x, \tau)$, the Hermite polynomial of the "first kind," the right side of (1) becomes $\lambda_\nu'(\tau)$. If $w_\tau(x, \tau)$ is positive, then $\lambda_\nu(\tau)$ is an increasing or decreasing function of τ according as $h_\nu(x, \tau)$ is positive or negative in the integration interval. If $\rho(x, \tau)$ is $[l_\nu(x, \tau)]^2[1-k(x-x_\nu)]$, k a parameter, the above results become a special case of $\int_a^b \rho(x)w_\tau(x, \tau)dx = \lambda_\nu(\tau)[k+L''(x_\nu)/L'(x_\nu)]x_\nu'(\tau)+\lambda_\nu'(\tau)$ for $k = -L''(x_\nu)/L'(x_\nu)$. Here $l_\nu(x, \tau)$ is the fundamental polynomial of the Lagrange interpolation (Szegö, *Orthogonal polynomials*). If $k \rightarrow +\infty$, then A. Markoff's theorem on the variation of the roots of the orthogonal polynomials with respect to a parameter is obtained. (Received March 21, 1947.)

212. Wilfred Kaplan: *The level curves of a harmonic function.*

The following theorem is established. If F is a regular curve-family filling the xy-plane, then there is a homeomorphism of the xy-plane either onto itself or onto the interior of the unit circle such that F is transformed onto the family of level curves of a harmonic function $u(x, y)$. A regular curve-family is here defined as one locally homeomorphic to a family of parallel lines. (Received January 29, 1947).

213. W. G. Leavitt: *A normal form for matrices whose elements are holomorphic functions.*

This paper considers transformations of type $T^{-1}AT$ of a matrix $A(z)$ whose elements are functions of a complex variable. The results obtained apply to a bounded region R in which all elements of $A(z)$ and all roots of its characteristic equation are holomorphic. The principal theorem states that there exists a matrix $T(z)$ nonsingular throughout R, transforming $A(z)$ into a normal form $U(z)$ whose elements are holomorphic over R and in which all elements below the main diagonal are identically zero. In the course of the proof it is shown that the Weyr characteristic and its associated Jordan form $J(z)$ are definable. A number of results are then established for matrices and vectors of holomorphic functions, leading to the proof that there exists a matrix $S(z)$ whose determinant is not identically zero over R satisfying the equation $AS=SJ$. Finally, $S(z)$ is used in the construction of a matrix $T(z)$ whose determinant is nonvanishing throughout R, and which transforms $A(z)$ into a normal form of type U. (Received March 17, 1947.)

214. D. H. Lehmer: *Approximations to the surface area and electrostatic capacity of the ellipsoid.*

Let a, b, c be the semi-axes of an ellipsoid. Consider all approximations to the surface and capacity of the ellipsoid by means of Minkowsky averages of the form $A\{(1/6)\sum a^\lambda b^\mu c^\nu\}^p$ where the sum extends over all six permutations of (a, b, c). For surface approximation $p=2(\lambda+\mu+\nu)^{-1}$ and $A=4\pi$; for capacity $p=(\lambda+\mu+\nu)^{-1}$ and $A=1$. In this paper, by the proper choice of λ, μ, ν, are found the best possible approximations of this type in the sense that the relative errors, when expanded in a double power series in the essential eccentricities of the ellipsoid, involve terms of the highest possible order. For both problems this order is 10. The λ, μ, ν are, in each case, real algebraic numbers of degree three. (Received March 21, 1947.)

215. D. C. Lewis: *Generalized orthonormality of polynomials.*

Let $a_0(x), a_1(x), \cdots, a_p(x)$ be $p+1$ monotonic nondecreasing functions defined for $a \leqq x \leqq b$. Corresponding to an arbitrary sufficiently regular function $f(x)$, introduce the polynomial $P_n(x)$ of degree not greater than n which renders $\sum_{k=0}^{p}\int_a^b(f^{(k)}(x) - P_n^{(k)}(x))^2 da_k(x)$ a minimum. In general $P_n(x)$ is unique, and (as the writer has previously shown) $f(x) - P_n(x) = (1/m!)\int_a^b L_n^m(x, t)df^{(m)}(t)$, where $m \geqq p \geqq 0$ and L_n^m is a certain function independent of f. In the present paper it is assumed that $a_p(x)$ has a nonvanishing derivative. The uniform tendency of $L_n^m(x, t)$ to zero as $n \to \infty$ (p and m are fixed) is established under certain general conditions. Other asymptotic properties of $L_n^m(x, t)$ are also studied. Methods of proof, adapted from the well known work of Dunham Jackson, are based on Bernstein's bound for the derivatives of a polynomial. Also introduced is a general theory of sets of polynomials $Q_0(x), Q_1(x), Q_2(x), \cdots$ (where $Q_n(x)$ is of degree n), such that $\sum_{k=0}^{p}\int_a^b Q_i^{(k)}(x)Q_j^{(k)}(x)da_k(x) = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. (Received March 20, 1947.)

216. Charles Loewner: *A topological characterization of a class of integral operators.*

A closed oriented curve γ in the x-y-plane will be called of non-negative circulation if its order relative to any point not on γ is non-negative. Let $k(t)$ be an L-integrable function of t with the period 2π. Consider the integral operator $y(t) = \int_0^{2\pi}k(\tau)x(t-\tau)d\tau$ applied to continuous functions $x(t)$ having the same period 2π. $y(t)$ is then also continuous of period 2π. The functions $x = x(t)$, $y = y(t)$ give a parametric representation of a closed oriented curve in the x-y-plane. Any curve obtained in this way will be called "generated by the kernel $k(t)$." Various problems of analysis and geometry lead to the question: Which kernels generate *only* curves of non-negative circulation? A complete answer is given in this paper: The class of kernels in question consists of those L-integrable functions $k(t)$ analytic in the open interval $0 < t < 2\pi$ whose derivative $k'(t)$ can be represented by a Laplace-Stieltjes integral $k'(t) = \int e^{-rt}d\alpha(r)$ $(-\infty < r < \infty)$ with a non-negative mass distribution $d\alpha(r) \geqq 0$. (Received March 12, 1947.)

217. E. R. Lorch: *On certain implications which characterize Hilbert space.*

Two principal characterizations of Hilbert space are given: (1) Let $\mathfrak{B}$ be a vector space with the property: There exists a real number α $(\alpha \neq 0, 1)$ such that if a vector pair f, g satisfies $|f+g| = |f-g|$ it follows that $|f+\alpha g| = |f-\alpha g|$. Then $\mathfrak{B}$ is a Hilbert space. (2) Let $\mathfrak{B}$ be a vector space with the property: If a vector pair f, g satisfies $|f| = |g|$ there follows that $|\alpha f+\alpha^{-1}g| \geqq |f+g|$ where α is an arbitrary nonzero real number. Then $\mathfrak{B}$ is a Hilbert space. Five new characterizations of Hilbert space are given each of which rests on Theorem 1 above. Among them are the following extensions of a theorem of Jordan and von Neumann: (3) If in $\mathfrak{B}$ there exists for arbitrary f, g a (nontrivial) relation involving the quantities $|f|$, $|g|$, $|f+g|$, and $|f-g|$, then $\mathfrak{B}$ is a Hilbert space. Note that the hypothesis merely calls for the existence of a relation. (4) If for arbitrary vectors f, g, h in $\mathfrak{B}$ with $f+g+h = 0$ it is true that $3\{|f|^2+|g|^2+|h|^2\} = |f-g|^2+|g-h|^2+|h-f|^2$, then $\mathfrak{B}$ is a Hilbert space. (Received February 17, 1947.)

218. C. W. Mathews: *Cauchy type double integral representations for functions of a complex variable.*

Consider the class, R, of functions of the complex variable which are continuous in a bounded domain, G, such that (1) $\limsup (1/|I|)\int_{(I)}f(z)dz<\infty$ (for intervals containing z_0) for all z_0 in $G-S$ where S is a denumerable infinity (at most) of segments parallel to the axes and (2) such that the partial derivatives of the real and imaginary parts of the function with respect to the real and imaginary parts of the variable exist almost everywhere in G. In a paper by Trjitzinsky (to appear in Journal de Mathématiques) conditions were found under which the function, $f(z)$, could be represented as a Cauchy type double integral. The author has determined conditions under which this integral equation is equivalent to a regular Fredholm integral equation. Hence, if $f(z)$ belongs to R and $K(z, \zeta)$ satisfies certain conditions while the regular Fredholm integral equation, $2\pi i\Phi(z)-\iint_G[\int_{(I)}K(z, \zeta)\Phi(\zeta)d\zeta_1d\zeta_2=-\psi'(z)$, where $\psi(z)=\int_{(I)}f(z)dz$ for I in G, has a solution, Φ, then $f(z)$ has the representation $f(z)=-\iint_G k(z, \zeta)\Phi(\zeta)/(\zeta-z)d\zeta_1d\zeta_2+a(z)$ where $a(z)$ is analytic, $\zeta=\zeta_1+i\zeta_2$, and $k(z, \zeta)/(\zeta-z)=1/(\zeta-z)+K(z, \zeta)$. (Received February 21, 1947.)

219. E. J. Mickle: *Metric foundations of continuous transformations.*

Let T be a continuous transformation from an analytic set A in a complete and separable metric space M into a metric space M^*, in which finite Carathéodory outer measures ν and ν^* are defined respectively. Let $N(p^*, E)$ designate the number of inverse points a point p^* of M^* has in a subset E of A. T is called of bounded variation if $N(p^*, E)$ is a ν^*-summable function and, if of bounded variation, is called absolutely continuous if sets of ν-measure zero are taken into sets of ν^*-measure zero. If T is of bounded variation, a completely additive class of subsets E of M containing closed sets can be found on which $\mu(E)$, the ν^*-integral of $N(p^*, E)$ over M^*, is a completely additive set function. The absolutely continuous part of μ can be expressed as the ν-integral of a point function which plays the role of the Jacobian of the transformation. Formulas (involving this Jacobian) for transforming definite integrals are obtained in this paper. (Received March 9, 1947.)

220. C. N. Moore: *Generalized limits in general analysis.* III.

In volumes 24 and 25 of Trans. Amer. Math. Soc. there were published two papers, entitled respectively: *Generalized limits in general analysis, First paper* and *Generalized limits in general analysis, Second paper*. It was the aim of these papers to develop a general theory in the field of general analysis, which would include as special cases results concerning summable series and their analogues for integrals. The applications of such a theory were illustrated by proving a general theorem concerning the equivalence of C and H summability and the analogous result for integrals. In the present paper a further application is made by proving a theorem which includes the regularity theorem for summability methods of a certain type, due to Silverman and Toeplitz, and the analogous theorem for integrals, due to Agnew. (Received March 8, 1947.)

221. John von Neumann and E. R. Lorch: *On the euclidean character of the perpendicularity relation.*

A proof is given of the following theorem: Let $\mathfrak{B}$ be a vector space such that whenever two vectors f and g satisfy $|f|=|g|$, it follows that $|\alpha f+\beta g|=|\beta f+\alpha g|$ for all real α and β. Then $\mathfrak{B}$ is a Hilbert space (of unspecified dimensionality). The theorem has been proved by F. A. Ficken (Ann. of Math. (2) vol. 45 (1944) pp. 362–366). The present proof which is very short introduces a notion of perpendicularity in vector spaces which renders transparent the geometric phenomena in which the problem is rooted. This same notion has application to the further study of the characterization of Hilbert space by means of implications which involve its norm. (Received February 17, 1947.)

222. T. G. Ostrom: *The solution of linear integral equations by means of Wiener integrals.*

Let the Fredholm solution of the integral equation $z(t)=x(t)+\int_0^1 K(t, s)x(s)ds$ be given by $x(t)=z(t)+\int_0^1 R(t, s)z(s)ds=G[z|t]$. By applying two linear transformations to the Wiener integral $\int_C^w G[z+y|t]d_w y$, the author shows that under fairly general conditions the solution can be expressed as a Wiener integral whose integrand is a functional involving only the fixed functions K and z, the variable function of integration, and the number t. The Fredholm determinant D (which is also expressible as a Wiener integral) appears as a constant factor and hence can be obtained without further computation by substitution in the integral equation. (Received March 15, 1947.)

223. Tibor Rado and E. J. Mickle: *A new geometrical interpretation of the Lebesgue area of a surface.*

Let S refer to a Fréchet surface of the type of the 2-cell in Euclidean 3-space E_3, let G_4 be the 4-dimensional space of all lines $g\in E_3$, and let m denote the properly normalized invariant measure in G_4. For each line $g\in E_3$ an essential intersection number $\kappa(g)$ is introduced in terms of ϵ-deformations of S. In this paper it is shown that $\int\kappa(g)dm$ taken over G_4 is equal to the Lebesgue area of S. (Received March 19, 1947.)

224. O. W. Rechard and P. V. Reichelderfer: *A new criterion for the extension of rectangle functions.*

Let $\phi(R)$ be a real, finite, non-negative function defined on the class K of rectangles R contained in a fixed oriented closed rectangle F. It is shown that ϕ admits a completely additive extension to an additive class of sets including all Borel sets if and only if it satisfies the following conditions: (i) if R_1 and R_2 are mutually exclusive rectangles in K contained in a rectangle R in K, then $\phi(R_1)+\phi(R_2)\leqq\phi(R)$; (ii) if R is a rectangle in K covered by two rectangles R_1 and R_2 in K, then $\phi(R)\leqq\phi(R_1)+\phi(R_2)$; (iii) if R is any rectangle in K and R_ϵ is the common part of F and the rectangle containing R with sides parallel to the corresponding sides of R at distance ϵ, then $\lim\phi(R_\epsilon)=\phi(R)$ as ϵ tends to zero through positive values. A similar result is established for functions of oriented open rectangles. In this case, R_ϵ is the rectangle contained in R with sides parallel to the corresponding sides of R at distance ϵ. Functions defined on all closed rectangles or on all open rectangles are also considered. (Received March 10, 1947.)

225. P. V. Reichelderfer: *The effect of a lipschitzian transformation on the area of a continuous surface.*

A transformation $X(x)$ from euclidean three-space into euclidean three-space is termed lipschitzian M if M is a positive constant such that the distance between the images $X(x_1)$ and $X(x_2)$ of any pair of points x_1 and x_2 does not exceed M times the distance between x_1 and x_2. Any continuous surface S is transformed by $X(x)$ into a continuous surface $X(S)$. If $X(x)$ is lipschitzian M, it is shown that the lebesgue area of $X(S)$ cannot exceed the square of M times the lebesgue area of S. Various applications are considered. (Received March 15, 1947.)

226. E. H. Rothe: *Extrema of functions in Banach spaces.* Preliminary report.

Let V be a bounded domain in a Banach space E. Let $I(x)$ be a real-valued function defined and (strongly) continuous in V. Moreover it is assumed that the first and second differential of $I(x)$ exist and are completely continuous and that the latter is "Hessian," that is, satisfies a certain condition which corresponds to the nonvanishing of the Hessian determinant in the finite-dimensional case. It is proved that then $I(x)$ is also weakly continuous. As a consequence $I(x)$ reaches a maximum and a minimum value in any weakly compact subset of V, and the existence of a maximum and minimum of $I(x)$ in any closed bounded set can be asserted for a number of Banach spaces important for applications to analysis, for example, for reflexive spaces (by the use of a theorem by Alaoglu, Ann. of Math. vol. 41 (1940)) or the Schauder spaces (Math. Ann. vol. 106 (1932) p. 663). (Received March 21, 1947.)

227. A. C. Schaeffer and D. C. Spencer: *A general class of problems in conformal mapping.*

Let S be the class of functions $f(z)$ which are regular and schlicht in the unit circle $|z|<1$ and are normalized by the condition that $f(0)=0, f'(0)=1$. Let R be a closed set in $|z|<1$, and let $\psi_\nu(\alpha)$ be a measure function defined in the space R. If F_ν is a function of $f\subset S$ and its derivatives up to the nth order and $P_\nu=\int F_\nu d\psi_\nu$ then (P_1, P_2, P_m) is a point in a euclidean space of m dimensions. The problem is to find the region of variability of this point. Several unsolved classical problems are special cases of this general problem. The region of variability of $f'(z_1)$ where $|z_1|<1$ is one example, and a differential equation is obtained which defines this region. The differential equation can be integrated explicitly in terms of elementary functions. (Received March 22, 1947.)

228. H. M. Schaerf: *Properties of measures biinvariant under a composition law.* Preliminary report.

Let the measure space R be the union of a sequence of measurable sets of finite measure. Let a composition law ascribe to every $x, y\in R$ an element $xy\in R$ so that Weil's condition M is satisfied and so that the measure of any measurable set X is invariant under the transformations rX, Xr^{-1} for every $r\in R$. The author gives a simple proof of the uniqueness of such a measure. He proves, moreover, that any set A, whose measure is not greater than that of a set B, can be split into a sequence of measurable sets, whose images under the composition law are disjoint subsets of B, and into a set of measure zero. Therefore, any value of $m(X)$ less than $m(B)$ is assumed on some subset of B. Furthermore, any one of two sets with equal measures can be split into a sequence of disjoint subsets and a set of measure zero so that subsets with equal subscripts are images of each other under the composition law. (Received March 20, 1947.)

229. H. M. Schaerf: *Two theorems on measure spaces with a composition law.* Preliminary report.

Let the measure space R be the union of a sequence of measurable sets of finite measure. Let a composition law ascribe to every x, $y\in R$ an element $xy\in R$ so that Weil's condition M is satisfied and so that, for every set X of measure zero and every element r of R, the set rX is of measure zero. By means of the generalized theorem of Fubini the following theorems are proved: (1) For any two sets A, B with positive measures there is an element r such that Ar^{-1} meets B in a set of positive measure. (2) If Z is a set of measure zero, then any completely additive function of a measurable set vanishes on Zr^{-1} for almost all $r\in R$ (generalization of a theorem of Wiener-Young). If to every element $x\in R$ a transformation t_x of R in R is ascribed, then generalizations of the above theorems are furnished by the composition laws $xy=t_x(y)$, and $xy=t_y(x)$. (Received March 20, 1947.)

230. I. M. Sheffer: *A limit theorem.*

A classical result in trigonometric series theory is that: If (i) $a_n\cos nx+b_n\sin nx\rightarrow 0$ for all x on an interval I, then $a_n\rightarrow 0$, $b_n\rightarrow 0$. In complex form (i) is replaced by (ii) $c_n\exp\{nx\}+d_n\exp\{-nx\}\rightarrow 0$. Here $\exp\{u\}=e^{iu}$. The present note extends this result in various directions. For example: Let $\{a_{s,n}\}$, $s=1,\cdots,k$, be complex number sequences, and let $\{r_{s,n}\}$ be real sequences with the following property: None of the sequences $\{r_{s,n}-r_{p,n}\}$ $(s\neq p)$ has zero as a limit point. If (iii) $\sum_{s=1}^{k}\alpha_{s,n}\exp\{r_{s,n}x\}\rightarrow 0$ for all x on I, then $a_{s,n}\rightarrow 0$ $(s=1,\cdots,k)$. (Received March 19, 1947.)

231. Y. C. Shen: *Interpolation to some classes of analytic functions by functions with pre-assigned poles.*

Let S_α be the class of functions $f(z)$ regular for $|z|<1$ and such that for some $\alpha>1$ the integral $\int_0^r\int_0^{2\pi}(1-\rho)^{\alpha-2}|f(\rho e^{i\theta})|^2\rho d\rho d\theta$ is bounded for $r\rightarrow 1$. Let a_{nk}, $k=1,2,\cdots,n$; $n=1,2,\cdots$, be a system of points in $|z|<1$, without points of accumulation in $|z|<1$. Let $\Phi_{nk}(z)=(1-\bar{a}_{nk}z)^{-\alpha}$, and let $f_n(z)$ denote the function of the form $\sum_1^n A_{nk}\Phi_{nk}(z)$ interpolating $f(z)$ at the points a_{nk}, $k=1,2,\cdots,n$. Necessary and sufficient conditions are obtained for the sequence a_{nk} so that for every $f\in S_\alpha$ the sequence $\{f_n(z)\}$ converges uniformly to f in every closed circle interior to $|z|<1$. (Received February 17, 1947.)

232. R. H. Stark: *Some classes of monotone functions.*

The purpose of this paper is the investigation of classes of monotone, continuous functions $h(x)$ made up from a system of basis functions $g_i(x)$ as sums of Hellinger integrals, $h(x)=\sum_{i=1}^{n}\int_a^x(df_i)^2/dg_i$. Classes defined by different basis functions are discussed. Criteria for determining whether a function belongs to a particular class are given. There is presented a method for constructing a set G of monotone continuous functions such that a continuum of classes without common elements is determined by choice of bases from G, and yet there exist functions outside of all classes determined by such a choice. (Received March 19, 1947.)

233. C. F. Stephens: *Concerning linear and nonlinear difference equations.*

The author considers the system of equations $u_i(x+1)=x^k\sum_{j=1}^{n}h_i^{(j)}(x)u_j(x)$

$+x^k g_i(u_1(x), \cdots, u_n(x); x)$ $(i=1, \cdots, n)$, where k is any integer or zero, $k_i^{(j)}(x)$ are bounded and continuous (not necessarily analytic) in the neighborhood of infinity, g_i are polynomials in the $u_i(x)$, but not in x, whose coefficients are continuous functions of x in the neighborhood of infinity. The nth order nonlinear difference equation corresponding to the above system, the nth order homogeneous linear difference equation and the case where $g_i=0$ are also considered. In all cases the author obtains unique particular solutions which are continuous and bounded in a certain domain extending to infinity on the left. The method used in this paper is based on some previous results of the author concerning the solutions of nonlinear difference equations for large values of the independent variable (Bull. Amer. Math. Soc. vol. 50 (1944) p. 343). (Received February 21, 1947.)

234. Walter Strodt (National Research Fellow): *Note on quasi-harmonic functions.* Preliminary report.

A quasi-harmonic function of x and y, as defined by Garrett Birkhoff, is a function satisfying the Gauss mean-value theorem on all circles of radius unity. This note answers affirmatively the question, raised by Birkhoff, whether there exist non-harmonic functions of x and y which are quasi-harmonic and also analytic in x and y. For example, $\exp(ax+by)$ is such a function if $a^2+b^2\neq 0$ and $F(a, b) = \int_0^{2\pi}[1-\exp(a\cos\theta+b\sin\theta)]d\theta=0$. For every fixed a there exist infinitely many b such that $F(a, b)=0$, since otherwise $F(a, b)$, which is an even transcendental entire function of b, of order unity, would be a polynomial times an exponential of an entire function, which is absurd. (Received March 17, 1947.)

235. L. V. Toralballa: *The integral in a normed division quasi-ring.*

Consider a system R with two operations, $+$ and $\times$, satisfying: (1) R is an additive Abelian group; (2) R is closed with respect to $\times$; (3) $\times$ is associative; (4) there exists a unity; (5) every nonzero element of R has a unique inverse; (6) to every element a of R corresponds a non-negative real number $|a|$, called its norm, satisfying (a) $|a|=0$ if and only if $a=0$; (b) $|a\cdot b|=|a|\cdot|b|$; (c) $|a|=|-a|$; (d) $|a+b| \leq |a|+|b|$; (e) there exists a real number $N>0$ such that $|ca-cb|\leq N|c(a-b)|$ always; (f) if cosine $(p, q)=(|p|^2+|q|^2-|p-q|^2)/2|p||q|$, $p\neq 0$, $q\neq 0$, there exists a real number $M>0$ such that $(ca, cb)\leq M(a, b)$ always. Consider functions f on R to R. In the norm topology one defines such notions as "curve" and "continuity." A regular curve C is one in which (1) C is rectifiable; (2) the limit of the ratio of the length of chord to that of the subtended arc is unity; (3) the scalar curvature is a continuous function of the arc length. The integral $\int_C f(\xi)d\xi$ is defined to be lim as norm of subdivision$\to 0$ of $\sum(z_\nu - z_{\nu-1})f(\xi_\nu)$. This is proved to exist whenever C is regular and f is continuous over C. (Received March 20, 1947.)

236. W. R. Utz: *On the decomposition of meromorphic functions.*

Let $f(z)$ be defined and meromorphic in a bounded and simply-connected region R having locally connected boundary $F(R)$. Let z_0 be a point of $F(R)$. If for each crosscut c subdividing R into subregions R_1 and R_2, where $z_0 \in F(R_1)-\bar{c}$, there exists a region $R_1' \subset R_1$ determined by a crosscut c', of R, and a rational function $g(\lambda)=w$, such that $z_0\in F(R_1')-\bar{c}'$ and $g^{-1}f(z)$ is 1-1 and meromorphic on R_1', then $f(z)$ is said to be *rationally-multivalent*. It is shown that a necessary and sufficient condition that $f(z)$ be decomposable into a 1-1, bounded and regular function on R, followed by a rational function, is that it be rationally-multivalent at each point of $F(R)$. This is a generalization of a result of L. H. Loomis (*The decomposition of meromorphic functions into rational functions of univalent functions*, Trans. Amer. Math. Soc. vol. 50 (1941) pp. 1–14). (Received March 7, 1947.)

237. J. L. Walsh: *The location of the critical points of simply and doubly periodic functions.*

By the use of conformal transformations, especially those involving the exponential function, known theorems (for instance that of Lucas) concerning the zeros of the derivative of a rational function and the critical points of a harmonic function yield new results on the zeros of the derivative of a periodic analytic function and the critical points of a periodic harmonic function. (Received March 6, 1947.)

238. Alexander Weinstein: *On Dirichlet's discontinuous factor and a class of many-valued functions.*

Continuing previous investigations (Bull. Amer. Math. Soc. vol. 53 (1947) p. 59) the paper deals with the function $\alpha(x, y) = \pi y^{-q} b^{-q} \int_0^\infty \exp(-xt) J_q(yt) J_{q+1}(bt) dt$ ($x \geqq 0$, $b > 0$, $-1 < q \leqq -1/2$), which represents a branch of a many-valued function with the ramification points $x = 0$, $y = \pm b$. For $q = -1/2$, $\alpha(x, y) = 2\int_0^\infty t^{-1} \exp(-xt) \cos(yt) \sin(bt) dt$. It is shown that, in this case, α is the angle at (x, y) subtended by the segment $(-b, b)$ of the y-axis. This angle $\alpha(x, y)$ is the streamfunction of doublets distributed along the segment, and $\pi^{-1}\alpha(0, y)$ is Dirichlet's discontinuous factor. (Received February 26, 1947.)

239. D. V. Widder: *Inversion formulas for convolution transforms.*

The author discovered earlier a linear differential operator of infinite order which inverts the Stieltjes transform, $f(x) = \int_0^\infty (x+t)^{-1}\phi(t)dt$. This is a convolution transform, $f(x) = \int_{-\infty}^\infty G(x-t)\phi(t)dt$, with $G(x) = (1+e^{-x})^{-1}$ after an exponential change of variable, and the inversion operator is $\pi^{-1} \sin \pi D$. Here D means differentiation with respect to x and the operator is interpreted by means of the infinite product expansion of the sine function. In the present paper the author generalizes this result, replacing $G(x)$ by any function whose bilateral Laplace transfunction is the reciprocal of an entire function $E(s) = s\prod_1^\infty [1-(s/a_k)^2]$, $\sum_1^\infty a_k^{-2} < \infty$. The inversion operator becomes $E(D)$. The method of Green's functions for differential systems is used. The kernel $G(x)$ is interpreted as a Green's function for a linear differential system of infinite order. (Received February 4, 1947.)

240. Albert Wilansky: *An application of Banach linear functionals to the theory of summability.* Preliminary report.

Normal conservative summability matrices are classified as co-regular, co-null. In particular multiplicative r ($r \neq 0$, $r = 0$) matrices are, respectively, co-regular, co-null. A co-null summability field cannot be a subfield of a co-regular field. Every field is a proper subfield of a co-null field. Theorems known for regular matrices (Mazur, Steinhaus, Agnew, Hill, and so on) are extended to conservative matrices, for example a co-regular matrix cannot sum all bounded sequences, Mazur's consistency theorem holds for co-regular matrices only. A normal matrix is shown to be of type M if and only if its columns and another sequence "span" (c). Two normal matrices with the same field are both or neither of type M, thus "of type M" may be applied to fields.

If F is called regular if a normal regular matrix exists with field F, a regular field is co-regular; F is regular if and only if $f(x)=\lim x_n$ is continuous on (c) using the "Mazur norm" $\|x\|=\sup\left|\sum_{k=0}^{\infty}a_{nk}x_k\right|$. Subfields of regular fields are regular. Other conditions are given. All Mazur metrics for a given field are equivalent and all fields are separable. A field is a non-reflexive Banach space. A normal conservative matrix sums only convergent sequences if and only if $\|A^{-1}\|<\infty$ where $\|A\|=\sup\sum_{k=0}^{\infty}|a_{nk}|$. (Received March 20, 1947.)

241. J. E. Wilkins: *Neumann series of Bessel functions.*

Suppose that $|f(t)|$ is integrable over $(0, a)$ and that $t^{-1/2}|f(t)|$ is integrable over (a, ∞) whenever $a>0$. Let $s=\sum_{n=0}^{\infty}a_{2n+1}J_{2n+1}(x)$, where $a_{2n+1}=\int_0^{\infty}t^{-1}f(t)J_{2n+1}(t)dt$, be the Neumann series associated with $f(x)$. It is shown that $s=\{f(x+)+f(x-)\}/2$ at each positive point x in a neighborhood of which $f(t)$ is of bounded variation if and only if $\int_1^{\infty}rJ_0(xr)dr\int_a^{N}f(t)J_0(tr)dt$ converges to zero as a approaches zero and N approaches ∞. (Received March 21, 1947.)

Applied Mathematics

242. J. W. Calkin: *Incipient shock waves in one dimension.*

The author considers the following one-dimensional model: a rectilinear cylinder, closed at one end by a piston, and containing a perfect gas. As is in effect known, while a discontinuous increase in piston velocity produces an immediate shock wave, starting at the piston face, a continuous acceleration results in a delayed shock which starts at a positive distance from the piston face, with initial velocity that of sound. The present paper is concerned with the boundary value problem to which the study of the subsequent motion leads, and provides procedures for approximating to the solution. A variant of the hypothesis of a perfect gas is also considered. (Received March 21, 1947.)

243. J. B. Díaz and H. J. Greenberg: *The determination of upper and lower bounds for the deflection of a clamped plate.*

A method is given for the determination of upper and lower bounds for the deflection w at any point of a thin elastic plate of arbitrary shape clamped along its edges and subjected to a distributed load. The method is based on the application of two variational principles: the first is that of minimum potential energy, the second is closely related to Castigliano's principle of minimum complementary energy but does not seem to have been used before in the present form. These principles yield inequalities for an integral of w. By considering two auxiliary loadings of the plate, in addition to the given loading, inequalities are obtained directly for w at any specified point. The bounds are obtained in terms of integrals of certain admissible functions. Explicit iterative formulas are given by means of which sequences of admissible functions can be utilized to successively improve the bounds. (Received March 21, 1947.)

244. J. B. Díaz and Alexander Weinstein: *Schwarz' inequality and the methods of Rayleigh-Ritz and Trefftz.*

It is shown that lower and upper bounds of a quadratic functional can be obtained by a simple and direct application of Schwarz' inequality and Green's formula, the results being equivalent to the application of the methods of Trefftz and Rayleigh-

Ritz. As an example, a lower bound for the Dirichlet integral $D(\phi)$ of a harmonic function ϕ with prescribed boundary values is derived by putting into the Schwarz' inequality $D^2(\phi, \psi) \leqq D(\phi)D(\psi)$ an arbitrary harmonic function ψ. This leads, by Green's formula, to the inequality $D(\phi) \geqq (\int_C \phi(\partial\psi/\partial n)ds)^2 D^{-1}(\psi)$, which is equivalent to the lower bound found by Trefftz (Proceedings of the Second International Congress for Applied Mechanics, Zürich, 1927, p. 132). (Received March 19, 1947.)

245. W. F. Eberlein: *A note on spheroidal wave functions.*

The equation defining the (axially symmetric) spheroidal wave functions can be written in the form $\{(1-x^2)u'\}'+(\lambda-\alpha^2x^2)u=0$, the boundary condition being that $u(x)$ remain finite at $x=\pm 1$. An asymptotic expansion for the characteristic values λ for $\alpha \gg 1$ is obtained. (Received March 21, 1947.)

246. Benjamin Epstein: *On the mathematical description of certain breakage mechanisms.*

It has been observed that the particle size distributions obtained from some breakage processes appear to be logarithmico-normal. In this paper a hypothetical breakage mechanism is constructed which has the property of yielding size distributions after breakage which will more closely approximate a logarithmico-normal distribution the longer the breakage process is continued. More precisely it is proved that if: (a) the probability of breakage of any piece during any step of the process is a constant independent of the size of the piece and the previous breakage history of the piece and (b) the distribution of pieces obtained from the application of a single breakage event to a given piece is independent of the dimension of the piece in the sense that the fraction by weight of material having dimension less than ky $(0 \leqq k \leqq 1)$ arising from the breakage of a unit mass of size y is independent of y, then the d.f. $F_n(x)$ after n steps in the breakage process is asymptotically logarithmico-normal. The proof depends essentially on the use of the Mellin transform in treating products of random variables. (Received February 18, 1947.)

247. H. J. Greenberg and Witold Hurewicz: *The stability of periodic dynamical processes subject to discontinuous impulses.*

The classical Poincaré-Liapunoff stability theory of periodic motions is extended so as to include the case of dynamical systems subjected to discontinuous impulses: for instance, mechanical systems involving colliding masses. The continuous phases of the motion are assumed to be governed by the equations (1): $\dot{x}_i=f_i(x_1, \cdots, x_m, t)$, $i=1, \cdots, m$, where the f_i are periodic functions of the time t. In addition, at any t_0 at which the variables x_i satisfy a relation $L(x_1, \cdots, x_m)=0$ where L is a given function, the x_i are assumed to have prescribed discontinuities $\Delta x_i=\phi_i(x_1, \cdots, x_m)$. With a given periodic motion of such a system a stability matrix is associated. If the characteristic values of this matrix are less than 1 the motion is stable; if one of these is greater than 1 the motion is unstable. Explicit expressions for the elements of this matrix are given in terms of a fundamental system of solutions of the "equations of variation" derived from (1) and of the values of the ϕ_i and their derivatives at the discontinuity points. The general theory is applied to the special case of a system of two degrees of freedom consisting of two linear colliding oscillators. (Received March 21, 1947.)

248. A. E. Heins: *Water waves over a channel with a dock.*

This paper is concerned with the solution of (*)$\Delta\phi(x, y)=0$ in the strip $-\infty<x<\infty$, $0\leqq y\leqq a$ subject to the conditions (a) $\partial\phi/\partial y=0$ at $y=0$ for all x, (b) $\partial\phi/\partial y=0$ at $y=a$ for all $x<0$, and (c) $\partial\phi/\partial y=\beta\phi$ $(\beta>0)$ for $y=a$ and all $x>0$. The velocity potential $\phi(x, y)$ is assumed to possess a logarithmic singularity at $x=0$, in order that (*) possess a travelling wave term for $x\gg 0$. The surface $y=a$, $x<0$ is the dock-like surface, while $y=a$, $x>0$ is the free surface. This problem is formulated as an integral equation of the Wiener-Hopf type in terms of a Green's function kernel which may be exhibited explicitly. The solution under the dock as well as under the free surface is found by a technique which is available for the solution of this integral equation. The equivalent three-dimensional problem is also treated. (Received March 1, 1947.)

249. Rufus Isaacs: *Recent progress in the theory of compressible fluids.*

Until recently progress toward a workable theory of compressible fluids has been barred by the staggering amount of computation required. Modern computing devices, together with theories aimed towards their usage, open up vast possibilities in this and other fields. Chaplygin showed that the differential equations for planar irrotational steady flows become linear when transferred to the hodograph plane. Here the potential and stream functions are functions of the velocity components instead of the point coordinates. Bergman modifies this approach by using not these functions and variables but related ones. For these equations—linear, but still formidable—he has devised an operator which transforms each analytic function of a complex variable into a solution. Analogously in the incompressible case (harmonic solutions) the operator is "Take the imaginary part." The operator involves a sequence of functions which must be known and tabulated in order to solve specific flow problems. The functions are defined recurrently by a relation involving both differentiation and integration. They are being calculated on the Automatic Sequence Controlled Computer at Harvard University in a current research program headed by Professor von Mises. The writer participated last summer and developed the method for this computation. (Received February 28, 1947.)

250. S. C. Kleene: *Analysis of lengthening of modulated repetitive pulses.*

Pulse lengtheners are circuits which lengthen pulses without changing the relative pulse amplitudes. The output of a pulse lengthener as a function of time t is $F(t)=f(\{t\})g(t-\{t\})$ where $f(t)=$modulating signal (here assumed periodic with fundamental frequency f_m), $g(t)=$unmodulated lengthened pulse (with repetition frequency f_0), and $\{t\}=[f_0t]/f_0$. The frequencies which may appear in $F(t)$ are then $|sf_0\pm pf_m|$ for s, $p=0, 1, 2, \cdots$. This was originally noted for the so-called ideal "box car" lengthener, for which $g(t)=1$ (in which case the frequencies sf_0, for $s=1, 2, \cdots$, are absent), by Ming-Chen Wang and G. E. Uhlenbeck (Radiation Laboratory classified report S-10 (May 16, 1944) pp. 103–104). The present paper (based on Naval Research Laboratory classified report R-2555 (July 2, 1945, declassified February 7, 1946)) contains a simple derivation of the Fourier expansion of $F(t)$ for a quite arbitrary $g(t)$, and formulas and graphs for calculating the amplitudes of the terms in the expansion for the family of pulse functions $g(t)=e^{-\alpha t}$ $(0<t<\beta/f_0)$, $g(t)=0$ $(\beta/f_0<t<1/f_0)$, including the ideal "box car" pulse for $\alpha=0$, $\beta=1$. (Received March 5, 1947.)

251. M. Z. Krzywoblocki: *On the two-dimensional steady flow of a compressible viscous fluid far behind a solid symmetrical body.*

Goldstein solved the equations of steady flow of an incompressible viscous fluid in a wake far behind a solid symmetrical body by application of the method of successive approximations. Very far downstream in a wake, if the motion were steady, the assumptions and approximations of the laminar boundary layer theory would be valid. In the present paper this method was applied to the solution of the equations of steady flow of a compressible viscous fluid in a wake far behind a solid symmetrical body. The following equations are taken into account: equations of motion, continuity, energy and state. The coefficients of viscosity and thermal conductivity are assumed to be functions of temperature. Just as in Goldstein's work only the first two approximations are successful, whereas the third one is not successful. (Received March 5, 1947.)

252. Isaac Opatowski: *Application of a theorem of Jacobi to compressible and rotational flows.*

Jacobi has given a method to determine the characteristics of the equation $\sum A^\lambda(x^1, x^2, x^3) \cdot \partial Y/\partial x^\lambda = 0$ if a solution of this equation of the type $Y(x^1, x^2, x^3) = \text{const.}$ and a function $M(x^1, x^2, x^3)$ such that $\sum \partial(MA^\lambda/\partial x^\lambda) = 0$ are known (*Vorlesungen über Dynamik*, 2d ed., 1884, pp. 74–79). His method is applied to determine the streamlines of compressible and rotational stationary flows. Then $M = \rho a^{1/2}$, where ρ is the density of the fluid and a is the discriminant of the first fundamental form related to the system of coordinates used (x^1, x^2, x^3). This gives equations of streamlines of rotational flows in the form: $S(x^1, x^2, x^3) = \text{const.}$, $W(x^1, x^2, x^3) = \text{const.}$, where S is the entropy and W is obtained explicitly by means of M (Jacobi's last multiplier). (Received March 22, 1947.)

253. Edmund Pinney: *Aerodynamically driven oscillations in suspension bridges.*

In the past hundred years many suspension bridges have been destroyed by wind driven oscillations, but most of these failures occurred so long ago and were so poorly observed that their cause remained uncertain. The spectacular and well observed failure of the Tacoma Narrows bridge on November 7, 1940 proved clearly the importance of such oscillations. The present paper develops a method of determining the wind speed at which oscillation sets in. Methods of obtaining symmetrical and asymmetrical vacuum mode forms and frequencies are given. Theodorsen type airforces are used, but a modification is given to include the effects of roadbed slots. A procedure is given to make use of airforce data obtained from wind tunnel section models. The theory is applied to the original Tacoma Narrows bridge and gives a "flutter speed" in close agreement with the observed wind speed at the time of failure. (Received March 20, 1947.)

254. A. H. Taub: *Orbits of charged particles in constant electromagnetic fields.*

The relativistic equations of motion of a particle of charge e and rest mass m_0 in an external electro magnetic field described by the anti-symmetric tensor $f_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ may be written as (1) $dV/d\tau = -\lambda FV$ where V is a matrix of one column $V = \|v^\mu\| = \|dx^\mu/d\tau\|$, $\lambda = e/m_0c$, c is the velocity of light and F is a four by four matrix determined in terms of the field tensor $f_{\mu\nu}$ by $F=\|f^{\mu}_{\nu}\|=\|g^{\mu\sigma}f_{\sigma\nu}\|$ where $g_{\mu\nu}$ are the components of the metric tensor which in the case of special relativity may be reduced to the form $g_{\mu\nu}=0$ for $\mu\neq\nu$, $c^{-2}g_{00}=-g_{11}=-g_{22}=-g_{33}=1$. In case F is a constant matrix, that is, its components are independent of x^{μ} or τ, the solution of (1) may be written as (2) $V=L(\tau)V_0$ where V_0 is a constant one column matrix and $L(\tau)$ is the one-parameter family of Lorentz matrices generated by the infinitesimal Lorentz matrix $1-d\lambda F$. Unpublished results of O. Veblen, J. W. Givens and the author give a complete classification of the Lorentz transformations in terms of the components of the tensor $f_{\mu\nu}$, as well as a closed expression for $L(\tau)$. The particle orbits may then be obtained by substituting (2) into the definition of V and integrating. In case f^{μ}_{ν} may be written as $f^{\mu}_{0\nu}+f^{\mu}_{1\nu}$ where $f^{\mu}_{0\nu}$ is a constant tensor and $f^{\mu}_{1\nu}$ is slowly varying, then approximations to the solution of (1) may be obtained by transforming (1) into an integral equation and using the classical Picard iteration process. (Received March 25, 1947.)

255. E. A. Trabant: *The Riemannian geometry of the symmetric top.*

The Riemannian geometry of the symmetric top with moment of inertia coefficients A, A, and B, using Euler's angles as coordinates, is developed. The following general theorem for the static space is proven. A necessary and sufficient condition in order that the static Riemannian space be an Einstein space of constant Riemannian curvature with the first covariant derivative of the Riemann symbols of the first kind equal to zero and which can be mapped conformally upon a 3-dimensional flat space is that $A=B$. (Received March 15, 1947.)

GEOMETRY

256. H. S. M. Coxeter: *Continuity in real projective geometry.*

In the presence of the usual axioms of incidence and separation (including one which ensures the compactness of collinear points), the following nonmetrical form of Cantor's axiom of closure suffices to characterize the real projective line: *Every monotonic sequence of points has a limit.* (The words "monotonic" and "limit" are defined in terms of separation alone.) It can be deduced that every point not belonging to a given harmonic net is the limit of a sequence of points of the net, whence the fundamental theorem follows at once. The axioms of Archimedes and Dedekind are likewise deducible. (Received March 22, 1947.)

257. John DeCicco: *Characterization of Halphen's theorem on central and parallel fields of force.*

Halphen showed that if the ∞^5 trajectories of a positional field of force are all plane curves, the lines of force are all straight lines concurrent in a fixed point which may be finite or at infinity, that is, the field of force is central or parallel. The author studies the problem of determining all the positional fields of force whose trajectories are general helices. (A helix is a curve drawn on any cylinder whatever, cutting the generators at a constant angle. In particular, if the curve cuts the generators orthogonally, the curve is plane.) It is proved that if the ∞^5 trajectories of a field of force are all helices, they must be all plane curves, and the field of force is central or parallel. Another characterization was obtained by Kasner. If each trajectory of a positional field of force lies on some sphere or plane, all the trajectories are plane curves, and the field of force is central or parallel. (Received March 6, 1947.)

258. V. G. Grove: *On the Darboux tangents.*

Abramescu (*Sur les tangents de Darboux d'une surface*, Annales Scientifiques Universitatea Jassey, section 1, vol. 27 (1941) pp. 283–288) gave a metric characterization of the cubic curve obtained by equating to zero the terms of the expansion of a surface at an ordinary point up to and including the third order. The author gives a projective characterization of such a cubic curve, and thereby gives a new projective characterization of the tangents of Darboux. By applying the method used in this characterization to the curve of intersection of the tangent plane with the surface, a new characterization of the second edge of Green is found. Finally a simple characterization of the Green-Fubini projective normal in terms of two binary forms is found. (Received March 20, 1947.)

259. C. C. Hsiung: *Differential geometry of a surface at a parabolic point.*

The purpose of this paper is to study the projective differential geometry of a surface in the neighborhood of a parabolic point, at which the two asymptotic tangents of the surface are coincident. The plane section of a surface made by its tangent plane at a parabolic point will be called *the tangential section*. According as the tangential section has a cusp or a tacnode at the parabolic point, we have five essentially different cases. For each case there is obtained a canonical power series expansion of the surface in the neighborhood of the point, together with a geometrical interpretation of the system of reference giving rise to the expansion. In a different manner, I. Popa (Rendiconti del Seminario Matematico della R. Universita di Roma (4) vol. 2 (1938) pp. 136–155) has derived another canonical expansion for one of the five cases, namely, that in which the tangential section has a cusp. (Received March 20, 1947.)

260. Edward Kasner and John DeCicco: *Differential geometric properties of the polar surfaces of a general algebraic surface.*

A general algebraic surface S_n of degree n has $(n-1)$ polar surfaces S_{n-1}, S_{n-2}, $\cdots$, S_1, where the rth polar S_{n-r} is of degree $n-r$, with respect to any point P of space. It is known that if the point P is on the surface S_n, the first polar S_{n-1} and all the other polar surfaces are tangent to S_n at P. The last polar S_1 is the tangent plane. It is shown that the directions of the lines of curvature at the point P of S_n and S_{n-r} are identical, and similarly for the asymptotic directions. The ratio of a principal curvature of S_{n-r} to that of S_n at the point P is $\rho=(n-r-1)/(n-1)$. The ratio of the mean curvatures is the same quantity ρ. The ratio of the gaussian curvatures is $\rho^2=(n-r-1)^2/(n-1)^2$. The ratio of the torsions of the asymptotic lines is also this number ρ. These results are extended to the case of higher order contact. Finally the case has been studied where P is a singular point of the algebraic surface S_n. (See forthcoming paper in Proc. Nat. Acad. Sci. U.S.A. concerning the polar theory of a general algebraic curve of the plane.) (Received March 6, 1947.)

261. B. J. Lockhart: *Covariants of a valence correspondence on an algebraic curve.*

Covariant correspondences of a given valence correspondence T are shown to satisfy certain relations. These relations are used to characterize the nature of T. A cyclic set of n points for T is defined as a set of n distinct points on the curve closed under T. The enumeration of such sets for arbitrary n is carried out for general symmetric and nonsymmetric correspondences. (Received February 28, 1947.)

262. R. M. Robinson: *On the decomposition of spheres.*

According to the "Banach-Tarski paradox" (Fund. Math. vol. 6 (1924) pp. 244–277), it is possible to cut a solid unit sphere into a finite number of pieces, and reassemble these by translation and rotation to form two solid unit spheres. Recently, Sierpinski (Fund. Math. vol. 33 (1945) pp. 228–234) showed that the total number of pieces may be taken as eight, three pieces being used to form one of the new spheres and five the other. In this paper, it is shown that the smallest possible total number of pieces is five; one of these pieces may be taken to consist of a single point. For the surface of the sphere, a similar result is true with four pieces. This follows from the fact that the surface of a sphere can be divided into two pieces, each of which can be subdivided into two pieces congruent to itself. More generally, the surface of a sphere may be decomposed into pieces satisfying any system of congruences, provided that it is not demanded, explicitly or implicity, that two complimentary portions of the surface be congruent. (Received March 3, 1947.)

LOGIC AND FOUNDATIONS

263. J. C. C. McKinsey and Alfred Tarski: *Some theorems about the sentential calculi of Lewis and Heyting.*

In this paper the authors prove certain theorems regarding systems of sentential calculus, by making use of results they have established elsewhere regarding closure algebras and Brouwerian algebras. Some of the results are new (in particular it is shown that there are infinitely many functions of one variable in the Heyting calculus, and various theorems about extensions of the Lewis system S4 are obtained); others have been stated without proof in the literature (in particular the authors establish some theorems due to Gödel, which enable one to translate the Heyting calculus into the Lewis system S4). (Received March 21, 1947.)

264. Ira Rosenbaum: *A method of determining the nth q-ary truth-function in m-valued logic.*

The following method is believed useful since truth-functional modes of propositional combination are very numerous in m-valued logic. There are, for example 19,683 distinct binary modes of combination in three-valued logic and m^{m^q} q-ary modes in m-valued logic. Denote these truth-functional modes of propositional combination by F_i, $i=1, 2, \cdots, m^{m^q}$. Let each function be regarded as determined by the truth-values it assigns to each combination of truth-values of its components or arguments. Let $\{n, r\}$ denote the truth-value assigned to the rth of the m^q q-ary truth-combinations by the nth q-ary truth-function, let $\iota x(\cdots)$ denote the one and only object x satisfying the condition $\cdots$, and let $[---]$ denote the integral part of $---$. Then the nth q-ary truth-function of m-valued logic is defined thus: $F_n = (\{n, 1\}, \{n, 2\}, \cdots, \{n, m^q\})$ where $\{n, 1\} = \iota x(1 \leq x \leq m \cdot\&\cdot x = n \pmod{m})$ and $\{n, k\} = \iota x(1 \leq x \leq m \cdot\&\cdot x = [(n - \{n, 1\})/m^{k-1} + 1] \pmod{m})$, $1 < k \leq m^q$. The above formulae determine the nth q-ary truth-function of m-valued logic without the construction of tables with m^{m^q} columns, provide a simple nomenclature for the numerous functions of m-valued logic, and are valid also for two-valued logic. A simple process for determining n, given the values $\{n, 1\}, \cdots, \{n, m^q\}$, is also available. (Received March 6, 1947.)

265. Ira Rosenbaum: *A method of determining the number n correlated with the given truth table of an arbitrary q-ary function in m-valued logic.*

In a previous paper, 1-1 correlation formulae were obtained between the numbers $1, 2, \cdots, m^{m^q}$ and the truth-tables of the m^{m^q} distinct q-ary truth-functional modes of propositional combination of m-valued logic. In the present paper a formula is given for proceeding in the reverse direction, that is, from the given truth-table of an arbitrary q-ary function to the number n correlated with that table. The original simple procedure for going from a given table to the number n associated with it lacked analytical representation. W. V. Quine suggested that this procedure was analogous to transforming the expression for an integer in the m-ary scale of notation into one in the denary scale of notation. This suggestion led to the following result. Let the m^q truth values in the given table be $V_1, V_2, \cdots, V_{m^q}$. From each of these values subtract one to obtain the values $W_1, W_2, \cdots, W_{m^q}$. The latter values are integers x of the range $0 \leq x \leq m-1$ rather than, like the V's, of the range $1 \leq x \leq m$. Hence the sequence of W's may be regarded as representing in the m-ary scale of notation an integer n. The mode of representation is indicated by the formula $n-1 = \sum_{j=0}^{c-1} W_{c-j} \cdot m^{c-1-j}$ in which $c = m^q$. (Received February 20, 1947.)

STATISTICS AND PROBABILITY

266. Z. W. Birnbaum: *Probabilities of sample-means for bounded random variables.*

Lower bounds are given for the probabilities $P(|\bar{X}_n| \geq t)$, where $t \leq a$, $\bar{X}_n = (1/n)\sum_{j=1}^{n} X_j$, and $X_1, X_2, \cdots, X_n$ is a sample of a continuous random variable with a probability density $f(X)$ such that: $f(X) = f(-X)$, $f(|X|)$ is a nonincreasing function of $|X|$, and $f(X) = 0$ for $|X| \geq a$. (Received March 21, 1947.)

TOPOLOGY

267. Felix Bernstein: *A lattice color problem.* Preliminary report.

The points of the lattice L of all points with coordinates which are integers or are centers of the elementary squares of L are considered as the regions of a color problem. In a partial set S two points A and B are called neighbors if AB does not contain another point of S and if the distance AB is equal either to 1 or to $2^{1/2}/2$ or to $2^{1/2}$. The number of colors required in order that two neighbors may be colored differently is obviously 5 or less. It is shown that 5 colors are necessary. For the proof two methods are used. The one method is based on the studying of color schemes at certain conveniently chosen boundaries in the manner introduced by G. D. Birkhoff. The other method is based on the study of the effect of the "centers" on the coloring of the total neighborhood. The efficiency of each method varies with the nature of the given configuration. (Received March 22, 1947.)

268. R. H. Bing: *A homogeneous indecomposable plane continuum.*

An example is given of a homogeneous bounded nondegenerate continuum which is not a circle. This answers the following question raised by Knaster and Kuratowski in Fund. Math. vol. 1 (1920) p. 223: If a nondegenerate bounded plane continuum is homogeneous, is it necessarily a simple closed curve? This example refutes a statement made in Theorem 5, C. R. Acad. Sci. Paris vol. 219 (1944) p. 543. (Received February 13, 1947.)

269. R. H. Bing: *Extending a metric.*

Suppose that r is a positive integer and $H_1, H_2, \cdots$ are collections of sets such that each pair of points that can be covered by a coherent collection of r or fewer elements of H_{i+1} can be covered by an element of H_i. If P and Q are two points whose sum cannot be covered by any element of H_s but which can be covered by a coherent collection of sets $h_1, h_2, \cdots, h_n$ belonging to $H_{\alpha(1)}, H_{\alpha(2)}, \cdots$, and $H_{\alpha(n)}$ respectively, then $2[1/r^{\alpha(1)}+1/r^{\alpha(2)}+\cdots+1/r^{\alpha(n)}]>1/r^s$. With the use of this result, it is found that if M is a given metric on a closed subset K of a metrizable space S, then a metric may be assigned to S that will preserve the given metric M on K. (Received March 19, 1947.)

270. Jean M. Boyer and D. W. Hall: *On Peano spaces as continuous images of intervals.*

Let M be a cyclic Peano space and A the set of all points of M which are interior points of arcs of M. Using the definition of M as the continuous image of the unit interval I it is shown that M is the closure of A. Standard arguments, using the same definition, can then be used to prove that A is a closed set. This result yields at once the three point theorem of W. L. Ayres. (Received March 20, 1947.)

271. C. H. Dowker: *Topologies of infinite complexes.*

Infinite complexes have been topologized in different non-equivalent ways by different authors; for example, the natural complex and the geometric complex of S. Lefschetz (*Topics in topology*, Princeton, 1942, p. 9) and the topological polyhedron of J. H. C. Whitehead (Proc. London Math. Soc. vol. 45 (1939) p. 316). It is shown that the three topological spaces, resulting from topologizing an infinite complex in each of these three ways, are of the same homotopy type. It follows that the topological invariance of homology groups, the Hopf mapping theorems, and the Alexandroff Überführungssatz hold equally for all three topologized complexes. (Received March 21, 1947.)

272. Beno Eckmann: *On infinite complexes with automorphisms.* Preliminary report.

Let K be a closure finite complex with an infinite group G of automorphisms without fixed cells, J an Abelian group. An n-cochain f^n in G, that is, a J-valued function of the n-cells c^n of K, is called *G-finite* if, for each c^n, $f(xc^n)=0$ for almost all $x \in G$ (for all except a finite number). With the ordinary co-boundary operation in K these cochains lead to cohomology groups β^n which in general depend upon K *and* G. The G-finite cocycles which are coboundaries of ordinary (infinite) cochains define a subgroup β_0^n of β^n. By methods similar to those used previously by the author (Comment. Math. Helv. vol. 18 (1946) pp. 232–282) the theorem is proved: When K is *acyclic* in dimensions $n<N$ (in the sense of finite chains), then β^n, $n<N$, and β_0^N are determined by the abstract structure of G and J; they may be described in a purely algebraic way by means of J-valued functions of several variables $\in G$. Geometric applications are based on the remark, that in case G has in K a finite fundamental

domain the β^n are the usual cohomology groups of K in the sense of *finite* cochains; a theorem results which for the dimension $n=1$ was already established by Specker (in an as yet unpublished paper). (Received March 21, 1947.)

273. D. W. Hall and D. C. Lewis: *On coloring six-rings.*

Let Q be a hexagon surrounded by a six-ring in a regular map M of n regions. Formulas have been developed (in terms of free chromatic polynomials for regular maps with fewer than n regions) for the number of ways $M-Q$ can be colored in such a manner that any prescribed color scheme appears in the ring. The formulas are valid for any number of assigned colors. These formulas are the basis for a general six-ring reduction formula and also for numerous inequalities involving chromatic polynomials. (Received February 20, 1947.)

274. O. H. Hamilton: *A fixed point theorem for upper semi-continuous transformations on n-cells for which the images of points are nonacyclic.*

It is shown that if T is an upper semi-continuous transformation of an n-cell M into a subset of itself such that for some positive real number d the image $T(P)$ of each point P of M under T is the boundary of a nondegenerate n-dimensional convex continuum containing a simple domain of diameter greater than d, then for some point Q of M, $T(Q)$ contains Q. (Received March 25, 1947.)

275. M. J. Norris: *A note on regular and completely regular topological spaces.*

Given a set of points and two topologies for this set, one topology is said to be stronger than the other if every open set in the latter topology is open in the former. E. Hewitt (Duke Math. J. vol. 10 (1943) pp. 309–333) has shown that the l.u.b. of a completely ordered family of regular (completely regular) topologies is also regular (completely regular). In this paper slight modifications of Hewitt's methods are used to remove the restriction of complete order. (Received March 21, 1947.)

276. M. J. Norris: *On the representation of real numbers by sequences of integers.*

It is well known that there exist one-to-one functions whose domain is the real number system and whose range is contained in the class of all infinite sequences of integers satisfying the following condition. Given a sequence S_0 and a positive ϵ, there exists a natural number N such that $f^{-1}(S)$ differs from $f^{-1}(S_0)$ by less than ϵ when S does not differ from S_0 in the first N positions. Representations by means of decimals or simple continued fractions furnish such functions. This raises the question as to whether or not there exist one-to-one functions with domain and range as above, such that, given a real number x_0 and a natural number N, there exists a positive ϵ such that $f(x)$ does not differ from $f(x_0)$ in the first N positions when x differs from x_0 by less than ϵ. By introducing the usual Cartesian product topology for the class of sequences, the question is easily answered in the negative. (Received March 21, 1947.)

277. H. M. Schaerf: *Generalization of the theory of distance sets to topological groups.* Preliminary report.

Hugo Steinhaus (Fund. Math. vol. 1 (1920) pp. 93–104) has found interesting theorems on sets of distances between the points of two sets A, B of positive Lebesgue measure. In this paper it is shown that Steinhaus' results are special cases of general theorems on sets $B^{-1}A$, where A and B are measurable subsets of positive measure of a topological group with a countable basis. Thereby, the measure function, $m(X)$, may be any one defined on the class of Borel sets, if only $m(gX)$ and $m(Xg)$ are for every group element g absolutely continuous functions of the Borel set X and if every point has a neighborhood with finite measure. (Received March 28, 1947.)

278. H. M. Schaerf: *On A. Weil's condition M.* Preliminary report.

The author proves that the class B of the Borel sets of a topological group G in any neighborhood space with open neighborhoods and with a countable basis satisfies Weil's condition M (that is, the smallest additive class of sets of $G\times G$, containing all combinatorial products $X\times Y$ with $X,\ Y\in B$, is invariant under the transformations $[(x,\ y)\rightarrow(y^{-1}x,\ y)]$ of $G\times G$ onto $G\times G$.) (Received March 20, 1947.)

279. H. M. Schaerf: *On sets of the form $B^{-1}A$ in topological groups.* Preliminary report.

Given a topological group G in a limit space and a measure function $m(X)$ defined on an additive class K of subsets of G and such that $m(Xg)$ is for every $g\in G$ an absolutely continuous function of the measurable set X. The following theorem is proved: Let A be a measurable set such that one of its neighborhoods is contained in a set of finite measure. Let B belong to the smallest additive class K_F of subsets of G, containing both all closed measurable sets and all sets of measure zero. Then the set $B^{-1}A$ contains a neighborhood of every element $g\in G$ such that $m(Ag\cap B)>0$. Therefore, the set $A^{-1}A$ contains a neighborhood of the group unity if $m(A)>0$ and $A\in K_F$. (Received March 28, 1947.)

280. H. M. Schaerf: *On the continuity of measurable mappings of neighborhood spaces.*

Let a measure be defined on an additive class K of subsets of a neighborhood space. Let T be a topological space satisfying the second axiom of countability. Finally, let $f(x)$ be a "measurable" mapping of a set $E\in K$ into T, that is, such a mapping that the image of any open set of T under f^{-1} is measurable. In this paper necessary and sufficient conditions are proved for the validity of each of the following assertions: (a) For every positive number ϵ there is a subset E_ϵ of E such that the measure of $E-E_\epsilon$ is less than ϵ and such that $f(x)$ is continuous on E_ϵ; (b) E_ϵ can be, moreover, taken to be a closed set; (c) $f(x)$ can be, moreover, assumed uniformly continuous on E_ϵ. (Received March 20, 1947.)

281. G. E. Schweigert and G. S. Young: *Remarks concerning invariants for certain finite transformations.* II.

The transformation $T_1(A)=B$ is exactly k to 1; $T_2(A)=B$ is interior and finite. Eleven types of curve, from the arc through rational curves and curves with property N, to curves of any dimension, are considered as to invariance under T_i and T_i^{-1}. Forty-two of these forty-four cases are settled. The negative cases can not be described here. Those with positive solutions admit the following remarks: Under T_2 only the simple closed curve is not invariant (this "backbone" justifies the choice of curves), most of these are invariant under more general transformations; at most

3 curves are invariant under all 4 transformations; many results are from the literature but require considerable adaptation; the rational curves are invariant under T_1 if locally connected. The unrestricted rational curves appear to be a difficult open question for T_1; otherwise (following Whyburn) they are invariant. See our previous abstract (Bull. Amer. Math. Soc. Abstract 52-5-443). (Received March 21, 1947.)

282. M. E. Shanks: *Decomposition homology groups*. Preliminary report.

To each open set O there corresponds a simple decomposition D whose nondegenerate slice is the complement of O. Thus to each open covering there corresponds a decomposition covering. A Čech type theory leads to the usual groups. The homology groups of a closed subset A and the homology groups mod A are obtained as the groups associated with the sublattices of the decomposition lattice "above" and "below" the decomposition whose only nondegenerate slice is A. By lifting the restriction to simple decompositions a large class of new groups is obtained. (Received March 21, 1947.)

283. S. M. Ulam and John von Neumann: *On the group of homeomorphisms of the surface of the sphere*.

It is proved that the group of all homeomorphisms of the sphere preserving the orientation (that is, the component of unity) is simple. More strongly one proves the existence of a number N such that for any two such homeomorphisms S and T, S different from the identity, there exist at most N conjugates of S whose product is T. The method of the proof can be used to establish analogous theorems about groups of homeomorphisms of other manifolds. (Received March 15, 1947.)

284. F. A. Valentine: *The determination of connected linear sections*.

Consider a continuum S in an n-dimensional Euclidean space R_n ($n \geqq 2$), a continuum being compact and connected. Let x be a point in R_n such that *each* hyperplane through x intersects S in a connected set, and let K denote the set of all such points x. Theorem: *Each component of the set K is convex*. The definition for K differs from that for the Kerneigebiet defined by Brunn (*Über Kerneigebiete*, Math. Ann. vol. 73 (1913) pp. 436–440), in that empty intersections are admissible, and hyperplanes replace the role played by straight lines. Necessary and sufficient conditions are found for a component of K to be closed. Among the theorems proved the following seemed to be of special interest. Designate the convex hull of S by $H(S)$, and let $B(H(S))$ be the boundary of $H(S)$. Definition: A component D of $H(S)-S$ is called a *dent* of S if $D \cdot B(H(S)) \neq 0$. Theorem in R_2, the plane: *Let S be a continuum in R_2 which has m dents. Then if N is the number of components of K the inequality $N \leqq (m^2+m+2)/2$ must hold.* (Received February 11, 1947.)

285. G. T. Whyburn: *On locally simple curves*.

A mapping $f(A)=B$ is locally simple (Morse-Heins) if it is (1-1) on some neighborhood of any point of A. It is shown that a continuum M has a locally simple representation on the circle (l.s.r.c.) if and only if it is the sum of a finite number of doubly extensible simple arcs, that is, arcs ab contained entirely in the interior of some other arc $a'b'$ in M. Every doubly extensible arc ab in a locally connected continuum M is contained in either a (a) simple closed curve, (b) θ-curve, (c) figure 8, (d) dumbbell curve, (e) lariat curve (circle plus radius), or (f) arc joining two end points of M. If

M is without end points, (e) and (f) may be omitted. Also if M is cyclic, ab lies in a cyclic graph in M. Thus a continuum has a l.s.r.c. if and only if it is the sum of a finite number of simple closed curves and dumbbells. A boundary curve B has a l.s.r.c. if and only if it has only a finite number of nodes and no end points. Further, if B satisfies this condition, *any* nonalternating light mapping of the circle onto B is locally simple. (Received March 3, 1947.)

286. G. T. Whyburn: *On n-arc connectedness.*

A simple elementary proof is given for the well known theorem that any two points of a locally connected generalized continuum M which are not separated in M by any set of less than n points can be joined in M by a set of n independent arcs. The proof represents an extension to arbitrary n of the inductive type of reasoning used by the author in simplifying the proof of the cyclic connectedness theorem. It includes an extension to arbitrary locally arcwise connected separable metric spaces S of the theorem of Menger-Nöbeling to the effect that if a compact locally connected space S is n-point strongly connected between two disjoint closed sets P and Q, then S contains a set of n disjoint arcs joining P and Q. (Received March 3, 1947.)

287. R. L. Wilder: *Recognition of manifolds by accessibility properties.*

Schoenflies used an accessibility property to characterize the S^1 in S^2. Generalizing the regular accessibility of Whyburn to a regular r-accessibility, Alexandroff characterized the S^1 in S^n. But regular r-accessibility is not strong enough to characterize the S^k, $k>1$, in S^n. For the latter purpose, if K and E are subsets of a space S, $K\cap E=0$, $p\in K$, then, using compact cycles and homologies, we define p to be (1) *r-accessible* from E if every r-cycle of $E\cup p$ bounds in $E\cup p$; (2) *semi-r-accessible* from E if there exists an f cos $\mathcal{E}$ of S such that every r-cycle of $E\cup p$ in an element of $\mathcal{E}$ bounds in $E\cup p$; (3) *locally r-accessible* from E if $E\cup p$ is r-lc at p. These are topologically invariant positional properties for closed subsets of S^n (or an n-gm). Using these (successively weaker, and all stronger than regular r-accessibility for $r>0$) properties, the general k-gcm can be characterized in the n-dimensional generalized manifold. For example, the only nondegenerate closed proper subsets of S^2 that are both 0- and 1-accessible are the S^0 and S^1; in S^3, if the boundary B of a simply-connected domain is a continuum and 0-, 1- and 2-accessible, then B is an S^2. These results will appear in Chapter 12 of the author's *Topology of manifolds*, to be submitted to the Colloquium Series of the Society. (Received March 20, 1947.)

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A SIMPLE PROOF THAT π IS IRRATIONAL

IVAN NIVEN

Let $\pi = a/b$, the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a-bx)^n}{n!},$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x),$$

the positive integer n being specified later. Since $n!f(x)$ has integral coefficients and terms in x of degree not less than n, $f(x)$ and its derivatives $f^{(j)}(x)$ have integral values for $x=0$; also for $x=\pi=a/b$, since $f(x)=f(a/b-x)$. By elementary calculus we have

$$\frac{d}{dx}\left\{F'(x)\sin x - F(x)\cos x\right\} = F''(x)\sin x + F(x)\sin x = f(x)\sin x$$

and

$$(1)\qquad \int_0^\pi f(x)\sin x\,dx = \left[F'(x)\sin x - F(x)\cos x\right]_0^\pi = F(\pi) + F(0).$$

Now $F(\pi)+F(0)$ is an *integer*, since $f^{(j)}(\pi)$ and $f^{(j)}(0)$ are integers. But for $0<x<\pi$,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (1) is *positive, but arbitrarily small* for n sufficiently large. Thus (1) is false, and so is our assumption that π is rational.

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ON THE NON-SUMMABILITY $(C, 1)$ OF FOURIER SERIES

M. L. MISRA

Let

$$\phi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt, \qquad \phi(0) = 0, \tag{1}$$

be the Fourier series associated with an even function $\phi(t)$ which is integrable (L) over the interval $(0, \pi)$ and defined outside this range by periodicity.

Lebesgue [2, pp. 561–562][1] proved that the series (1) is summable $(C, 1)$ to zero at the point $t=0$ if, as $t\rightarrow 0$,

$$\int_0^t | \phi(t) | dt = o(t). \tag{2}$$

If, however, a condition weaker than (2) is satisfied, namely

$$\int_0^t \phi(t)dt = o(t), \tag{3}$$

as $t\rightarrow 0$, the series (1) is not necessarily summable $(C, 1)$. Hahn [1] gave an example to prove that the series (1) is not summable $(C, 1)$, though the condition (3) is satisfied but not (2). Prasad [3] has investigated whether, at the point $t=0$, the series (1) would be summable $(C, 1)$ if a condition stronger than (3) is satisfied, namely

$$\int_0^\pi \frac{\phi(t)}{t} dt = \lim_{t\rightarrow 0} \int_t^\pi \frac{\phi(t)}{t} dt = s, \tag{4}$$

say, exists as a non-absolutely convergent integral, and he has constructed an example to show that even when (4) is satisfied the series (1) is not necessarily summable $(C, 1)$. As the condition (4) implies (3), Prasad's example includes that of Hahn.

The object of the present note is to construct an example to prove that the series (1) is not necessarily summable $(C, 1)$ even though the condition

$$\phi_1(t) \equiv \int_t^\pi \frac{\phi(t)}{t} dt - s = o\left(1/\log\log\frac{1}{t}\right), \tag{5}$$

as $t\rightarrow 0$, is satisfied.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

In view of the fact that the condition (5) is more stringent than (4) and (3), this example would be more far-reaching than those of Prasad and Hahn.

I am much indebted to Dr. B. N. Prasad for his kind interest and advice in the preparation of this note.

EXAMPLE. Let

$$\lambda_r = 3^{r^2}, \qquad t_r = \pi/\lambda_r, \qquad t_0 = \pi, \qquad c_r = 1/(2r-1)^{1/2},$$

for $r = 1, 2, 3, \cdots$. We define an even function $\phi(t)$ in $(0, \pi)$ by

$$\phi(0) = 0, \qquad \phi(t) = -c_r\lambda_r t \cos \lambda_r t, \tag{6}$$

for $t_r < t \leqq t_{r-1}$ and by periodicity outside. Then

$$\begin{aligned}\int_{t_r}^{t_{r-1}} |\phi(t)|\, dt &= c_r\lambda_r \int_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} t\,|\cos \lambda_r t|\, dt \\ &= \frac{c_r}{\lambda_r}\int_{\pi}^{3^{2r-1}\pi} u\,|\cos u|\, du \\ &= O\left(\frac{3^{4r-2}}{\lambda_r}\right) = O\left(\frac{1}{3^{r^2-4r+2}}\right).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{\pi} |\phi(t)|\, dt &= O(1) + \sum_{r=1}^{\infty}\int_{t_r}^{t_{r-1}} |\phi(t)|\, dt \\ &= O(1) + \sum_{r=1}^{\infty} O\left(\frac{1}{3^{r^2-4r+2}}\right) \\ &= O(1),\end{aligned}$$

so that $\phi(t)$ is integrable (L) in $(0, \pi)$. Again

$$\int_{t_r}^{t_{r-1}} \frac{\phi(t)}{t}\, dt = -c_r\lambda_r \int_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} \cos \lambda_r t dt = -c_r[\sin \lambda_r t]_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} = 0.$$

For $t_r < t < t_{r-1}$, we have

$$\int_t^{t_{r-1}} \frac{\phi(t)}{t}\, dt = -c_r[\sin \lambda_r t]_t^{\pi/\lambda_{r-1}} = c_r \sin \lambda_r t,$$

so that, taking $s = 0$, we have, for $t_r < t < t_{r-1}$,

$$\phi_1(t) = \int_t^{\pi} \frac{\phi(t)}{t}\, dt = \int_t^{t_{r-1}} \frac{\phi(t)}{t}\, dt = c_r \sin \lambda_r t.$$

Now

$$\left|\log\log\frac{1}{t}\,\phi_1(t)\right| \leqq C_r\log\log\frac{1}{t_r} = C_r\log\log\frac{\lambda_r}{\pi}$$

$$= \frac{1}{(2r-1)^{1/2}}\log\log\frac{3^{r^2}}{\pi}$$

$$\leqq \frac{1}{(2r-1)^{1/2}}\left[2\log r+\log\log 3\right],$$

and hence, as $t\to 0$,

(7) $$\phi_1(t) = o\left(1/\log\log\frac{1}{t}\right).$$

Now the necessary and sufficient condition that the Fourier series (1) be summable $(C, 1)$ is

$$\frac{1}{n}\int_0^\pi \phi(t)\sin^2\frac{nt}{2}\cdot\frac{dt}{t^2} = o(1), \qquad \text{as } n\to\infty,$$

or,

$$\frac{1}{n}\left[-\sin^2\frac{nt}{2}\cdot\frac{1}{t}\int_t^\pi\frac{\phi(t)}{t}\,dt\right]_0^\pi + \frac{1}{2}\int_0^\pi\phi_1(t)\left[\frac{\sin nt}{t}-\frac{2}{n}\,\frac{\sin^2(nt/2)}{t^2}\right]dt = o(1).$$

This is $I_n+J_n=o(1)$, with

$$I_n = \int_0^\pi \phi_1(t)\,\frac{\sin nt}{t}\,dt,$$

$$J_n = \frac{1}{n}\int_0^\pi \phi_1(t)\left(\frac{\sin (nt/2)}{t}\right)^2dt.$$

Also

$$J_n = \frac{1}{n}\int_0^{\pi/n} o\left(1/\log\log\frac{1}{t}\right)dt + \frac{1}{n}\int_{\pi/n}^\pi o\left(1/\log\log\frac{1}{t}\right)\frac{dt}{t^2}$$

$$= o(1) + \frac{1}{n}\int_{\pi/n}^\pi o\left(\frac{1}{t^2}\right)dt = o(1),$$

if (5) is satisfied.

Hence in order to show that the Fourier series associated with the function $\phi(t)$ defined by (6) is not summable $(C,1)$ at $t=0$, although the condition (7) is satisfied, we have only to prove that the integral

$$I_n = \int_0^\pi \phi_1(t)\,\frac{\sin nt}{t}\,dt$$

tends to ∞ as n assumes successively the values of integers

$$n = \lambda_r = 3^{r^2}, \qquad r = 1, 2, 3, \cdots,$$

that is

$$I_{\lambda_r} \to \infty, \qquad \text{as } r \to \infty.$$

Now

$$\begin{aligned} I_{\lambda_r} &= \int_0^\pi \phi_1(t)\,\frac{\sin \lambda_r t}{t}\,dt \\ &= \left[\int_0^{t_r} + \int_{t_r}^{t_{r-1}} + \sum_{i=1}^{r-1}\int_{t_i}^{t_{i-1}}\right]\phi_1(t)\,\frac{\sin \lambda_r t}{t}\,dt \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say, and

$$I_1 = \int_0^{t_r} o\left(1/\log\log\frac{1}{t}\right)\frac{\sin\lambda_r t}{t} = \lambda_r\int_0^{\pi/\lambda_r} o(1)dt = o\,(1),$$

as $r\to\infty$.

$$\begin{aligned} I_2 &= c_r\int_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} \frac{\sin^2 \lambda_r t}{t}\,dt = \frac{c_r}{2}\int_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} \frac{1-\cos 2\lambda_r t}{t}\,dt \\ &= \frac{c_r}{2}\log\frac{\lambda_r}{\lambda_{r-1}} - \frac{c_r}{2}\int_{2\pi}^{2\cdot 3^{2r-1}\pi}\frac{\cos u}{u}\,du \\ &= \frac{1}{2(2r-1)^{1/2}}\log 3^{2r-1} - \frac{1}{2}\cdot\frac{1}{(2r-1)^{1/2}}\,O(1) \\ &= \frac{(2r-1)^{1/2}}{2}\log 3 + o(1). \end{aligned}$$

Thus

$$I_2 \to \infty, \qquad \text{as } r \to \infty.$$

Again

$$\begin{aligned} I_3 &= \sum_{i=1}^{r-1} c_i \int_{t_i}^{t_{i+1}} \frac{\sin \lambda_i t \sin \lambda_r t}{t}\, dt \\ &= \sum_{i=1}^{r-1} \frac{c_i}{2} \int_{t_i}^{t_{i+1}} \frac{\cos (\lambda_r - \lambda_i)t - \cos (\lambda_r + \lambda_i)t}{t}\, dt \\ &= \sum_{i=1}^{r-1} \frac{c_i}{t_i}\, O\left(\frac{1}{\lambda_r - \lambda_i}\right) = \sum_{i=1}^{r-1} \frac{1}{(2i-1)^{1/2}}\, O\left(\frac{\lambda_i}{\lambda_r - \lambda_i}\right) \\ &= O\left\{\frac{r3^{(r-1)^2}}{3^{r^2} - 3^{(r-1)^2}}\right\} = O\left(\frac{r}{3^{2r-1} - 1}\right) = o(1), \end{aligned}$$

as $r \to \infty$.

Combining our results, we have

$$I_{\lambda_r} \to \infty, \qquad \text{as } r \to \infty.$$

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NOTE ON THE DERIVATIVES OF FUNCTIONS ANALYTIC IN THE UNIT CIRCLE

J. L. WALSH

In a recent paper,[1] W. Seidel and the present writer established distortion theorems for various classes of functions analytic in the unit circle; more explicitly, established relations between the derivatives of functions and their radii of univalence and of p-valence, with particular reference to behavior as a point approaches the circumference. Classes studied in detail were functions respectively univalent, bounded, omitting two values, p-valent, and having a bounded radius of univalence. The last-named class clearly includes the class of functions $f(z)$ each analytic in $|z|<1$ and transforming $|z|<1$ onto a Riemann configuration of finite area. It is the primary object of the present note to study this included class to the same end more effectively by other methods. Terminology and notation are uniform with the paper referred to; unless otherwise specified, references in the present note are to that paper; we shall also have occasion to use the results of a subsequent paper by Loomis.[2]

We denote by K_M the class of functions $f(z)$ analytic in $|z|<1$ and transforming the region $|z|<1$ into a region whose area (counted according to the multiplicity of covering) is not greater than πM^2:[3]

$$\iint_{|z|<1} |f'(z)|^2 dS \leq \pi M^2. \tag{1}$$

If we set $f(z)=\sum_0^\infty a_n z^n$, inequality (1) becomes

$$\sum_1^\infty n\,|a_n|^2 \leq M^2, \tag{2}$$

whence we have

$$|a_n| \leq \frac{M}{n^{1/2}}. \tag{3}$$

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[1] Trans. Amer. Math. Soc. vol. 52 (1942) pp. 128–216.

[2] Bull. Amer. Math. Soc. vol. 48 (1942) pp. 908–911.

[3] *Added in proof.* This class of functions has recently been studied by P. Montel (Publicaciones del Instituto de Matemática de la Universidad Nacional del Litoral, Rosario, vol. 6 (1946) pp. 273–286), who obtains sharp inequalities improving (12) and (13) below, but does not prove any of the italicized theorems of the present note. Compare also T. H. Gronwall, Ann. of Math. vol. 16 (1914) pp. 72–76.

We proceed now to establish the following theorem.

THEOREM 1. *If $f(z)$ is of class K_M, with $w_0=f(z_0)$, we have*

$$D_1(w_0) \leqq |f'(z_0)|(1-|z_0|^2) \leqq [6MD_1(w_0)]^{1/2}. \tag{4}$$

The first inequality in (4) has already been established (loc. cit. §4). We proceed to prove the second inequality in (4).

The proof is by the method of Landau employed in the previous paper (§18). Let us assume first $f(0)=0$, $f'(0)=1$, from which it follows by (3) that the series $\sum a_n z^n$ is term by term not greater in absolute value than the series

$$z+Mz^2+Mz^3+Mz^4+\cdots; \tag{5}$$

the latter series has the sum ($|z|<1$)

$$z+\frac{Mz^2}{1-z}. \tag{6}$$

Of course we have $M\geqq 1$.

For the value $r=1/(4M)$ we have

$$r-\max_{|z|=r}|f(z)-z| \geqq r-\frac{Mr^2}{1-r} \geqq r-\frac{4Mr^2}{3}=\frac{1}{6M}=\phi(M). \tag{7}$$

Thus on the circle $|z|=r$ we have

$$\left|\frac{f(z)-z}{z-w}\right|<1$$

for $|w|<\phi(M)$, so by Rouché's theorem $f(z)$ takes on the value w precisely as many times in $|z|<r$ as does the function z, namely once. That is to say, the map $w=f(z)$, $|z|<1/(4M)$, covers smoothly the circle $|w|<\phi(M)=1/(6M)$. Thus we have

$$D_1(0) \geqq 1/6M. \tag{8}$$

Given an arbitrary z_0, with $|z_0|<1$, the case $f'(z_0)=0$ trivially implies (4), with each member zero; in the case $f'(z_0)\neq 0$ we construct the frequently used auxiliary function

$$\psi(z)=\frac{f((z+z_0)/(1+\bar{z}_0 z))-f(z_0)}{f'(z_0)(1-|z_0|^2)}. \tag{9}$$

We have $\psi(0)=0$, $\psi'(0)=1$. Since the transformation $z'=(z+z_0)/(1+\bar{z}_0 z)$ maps $|z|<1$ smoothly onto $|z'|<1$, it is clear that the Riemann configuration which is the image of $|z|<1$ under the transformation $w=f(z')$ is the same as the image of $|z|<1$ under the transformation $w=f(z)$; consequently $\psi(z)$ is of class $K_{M'}$, with

$$M'=\frac{M}{|f'(z_0)|(1-|z_0|^2)}.$$

By inequality (8) we may now write

$$D_1(w_0)\geqq\frac{|f'(z_0)|(1-|z_0|^2)}{6M},$$

where $D_1(w_0)$ here refers to the image of $|z|<1$ under the transformation $w=\psi(z)$. By equation (9) we obviously have

$$D_1(w_0)\big|_{w=\psi(z)}=D_1(w_0)\big|_{w=f(z)}+|f'(z_0)|(1-|z_0|^2),$$

which implies the second inequality in (4) and completes the proof of Theorem 1.

The order 1/2 as $D_1(w_0)$ in (4) approaches zero cannot be improved; compare loc. cit. §10.

This method just used yields a slightly more favorable inequality than (4). Series (5) can be replaced by

$$z+\frac{Mz^2}{2^{1/2}}+\frac{Mz^3}{2^{1/2}}+\frac{Mz^4}{2^{1/2}}+\cdots=z+\frac{Mz^2}{2^{1/2}(1-z)}. \tag{10}$$

Then the latter part of (7) can be replaced by

$$r-\frac{Mr^2}{2^{1/2}(1-r)}\geqq r-\frac{4Mr^2}{3\cdot 2^{1/2}}=\frac{6-2^{1/2}}{24M}=\phi(M).$$

The second member of (8) is $\phi(M)$, so in (4) the number 6 may be replaced by

$$\frac{24}{6-2^{1/2}}=\frac{72+12\cdot 2^{1/2}}{17}=5.24.$$

We remark too that in (10) the number M can be replaced by $(M^2-1)^{1/2}$ with a slight improvement in (4).

In extending Theorem 1 to higher derivatives, we shall prove the following theorem.

THEOREM 2. *If the function* $f(z)=\sum_1^\infty a_nz^n$ *is of class* K_M, *then we have*

$$D_p(0)\leqq\sum_1^p|a_n|\leqq 2^p\cdot A_p(M\log 2)^{p/(p+1)}[D_p(0)]^{1/(p+1)}, \tag{11}$$

$$A_p=(p+1)[(p+1)/p]^{p/(p+1)}(4p)^p.$$

The first inequality in (11) is established by Loomis (loc. cit.), a sharpening of an inequality due to Seidel and Walsh. To establish the second inequality we write for $|z_0| < 1$, with the double integral taken over $|z - z_0| < 1 - |z_0|$,

$$(12) \qquad |f'(z_0)|^2 = \frac{1}{\pi(1 - |z_0|)^2}\left|\iint [f'(z)]^2 dS\right| \leq \frac{M^2}{(1 - |z_0|)^2}.$$

If we assume $f(0) = 0$, which here involves no loss of generality, we have

$$(13) \qquad |f(z)| = \left|\int_0^z f'(t)dt\right| \leq \int_0^r \frac{M\,dt}{1 - t} = -M \log (1 - r), \qquad |z| = r < 1.$$

The function $F(z) \equiv f(z/2) \equiv a_1(z/2) + a_2(z/2)^2 + \cdots$ is analytic in the circle $|z| < 1$ and by (13) is of modulus not greater than $M \log 2$ there. Consequently we have (Loomis, loc. cit.)

$$(14) \qquad \frac{|a_1|}{2} + \frac{|a_2|}{2^2} + \cdots + \frac{|a_p|}{2^p} \leq A_p(M \log 2)^{p/(p+1)}[D_p(0)]^{1/(p+1)},$$

where $D_p(0)$ refers to the largest m-sheeted circle ($m \leq p$) with center O and contained in the image of $|z| < 1$ under the transformation $w = F(z)$. Inequality (14), valid where $D_p(0)$ refers to the image of $|z| < 1/2$ under the transformation $w = f(z)$, is a fortiori valid if $D_p(0)$ refers to the image of $|z| < 1$ under the transformation $w = f(z)$, and (11) follows at once.

If inequality (11) is rewritten so as to apply to the function

$$g(\zeta) = f\left(\frac{\zeta + z}{1 + \zeta z}\right), \qquad |z| < 1, \ |\zeta| < 1,$$

it appears (Siedel and Walsh, loc. cit. Chapter I, Lemma 2) that $D_p(w_n) \to 0$ is a necessary and sufficient condition for the relations

$$f^{(m)}(z_n)(1 - |z_n|^2)^m \to 0, \qquad m = 1, 2, \cdots, p.$$

For functions $f(z)$ of class K_M, this remark establishes the relation $D_p(w_n) \to 0$ as a consequence of $w_n = f(z_n)$, $|z_n| \to 1$, by virtue of the following theorem:

THEOREM 3. *If $f(z)$ belongs to K_M, then we have for every k*

$$(15) \qquad \lim_{|z| \to 1} f^{(k)}(z)(1 - |z|^2)^k = 0;$$

the approach is uniform in the sense that if $\epsilon>0$ *is given, there exists* δ_ϵ *such that* $1-|z|<\delta_\epsilon$ *implies* $|f^{(k)}(z)(1-|z|^2)^k|<\epsilon$.

Of course the uniformity of the approach is a consequence of the existence of the limit. We introduce the notation

$$\iint_{r<|z|<1} |f'(z)|^2 dS = \Phi(r), \tag{16}$$

whence $\lim_{r\to 1}\Phi(r)=0$ monotonically. If ρ $(>1/2)$ is arbitrary, $\rho<|z_0|<1$, we choose $r\leqq 2\rho-1$, and the integral in (12) is in absolute value not greater than $\Phi(r)$, so we have

$$|f'(z_0)|^2 \leqq \frac{\Phi(r)}{\pi(1-|z_0|)^2},$$

from which (15) follows uniformly, for the value $k=1$.

We now take $|z|>1/3$, $\rho=(1-|z|)/2$, and use Cauchy's integral formula

$$f'(z)=\frac{1}{2\pi i}\int_{|t-z|=\rho}\frac{f'(t)dt}{t-z}.$$

Successive differentiation yields

$$f^{(k)}(z)=\frac{(k-1)!}{2\pi i}\int_{|t-z|=\rho}\frac{f'(t)dt}{(t-z)^k},$$

whence we have

$$|f^{(k)}(z)| \leqq (k-1)![\max|f'(t)|, \text{ for } |t-z|=\rho]/\rho^{k-1},$$

and by the definition of ρ,

$$(1-|z|)^{k-1}|f^{(k)}(z)| \leqq 2^{k-1}\cdot(k-1)![\max|f'(t)|, \text{for } |t-z|=\rho].$$

We have further $(1-|t|)\geqq(1-|z|)/2$, so we deduce

$$\begin{aligned}&(1-|z|)^k|f^{(k)}(z)|\\ &\qquad\leqq 2^{k-1}(k-1)!(1-|z|)[\max|f'(t)|, \text{ for } |t-z|=\rho]\\ &\qquad\leqq 2^k(k-1)![\max|f'(t)|(1-|t|), \text{ for } |t-z|=\rho],\end{aligned}$$

which implies (15) uniformly and establishes Theorem 3.

It may be noted that we have derived (15) for $k>1$ merely from (15) for $k=1$ as a consequence only of the analyticity of $f(z)$ in $|z|<1$.

For the class K_M we can make no deductions regarding the rapidity of approach to zero in (15):

THEOREM 4. *Let the function $Q(r)$ be defined and positive for $0<r<1$, with $\lim_{r\to 1}Q(r)=0$. Let the positive integer m and the positive number M be given. Then there exists a function $f(z)$ of class K_M, continuous and schlicht for $|z|\leq 1$, and there exists a sequence of points z_k with $0<z_k<1$, $z_k\to 1$, such that we have*

$$\lim_{k\to\infty}\frac{f^{(m)}(z_k)(1-|z_k|^2)^m}{Q(|z_k|)}=\infty.$$

It suffices to choose here a suitable constant multiple of the function $f(z)$ previously constructed (loc. cit. §9).

It is clear from Theorem 1 that the two conditions

$$D_1(w_n)\to 0,\qquad w_n=f(z_n), \tag{17}$$

$$f'(z_n)(1-|z_n|^2)\to 0, \tag{18}$$

for a function $f(z)$ of class K_M are equivalent in the sense that each implies the other. If z_n is a sequence ($|z_n|<1$) which has no limit point in a zero of $f'(z)$ interior to $|z|=1$, it is clear that (18) implies

$$|z_n|\to 1; \tag{19}$$

reciprocally (19) implies (18) by virtue of Theorem 3. That is to say, for a function of class K_M these three conditions are all equivalent, except that (17) and (18) are trivially fulfilled for any sequence or subsequence z_n which approaches a zero of $f'(z)$ interior to $|z|=1$.[4]

Theorem 3 asserts that if $f(z)$ belongs to K_M, then (19) implies (15) for every k. We have remarked that Theorem 2 as applied to the function

$$f\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right)$$

shows that

$$\lim_{n\to\infty}D_p(w_n)=0 \tag{20}$$

is equivalent to (15) for $k=1, 2, \cdots, p$. Thus for a function $f(z)$ of class K_M, the three relations (19), (20), and (15) for $k=1, 2, \cdots, p$ are all equivalent except that the latter two relations are trivially fulfilled for any sequence or subsequence z_n which approaches a common zero of $f^{(k)}(z)$, $k=1, 2, \cdots, p$, interior to $|z|=1$.

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[4] In the study of the equivalence of the relations (17), (18), and (19) for bounded univalent functions (loc. cit. p. 211), the phrase "every point of R_{r_1}" should read "every point of R exterior to R_{r_1}."

In considering the relationships between (17), (18), and (19), it is of interest to study the geometric significance of (17) independently of the class K_M. We shall prove:

THEOREM 5. *Let the function* $w=f(z)$ *be analytic interior to* $|z|<1$, *and map* $|z|<1$ *onto the Riemann configuration* R. *A necessary and sufficient condition that* (19) *imply* (17) *with* $w_n=f(z_n)$ *is that there exist no* δ (>0) *to which correspond an infinity of mutually nonoverlapping smooth circles* $C_1, C_2, \cdots$ *of radius* δ *in* R.

If for some δ (>0) such circles C_n do exist, their respective centers w_n (considered as points of R, not merely as values of the complex variable) have no limit point in R, and the corresponding points z_n have no limit point interior to $|z|<1$; thus (19) is satisfied but not (17), so (19) does not imply (17). Conversely, let us assume that (19) does not imply (17); we suppose (19) satisfied for a particular sequence z_n, but suppose that (17) is not satisfied; we shall show the existence of a suitable δ (>0) and corresponding circles C_n in R. Choose a subsequence of the z_n so that for a suitably chosen δ (>0) the values $D_1(w_n)$ are all greater than or equal to 2δ; we change the notation if necessary so as to have $D_1(w_n)\geqq 2\delta$ for every n. If any subsequence Γ_{n_k} of these circles Γ_n (with respective centers w_n and radii $D_1(w_n)$) have centers w_{n_k} in R which approach as limit a point w_0 of R, then (loc. cit. §15) by the continuity of $D_1(w)$ as a function of w, we have $D_1(w_0)\geqq 2\delta$; consequently we have $w_{n_k}\to w_0$, $z_{n_k}\to z_0$ with $|z_0|<1$, contrary to hypothesis. At most a finite number of the points $w_2, w_3, \cdots$ lie interior to the circle γ_1 whose center is w_1 and radius 2δ; suppose none of the points w_n for $n\geqq N_1$ lies interior to γ_1. At most a finite number of the points w_n for $n>N_1$ lie interior to the circle γ_2 whose center is w_{N_1} and radius 2δ; suppose none of the points w_n for $n\geqq N_2$ $(>N_1)$ lies interior to γ_2. Let γ_3 be the circle whose center is w_{N_2} and radius 2δ. We continue this process indefinitely, and find a sequence of circles $\gamma_1, \gamma_2, \cdots$; all of radius 2δ; the center of γ_k cannot lie interior to any of the circles $\gamma_1, \gamma_2, \cdots, \gamma_{k-1}$. The circles C_k concentric with the circles γ_k and having the common radius δ lie in R and no two of them overlap. The proof is complete.

The condition of Theorem 5 is equivalent to the condition that *there exist no* ϵ (>0) *such that for every* $n=1, 2, 3, \cdots$, *the image* R_n *of the annulus* $1>|z|>1-1/n$ *under the transformation* $w=f(z)$ *contains a smooth circle of radius* ϵ. Let each R_n contain a smooth circle Γ_n of radius ϵ; then R_n contains the concentric circle C_n of radius $\delta=\epsilon/2$. Points of each C_n lie in but a finite number of regions R_k, so there exist among the C_n an infinite number of nonoverlapping smooth circles in R, each of radius δ. Conversely, let there exist an infinity of nonoverlapping smooth circles Γ_n of radius δ (>0) in R; let C_n denote the concentric circles of radius $\epsilon=\delta/2$, and w_n denote the respective centers. Any closed region $R-R_k$ contains points belonging to at most a finite number of the circles C_n, for otherwise there exists a point z_0 with $|z_0|\leqq 1-1/k$ such that every neighborhood of z_0 contains points z corresponding to points w of an infinity of those circles C_n; every neighborhood in R of the point $w_0=f(z_0)$ then contains points of an infinity of the circles C_n, which is impossible. Consequently each R_k contains at least one C_n; the equivalence is established.

As a consequence of Theorem 5 we prove:

COROLLARY 1. *Let the function* $w=f(z)$ *be analytic interior to* $|z|<1$, *and map* $|z|<1$ *onto the Riemann configuration* R. *A necessary and sufficient condition that* (19) *imply* (20) *for each* p *is that there exist no* δ (>0) *to which correspond an infinity of mutually nonoverlapping smooth circles of radius* δ *in* R.

We have the inequality

$$D_1(w)\leqq D_p(w),$$

so if (19) implies (20) for a single value of p, then (19) also implies (17), and by Theorem 5 there exists no δ (>0) to which correspond an infinity of mutually nonoverlapping smooth circles of radius δ in R. Conversely, for given p suppose (19) does not imply (20); we assume (19) satisfied for a particular sequence z_n but assume (20) for a particular p not satisfied; we shall show the existence of a suitable δ and the corresponding circles. By the method of proof of Theorem 5 there exists a δ (>0) and a sequence of m-sheeted circles ($m\leqq p$) $\Gamma_1, \Gamma_2, \cdots$, of radius not less than 2δ, which belong to R and are mutually nonoverlapping. Each circle Γ_j contains at least one smooth circle C_j of radius δ;[5] for an m-sheeted circle has branch points of total multiplicity $m-1$; a branch point of order $m-1$ lies in m sheets, a branch point of order $m-2$ lies in $m-1$ sheets, $\cdots$, a branch point of order 1 lies in 2 sheets; at least two sheets of Γ_j contain but one branch point; and each of these two sheets contains a smooth circle of radius δ. The circles C_j are mutually nonoverlapping, so Corollary 1 is established. If for any function analytic in $|z|<1$ condition (19) implies (17), then (19) also implies (20) for every p.

The condition of Theorem 5 is obviously satisfied by any function of class K_M, and by functions bounded or unbounded of many other types. Moreover we prove the following corollary.

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[5] This fact follows also from a lemma due to Grünwald and Turán as sharpened by Grünwald and Várnonyi; see Acta Univ. Szeged. vol. 8 (1936–1937) pp. 236–240.

COROLLARY 2. *If* $f(z)$ *is analytic and schlicht in* $|z| < 1$, *and maps* $|z| < 1$ *onto a region* R, *then a necessary and sufficient condition for the equivalence of* (17), (18), *and* (19) *is that there exist no* δ (>0) *to which correspond an infinity of mutually nonoverlapping smooth circles* C_n *of radius* δ *in* R.

We have (loc. cit. §4) the inequalities

$$D_1(w_0) \leqq |f'(z_0)|(1 - |z_0|^2) \leqq 4D_1(w_0), \tag{21}$$

from which the equivalence of (17) and (18) follows. The derivative $f'(z)$ is different from zero at all points in $|z| < 1$, so (18) must imply (19). Corollary 2 now follows from Theorem 5.

For an *arbitrary* function $f(z)$ analytic in $|z| < 1$, the existence of $\delta(>0)$ and corresponding mutually nonoverlapping smooth circles C_j in R of radius δ implies

$$\limsup_{|z| \to 1} |f'(z)|(1 - |z|^2) > 0,$$

by the first of inequalities (21); see loc. cit. §4.

HARVARD UNIVERSITY

A NOTE ON THE MINIMUM MODULUS OF A CLASS OF INTEGRAL FUNCTIONS

S. M. SHAH

A well known theorem due to Littlewood, Wiman, and Valiron[1] states that for any integral function of order less than one-half,

$$\log m(r) > (\text{a positive constant}) \log M(r),$$

on a sequence of circles of indefinitely increasing radius. I consider in this note a class of integral functions which have this property and prove the following theorem.

THEOREM 1. *Hypothesis*:

(1) *(R_n) is any sequence of positive numbers such that $R_1 > 1$, $R_n/R_{n-1} \geqq \lambda > 1$.*

(2) *(p_n) is any sequence of positive integers.*

(3) *$a_{11}, a_{12}, \cdots, a_{1p_1}, a_{21}, \cdots, a_{2p_2}, \cdots$ are a set of points such that $0 < |a_{11}| \leqq |a_{12}| \leqq \cdots$ and such that a finite number $a_{n1}, \cdots, a_n p_n$ lie inside the ring $(R_n - R_n^\alpha < |z| < R_n)$ where $0 < \alpha < 1$.*

(4) *μ_n is a sequence of positive integers such that $\sum_1^\infty p_n/\beta^{\mu_n}$ is convergent, β being any constant greater than one.*

(5) *The exponent of convergence of the points*

$$a_{nr} \exp (2\pi i\nu/\mu_n),$$

where $r = 1, 2, \cdots, p_n$; $\nu = 0, 1, 2, \cdots, \mu_n - 1$; $n = 1, 2, 3, \cdots$, is ρ $(0 \leqq \rho < \infty)$.

(6)[2] *Lower bound $\{\mu_n\} \geqq 1 + \rho$.*

Conclusion:

(7) *The canonical product*

$$f(z) = \prod_{n=1}^{\infty} \prod_{s=1}^{p_n} \left\{1 - \frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right\} \tag{8}$$

formed with these points as zeros is of order ρ; and the values of $r = |z|$ for which the inequality

$$m(r, f) > CM(r, f),$$

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[1] G. Valiron, *Lectures on the general theory of integral functions*, pp. 128–130.

[2] It is possible to choose R_n, p_n, and so on, satisfying the conditions (1) to (6). Example: $R_n = 2^{2n}$; $p_n = n^2 2^n$; $\mu_n = 2^n$. Here $\rho = 1$.

where $C = C(\lambda, \epsilon) > 0$, *is satisfied form a set of upper density greater than* $1 - 1/\lambda - \epsilon$.

THEOREM 2. *If* (1), (2), (3), (4), (5), *and* (6) *hold and if* $\rho > 0$ *and if further*[2]

$$\sum_{n=1}^{N} \mu_n p_n / R_N^{\rho} \to \infty \qquad \text{with } N \to \infty \tag{9}$$

then

$$\limsup_{r\to\infty} \log m(r, f)/r^{\rho} = \infty,$$

where f is the canonical product (8); *and the values of r for which* $\log m(r, f) > \Delta r^{\rho}$ *where* Δ *is any arbitrarily large constant form a set of upper density greater than* $1 - 1/\lambda - \epsilon$.

THEOREM 3. *Hypothesis: Let* $\rho > 0$ *be nonintegral and* (1), (2), (3), (4), *and* (5) *hold.*[3]

Conclusion:

(10) *Any integral function of order* ρ *with exactly these zeros will be of the form*

$$F(z) = e^{g(z)} P(z) \prod_{n=n_1}^{\infty} \prod_{r=1}^{p_n} \left\{1 - \frac{z^{\mu_n}}{a_{nr}^{\mu_n}}\right\}, \tag{11}$$

where $g(z)$ *is a polynomial of degree not exceeding* ρ, $P(z)$ *a polynomial;*[4] *and the values of r for which*

$$\log m(r, F) > (1 - \epsilon) \log M(r, F)$$

holds will form a set of upper density greater than $1 - 1/\lambda - \epsilon$.

THEOREM 4. *If* $\rho > 0$ *and* (1), (2), (3), (4), (5), *and* (9) *hold*[5] *then conclusion* (10) *holds.*

THEOREM 5. *If* (1), (2), (3), (4), (5), *and* (6) *hold and if* $m_\sigma(r)$ *and* $M_\sigma(r)$ *denote the lower and upper bounds of* $|f(z)|$, *where* $f(z)$ *is the canonical product* (8), *of order* ρ $(0 \leqq \rho < \infty)$ *in the annulus* $r \leqq |z| \leqq r + r^{\sigma}$ $(\sigma < 1 - \rho)$ *then the values of r for which*[6]

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[3] For instance $R_n = 2^{2n}$, $p_n = n$; $\mu_n = 2^{7(n-1)}$. Here $\rho = 7/2$.

[4] $P(z)$ is a polynomial having zeros at points $a_{nr} \exp(2\pi i\nu/\mu_n)$, $r = 1, 2, \cdots, p_n$; $\nu = 0, 1, 2, \cdots, \mu_n - 1$ and $n = 1, 2, \cdots, n_1 - 1$ only.

[5] See footnotes 2 and 3.

[6] For a number of results on the flat regions of integral functions, see J. M. Whittaker, *A property of integral functions of finite order*, Quart. J. Math. Oxford Ser. vol. 2 (1931) pp. 252–258; B. J. Maitland, *The flat regions of integral functions of finite order*, ibid. vol. 15 (1944) pp. 84–96; and the references mentioned in the paper of Maitland.

$$m_e(r) > C_1 M_e(r),$$

where[7] $C_1 = C_1(\lambda, \epsilon) > 0$, *holds form a set of upper density greater than* $1 - 1/\lambda - \epsilon$.

PROOF OF THEOREM 1. Let $|z| = R = \lambda^{\gamma} R_k$ $(0 < \gamma < 1)$, where k is so large that

$$\lambda^{\gamma} R_k < R_{k+1} - R^{\alpha}_{k+1},$$

$f(z) = P_1 P_2$, where

$$P_1 = \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{1 - \frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right\},$$

$$P_2 = \prod_{n=k+1}^{\infty} \prod_{s=1}^{p_n} \left\{1 - \frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right\},$$

$$\begin{aligned} |P_1| &\leqq \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{1 + \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}\right\} \\ &= \left(\prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \right) \left(\prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{1 + \frac{|a_{ns}|^{\mu_n}}{R^{\mu_n}}\right\} \right) \\ &= P_{11} P_{12}, \end{aligned}$$

say. Now $|a_{ns}| < R_n$,

$$|P_{12}| \leqq \prod_{1}^{k} \left\{1 + \left(\frac{R_n}{R}\right)^{\mu_n}\right\}^{p_n},$$

and $R_n/R \leqq 1/\lambda^{\gamma} < 1$ for $n = 1, 2, \cdots, k$, and $\sum p_n/\lambda^{\gamma \mu_n}$ is convergent. Hence

$$|P_{12}| \leqq C_2,$$

$$|P_2| \leqq \prod_{n=k+1}^{\infty} \prod_{s=1}^{p_n} \left\{1 + \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}\right\},$$

where $|a_{ns}| \geqq |a_{k+1,s}| \geqq R_{k+1} - R^{\alpha}_{k+1}$,

$$\frac{R}{|a_{ns}|} \leqq \frac{R}{R_{k+1} - R^{\alpha}_{k+1}} \sim \frac{\lambda^{\gamma} R_k}{R_{k+1}} \leqq \frac{1}{\lambda^{1-\gamma}},$$

and $\sum p_n/\lambda^{(1-\gamma)\mu_n}$ is convergent. Hence

$$|P_2| \leqq C_3$$

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[7] $C, C_1, C_2, \cdots$ denote finite positive (nonzero) constants.

and so

$$M(R) \leqq C_2C_3 \prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}.$$

Further

$$\begin{aligned} |P_1| &= \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left| 1 - \frac{z^{\mu_n}}{a_{ns}^{\mu_n}} \right| \\ &\geqq \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{ \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} - 1 \right\} \\ &\geqq \left(\prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \right) \left(\prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{ 1 - \frac{|a_{ns}|^{\mu_n}}{R^{\mu_n}} \right\} \right) \\ &= P_{11}P_{14} \end{aligned}$$

say. Since $\sum p_n/\lambda^{\gamma\mu_n}$ is convergent and

$$|P_2| \geqq \prod_{n=k+1}^{\infty} \prod_{s=1}^{p_n} \left\{ 1 - \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \right\} \geqq C_5,$$

$$m(R) \geqq C_4C_5 \prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \cdots, \tag{12}$$

which gives that $m(R) \geqq C_6M(R)$ where $C_6 = C_6(\lambda, \gamma)$. Now given $\epsilon > 0$ let $\epsilon_1 = \epsilon\lambda^2/(\lambda+1+\epsilon\lambda)$. Writing $\lambda^\gamma = \theta$ and $R = \theta R_k$, where $1+\epsilon_1 \leqq \theta \leqq \lambda - \epsilon_1$ and $k \geqq K$, K being so large that $R_K(\lambda - \epsilon_1) < R_{K+1} - R_{K+1}^{\alpha}$, we get $m(R) \geqq C(\lambda, \epsilon)M(R)$. This inequality holds good over a set of upper density greater than

$$\frac{(\lambda - \epsilon_1) - (1 + \epsilon_1)}{\lambda - \epsilon_1} = 1 - \frac{1}{\lambda} - \epsilon.$$

Proof of Theorem 2. We know from (12) that $m(R, f) \geqq C_4C_5X$, where

$$X = \prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \geqq \lambda^{(\gamma \sum_1^k \mu_n p_n)}$$

$$\begin{aligned} \log m(R, f) &\geqq \log (C_4C_5) + \log X \geqq \log (C_4C_5) + \gamma \log \lambda \left(\sum_1^k \mu_n p_n \right) \\ &> \Delta R^\rho \qquad \text{for all large } R. \end{aligned}$$

Hence $\limsup_{r \to \infty} \log m(r, f)/r^\rho = \infty$. Further, the values of r for which $\log m(r, f) > \Delta r^\rho$ form a set of upper density greater than $1 - 1/\lambda - \epsilon$.

PROOF OF THEOREM 3. Given $\epsilon>0$, let $\epsilon_2=\epsilon/(2-\epsilon)$. Since

$$\sum \mu_n p_n/(R_n - R_n^{\alpha})^{\rho-\epsilon_3}$$

is divergent we have

$$\mu_n p_n \geqq R_n^{\rho-\epsilon_4} \qquad \text{or } n = k_1, k_2.$$

Let $|z|=R=\lambda^{\gamma}R_k$ $(0<\gamma<1$ and $1+\epsilon_1\leqq\lambda^{\gamma}\leqq\lambda-\epsilon_1)$, where k takes the values $k_1, k_2, \cdots$. If $X=\prod_{n=n_1}^{k}\prod_{s=1}^{p_n}R^{\mu_n}/|a_{ns}|^{\mu_n}$ then $X\geqq\exp\{\gamma\log\lambda\sum_{n_1}^{k}\mu_n p_n\}$ and so $\log X\geqq C_6\sum_{n_1}^{k}\mu_n p_n\geqq C_6 R_k^{\rho-\epsilon_4}=C_7R^{\rho-\epsilon_4}$. Choosing k and hence R sufficiently large we have, as in Theorem 1,

$$\begin{aligned} m(R, F) &> C_8 \exp\{\log X - C_9 R^{[\rho]}\},\\ \log m(R, F) &> \log C_8 + \log X - C_9 R^{[\rho]}\\ &> (1-\epsilon_2)\log X. \end{aligned}$$

Similarly $\log M(R, F)<(1+\epsilon_2)\log X$ which gives

$$\frac{\log m(R, F)}{\log M(R, F)} > \frac{1-\epsilon_2}{1+\epsilon_2} = 1-\epsilon.$$

As in Theorem 1, this result holds for values of R forming a set of upper density greater than $1-1/\lambda-\epsilon$.

Theorem 4 can be similarly proved.

PROOF OF THEOREM 5. We know that for $|z|=R=\lambda^{\gamma}R_k$ $(0<\gamma<1$, $1+\epsilon_1\leqq\lambda^{\gamma}\leqq\lambda-\epsilon_1)$

$$m(R, f) \geqq C_4C_5\prod_{n=1}^{k}\prod_{s=1}^{p_n}\frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}.$$

We can choose k so large that $R'=R+R^{\sigma}<\lambda^{\gamma+\epsilon_5}R_k$, where $\gamma+\epsilon_5<1$,

$$R' < R_{k+1} - R_{k+1}^{\alpha}.$$

Now

$$M(R', f) < C_{10}\prod_{n=1}^{k}\prod_{s=1}^{p_n}\frac{R'^{\mu_n}}{|a_{ns}|^{\mu_n}}$$

and therefore

$$\frac{m(R, f)}{M(R', f)} > \frac{C_4C_5}{C_{10}}\left(\frac{R}{R'}\right)^{\sum_{n=1}^{k}\mu_n p_n}.$$

Now $Y=(R'/R)^{-\sum_1^k\mu_n p_n}=(1+R^{\sigma-1})^{-\sum_1^k\mu_n p_n}$. Further $\sum_1^k\mu_n p_n<(C_{11}\log R)R^{\rho+\epsilon_6}<R^{\rho+\epsilon_7}$ for all large R. Hence $Y>\exp\{-R^{\rho+\epsilon_7}$

$\log(1+R^{\sigma-1})\}$ and $R^{\rho+\epsilon_7}\log(1+R^{\sigma-1})\sim R^{\rho+\epsilon_7+\sigma-1}\to 0$ as $R\to\infty$, since $\sigma<1-\rho$ and ϵ_7 can be chosen so small that $\sigma<1-\rho-\epsilon_7$. Hence $Y>1/2$ for all large R and so

$$\frac{m(R,f)}{M(R',f)}>\frac{C_4C_5}{2C_{10}}.$$

Further

$$\frac{m(R',f)}{M(R',f)}>C_{11}.$$

Hence

$$\frac{m_r(R)}{M_r(R)}=\min\left\{\frac{m(R)}{M(R')},\ \frac{m(R')}{M(R')}\right\}\geqq\min\left\{\frac{C_4C_5}{2C_{10}},\ C_{11}\right\}\geqq C_1.$$

The values of R for which this result holds form a set of upper density greater than $1-1/\lambda-\epsilon$.

Added in proof. The positive numbers ϵ and ϵ_4 are chosen so small that

$$1/\lambda+\epsilon<1;\qquad [\rho]+\epsilon_4<\rho.$$

In the proof of Theorem 1 we showed that

$$M(R)\leqq C_2C_3P_{11};\qquad m(R)\geqq C_4C_5P_{11},$$

both relations holding for all R such that

$$(1+\epsilon_1)R_k\leqq R\leqq(\lambda-\epsilon_1)R_k\qquad (k>K).$$

Here

$$C_2=\prod_{n=1}^{\infty}\left\{1+\left(\frac{1}{1+\epsilon_1}\right)^{k_n}\right\}^{p_n},\ C_4=\prod_{n=1}^{\infty}\left\{1-\left(\frac{1}{1+\epsilon_1}\right)^{k_n}\right\}^{p_n},$$

$$C_3=\prod_{n=1}^{\infty}\left\{1+\left(1-\frac{\epsilon_1}{2\lambda}\right)^{k_n}\right\}^{p_n},\ C_5=\prod_{n=1}^{\infty}\left\{1-\left(1-\frac{\epsilon_1}{2\lambda}\right)^{k_n}\right\}^{p_n}.$$

If $C=C_4C_5/C_2C_3$ we have

$$m(R)\geqq CM(R),$$

the inequality holding over a set of upper density greater than $1-1/\lambda-\epsilon$. If we further suppose that $\lambda=R_n/R_{n-1}$ $(n=2,3,\cdots)$, then this inequality holds good over a set of upper density greater than $1-\lambda\epsilon(1+\epsilon)/(\lambda-1)$.

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MEAN VALUES OF PERIODIC FUNCTIONS

PAUL CIVIN

Let L^p denote the class of complex measurable functions of period 2π for which $M_p(f) < \infty$, where

$$M_p(f) = \left(\int_0^{2\pi} |f(x)|^p dx \right)^{1/p} \qquad (1 \leqq p < \infty), \tag{1}$$

$$M_\infty(f) = \operatorname*{ess.\,sup}_{0 \leqq x \leqq 2\pi} |f(x)|. \tag{2}$$

Let $K_{m,p}$ denote the subset of L^p whose elements, $f(x)$, have a Fourier series of the form

$$\sum_{n=m}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (m \geqq 1). \tag{3}$$

The functions of $K_{m,p}$ and their Fourier series (3) are transformed by the real number δ and the sequence of real numbers $\lambda = \{\lambda(n)\}$ into the series

$$\begin{aligned} &\sum_{n=m}^{\infty} \lambda(n) \left\{ a_n \cos\left(nx + \frac{\delta\pi}{2}\right) + b_n \sin\left(nx + \frac{\delta\pi}{2}\right) \right\} \\ &= \sum_{n=m}^{\infty} \lambda(n) \left\{ \left(a_n \cos\frac{\delta\pi}{2} + b_n \sin\frac{\delta\pi}{2} \right) \cos nx \right. \\ &\qquad \left. + \left(b_n \cos\frac{\delta\pi}{2} - a_n \sin\frac{\delta\pi}{2} \right) \sin nx \right\}. \end{aligned} \tag{4}$$

A slight modification of the well known result[1] [5, pp. 100 ff.][2] for the case in which $\delta = 0$ shows that if

$$\sum_{n=m}^{\infty} \lambda(n) \cos\left(nx - \frac{\delta\pi}{2}\right) = \sum_{n=m}^{\infty} \lambda(n) \left\{ \cos\frac{\delta\pi}{2} \cos nx + \sin\frac{\delta\pi}{2} \sin nx \right\} \tag{5}$$

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[1] Although the convention is adopted in *Trigonometrical series* that $f(x)$ is real, the results of the sections of *Trigonometrical series* to which reference is made in this note hold for complex $f(x)$.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

is a Fourier or even a Fourier-Stieltjes series then (4) is the Fourier series of a function of L^p, $1 \leqq p \leqq \infty$. Throughout the sequel it is assumed that (5) is a Fourier series. Series (4) is therefore the Fourier series of a function $g(x) \in L^p$ and is of the form of (3), hence $g(x) \in K_{m,p}$. The transformation determined by the number δ and the sequence λ is thus a transformation of $K_{m,p}$ onto itself.

The objective of the present note is to establish an inequality between the means $M_p(f)$ and $M_p(g)$ which holds for all $f(x) \in K_{m,p}$. For the essentially bounded case this has been done by B. v. Sz. Nagy [3], and for completeness the result is stated as a lemma.

LEMMA 1 [Sz. Nagy]. *If* $f(x) \in K_{m,\infty}$, *then* (4) *is the Fourier series of a continuous function* $g(x) \in K_{m,\infty}$ *and*

$$M_\infty(g) \leqq A(\lambda, \delta, m) M_\infty(f), \tag{6}$$

where $A(\lambda, \delta, m)$ *is a function only of the indicated variables and not of the particular* $f(x) \in K_{m,\infty}$.

The notation $A(\lambda, \delta, m)$ will be used throughout the sequel to denote the smallest possible function which will satisfy (6) for all $f(x) \in K_{m,\infty}$.

LEMMA 2. *If* $f(x) \in K_{m,2}$, *then* (4) *is the Fourier series of a function* $g(x) \in K_{m,2}$ *and*

$$M_2(g) \leqq \Lambda(m) M_2(f), \tag{7}$$

where $\Lambda(m) = \max_{m \leqq n} |\lambda(n)|$.

The Riesz-Fischer theorem asserts that (4) is the Fourier series of a function $g(x) \in L^2$ and that

$$\begin{aligned} M_2(g) &= \left\{\pi \sum_{n=m}^{\infty} (\lambda(n))^2 (|a_n|^2 + |b_n|^2)\right\}^{1/2} \\ &\leqq \Lambda(m) \left\{\pi \sum_{n=m}^{\infty} (|a_n|^2 + |b_n|^2)\right\}^{1/2} = \Lambda(m) M_2(f). \end{aligned}$$

If $\Lambda(m) = |\lambda(r)|$ $(r \geqq m)$, then $f(x) = \cos rx$ gives equality in (7).

It is now possible to state the principal theorem.

THEOREM 1. *If* $f(x) \in K_{m,p}$, *then* (4) *is the Fourier series of a function* $g(x) \in K_{m,p}$ *and*

$$M_p(g) \leqq \Lambda^{2/p}(m) A^{(p-2)/p}(\lambda, \delta, m) M_p(f) \qquad (2 \leqq p < \infty), \tag{8}$$

$$M_p(g) \leqq B_p \Lambda^{2/p'}(m) A^{(p'-2)/p'}(\lambda, -\delta, m) M_p(f) \qquad (1 < p \leqq 2) \tag{9}$$

where $\Lambda(m)$ *and* $A(\lambda, \delta, m)$ *are defined in Lemmas* 1 *and* 2, B_p *is a constant depending only on* p *and not on the particular* $f(x) \in K_{m,p}$, *and* $p' = p/(p-1)$.

The transformation of $f(x)$ into $g(x)$, or of series (3) into series (4), is a linear transformation of $K_{m,2}$ onto itself, and also of $K_{m,\infty}$ onto itself. The direct application of an interpolation scheme for L^p fails in the attempt to establish (8) since the space $K_{m,p}$ is a nondense linear subspace of L^p. However, the proof of the interpolation result for L^p as given in *Trigonometrical series* [5, p. 198 ff.] carries through for the space $K_{m,p}$ on the basis of the following lemma.

LEMMA 3. *The step functions of* $K_{m,p}$ *are dense in* $K_{m,p}$ *in the metric of* L^p $(1 < p < \infty)$.

The step functions of $K_{m,p}$ are those functions of $K_{m,p}$ which assume only a finite number of values and assume each of these values on a finite sum of intervals in $(0, 2\pi)$. Suppose $f(x) \in K_{m,p}$ and η is a positive number. The density of the continuous functions of L^p requires the existence of a continuous function $h(x) \sim c_0/2 + \sum_{n=1}^{\infty}(c_n \cos nx + d_n \sin nx)$ such that $M_p(f-h) < \eta$ and $|c_0|/2 + \sum_{n=1}^{m-1}(|c_n| + |d_n|) < \eta$. The function $k(x) = h(x) - c_0/2 - \sum_{n=1}^{m-1}(c_n \cos nx + d_n \sin nx)$ is therefore a continuous function of class $K_{m,p}$ and $M_p(f-k) \leq M_p(f-h) + \eta(2\pi)^{1/p} < 8\eta$. Hence the continuous functions of $K_{m,p}$ are dense in $K_{m,p}$. It is sufficient therefore to show that the continuous functions of L^p can be approximated uniformly by step functions of $K_{m,p}$.

Consider first a continuous $f(x) \in K_{1,p}$. For any positive η, there is a step function $s(x)$ such that $|f(x) - s(x)| \leq \eta$ for all x. If $c = (1/2\pi)\int_0^{2\pi} s(x)dx$, then since $\int_0^{2\pi} f(x)dx = 0$, $|c| \leq (1/2\pi)\int_0^{2\pi} |s(x) - f(x)| dx \leq \eta$. The step function $t(x) = s(x) - c$ is therefore in $K_{1,p}$ and $|f(x) - t(x)| \leq 2\eta$.

Suppose next that it has been demonstrated that the continuous functions of $K_{r,p}$ can be uniformly approximated by step functions of $K_{r,p}$ for $1 \leq r < m$. Since $K_{r,p} \supset K_{s,p}$ if $r < s$, any continuous function of $K_{m,p}$ can be uniformly approximated by step functions whose Fourier coefficients of order less than $(m-1)$ vanish. Hence if $f(x)$ is a continuous function of $K_{m,p}$ and η is a positive number, there is a step function $s(x) \in K_{m-1,p}$ such that $|f(x) - s(x)| < \eta$ for all x. Suppose that $c = (1/4)\int_0^{2\pi} s(x) \cos (m-1)x dx$ and $d = (1/4)\int_0^{2\pi} s(x) \sin (m-1)x dx$. Since $f(x) \in K_{m,p}$, both $|c| < 2\eta$ and $|d| < 2\eta$. Suppose the function $t(x) = s(x) - c \operatorname{sgn} \cos (m-1)x - d \operatorname{sgn} \sin (m-1)x$, where $\operatorname{sgn} u = 0$ if $u = 0$ and $\operatorname{sgn} u = u/|u|$ if $u \neq 0$. It can be shown by direct calculation that the step function $t(x) \in K_{m,p}$. Since $|f(x) - t(x)| \leq |f(x) - s(x)|$

$+|c \operatorname{sgn} \cos (m-1)x| + |d \operatorname{sgn} \sin (m-1)x| < 5\eta$, the function $t(x)$ gives the desired uniform approximation.

In order to establish (9), it is first noted [5, p. 105] that since $g(x) \in L^p$,

$$M_p(g) = \sup \left| \int_0^{2\pi} g(x)\overline{h(x)}dx \right| \tag{10}$$

with the supremum taken over all $h(x)$ for which $M_{p'}(h) \leqq 1$. Hence if η is a positive number there is an $h(x)$ for which

$$M_{p'}(h) \leqq 1$$

and

$$M_p(g) - \eta \leqq \left| \int_0^{2\pi} g(x)\overline{h(x)}dx \right| . \tag{11}$$

Suppose that $h(x) \sim r_0/2 + \sum_{n=1}^{\infty}(r_n \cos nx + s_n \sin nx)$ and that $h_m(x) \sim \sum_{n=m}^{\infty}(r_n \cos nx + s_n \sin nx)$. A double application of Parseval's relation for functions of L^p and $L^{p'}$ shows that

$$\begin{aligned} \int_0^{2\pi} g(x)\overline{h(x)}dx &= \pi \sum_{n=m}^{\infty} \left\{ \lambda(n) \left(a_n \cos \frac{\delta\pi}{2} + b_n \sin \frac{\delta\pi}{2} \right) \overline{r_n} \right. \\ &\qquad \left. + \lambda(n) \left(b_n \cos \frac{\delta\pi}{2} - a_n \sin \frac{\delta\pi}{2} \right) \overline{s_n} \right\} \\ &= \pi \sum_{n=m}^{\infty} \left\{ \lambda(n) \left(\overline{r_n} \cos \frac{\delta\pi}{2} - \overline{s_n} \sin \frac{\delta\pi}{2} \right) a_n \right. \\ &\qquad \left. + \lambda(n) \left(\overline{s_n} \cos \frac{\delta\pi}{2} + \overline{r_n} \sin \frac{\delta\pi}{2} \right) b_n \right\} \\ &= \int_0^{2\pi} \overline{H(x)} f(x) dx, \end{aligned} \tag{12}$$

where

$$\begin{aligned} H(x) &\sim \sum_{n=m}^{\infty} \lambda(n) \left\{ \left(r_n \cos \frac{\delta\pi}{2} - s_n \sin \frac{\delta\pi}{2} \right) \cos nx \right. \\ &\qquad \left. + \left(s_n \cos \frac{\delta\pi}{2} + r_n \sin \frac{\delta\pi}{2} \right) \sin nx \right\} \\ &= \sum_{n=m}^{\infty} \lambda(n) \left\{ r_n \cos \left(nx - \frac{\delta\pi}{2} \right) + s_n \sin \left(nx - \frac{\delta\pi}{2} \right) \right\}. \end{aligned}$$

Thus $H(x)$ is the transform of $h_m(x)$ which is obtained by use of the number $-\delta$ and the sequence λ. The application of Hölder's inequality followed by the use of (8) then shows that

$$\left|\int_0^{2\pi}\overline{H(x)}f(x)dx\right| \leqq M_{p'}(H)M_p(f) \tag{13}$$

$$\leqq M_p(f)\Lambda^{2/p'}(m)A^{(p'-2)/p'}(\lambda, -\delta, m)M_{p'}(h_m).$$

A well known result of M. Riesz [1] implies that

$$M_{p'}(h_m) \leqq B_p M_{p'}(h) \tag{14}$$

where B_p depends only on p and not on the functions involved. The combination of formulas (10) through (14) then shows that

$$M_p(g) - \eta \leqq B_p\Lambda^{2/p'}(m)A^{(p'-2)/p'}(\lambda, -\delta, m)M_p(f)$$

and (9) follows since η was arbitrary.

The result of Theorem 1 will now be applied to integrals of functions of $K_{m,p}$. It is convenient to use the definition of the integral of order α which is due to Weyl [4]. For any positive α, for $f(x)\in K_{m,p}$ and with Fourier series (3), the integral of order α, $f_\alpha(x)$, is defined as

$$f_\alpha(x) = \sum_{n=m}^{\infty}\frac{1}{n^\alpha}\left\{a_n\cos\left(nx - \frac{\alpha\pi}{2}\right) + b_n\sin\left(nx - \frac{\alpha\pi}{2}\right)\right\}.$$

Thus $f_\alpha(x)$ is the transform of $f(x)$ of the type of (4) with $\delta = -\alpha$ and the sequence $\{\lambda(n)\} = \{n^{-\alpha}\}$. Various results are known concerning the relationship between $M_\infty(f_\alpha)$ and $M_\infty(f)$, the most inclusive of which is that of Sz. Nagy [3] who shows that

$$A(\lambda, \delta, m) = A(\{n^{-\alpha}\}, -\alpha, m)$$

$$\leqq (4/\pi m^\alpha)\left\{\left|\cos\frac{\alpha\pi}{2}\right|\sum_{v=0}^{\infty}(-1)^v(2v+1)^{-(1+\alpha)}\right.$$

$$\left. + \left|\sin\frac{\alpha\pi}{2}\right|\sum_{v=0}^{\infty}(2v+1)^{-(1+\alpha)}\right\}$$

$$\leqq (4/\pi m^\alpha)H(\alpha).$$

It can also be seen from [3] that

$$A(\{n^{-\alpha}\}, \alpha, n) \leqq (4/\pi m^\alpha)H(\alpha).$$

A direct application of Theorem 1 yields the following theorem.

THEOREM 2. *If $f(x)\in K_{m,p}$, and $f_\alpha(x)$ is its integral of order α (α not necessarily integral) then*

$$
(15)\qquad \begin{aligned} M_p(f_\alpha) &\leqq m^{-\alpha}(4H(\alpha)/\pi)^{(p-2)/p}M_p(f) && (2 \leqq p),\\ M_p(f_\alpha) &\leqq B_p m^{-\alpha}(4H(\alpha)/\pi)^{(2-p)/p}M_p(f) && (1 < p \leqq 2) \end{aligned}
$$

where B_p *is the constant of Theorem* 1 *and*

$$
H(\alpha) = \left|\cos\frac{\alpha\pi}{2}\right| \sum_{\nu=0}^{\infty} (-1)^\nu(2\nu+1)^{-(1+\alpha)} + \left|\sin\frac{\alpha\pi}{2}\right| \sum_{\nu=0}^{\infty} (2\nu+1)^{-(1+\alpha)}.
$$

A result of Schmidt [2] shows that for the real functions of $K_{1,p}$ and α integral the coefficient of $M_p(f)$ in (15) is not the best possible.

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SOME ASYMPTOTIC FORMULAS FOR MULTIPLICATIVE FUNCTIONS

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The present paper contains several asymptotic formulas for the sum of multiplicative functions. A function $f(n)$ is called multiplicative if $f(a \cdot b) = f(a) \cdot f(b)$ for $(a, b) = 1$. We assume $f(n) > 0$. In this paper $f(n)$, $f_1(n)$ will always denote multiplicative functions. First we prove the following theorem.

THEOREM 1.[1] *Assume that the two series*

$$(1) \qquad -\sum_{p,\alpha} \frac{f(p^\alpha) - 1}{p^\alpha}, \qquad \sum_{p,\alpha} \frac{(f(p^\alpha) - 1)^2}{p^\alpha},$$

converge; then $f(n)$ has a mean value, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum f(m)$$

exists and is not equal to zero.

This result was conjectured in a slightly more special form at the end of my paper *Some remarks on additive and multiplicative functions.*[2]

REMARK. The convergence of (1) is the necessary and sufficient condition for the existence of the distribution function of $f(n)$.[3]

For the sake of simplicity we assume $f(p^\alpha) = f(p)$. Then we prove

$$(2) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{m \leq n} f(m) = \prod_p \left(1 + \frac{f(p) - 1}{p}\right).$$

It easily follows from (1) that the product on the right side of (2) converges and thus the value of the limit is not 0.

We easily obtain from (1) that for every $\epsilon > 0$

$$\sum_{|f(p)-1|>\epsilon} \frac{1}{p} < \infty.$$

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[1] This result generalizes a result of Wintner, Amer. J. Math. vol. 67 (1945) pp. 481–485.

[2] Bull. Amer. Math. Soc. vol. 52 (1946) pp. 527–537.

[3] P. Erdős and A. Wintner, Amer. J. Math. vol. 61 (1939) pp. 713–721. See also footnote 2.

Thus by arguments used in previous papers[4] we can assume, for the sake of simplicity (without loss of generality), that $f(p)\to 1$ as $p\to\infty$.

Define

$$f_1(m) = \prod_{p|m,\, p\le p_k} f(p).$$

Also $M(x, n)$ and $M_k(x, n)$ (x large) denote the number of integers $m\le n$ for which $f(m)\ge x$, $f_k(m)\ge x$, respectively. Further let

$$A(m) = \prod_{p^\alpha\|m}{}' p^\alpha, \qquad B(m) = \prod_{p^\alpha\|m}{}'' p^\alpha \qquad (m = A(m)\cdot B(m)).$$

Here $p^\alpha\|m$ means that $p^\alpha|m$ and $p^{\alpha+1}\nmid m$, and the prime denotes that the product is extended over the $p\le n^{1/x^{10}}$, and the double prime denotes that the product is extended over the $p>n^{1/x^{10}}$.

First we have to prove some lemmas.

Lemma 1. *The number N of integers $m\le n$ with*

$$A(m)\ge n^{1/2}$$

is $o(n/x^4)$ (we assume that $x\to\infty$).

In the product $\prod_{m\le n}A(m)$, the prime p occurs as many times as p divides n (and $p\le n^{1/x^{10}}$). Hence,

$$\prod_{m\le n} A(m) < \prod_p{}' p^{n/(p-1)} < \exp\left(\frac{cn\log n}{x^{10}}\right)$$

since $\sum_{p\le n}'\log p/p < c\log x$ (the prime means $p\le n^{1/x^{10}}$). Multiplying together the inequalities $n^{1/2}\le A(m)$ we have

$$n^{N/2}\le \prod_{m\le n} A(m) < \exp\left(\frac{cn\log n}{x^{10}}\right),$$

or

$$N < c\frac{2n}{x^{10}},$$

which proves the lemma.

Lemma 2. *The number N' of integers $m\le n$ with $f(B(m))\ge x^{1/2}$ is $o(n/x^4)$.*

---

[4] P. Erdős, J. London Math. Soc. vol. 10 (1935) p. 124. (By the method used there we can show that without loss of generality we can neglect a sequence of primes p_i with $\sum 1/p_i<\infty$.)

Since $f(p)\to 1$ and all prime factors of $B(m)$ are greater than $n^{1/x^{10}}$, we obtain that if $p\mid B(m)$, $f(p)<1+\epsilon$. Thus if $f(B(m))\geqq x^{1/2}$ we conclude that m has at least $100\log x$ prime factors greater than $n^{1/x^{10}}$. Thus on the one hand

$$\sum_{f(B(m))\geqq x^{1/2}}^{*}\frac{n}{B(m)}> N' ;$$

on the other hand, where the star indicates that each $B(m)$ is counted only once,

$$\sum_{f(B(m))\geqq x^{1/2}}^{*}\frac{1}{B(m)}<\left(\sum{}'\frac{1}{p}\right)^{u}\Big/u!,$$

where $u=[100\log x]$ and the prime indicates that $n^{1/x^{10}}<p\leqq n$. Now

$$\sum{}'\frac{1}{p}<\log\log n-\log\log\left(n^{1/x^{10}}\right)+o(1)<11\log x;$$

hence

$$N'<n\frac{(11\log x)^{u}}{u!}=o\left(\frac{n}{x^{4}}\right),$$

which proves the lemma.

Lᴇᴍᴍᴀ 3. *There exists an absolute constant* c (*independent of* k) *such that for all* x *and* n

$$M(x,n)<cn/x^{3}\quad\textit{and}\quad M_{k}(x,n)<cn/x^{3}.$$

Rᴇᴍᴀʀᴋ. Lemma 3 is not trivial only for large x and n. It will be clear from the proof that the lemma is true with an arbitrary t instead of 3. It will be clear from the proof that it suffices to consider $M(x,n)$.

Suppose the lemma is false. Then we clearly can assume that there exist infinite sequences x_i and n_i such that

$$M(x_i,n_i)>1/x_i^{4},\qquad x_i\to\infty,\ n_i\to\infty. \tag{3}$$

Let $a_1<a_2<\cdots<a_k\leqq n_i$ be the integers not greater than n_i with $f(a_j)\geqq x$. For simplicity of notation we replace x_i by x and n_i by n where there is no danger of confusion. We obtain from Lemmas 1 and 2 that there exist at least $n/2x^4$ a's for which

$$f(A(a_j))>x^{1/2},\quad A(a_j)<n^{1/2}\quad(\text{since } f(m)=f(A(m))f(B(m))). \tag{4}$$

Denote these a's by a^{+}. Consider the integers $m\leqq n$ for which $A(m)=t$. We must have $m=tv\leqq n$, where v is not divisible by any $p\leqq n^{1/x^{10}}$, since these primes are included in t, and $v \leqq n/t$. The number of such integers v is equal to the number of integers $m \leqq n$ with $A(m) = t$. Since we are later going to let t run through the $A(a_j^+)$, it is permissible to assume $t < n^{1/2}$ by (4). Then Brun's method[5] yields the result

$$\sum_{A(m)=t} 1 \leqq c\frac{n}{t}\prod_p{}'\left(1-\frac{1}{p}\right) < c\,\frac{nx^{10}}{t\log n}; \tag{5}$$

the prime indicates that $p \leqq n^{1/x^{10}}$. Now letting t run through the $A(a_j^+)$ we obtain that the number of a_j^+'s satisfying (4) is less than

$$\sum{}'(c/A(a_j^+))(nx^{10}/\log n) \tag{6}$$

(the prime indicates that each $A(a_j^+)$ is counted only once); on the other hand the number of a_j^+'s is equal to or greater than $n/2x^4$ as stated in the lines preceding (4). Hence

$$\sum{}'\frac{c}{A(a_j^+)}\,\frac{nx^{10}}{\log n} > \frac{1}{2}\,\frac{n}{x^4}. \tag{7}$$

Let now N tend to infinity. As above we have that the number of integers $m \leqq N$ for which $A(m) = t$ equals

$$(1+o(1))\,\frac{N}{t}\prod_p{}'\left(1-\frac{1}{p}\right) > \frac{cN}{t}\,\frac{x^{10}}{\log n},$$

with a new value for the constant c. Thus we obtain from (7) that the number of integers $m \leqq N$ with $A(m) = A(a_j^+)$, $j = 1, 2, \cdots$, is greater than cN/x^4. But for these integers we have by (4)

$$f_k(m) = f(A(m)) \geqq x^{1/2} \qquad (k = \pi(n^{1/x^{10}})). \tag{8}$$

Thus for all sufficiently large N

$$M_k(x_1^{1/2}, N) > cN/x_1^4 \qquad (x_1 = x) \tag{9}$$

where c is an absolute constant and x_1 and k are independent of N. From (9) we obtain by a simple calculation that

$$\sum_{m=1}^{N} (f_k(m))^{10} > cx_1N \tag{10}$$

and x_1 can assume arbitrarily large values. Now we shall show that (10) is false; in fact we prove the following lemma.

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[5] V. Brun, *Le crible d'Eratosthene et le théorème de Goldbach*, Skrifter udgivne af Videnskabs selskabet Kristiania, Si Matematisk-Naturvidenskapelig Klasse vol. 3 (1920).

LEMMA 4. *Put* $h_t(p) = (f(p))^t - 1$. We have

$$(11) \qquad \sum_{m=1}^{N} f_k(m)^t = (1 + o(1))N \prod_{p \le p_k}\left(1 + \frac{h_t(p)}{p}\right) < cN,$$

where $c = c(t)$ *is independent of* k.

We have

$$\sum_{m=1}^{N} f_k(m)^t = \sum_{m=1}^{N} \prod_{p \mid m,\, p \le p_k} (1 + h_t(p)) = \sum_{d}{}' \left[\frac{N}{d}\right] \prod_{p \mid d} h_t(p)$$
$$= N \sum_{d}{}' \frac{\prod_{p \mid d} h_t(p)}{d} + O(1) = (1 + o(1))N \prod_{p \le p_k}\left(1 + \frac{h_t(p)}{p}\right),$$

where the dash indicates that d is squarefree and that all prime factors of d are not greater than p_k, and the error term $O(1)$ depends on k but not on N (the number of terms in $\sum'$ is bounded, and the bound depends on k but not on N). The second inequality of (11) follows easily from (1). This proves Lemma 4, and since (11) contradicts (10), Lemma 3 is also proved.

Now we can prove Theorem 1. For $t = 1$ we obtain from (11)

$$(12) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} f_k(m) = \prod_{p \le p_k}\left(1 + \frac{f(p) - 1}{p}\right).$$

Further from (1) we have

$$(13) \qquad \lim_{k \to \infty} \prod_{p \le p_k}\left(1 + \frac{f(p) - 1}{p}\right) = \prod_{p=2}^{\infty}\left(1 + \frac{f(p) - 1}{p}\right).$$

Thus to prove (2) (that is, Theorem 1) it will suffice to show that for every ϵ there exists a k_0 so that for $k > k_0$ and all sufficiently large n

$$(14) \qquad \frac{1}{n}\left|\sum_{m=1}^{n} f(m) - f_k(m)\right| < \epsilon.$$

Write

$$(15) \qquad \sum_{m=1}^{n} (f(m) - f_k(m)) = \sum\nolimits_1 + \sum\nolimits_2,$$

where in $\sum_1$ the summation is extended over the $m \le n$ for which $f(m) \le t$ and $f_k(m) \le t$ (t fixed), and in $\sum_2$ the summation is extended over the remaining $m \le n$.

Consider $\sum_2$. Divide the integers m appropriate to $\sum_2$ into classes

m_i $(i=t, t+1, \cdots)$ such that $i \leqq f(m_i) < i+1$. Then applying Lemma 3, we get

$$(16)\quad \begin{aligned} \sum\nolimits_2 &\leqq \sum_{f(m)\geqq t} f(m) + \sum_{f_k(m)\geqq t} f_k(m) = \sum_{i=t}^{\infty} f(m_i) + \sum_{k=t}^{\infty} f_k(m_i) \\ &< 2cn \sum_{i=t}^{\infty} \frac{i+1}{i^3} < \frac{\epsilon}{2} n, \end{aligned}$$

for t sufficiently large. We now fix t_0 so that for all $t > t_0$, the last inequality of (16) holds.

Now we estimate $\sum_1$. Let

$$\sum\nolimits_1 = \sum\nolimits_1' + \sum\nolimits_1'',$$

where in $\sum_1'$ we impose the condition $|f(m) - f_k(m)| > \epsilon/4$, and in $\sum_1''$ we require $|f(m) - f_k(m)| \leqq \epsilon/4$. Then in $\sum_1''$ there are at most n summands each numerically less than $\epsilon/4$, so this sum is numerically less than $\epsilon n/4$. In $\sum_1'$ each summand exceeds $\epsilon/4$ in absolute value but is less than t by our definition of $\sum_1$. Moreover by taking k sufficiently large we can insure that the number of summands in $\sum_1'$ will be less than $\epsilon n/4t$. For it is well known that both $f_k(m)$ and $f(m)$ have asymptotic distribution functions and that the asymptotic distribution function of $f_k(m)$ tends to that of $f(m)$ as $k \to \infty$.[6] But this implies that $f_k(n)$ tends to $f(n)$ in relative measure, that is, that the number of integers not exceeding n for which $|f(n) - f_k(n)| > \epsilon$ tends to zero with $1/k$. Hence, we have the estimate

$$\left|\sum\nolimits_1\right| = \left|\sum\nolimits_1'\right| + \left|\sum\nolimits_1''\right| < \frac{\epsilon n}{4t}\cdot t + \frac{\epsilon n}{4} < \frac{\epsilon n}{2},$$

and

$$(17)\quad \left|\sum_{m=1}^{n} (f(m) - f_k(m))\right| = \left|\sum\nolimits_1\right| + \left|\sum\nolimits_2\right| < \epsilon n.$$

This completes the proof.

It seems possible that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(m)$$

exists if we only assume that $\sum f(p)/p$ converges (some assumption like $|f(p)| < c$ might also be necessary). At present I am unable to decide this question. The present proof breaks down since Lemma 4 is false if $\sum f(p)/p$ converges and $\sum (f(p))^2/p$ diverges.

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[6] Ibid. footnote 3.

Ramanujan[7] conjectured that for any $\alpha > 0$, $\beta > 0$

$$\sum_{\nu=1}^{n} \sigma_\alpha(\nu)\sigma_\beta(n-\nu)$$
$$= (1+o(1)) \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)} \sigma_{\alpha+\beta+1}(n),$$

where $\sigma_\alpha(n)$ denotes the sum of the αth power of the divisors of n.

Ingham[8] proved this conjecture; he also found analogous asymptotic formulas for $\sum_{\nu=1}^{n} \sigma_\alpha(\nu)\sigma_\alpha(\nu+k)$, $\sum_{\nu=1}^{n} \phi(\nu)\phi(\nu+k)$, $\sum_{\nu=1}^{n} d(\nu)d(n-\nu)$, $\sum_{\nu=1}^{n} d(\nu)d(\nu+k)$.

We are going to generalize these results. First we prove the following theorem.

THEOREM 2. *Assume that* $f^{(1)}(m)$ *and* $f^{(2)}(m)$ *satisfy* (1), *also* $f^{(i)}(p^\alpha) = f^{(i)}(p)$, $i=1, 2$. *Put* $f^{(i)}(p) - 1 = g^{(i)}(p)$, $i = 1, 2$. *Then*

$$\sum_{\nu=1}^{n} f^{(1)}(\nu) f^{(2)}(n-\nu) = (1+o(1))An,$$

$$\sum_{\nu=1}^{x} f^{(1)}(\nu) f^{(2)}(\nu+n) = (1+o(1))Ax,$$

where

$$A = \prod_{p \nmid n} \left(1 + \frac{g^{(1)}(p) + g^{(2)}(p)}{p}\right) \cdot \prod_{p|n} \left(1 + \frac{g^{(1)}(p) + g^{(2)}(p) + g^{(1)}(p)g^{(2)}(p)}{p}\right).$$

REMARKS. (1) It clearly follows from (1) that the product for A converges. (2) The assumption $f(p^\alpha) = f(p)$ can be omitted without any difficulty but the expression for A then becomes much more complicated.

We have by a simple calculation ($g_k(p) = f_k(p) - 1$)

$$\sum_{\nu=1}^{n} f_k^{(1)}(\nu) f_k^{(2)}(n-\nu) = \sum_{\nu=1}^{n} \left(\prod_{p|\nu} (1 + g_k^{(1)}(p)) \prod_{p|n-\nu} 1 + g_k^{(2)}(n-\nu) \right)$$
$$= \sum_{d_1, d_2 \geq 1} \left[\frac{n}{d_1 d_2}\right]' \prod_{p|d_1} g_k^{(1)}(p) \prod_{p|d_2} g_k^{(2)}(p),$$

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[7] *Collected papers*, p. 137.

[8] J. London Math. Soc. vol. 2 (1927) pp. 202–208.

where $[n/d_1d_2]$ denotes the number of solutions of $n=d_1x+d_2y$ in positive integers x and y. Clearly $|n/d_1d_2-[n/d_1d_2]'|\leqq 1$. Thus a simple calculation shows that

$$\sum_{\nu=1}^{n} f_k^{(1)}(\nu)f_k^{(2)}(n-\nu) = (1+o(1))n \prod_{p\nmid n,\, p\leqq p_k}\left(\frac{1+g^{(1)}(p)+g^{(2)}(p)}{p}\right)$$
$$\cdot\prod_{p\mid n,\, p\leqq p_k}\left(1+\frac{g^{(1)}(p)+g^{(2)}(p)+g^{(1)}(p)g^{(2)}(p)}{p}\right).$$

Since the product for A converges our proof will be complete if we show that for sufficiently large k

$$(18)\qquad \left|\sum_{\nu=1}^{n}\{f^{(1)}(\nu)f^{(2)}(n-\nu)-f_k^{(1)}(\nu)f_k^{(2)}(n-\nu)\}\right|<\epsilon n.$$

We suppress the proof of (18) since it is almost identical with that of (14); this completes the proof of the first half of Theorem 2. The proof of the second half is similar.

By the same method we can prove

$$(19)\qquad \begin{aligned}\sum F^{(1)}(\nu)F_k^{(2)}(n-\nu) &= (1+o(1))A\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}n^{\alpha+\beta+1},\\ \sum_{\nu=1}^{x}F^{(1)}(\nu)F^{(2)}(\nu+n) &= (1+o(1))\frac{A}{\alpha+\beta+1}x^{\alpha+\beta+1};\end{aligned}$$

where $F^{(1)}(n)=n^{\alpha}f^{(1)}(n)$, $F^{(2)}(n)=n^{\beta}f^{(2)}(n)$, $\alpha>0$, $\beta>0$ are arbitrary real numbers. (19) contains all the results of Ingham except those on $d(n)$ (we of course have to drop the assumption $f(p^{\alpha})=f(p)$).

Carlitz proved the following theorem:[9] Let

$$f(p)=1+o(p^{-1/2+\epsilon}).$$

Then

$$(20)\qquad \sum{}' f(n_1)f(n_2)\cdots f(n_\nu)=Cn^{\nu-1}+O(n^{\nu-3/2+\epsilon});$$

the prime means that the summation is extended over all partitions of n into v summands. The value of C is given by a complicated expression.

By the method we used in proving Theorem 2 we can prove the following theorem.

THEOREM 3. *Let* $f^i(n)$ *satisfy* (1), $i=1, 2, \cdots, \nu$. *Then*

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[9] Quart. J. Math. Oxford Ser. vol. 2 (1931) pp. 97–106; see also vol. 3 (1932) pp. 273–290.

$$\sum{}' f^{(1)}(n_1) f^{(2)}(n_2) \cdots f^{(r)}(n_r) = (1 + o(1)) D n^{r-1},$$

also

$$\sum_{m=1}^{n} f^{(1)}(m + k_1) f^{(2)}(m + k_2) \cdots f^{(r)}(m + k_r) = (1 + o(1)) E n,$$

D and E are given by a complicated expression.

STANFORD UNIVERSITY

ON A CLASS OF TAYLOR SERIES

V. F. COWLING

1. **Introduction.** Consider the Taylor series $\sum_{n=0}^{\infty} a_n z^n$. Suppose that the singularities of the function defined by the series all lie in certain regions of the complex plane and that the coefficients possess certain arithmetical properties. Mandelbrojt[1] has shown that under restrictions of this nature it is possible to predict the form of the function defined by the series. This note is concerned with the establishing of a new method to obtain more general results of this nature.

2. **The method.** The method that is employed here is an adaptation of a method used by Lindelöf [2] in the problem of representation of a function defined by a series.

Let $f(z)$ be regular in a region D of the complex plane. Suppose that there exists a linear transformation $t = h(z)$ which maps the region of regularity into a region which includes the unit circle of the t-plane in its interior. Let $z = g(t)$ be the inverse of this transformation. Then $F(t) = f(g(t))$ is regular in this region in the t-plane. For this note it is convenient to suppose that $z = 0$ corresponds to $t = 0$ in the mapping. We may expand $g(t)$ in a Taylor series about $t = 0$ and obtain

$$z = b_1 t + b_2 t^2 + \cdots \tag{2.1}$$

convergent for t in absolute value sufficiently small. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.2}$$

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[1] See, Mandelbrojt [3]. Numbers in brackets refer to the bibliography at the end of the paper.

be the element of $f(z)$ at the origin. For t in absolute value sufficiently small we may substitute (2.1) in (2.2) and obtain

$$F(t) = f(g(t)) = \sum_{n=0}^{\infty} C_n t^n. \tag{2.3}$$

We have seen, however, that $F(t) = f(g(t))$ is regular in a region in the t-plane which includes the unit circle in its interior. Hence the radius of convergence of (2.3) is greater than one. Therefore we may write

$$\limsup_{n\to\infty} (|C_n|)^{1/n} < 1. \tag{2.4}$$

As the C_n are polynomial combinations of the a_n and b_n we see that under certain circumstances (2.4) may imply $C_n = 0$ for n greater than some n_0. For example, if the C_n are all integers (2.4) implies the existence of an n_0 such that $C_n = 0$ for $n > n_0$. It is also clear that if $C_n = C_n' + iC_n''$ where C_n' and C_n'' are integers that the conclusion $C_n = 0$, $n > n_0$, still holds. Under these circumstances we obtain upon substituting $t = h(z)$ in (2.3)

$$f(z) = \sum_{n=0}^{n_0} C_n [h(z)]^n.$$

3. **Applications.** We now proceed to the proof of the principal theorem.

THEOREM 3.1. *If the series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *has rational coefficients such that there exists an integer* L *for which the quantities*

$$a_0,\ a_1 L,\ a_2 L^2,\ \cdots,\ a_n L^n, \cdots$$

are integers, and if the function defined by the series is regular exterior to and on the circumference of a circle with center $(LK'/(K'^2+K''^2-1),$ $LK''/(K'^2+K''^2-1))$ *and radius* $L/(K'^2+K''^2-1)$ *where* K' *and* K'' *are integers,* $K'^2+K''^2 \neq 1$, *then the function defined by the series is of the form*

$$\frac{P(z)}{(L/(K'+iK'') - z)^{n_0}}$$

where $P(z)$ *is a polynomial and* n_0 *is a positive integer.*

It is easily shown that the transformation

$$t = \frac{z}{L - (K' + iK'')z} \tag{3.1}$$

maps the region of regularity into a region containing the unit circle of the t-plane in its interior. By substituting the solution of (3.1) for z in (2.2) and setting $K=K'+iK''$ we have

$$F(t) = f(g(t)) = \sum_{n=0}^{\infty} a_n \frac{L^n t^n}{(1+Kt)^n}$$

$$= \sum_{n=0}^{\infty} a_n L^n t^n \sum_{m=0}^{\infty} C_{-n,m}(Kt)^m.$$

Then since the series are absolutely convergent, we obtain

$$F(t) = \sum_{n=0}^{\infty} [a_n L^n C_{-n,0} + C_{-n,1} K L^{n-1} a_{n-1} + \cdots$$
$$+ a_0 C_{-n,n} K^n] t^n,$$

$$(3.2)\quad F(t) = \sum_{n=0}^{\infty} C_n t^n.$$

Here $C_n = C_n' + iC_n''$ where C_n' and C_n'' are integers, for the $C_{-n,m}$ are binomial coefficients and $a_n L^n$ is an integer for all $n \geqq 0$ by hypothesis. Also K' and K'' are integers. Hence, from the discussion in §2 it follows that

$$\limsup_{n\to\infty} |C_n|^{1/n} < 1$$

implies the existence of a number n_0 such that $C_n = 0$ for $n > n_0$. Therefore upon substituting (3.1) in (3.2) we have

$$f(z) = \sum_{n=0}^{n_0} C_n \left(\frac{z}{L-Kz}\right)^n = \frac{P(z)}{(L/(K'+iK'') - z)^{n_0}}.$$

This completes the proof of the theorem. If now we choose $K''=0$ and $K'=L$ we have the theorem of Mandelbrojt [3]. This proof of Mandelbrojt's theorem has some points in common with a proof of the same theorem due to Achyser [1].[3] If in addition the a_n are all integers we may set $L=1$ and have a new theorem.

We now proceed to the proof of the following theorem.

THEOREM 4.1. *If the series* $\sum_{n=0}^{\infty} a_n z^n$ *has rational coefficients such that there exists an integer* L *for which the quantities*

$$a_0,\ a_1L,\ a_2L^2,\ \cdots,\ a_nL^n,\ \cdots$$

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[3] The author is indebted to the referee for this observation.

are integers and if the function defined by the series is regular in the half-plane $R(z) \leq L/2$ including the point at infinity then the series defines a function of the form

$$\frac{P(z)}{(L-z)^{n_0}}$$

where $P(z)$ is a polynomial and n_0 is a positive integer.

From the hypothesis it is easily seen that the transformation

$$t = \frac{z}{L - z} \tag{3.3}$$

maps the region of regularity into a region which includes the unit circle in the t-plane in its interior. Upon solving (3.3) for z and substituting in (2.2) we obtain

$$\begin{aligned} F(t) &= f\left(\frac{Lt}{1+t}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{Lt}{1+t}\right)^n \\ &= \sum_{n=0}^{\infty} a_n t^n L^n \sum_{m=0}^{\infty} C_{-n,m}(t)^m = \sum_{n=0}^{\infty} C_n t^n. \end{aligned} \tag{3.4}$$

Then by the same arguments employed in Theorem (3.1) it follows that there exists an n_0 such that $C_n = 0$ for $n > n_0$. Therefore by substituting from (3.3) in (3.4) we have

$$f(z) = \sum_{n=0}^{n_0} C_n \left(\frac{z}{L-z}\right)^n = \frac{P(z)}{(L-z)^{n_0}},$$

where $P(z)$ is a polynomial and n_0 is a positive integer. This completes the proof of the theorem.

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LEHIGH UNIVERSITY

ON REAL CONTINUOUS SOLUTIONS OF ALGEBRAIC DIFFERENCE EQUATIONS

S. M. SHAH

1. **Introduction.** This paper considers the behaviour at infinity of real continuous solutions of algebraic difference equations

$$P(y(x+m), \cdots, y(x), x) = 0 \tag{1}$$

where P is a polynomial with real coefficients in its arguments $y(x+m), \cdots, y(x)$ and x. The problem was first treated by Lancaster,[1] who obtained an upper bound for the rate of increase of the solutions of algebraic difference equations of a given order and pointed out the surprising dissimilarity with the known rates of increase for solutions of differential equations of the same order.

The main object of this paper is to show that any real continuous solution of an algebraic difference equation of the first order

$$P(y(x+1), y(x), x) = 0 \tag{2}$$

satisfies the inequality

$$\liminf_{x\to\infty} \log\log |y(x)|/x < \infty. \tag{3}$$

This is an improvement over the results of Lancaster, who proved that a continuous solution $y(x)$ cannot equal or exceed $ce_2(xl_nx)$ for all $x>x_0$, where $l_n(x)$ is the nth iterate of $\log x$, $e_2(x)=e^{e^x}$, n is any fixed positive integer, and c is any positive constant. The essential difference in the rates of increase of the solutions of algebraic differential and difference equations is clearly emphasized when this new result is compared with the results of Borel, who showed that any real continuous solution $y(x)$ of an algebraic differential equation of the first order satisfies the inequality

$$\limsup_{x\to\infty} \log\log |y(x)|/\log x < \infty. \tag{4}$$

Simple examples reveal that relation (3) is the *best possible* result. First this is the best possible limit, for when $y(x)=a^{b^x}$, a solution of

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[1] Otis E. Lancaster, *Some results concerning the behavior at infinity of real continuous solutions of algebraic difference equations*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 169–177. References to the papers of Borel, Vijayaraghavan and others will be found in this paper.

$y(x+1)-(y(x))^b=0$, we have

$$\lim_{x\to\infty} \log\log |y(x)|/x = \log b.$$

Since b is any rational number, $\log b$ can be made to exceed any finite value. Second, it is not possible to replace limit inferior by limit or limit superior without restrictive hypothesis on $y(x)$, for if $\psi(x)$ is any increasing continuous function tending to infinity with x, a solution of

$$(y(x) - x^b)(y(x+1) - (x+1)^b) = 0 \tag{5}$$

can exceed $\psi(x)$ for a sequence of values of x tending to infinity. The difference equation (5) is satisfied by the continuous function

$$\begin{aligned} y(x) &= x^b && (\text{for } 2n - 1 \leqq x \leqq 2n) \\ &= (2n)^b + 2\{\phi(2n+1/2) - (2n)^b\}(x - 2n) && \\ & && (\text{for } 2n \leqq x \leqq 2n + 1/2) \\ &= (2n+1)^b + 2\{(2n+1)^b - \phi(2n+1/2)\}(x - 2n - 1) && \\ & && (\text{for } 2n + 1/2 \leqq x \leqq 2n + 1) \end{aligned}$$

($n=1, 2, 3, \cdots$) where ϕ is an arbitrary function. Let $\phi(x)=e_3(\psi(x))$. Then

$$\lim_{n\to\infty} \frac{\log\log y(2n+1/2)}{\psi(2n+1/2)} = \lim_{n\to\infty} \frac{e^{\psi(2n+1/2)}}{\psi(2n+1/2)} = \infty,$$

which establishes the above statement.

2. **Statements of the theorems.** In §§4–6 we shall prove the following three theorems on the algebraic difference equations of the first order. In §§7–10 we consider equations of the mth order.

THEOREM 1. *If $y(x)$ be a real continuous solution of a first order algebraic difference equation $P=0$, then $|y(x)|$ cannot equal or exceed $e_2(Ax)$ for all*[1] *$x>x_0(A)$, A being a positive constant depending on the given equation $P=0$.*

In what follows we suppose throughout that $y(x)$ is real and continuous for $x>x_0$.

THEOREM 2. *If $y(x)$, a solution of an algebraic difference equation of the first order, be such that*[2]

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[1] x_0 is not necessarily the same at each occurrence.

[2] The condition (6) can be replaced by a less stringent one, $\log |y(x)|/\log x>A_1$ for all $x>x_0(A_1)$, where A_1 is a constant depending on the given equation.

$$\log |y(x)| / \log x \to \infty \qquad \text{as } x \to \infty \tag{6}$$

then $|y(x)| < e_2(Ax)$ *for all* $x > x_0(A)$, *where* A *is a positive constant depending on the given equation.*

THEOREM 3. *If* $y(x)$, *a solution of an algebraic difference equation of the first order, be such that*

$$y(x) \text{ is monotonic} \qquad \text{for } x > x_0 \tag{7}$$

then $|y(x)| < e_2(Ax)$ *for all* $x > x_0(A)$, *where* A *is positive constant depending on the given equation.*

3. **Lemmas.** We require three lemmas.

LEMMA 1. *If* $f(x)$ *is continuous*[4] *and non-negative for* $x > x_0$ *and if*

$$\limsup_{x\to\infty} \log\log f(x)/x \geqq A > 0 \tag{8}$$

then there exists a sequence of numbers $x_1, x_2, \cdots, x_n \to \infty$ *such that*

$$f(x+1) \geqq \{f(x)\}^B \tag{9}$$

for $x = x_1, x_2, \cdots$; B *is any positive number less than* $\exp(A)$.

For if the lemma is false then for all $x \geqq X_0 > x_0$

$$f(x+1) < \{f(x)\}^B$$

and so $f(X_0 + p) < \{f(X_0)\}^{B^p}$. Hence

$$\limsup_{p\to\infty} \log\log f(X_0 + p)/(X_0 + p) \leqq \log B < A,$$

which contradicts the hypothesis and so the lemma is proved.

LEMMA 2. *If* $f(x)$ *is continuous and nondecreasing for* $x > x_0$ *and if* (8) *holds then there exists a sequence of numbers* $x_1, x_2, \cdots, x_n \to \infty$ *such that*[5]

$$f(x) \geqq e_2(\phi(x)), \tag{10.1}$$

$$f(x+1) \geqq \{f(x)\}^B \tag{10.2}$$

for $x = x_1, x_2, x_3, \cdots$; B *is any positive number less than* $\exp(A)$ *and* $\phi(x)$ *is any nondecreasing continuous function such that* $\lim_{x\to\infty} \phi(x)/x = 0$.

PROOF. If $f(x) \geqq e_2(\phi(x))$ for all large x, then the result follows from

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[4] Continuity of $f(x)$ is utilized in Lemma 2.

[5] We prove a result more general than the one we require later on.

Lemma 1. Suppose therefore that $f(x) < e_2(\phi(x))$ for a sequence of values of x tending to infinity. Let

$$F(x) = f(x)/e_2(\phi(x))$$

and $X > x_0$ be an arbitrary large number such that $F(X) < 1$. Let $\log B < C < A$. Take $Y > X$ such that $F(Y) > 1$ and $\log\log f(Y) > CY + \phi(Y)$. This is possible since $\limsup_{x\to\infty} \log\log f(x)/x \geqq A > C$. Let Z be the largest number such that $X < Z < Y$ and $F(Z) = 1$. Since $F(x)$ is continuous for $x \geqq X$, Z exists, and $F(x) \geqq 1$ for $Z \leqq x \leqq Y$. Consider $f(Z), f(Z+1), \cdots, f(Z+m)$ where $Y \leqq Z+m < Y+1$.

$$\begin{aligned}\log\log f(Z+m) &\geqq \log\log f(Y)\\ &> CY + \phi(Y)\\ &> C(Z+m-1) + \phi(Z)\\ &= C(Z+m-1) + \log\log f(Z).\end{aligned}$$

Hence

$$\log\log f(Z+m) - \log\log f(Z) > C(Z+m-1) > Cm.$$

Hence at least one of the values

$$\log\log f(Z+n) - \log\log f(Z+n-1) \qquad (n = 1, 2, \cdots, m)$$

is greater than C. If $\log\log f(Z+N) - \log\log f(Z+N-1) > C$ then

$$f(Z+N-1) \geqq e_2(\phi(Z+N-1))$$

and

$$f(Z+N) \geqq \{f(Z+N-1)\}^{e^C} > \{f(Z+N-1)\}^B$$

which proves the lemma.

Lemma 3. *If $f(x)$ is continuous and nondecreasing for $x > x_0$ and if* (8) *holds, then*[5] *there exists a sequence of numbers $x_1, x_2, \cdots, x_n \to \infty$ such that*

$$f(x+1) \geqq e_2(\phi(x)), \tag{11.1}$$

$$f(x+1) \geqq \{f(x)\}^B, \tag{11.2}$$

$$f(x+2) \geqq \{f(x+1)\}^B, \tag{11.3}$$

for $x = x_1, x_2, \cdots$; $\phi(x)$ is any increasing continuous function such that $\lim_{x\to\infty}\phi(x)/x = 0$, *and* $0 < B < \exp(A/2)$.

Proof. Let $0 < \epsilon < 1 - (2 \log B/A)$. By Lemma 2 we can choose a sequence $X_1, X_2, \cdots$ such that

$$f(x) \geqq e_2(\phi(x)),$$

$$\log\log f(x+1) - \log\log f(x) > A\ (1-\epsilon)$$

for $x = X_1, X_2, \cdots$. Let ξ_r be such that $X_r < \xi_r < 1 + X_r$ and

$$\log\log f(\xi_r) = (1/2)(\log\log f(1+X_r) + \log\log f(X_r)).$$

ξ_r exists since $f(x)$ is continuous and nondecreasing. Since $\xi_r - 1 < X_r < \xi_r < 1 + X_r < 1 + \xi_r$ we have

$$\begin{aligned}\log\log f(\xi_r) &- \log\log f(\xi_r - 1)\\ &\geqq \log\log f(\xi_r) - \log\log f(X_r) > A(1-\epsilon)/2,\\ \log\log f(1+\xi_r) &- \log\log f(\xi_r)\\ &\geqq \log\log f(1+X_r) - \log\log f(\xi_r) > A(1-\epsilon)/2.\end{aligned}$$

Hence writing x_r for $\xi_r - 1$ and $d = \exp\ (A(1-\epsilon)/2)$, we have

$$f(1+x_r) > (f(x_r))^d, f(2+x_r) > (f(1+x_r))^d.$$

Also

$$f(1+x_r) = f(\xi_r) \geqq f(X_r) \geqq e_2(\phi(X_r)) \geqq e_2(\phi(\xi_r - 1)) = e_2(\phi(x_r)).$$

Hence the lemma is proved.

4. Proof of Theorem 1. Let the given equation be (2). Among the terms of the polynomial P there is a term

$$T' = a' x^{\alpha'} y(x)^{\beta_0'} y(x+1)^{\beta_1'} \tag{12}$$

which has the property that if

$$T = a x^{\alpha} y(x)^{\beta_0} y(x+1)^{\beta_1} \tag{13}$$

is any other term of the polynomial, then in the sequence of differences

$$\beta_1' - \beta_1, \qquad \beta_0' - \beta_0, \qquad \alpha' - \alpha$$

the first nonzero term is positive. T' is called the principal term. Divide the equation (2) by T'. Then all terms except one T'/T' are of the form T/T' where

$$T/T' = b x^{\alpha-\alpha'} \{y(x)\}^{\beta_0-\beta_0'} \{y(x+1)\}^{\beta_1-\beta_1'}.$$

These ratios T/T' are of three possible types

$$b\left\{\frac{x^{K_0} y(x)^{K_1}}{y(x+1)}\right\}^{\beta_1'-\beta_1}, \qquad \beta_1' > \beta_1, \tag{14}$$

$$(15) \qquad b\left\{\frac{x^{K_1}}{y(x)}\right\}^{\beta_0'-\beta_0}, \qquad \beta_0' > \beta_0,$$

$$(16) \qquad bx^{\alpha-\alpha'}, \qquad \alpha' > \alpha.$$

Let K be the max $|K_1|$ for all the ratios of type (14) and let[6]

$$A = 1 + \max(1, \log K).$$

(i) First, assume that $e_2(Ax)$ is a solution of (2). If we replace $y(x)$ by $e_2(Ax)$, the limit of these expressions (14), (15) and (16), as x tends to infinity, is zero. Hence we have a contradiction to the assumption that $e_2(Ax)$ is a solution of (2). For there exists an x_0 such that for $x > x_0$, the sum of all the terms T/T' is less than one in absolute value whereas $T'/T' = 1$.

(ii) Second, assume that a solution $y(x)$ is greater than or equal to $e_2(Ax)$ for all $x > x_0$. Then the ratios T/T' are again of the three types (14), (15) and (16). The ratios of types (15) and (16) approach zero as x tends to infinity; and the ratios of type (14) are in absolute value less than

$$\left\{\frac{y(x)^{K+\epsilon}}{y(x+1)}\right\}^{\beta_1'-\beta_1} \qquad (x > X_0(\epsilon))$$

and the last expression, by Lemma 1, tends to zero as $x = x_n \to \infty$; for $K + \epsilon < eK/2 = B$ and $eK/2 < eK = e^{1+\log K} \leq e^A$. Hence we have a contradiction to the assumption that $y(x)$ is greater than or equal to $e_2(Ax)$ for all $x > x_0$.

(iii) Third, assume that $y(x)$ is less than or equal to $-e_2(Ax)$ for all $x > x_0$. Let $y(x) = -Y(x)$. The argument in (i) and (ii) applies to the transformed equation and so we get a contradiction to the assumption (iii). Since $y(x)$ is continuous, the theorem follows.

5. **Proof of Theorem** 2. By hypothesis (6), $y(x)$ is either positive or negative for all $x > x_0$.

If $y(x) \geq e_2(Ax)$ for a sequence of values of x tending to infinity, A being the constant defined in the proof of Theorem 1, then by Lemma 1, we can find a sequence $x_1, x_2, \cdots, x_n \to \infty$ such that

$$y(x+1) \geq \{y(x)\}^B$$

where $B < \exp(A)$, for $x = x_1, x_2, \cdots$. The ratios of type (16) tend to zero as x tends to infinity and by hypothesis (6), the ratios of type (15) tend to zero as x tends to infinity. Further the ratios of type (14) tend to zero as $x=x_n\to\infty$. Hence we have a contradiction and so $y(x)<e_2(Ax)$ for $x>x_0$.

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[6] It is possible to choose a lower value of A. For instance in Theorems 1, 2, 3, we may take A to be any number greater than max ($\log K$, 0).

If $y(x)\leqq -e_2(Ax)$ for a sequence of values of x tending to infinity, then $y(x)$ is negative for $x>x_0$ and we take $y(x)=-Y(x)$. The above argument then applies and the theorem follows.

6. **Proof of Theorem** 3. We may suppose that $y(x)$ is nondecreasing (for otherwise we can transform the equation by taking $y(x)=-Y(x)$) for $x>x_0$. If $y(x)\geqq e_2(Ax)$ for a sequence of values of x tending to infinity, A being the constant defined in the proof of Theorem 1, then by Lemma 2, we can find a sequence $x_1, x_2, \cdots, x_n\to\infty$ such that

$$y(x)\geqq \exp(x), \qquad y(x+1)\geqq (y(x))^B,$$

where $0<B<\exp(A)$, for $x=x_1, x_2, \cdots$. Hence the ratios of types (14), (15) and (16) tend to zero as $x=x_n\to\infty$. Hence we have a contradiction and so $y(x)<e_2(Ax)$ for $x>x_0$. Since $y(x)$ is nondecreasing

$$|y(x)|<e_2(Ax) \qquad \text{for } x>x_0.$$

7. **Equations of the** m**th order.** We now consider algebraic difference equations

$$(1)\quad P(y(x+m),\cdots,y(x),x)=\sum ax^\alpha y(x)^{\beta_0}\cdots y(x+m)^{\beta_m}=0$$

of order $m>1$. Let the principal term be

$$(17)\qquad T'=a'x^{\alpha'}y(x)^{\beta_0'}\cdots y(x+m)^{\beta_m'}$$

so that in the sequence of differences

$$\beta_m'-\beta_m,\ \beta_{m-1}'-\beta_{m-1},\ \cdots,\ \beta_0'-\beta_0,\ \alpha'-\alpha$$

the first nonzero term is positive.

We can show, as in first part of Theorem 1, that $ce_2(Ax)$, *where c is any finite constant and A a constant depending on $P=0$, cannot be a solution of $P=0$.*

THEOREM 4. *Let $P=0$ be an algebraic difference equation of the mth order. If the terms of the equation are so related that when*

$$(18)\qquad \beta_m'=\beta_m \quad \textit{then} \quad \beta_i'\geqq\beta_i \qquad (i=0,1,2,\cdots,m-1)$$

and if $y(x)$ be a solution of the equation then $|y(x)|$ cannot exceed $e_2(Ax)$ for all $x>x_0(A)$, A being a positive constant depending on $P=0$.

THEOREM 5. *Let $P=0$ be an algebraic difference equation of the mth order. If the terms of the equation are so related that when $\beta_m'=\beta_m$ then* *either* $\beta'_{m-1} > \beta_{m-1}$ *or* $\beta'_{m-1} = \beta_{m-1}$ *and* $\beta'_i \geqq \beta_i$ $(i = 0, 1, 2, \cdots, m-2)$ *and if* $y(x)$ *be a solution satisfying conditions* (6) *and* (7), *then*

$$|\,y(x)\,| < e_2(Ax) \qquad \textit{for all } x > x_0(A),$$

A being a positive constant depending on $P = 0$.

COROLLARY. *If* $y(x)$, *a solution of an algebraic difference equation of the second order, satisfies the conditions* (6) *and* (7), *then*

$$|\,y(x)\,| < e_2(Ax) \qquad \textit{for all } x > x_0(A),$$

where A is a constant depending on the given equation.

THEOREM 6. *If* $y(x)$, *a solution of an algebraic difference equation of the second order, be monotonic for* $x > x_0$, *then*

$$\liminf_{x\to\infty} \log\log |\,y(x)\,| / x < \infty. \tag{19}$$

THEOREM 7. *If* $y(x)$, *a solution of an algebraic difference equation of the mth order, be such that*

$$|\,y(x+1)\,| \geqq c\,|\,y(x)\,|^{A_2} \tag{20}$$

where c is any positive constant and A_2 *is a positive constant depending on the given equation, for* $x = a, a+1, a+2, \cdots$, *then*

$$\liminf_{x\to\infty} \log |\,y(x)\,| / \log x < A_2. \tag{21}$$

8. **Proof of Theorem 4.** Under the hypothesis, the ratios T/T' take the forms

$$b\left\{\frac{x^{K_0}y(x)^{K_1}\cdots y(x+m-1)^{K_m}}{y(x+m)}\right\}^{\beta'_m-\beta_m} \qquad (\beta'_m > \beta_m), \tag{22}$$

$$b\left\{\frac{x^{k_0}}{y(x)^{k_1}\cdots y(x+m-1)^{k_m}}\right\} \tag{23}$$

where $k_1, k_2, \cdots$ are all non-negative constants and k_0 is negative if all $k_1, k_2, \cdots, k_m$ are zero. Let M_i be the maximum value of all $|K_i|$ for the ratios of type (22) and let max $((M_0+M_1+M_2+\cdots+M_m), e) = K$ and[6] $A = 1+\log K$.

If $y(x) \geqq e_2(Ax)$ for all $x > x_0$, then the ratios (23) all tend to zero as x tends to infinity, and so some of the ratios of type (22) do not tend to zero. Hence

$$\frac{x^{M_0}y(x)^{M_1}\cdots y(x+m-1)^{M_m}}{y(x+m)} > \frac{1}{N} > 0.$$

That is,

$$y(x+m) < Nx^{M_0}y(x)^{M_1}\cdots y(x+m-1)^{M_m} \qquad \text{for } x \geqq x_0.$$

Hence

$$\lim_{p\to\infty} \frac{y(x_0+p+m)}{e_2(A(x_0+p+m))}$$

$$\leqq \lim_{p\to\infty} \frac{\{(x_0+p)N\cdots y(x_0+m-1)\}^{(K+1)^{p+2}}}{e_2(A(x_0+p+m))} = 0.$$

But this is contrary to the hypothesis that $y(x) \geqq e_2(Ax)$ for all $x > x_0$, and hence $y(x)$ does not exceed $e_2(Ax)$ for all $x > x_0$.

If $y(x) \leqq -e_2(Ax)$ for all $x > x_0$, then we take $y(x) = -Y(x)$ and argue as above. Since $y(x)$ is continuous the theorem follows.

9. **Proof of Theorem** 5. The ratios T/T' take the forms

$$b\left\{\frac{x^{K_0}y(x)^{K_1}\cdots y(x+m-1)^{K_m}}{y(x+m)}\right\}^{\beta'_m-\beta_m}, \qquad \beta'_m > \beta_m, \tag{24}$$

$$b\left\{\frac{x^{p_0}y(x)^{p_1}\cdots y(x+m-2)^{p_{m-1}}}{y(x+m-1)}\right\}^{\beta'_{m-1}-\beta_{m-1}}, \quad \beta'_{m-1} > \beta_{m-1}, \tag{25}$$

$$b\left\{\frac{x^{k_0}}{y(x)^{k_1}\cdots y(x+m-2)^{k_{m-1}}}\right\}, \tag{26}$$

respectively, where $k_1, k_2, \cdots$ are all non-negative constants and k_0 is negative if all $k_1, k_2, \cdots, k_{m-1}$ are zero. Let

$$|K_1| + |K_2| + \cdots + |K_m| = A_m,$$
$$|p_1| + |p_2| + \cdots + |p_{m-1}| = B_m,$$

and let α_1 be max A_m for all the ratios of type (24) and let α_2 be max B_m for all the ratios of type (25) and let[6]

$$A = 1 + 2\max(1, \log\alpha_1, \log\alpha_2).$$

We may suppose that $y(x)$ is nondecreasing for $x > x_0$. If $y(x) \geqq e_2(Ax)$ for a sequence of values of x tending to infinity, then by Lemma 3

$$y(x+1) \geqq \exp(x), \quad y(x+1) \geqq (y(x))^B, \quad y(x+2) \geqq (y(x+1))^B,$$

where $0 < B < \exp(A/2)$, for $x = x_1, x_2, \cdots, x_n \to \infty$. By hypothesis (6) all the ratios of type (26) approach zero as x tends to infinity. Further for $x > x_0(\epsilon)$

$$x^{K_0}y(x)^{K_1}\cdots y(x+m-1)^{K_m} < y(x+m-1)^{\alpha_1+\epsilon},$$
$$x^{p_0}y(x)^{p_1}\cdots y(x+m-2)^{p_{m-1}} < y(x+m-2)^{\alpha_2+\epsilon}$$

so all the ratios of types (24) and (25) tend to zero as $x=x_n+2-m$ tends to infinity. Hence we have a contradiction and so

$$y(x) < e_2(Ax) \qquad \text{for } x > x_0.$$

Since $y(x)$ is nondecreasing

$$|\,y(x)\,| < e_2(Ax) \qquad \text{for } x > x_0.$$

The corollary follows immediately from the theorem. The proof of Theorem 6 is similar to the proofs of the preceding theorems.

10. **Proof of Theorem** 7. The ratios T/T' take the forms

$$(27.1) \qquad b\left\{\frac{x^{k_{00}}y(x)^{k_{01}}\cdots y(x+m-1)^{k_{0,m}}}{y(x+m)}\right\}^{\beta_m'-\beta_m} \qquad (\beta_m' > \beta_m),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(27.r+1) \quad b\left\{\frac{x^{k_{r0}}\cdots y(x+m-r-1)^{k_{r,m-r}}}{y(x+m-r)}\right\}^{\beta_{m-r}'-\beta_{m-r}}$$
$$(\beta_{m-r}' > \beta_{m-r}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(27.m+1) \qquad b\left\{\frac{x^{k_{m0}}}{y(x)}\right\}^{\beta_0'-\beta_0} \qquad (\beta_0' > \beta_0),$$

$$(27.m+2) \qquad bx^{\alpha-\alpha'} \qquad \alpha' > \alpha.$$

Let $m(r, s)$ be the maximum value of $|k_{rs}|$ for all the ratios of type (27.r+1) and let

$$A_2 = 2 + \max m(r, s) \qquad (0 \leqq r,\ 0 \leqq s,\ r+s \leqq m).$$

Suppose if possible

$$\liminf_{x\to\infty} \log|\,y(x)\,|/\log x \geqq A_2$$

then

$$|\,y(x)\,| \geqq x^{A_2-\epsilon} \qquad \text{for all } x > x_0(\epsilon).$$

Let $x=x_n=a+n$. Then

$$\left|\frac{y(x+r-1)^{1+k_{0r}}}{y(x+r)}\right| < c_1 = \text{constant}$$

for $x = x_n$. Further

$$\{x^{bm}/y(x)\}$$

tends to zero as $x = x_n \to \infty$. Hence all the ratios T/T' of type (27.1) tend to zero as $x = x_n \to \infty$. Similarly all the ratios of types (27.2) $\cdots$ (27.m+2) tend to zero as $x = x_n \to \infty$. Hence we have a contradiction and so the theorem is proved.

Remark. If $y(x)$ satisfies the conditions

$$(28) \qquad |y(x+1)| \geqq c|y(x)|^{A_2} \qquad \text{for all } x > x_0,$$

A_2 being the constant defined in the proof of Theorem 7, and

$$(29) \qquad |y(x)| \to \infty \qquad \text{as } x \to \infty,$$

we would get, as in Theorem 7, that

$$(30) \qquad \limsup_{x\to\infty} \log|y(x)|/\log x < A_2.$$

But from (28) and (29) we have

$$(31) \qquad \liminf_{x\to\infty} \log\log|y(x)|/x \geqq \log A_2$$

which contradicts (30). Hence if $y(x)$ be a solution of an algebraic difference equation of the mth order, $y(x)$ cannot satisfy both the conditions (28) and (30).

MUSLIM UNIVERSITY

INNER PRODUCTS IN NORMED LINEAR SPACES

ROBERT C. JAMES

Let T be any normed linear space [1, p. 53].[1] Then an ***inner product*** is defined in T if to each pair of elements x and y there is associated a real number (x, y) in such a way that $(x, y) = (y, x)$, $\|x\|^2 = (x, x)$, $(x, y+z) = (x, y) + (x, z)$, and $(tx, y) = t(x, y)$ for all real numbers t and elements x and y. An inner product can be defined in T if and only if any two-dimensional subspace is equivalent to Cartesian space [5]. A complete separable normed linear space which has an inner product and is not finite-dimensional is equivalent to (real) Hilbert space,[2] while every finite-dimensional subspace is equivalent to Euclidean space of that dimension. Any complete normed linear space T which has an inner product is characterized by its (finite or transfinite) cardinal "dimension-number" n. It is equivalent to the space of all sets $x = (x_1, x_2, \cdots)$ of n real numbers satisfying $\sum_i x_i^2 < +\infty$, where $\|x\| = (\sum_i x_i^2)^{1/2}$ [7, Theorem 32]. Various necessary and sufficient conditions for the existence of an inner product in normed linear spaces of two or more dimensions are known. Two such conditions are that $\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2]$ for all x and y, and that $\lim_{n\to\infty} \|x+ny\| - \|nx+y\| = 0$ whenever $\|x\| = \|y\|$ ([5] and [4, Theorem 6.3]). A characterization of inner product spaces of three or more dimensions is that there exist a projection of unit norm on each two-dimensional subspace [6, Theorem 3]. Other characterizations valid for three or more dimensions will be given here, expressed by means of orthogonality, hyperplanes, and linear functionals.

A hyperplane of a normed linear space is any closed maximal linear subset M, or any translation $x + M$ of M. A hyperplane is a supporting hyperplane of a convex body S if its distance from S is zero and it does not contain an interior point of S; it is tangent to S at x if it is the only supporting hyperplane of S containing x [8, pp. 70–74]. It will be said that an element x_0 of T is orthogonal to y ($x_0 \perp y$) if and only if $\|x_0 + ky\| \geq \|x_0\|$ for all k, which is equivalent to requiring the existence of a nonzero linear functional f such that $f(x_0) = \|f\|\,\|x_0\|$ and $f(y) = 0$, or that $x_0 + y$ belong to a supporting hyperplane of the sphere

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[1] Numbers in brackets refer to the references at the end of the paper.

[2] "Equivalent" meaning isometric under a linear transformation [1, p. 180]. The equivalence to (real) Hilbert space follows by reasoning similar to that of [10, pp. 3–16].

$\|x\| \leq \|x_0\|$ at the point x_0 [4, Theorem 2.1 and §5]. In a space with an inner product, $x \perp y$ if and only if $(x, y) = 0$.

Orthogonality is said to be ***additive on the right*** if and only if $z \perp x$ and $z \perp y$ imply $z \perp x+y$. Clearly $x \perp x$ implies $x = 0$, while $x \perp y$ implies $ax \perp by$ for any numbers a and b. Every element is orthogonal to at least one hyperplane through the origin, this hyperplane being unique for any given element if and only if: (1) For any x $(\neq 0)$ and y there is a unique number a with $x \perp ax+y$; (2) The unit sphere $\|x\| \leq 1$ of T has a tangent hyperplane at each point; (3) The norm is Gateaux differentiable; or (4) Orthogonality is additive on the right [4, Theorems 4.2, 5.1].

Orthogonality is said to be ***additive on the left*** if and only if $x \perp z$ and $y \perp z$ imply $x+y \perp z$. Orthogonality is not symmetric in general, and there does not necessarily exist a hyperplane orthogonal to a given element (Theorems 1 and 5). Additivity on the left does not imply strict convexity,[3] nor conversely, but a normed linear space is strictly convex if and only if: (1) For any x $(\neq 0)$ and y there is a unique number a with $ax+y \perp x$; or (2) No supporting hyperplane has more than one point of contact [4, Theorems 4.3, 5.2].

Birkhoff has shown that an inner product can be defined in a normed linear space of three or more dimensions if orthogonality is symmetric and unique.[4] An equivalent condition is that $N_+(x; y) = 0$ whenever $N_+(y; x) = 0$, where $N_+(x; y) = \lim_{h \to +0} [\|x+hy\| - \|x\|]/h$ exists because of the convexity of the function $f(h) = \|x+hy\|$ [4, Theorem 6.2]. It is possible to show by a purely geometric argument that in a space of three or more dimensions orthogonality must be unique if it is symmetric, but this follows more easily from known facts about projections in normed linear spaces:

THEOREM 1. *Orthogonality is symmetric in a normed linear space T of three or more dimensions if and only if an inner product can be defined in T.*

PROOF. Let x_1 and x_2 be any two elements of a three-dimensional subspace T_0 of T. Then there is an element $y \in T_0$ orthogonal to the linear hull H_0 of x_1 and x_2 [4, Theorem 7.1]. If orthogonality is symmetric, then $H_0 \perp y$. Hence if a projection of T_0 on H_0 is defined by $z = P(z) + a_z y$, where $P(z) \in H_0$, then $\|P(z)\| \leq \|z\|$ for all z and $\|P\| = 1$. But it is known that an inner product can be defined in a normed linear space of three or more dimensions if there is a projection of norm one on any given closed linear subspace [6, Theorem 3]. Thus an inner product can be defined in any three-dimensional subspace of T and hence in T itself [5].

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[3] A normed linear space is strictly convex if $\|x+y\| = \|x\| + \|y\|$ and $y \neq 0$ imply $x = ty$ for some t.

[4] See [2]. With symmetry, uniqueness means the uniqueness for any x $(\neq 0)$ and y of the number a for which $x \perp ax+y$.

For elements x and y of a normed linear space, $x \perp y$ if and only if there is a nonzero linear functional f such that $f(x)=\|f\|\,\|x\|$ and $f(y)=0$, while $ax+y\perp x$ if and only if $\|kx+y\|$ is minimum for $k=a$ [4, Theorems 2.1, 2.3]. Also, the set H of all z satisfying $f(z)=\|f\|$ is a supporting hyperplane of the unit sphere at x if $f(x)=\|f\|$ and $\|x\|=1$, while any supporting hyperplane can be defined by such an equation (see Mazur [8, p. 71]). Also, H is said to be parallel to an element y if and only if $f(y)=0$ (that is, the line $\{ky\}$ does not intersect H). Interpretations of Theorem 1 by means of linear functionals and hyperplanes therefore give the following necessary and sufficient conditions for the existence of an inner product in a normed linear space of three or more dimensions:

(1) *For any elements x and y, the existence of a nonzero linear functional f with $f(x)=\|f\|\,\|x\|$ and $f(y)=0$ implies the existence of a nonzero linear functional g with $g(y)=\|g\|\,\|y\|$ and $g(x)=0$.*

(2) *For any elements x and y, $\|kx+y\|$ is minimum when $k=-f(y)/f(x)$ if f is a linear functional with $f(x)=\|f\|\,\|x\|$.*

(3) *The existence of a supporting hyperplane of the unit sphere at x parallel to y ($\|x\|=\|y\|=1$) implies the existence of a supporting hyperplane at y parallel to x.*

There are infinitely many different normed linear spaces of two dimensions in which orthogonality is not symmetric [2, Theorem 4]. If an isomorphism $ax+by\leftrightarrow(a, b)$ is set up between the Cartesian plane and a two-dimensional normed linear space containing x and y ($\|x\|=\|y\|=1$) and if C is the "unit pseudo-circle" of all points (a, b) for which $\|ax+by\|=1$, then orthogonality is symmetric in T if and only if the line through the origin parallel to any supporting line of C at any point p cuts C in a point at which there is a supporting line parallel to the line from p to the origin. Let B_r ($r\geqq 1$) be the normed linear space of pairs $(x_1, x_2)=x$ of real numbers, where $\|x\|^r=(|x_1|^r+|x_2|^r)$ if x_1 and x_2 are of the same sign, and $\|x\|^s=(|x_1|^s+|x_2|^s)$ otherwise, where $s=r/(r-1)$. It can easily be verified that orthogonality is symmetric in B_r for $r\geqq 1$, and that it is unique except in the limiting case $r=1$. Thus orthogonality can be symmetric and not unique in a two-dimensional space.

THEOREM 2. *An inner product can be defined in a normed linear space of three or more dimensions if and only if orthogonality is additive on the left.*

PROOF. Let T be a normed linear space of three or more dimensions, and x_1 and x_2 be any two elements. Then there are hyperplanes H_1 and H_2 with $x_1 \perp H_1$ and $x_2 \perp H_2$. Let $M = H_1 \cap H_2$. If orthogonality is additive on the left, then $ax_1 + bx_2 \perp M$ for all a and b, and any element z has a unique representation in the form $z = P(z) + y$, where $y \in M$ and $P(z) = ax_1 + bx_2$. Also, $\|z\| \geqq \|P(z)\|$ for all z, and $\|P\| = 1$. Since there is a projection of norm one on any given two-dimensional linear subspace of T, it follows as for Theorem 1 that an inner product can be defined in T [6, Theorem 3].

The conclusion of the above theorem is not valid without the assumption that the space be of more than two dimensions, since it is clear that for a two-dimensional normed linear space orthogonality is additive on the left if and only if for any x ($\neq 0$) there is a unique nonzero element orthogonal to x. It therefore follows that orthogonality is additive on the left in a two-dimensional normed linear space if and only if the space is strictly convex [4, Theorem 4.3].

If L is a closed linear set in a Banach space B, then the ***normal projection*** of x on L is said to be the element u for which $x - u \perp L$, or for which $\|x - u\|$ is the distance from x to L. If L is finite-dimensional, or if the unit sphere of B is weakly compact, then normal projection is defined for all x and L [4, Theorem 7.2]. It was shown by Fortet [3, p. 45] that if orthogonality is symmetric in a uniformly convex Banach space, then normal projection is a continuous linear operation and the set H of points y with $y \perp x$ is linear and closed. However, it follows from the above theorems that H is linear for all x only if an inner product can be defined in the space R and that the existence of an inner product follows from symmetry of orthogonality. Also, $x \perp L$ if and only if there is a linear functional f with $f(x) = \|f\| \|x\|$ and $f(L) = 0$ [4, Theorem 2.1]. The following characterizations of inner product spaces of three or more dimensions are therefore direct consequences of Theorem 2.

(4) *The existence of a linear functional F with $F(x+y) = \|F\| \|x+y\|$ and $F(z) = 0$ whenever x, y, and z are such that there are linear functionals f and g with $f(x) = \|f\| \|x\|$, $g(y) = \|g\| \|y\|$, and $f(z) = g(z) = 0$.*

(5) *That normal projection be a linear operation.*

If a complete normed linear space has an inner product, then any linear functionals f and g can be written in the form $f(u) = (x, u)$ and $g(u) = (y, u)$, for some elements x and y [7, Theorem 11]. Then F of (4) can be taken as $f + g$. For any linear functional $G = Af + Bg$, there are then numbers a and b such that $G(ax + by) = \|G\| \|ax + by\|$. This condition is also sufficient for an inner product:

Theorem 3. *An inner product can be defined in a normed linear space T of three or more dimensions if and only if it follows from $f(x)=\|f\|\,\|x\|$ and $g(y)=\|g\|\,\|y\|$ for linear functionals f and g and elements x and y of T that there are numbers a and b such that $f(ax+by)+g(ax+by)=\|f+g\|\,\|ax+by\|$* and $ax+by\neq 0$.

Proof. First note that if for some x there are two nonzero linear functionals F and G with $F(x)=\|F\|\,\|x\|$ and $G(x)=\|G\|\,\|x\|$, then the assumption of the theorem would imply that $|h(x)|=\|h\|\,\|x\|$ if $h=\|G\|F-\|F\|G$. But this is clearly impossible unless $h=0$, or $\|G\|F=\|F\|G$. Thus two independent linear functionals cannot take on their maximum in the unit sphere $\|x\|\leq 1$ at the same point, which is known to imply that the unit sphere has a tangent hyperplane at each point [4, Theorem 5.1]. Now suppose that $x\perp z$ and $y\perp z$, and let T_0 be the linear hull of x, y, and z. There are then two linear functionals f and g with $f(x)=\|f\|\,\|x\|$, $g(y)=\|g\|\,\|y\|$, and $f(z)=g(z)=0$ [4, Theorem 2.1]. If x and y are not linearly independent, then $x+y\perp z$. Let x and y be linearly independent and suppose that for $u=x+y$ there are no numbers A and B satisfying $|Af(u)+Bg(u)|=\|Af+Bg\|\,\|u\|$. Let C be the curve of all elements $ax+by$ with $\|ax+by\|=1$. Then there are elements x' and y' on either side of $(x+y)/\|x+y\|$ and in C for which there are linear functionals $f'=A_1f+B_1g$ and $g'=A_2f+B_2g$ with $f'(x')=\|f'\|\,\|x'\|$ and $g'(y')=\|g'\|\,\|y'\|$, but such that none of the linear functionals $Af'+Bg'$ satisfy $|Af'[rx'+(1-r)y']+Bg'[rx'+(1-r)y']|=\|Af'+Bg'\|\,\|rx'+(1-r)y'\|$ for any r with $0<r<1$. For each such r, there is a number α_r for which $[rx'+(1-r)y'+\alpha_r z]/\|rx'+(1-r)y'+\alpha_r z\|=v\perp z$ [4, Theorem 2.3]. If h is a linear functional defined in T_0 for which $h(v)=\|h\|\,\|v\|$ and $h(z)=0$, and if A_r and B_r are such that $A_rf'(z_0)+B_rg'(z_0)=0$ for some $z_0\in T_0$ for which $h=0$ but not both f' and g' are zero, then h and $A_rf'+B_rg'$ are both zero at z_0 and z and hence are multiples of each other on T_0. Then if a_r and b_r are chosen by the assumptions of the theorem so that $\|a_rx'+b_ry'\|=1$ and $|A_rf'(a_rx'+b_ry')+B_rg'(a_rx'+b_ry')|=\|A_rf'+B_rg'\|$, it follows that h is a multiple of $A_rf'+B_rg'$ and that $h(a_rx'+b_ry')|=\|h\|\,\|a_rx'+b_ry'\|$. Thus the unit sphere S contains the straight lines l_r between $a_rx'+b_ry'$ and v, since the unit sphere is convex and the tangent hyperplane defined by $h(x)=\|h\|$ contains $a_rx'+b_ry'$ and v. This tangent hyperplane at v then contains this line, but does not contain a point of C between x' and y'. But there are also tangent hyperplanes at x' and y' parallel to z, while $a_rx'+b_ry'$ is by assumption not of the form $[rx'+(1-r)y']/\|rx'+(1-r)y'\|$ for any r satisfying $0<r<1$. This implies that the tangent hyperplane at v contains either x' or y' and is coincident with the tangent hyperplane at x' or y', respectively. Letting r vary from 0 to 1, it now follows from the convexity of S that the tangent hyperplanes at x' and at y' have a common point of contact and must therefore coincide, since S has a tangent hyperplane at each point. This tangent hyperplane then contains the line from x' to y', and $f'(x+y)=\|f'\|\,\|x+y\|$, contrary to assumption. Hence there are numbers A and B with $|Af(x+y)+Bg(x+y)|=\|Af+Bg\|\,\|x+y\|$. Since $Af(z)+Bg(z)=0$, this implies that $x+y\perp z$ and that orthogonality is additive on the left. It now follows from Theorem 2 that an inner product can be defined in T.

For any element x of a normed linear space there is always a hyperplane H through the origin with $x\perp H$. However, for no hyperplane H of the space[5] C of continuous functions is there an element $f\in C$ with $H\perp f$. This follows from the fact that $g\perp f$ if and only if $\min_A gf\leqq 0\leqq \max_A gf$, where A is the set of all t with $|g(t)|=\|g\|$ [4, §4]. If T is one of the spaces[5] (s), (m), (c), or $l^{(p)}(p\geqq 1)$, then clearly $H\perp x$ for an infinite number of different hyperplanes H and elements x. If a normed linear space is strictly convex, then for no element x is there more than one hyperplane H with $H\perp x$, while no hyperplane is orthogonal to more than one element if the norm of T is differentiable [4, Theorems 4.2, 4.3]. This difference is the reason for the lack of similarity between the proofs of the following theorems.

THEOREM 4. *An inner product can be defined in a normed linear space of three or more dimensions if and only if each hyperplane through the origin is orthogonal to at least one element.*

PROOF. Let x_1 and x_2 be any two elements of a normed linear space T of three or more dimensions, and let P_0 be the linear hull of x_1 and x_2. By well-ordering the set of all linear subspaces M of T for which $P_0\perp M$, it follows that there is a linear subspace $\bar{M}$ of T such that $P_0\perp \bar{M}$ and $\bar{M}$ is not contained properly in any other such linear subspace. Then it is clear that $\bar{M}$ is closed. Hence if the linear hull H of P_0 and $\bar{M}$ were not T, there would be a hyperplane through the origin which contains P_0 and $\bar{M}$. If every hyperplane through the origin is orthogonal to some element, then there would be an element x such that $H\perp x$. But if $y=x_p+x_m+kx$, where $x_p\in P_0$ and $x_m\in\bar{M}$, then $\|y\|\geqq\|x_p+x_m\|\geqq\|x_p\|$, since $(x_p+x_m)\perp x$ and $x_p\perp x_m$. Thus P_0 would be orthogonal to the linear hull of $\bar{M}$ and x. Hence the linear hull of P_0 and $\bar{M}$ must be T. A projection $P(z)$ of T on P_0 can now be defined by $z=P(z)+z_m$, where $P(z)\in P_0$ and $z_m\in\bar{M}$. Since $\|P\|=1$, it follows that there is a projection of unit norm on any given two-dimensional linear subspace of T and hence (as in the proof of Theorem 1) that an inner product can be defined in T.

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[5] The notation is that of Banach [1, pp. 10–12].

THEOREM 5. *An inner product can be defined in a normed linear space T of three or more dimensions if and only if for any $x\in T$ there is a hyperplane H through the origin with $H \perp x$.*

PROOF. Suppose x, y, and z are any three elements of T with $x\perp z$ and $y\perp z$. If T is strictly convex, then for any u and v of T there is a unique a such that $au+v\perp u$ [4, Theorem 4.3]. Hence if H is a hyperplane through the origin with $H\perp z$, and if T is strictly convex, then $x\in H$ and $y\in H$. Thus $x+y\in H$ and $x+y\perp z$, orthogonality is additive on the left, and an inner product can be defined in T. Now suppose T is not strictly convex. Then there are elements x and y and a linear functional f with $f(x)=f(y)=\|f\|$ and $\|x\|=\|y\|=1$ [9, Theorem 6]. Let z be any other element of unit norm not in the linear set generated by x and y and let S_0 be the unit sphere of the space T_0 generated by x, y, and z. Let P_0 be the set of all points $u\in S_0$ for which $\|u\|=1$ and $f(u)=\|f\|$. Then P_0 contains the line from x to y, and is itself either a straight line segment or a section of a plane. Let L_0 be the hyperplane of T_0 with $P_0\perp L_0$, where L_0 contains all points at which f is zero. Then for any v and each number a there is a hyperplane H_a of T_0 with $H_a\perp v+ax$. As $a\to+0$ (or as $a\to-0$), the planes H_a will have at least one limit H_+ (or H_-) in the sense that there exist sequences $\{a_i\}$ and $\{b_i\}$, with $a_i\to+0$ and $b_i\to-0$, $\lim_{a_i\to+0}\rho(w, H_{a_i})=0$ and $\lim_{b_i\to-0}\rho(w, H_{b_i})=0$, if w is any fixed element of H_+ or H_-, respectively. Since at each point of unit norm in H_a there is a supporting plane of S_0 parallel to $v+ax$, it follows that if $v\in L_0$ then neither H_+ nor H_- crosses P_0, and P_0 consists of those and only those points of the surface of S_0 in a region containing x and bounded by H_+, H_-, and the two supporting lines of P_0 parallel to v. But this is possible for arbitrary $v\in L_0$ only if P_0 is a point.

Theorems 3–5 can be given direct interpretations by means of supporting hyperplanes of the unit sphere S, as was done for Theorem 1 to get (3). The first of these interpretations can be changed somewhat to give the following nontrivial consequence of Theorem 3.

THEOREM 6. *An inner product can be defined in a Banach space if every supporting hyperplane of the unit sphere S has a point of contact and the existence of supporting hyperplanes H_1 and H_2 at points x and y of S imply that any supporting hyperplane H_3 of S satisfying $H_1\cap H_2\cap H_3=0$ have a point of contact which is in the linear hull of x and y.*

PROOF. First suppose that there is an element x and nonzero linear functionals f_1 and f_2 such that $\|x\|=1$, $f_1(x)=\|f_1\|$, and $f_2(x)=\|f_2\|$. Then x is in both of the supporting hyperplanes H_1 and H_2 of S, where H_1 and H_2 are defined by $f_1(z)=\|f_1\|$ and $f_2(z)=\|f_2\|$. If L is the set of points at which $f_1=f_2=0$, then $H_1 \cap H_2=x+L$. If f_1 and f_2 are linearly independent, then the linear hull of x and L is not the whole space and there is a nonzero linear functional f_3 which is zero on x and L. Let H_3 be defined by $z \in H_3$ if and only if $f_3(z)=\|f_3\|$. Then clearly $H_1 \cap H_2 \cap H_3=0$. But the second hypothesis of the theorem would imply that $x \in H_3$, or $f_3(x)=\|f_3\|$, which contradicts $f_3(x)=0$. Therefore f_1 and f_2 are linearly dependent.

Now suppose that $\|x\|=\|y\|=1$, $f_1(x)=\|f_1\|$, and $f_2(y)=\|f_2\|$. If $f_3=f_1+f_2$, and H_1, H_2, H_3 are defined by $f_i(z)=\|f_i\|$ $(i=1, 2, 3)$, then $x \in H_1$ and $y \in H_2$. If $H_1 \cap H_2 \cap H_3 \neq 0$, then there exists an element w such that $f_1(w)=\|f_1\|$, $f_2(w)=\|f_2\|$, and $f_1(w)+f_2(w)=\|f_1+f_2\|$. Thus $\|f_1+f_2\|=\|f_1\|+\|f_2\|$. Since every linear functional in T takes on its maximum in the unit sphere, H_3 contains a point z of norm 1. Then $f_1(z)+f_2(z)=\|f_1+f_2\|=\|f_1\|+\|f_2\|$. Therefore $f_1(z)=\|f_1\|$ and $f_2(z)=\|f_2\|$. Hence f_1 and f_2 must be linearly dependent, and $f_1(x)+f_2(x)=\|f_1+f_2\|$. If $H_1 \cap H_2 \cap H_3=0$, then H_3 has a point of contact $ax+by$ $(\|ax+by\|=1)$ and $f_3(ax+by)=\|f_3\|\ \|ax+by\|$. Thus it follows from Theorem 3 that an inner product can be defined.

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TERNARY BOOLEAN ALGEBRA

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1. Introduction. The present paper[1] is concerned with a ternary operation in Boolean algebra. We assume a degree of familiarity with the latter [1, 2],[2] and by the former we shall mean simply a function of three variables defined for elements of a set K whose values are also in K. Ternary operations have been discussed in groupoids [4] and groups [3]; in Boolean algebra an operation different from the one introduced here was discussed by Whiteman [5].

By a simple set of postulates (§2), we define a ternary system, which we call a *ternary Boolean algebra*, from which Boolean algebras are obtained by specialization of the ternary operation, and which itself may be considered as a more general binary system with as many binary operations as elements, each having the properties of the Boolean operations (§4). The ternary Boolean algebra is homogeneous and there is a one-to-one correspondence between distinct ternary algebras and abstract Boolean algebras (§5, §7); thus the ternary algebra provides a new postulational approach to Boolean algebra. The ternary operation has a unique realization in Boolean algebra (§6). Other applications of ternary operations and the matter of a valid representation for ternary Boolean algebra are left to a subsequent paper.

2. Postulates for ternary Boolean algebra. Let K be a system consisting of a set of elements $a, b, \cdots$, and two operations under which the system is closed, one ternary, a^bc, and the other unitary a'. These satisfy the following relations for all a, b, c, d, and e:

$$a^b(c^de) = (a^bc)^d(a^be), \tag{2.1}$$

$$a^bb = b^ba = b, \tag{2.2}$$

$$a^bb' = b'^ba = a. \tag{2.3}$$

The system thus defined we shall call a *ternary Boolean algebra*.

It is easily verified that the following function in Boolean algebra satisfies the postulates:

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

$$(a \cap b) \cup (b \cap c) \cup (c \cap a). \tag{2.4}$$

The system K possesses a realization therefore and the postulates are consistent. By specifying zero and unit elements, 0 and 1, and then restricting the ternary product $a^b c$ to $a \cap b = a^0 b$ and $a \cup b = a^1 b$, Boolean algebra arises as a special case of ternary Boolean algebra (cf. §4).

The principal property of K is expressed by postulate (2.1); it is what we shall call *distributive*. The other two postulates are reducibility conditions, which state that under certain conditions the ternary product becomes a function of only one of the three elements involved. They are analogous to the identity and absorption relations in binary Boolean algebra.

The propositions of Boolean algebra exhibit a duality in the operations. In the ternary algebra this is replaced by homogeneity; we do not have elements with special significance such as 0 and 1. The fact that K is homogeneous (that is, that all elements have equivalent properties) will be discussed in detail later.

3. Theorems in ternary Boolean algebra.

A few basic theorems that we require are proved here; the proofs are for the most part analogous to those of the corresponding theorems in Boolean algebra.

THEOREM 3.1. *For each element b the element b' is unique.*

For if for some b there were two distinct complements b_1' and b_2' we have by postulate (2.3):

$$b_1' = b_1'{}^b b_2' = b_2'.$$

THEOREM 3.2. *The idempotent law holds, $a^b a = a$.*

$$\begin{aligned} a^b a &= (a^b b')^b (a^b b') && \text{by } (2.3) \\ &= a^b (b'{}^b b') && (2.1) \\ &= a^b b' && (2.3) \\ &= a && (2.3). \end{aligned}$$

THEOREM 3.3. *The associative law, $a^b (c^b d) = (a^b c)^b d$, holds for any derived binary operation b.*

The proof requires a lemma, $(a^b c)^b a = a^b c$:

$$\begin{aligned} (a^b c)^b a &= (a^b c)^b (a^b b') = a^b (c^b b') \\ &= a^b c. \end{aligned} \tag{3.31}$$

Then the theorem follows by:

$$\begin{aligned} a^{b}(c^{b}d) &= (a^{b}c)^{b}(a^{b}d) && (2.1)\\ &= [(a^{b}c)^{b}a]^{b}[(a^{b}c)^{b}d] && (2.1)\\ &= (a^{b}c)^{b}[(a^{b}c)^{b}d] && (3.31)\\ &= [(a^{b}c)^{b}b']^{b}[(a^{b}c)^{b}d] && (2.3)\\ &= (a^{b}c)^{b}d && (2.1)(2.3). \end{aligned}$$

THEOREM 3.4. $(a')'=a$.

We use the lemma $a^{b'}b=a$:

$$a^{b'}b = (a^{b}b')^{b'}(a^{b}b) = a^{b}(b'^{b'}b) = a.$$

The theorem may be proved using (2.3):

$$(a')' = (a')'^{a'}a = a.$$

THEOREM 3.5. $a^{b}a'=b$.

$$\begin{aligned} a^{b}a' &= a^{b}(b^{a'}a') && (2.2)\\ &= (a^{b}b)^{a'}(a^{b}a') && (2.1)\\ &= b^{a'}(a^{b}a') && (2.2)\\ &= (b^{a'}a)^{b}(b^{a'}a') && (2.1)\\ &= b^{b}(b^{a'}a') && (2.3)(3.4)\\ &= b && (2.2). \end{aligned}$$

THEOREM 3.6. *The operation $a^{b}c$ is commutative in any pair of elements.*

(a)

$$\begin{aligned} a^{b}c &= a^{b}(a^{c}a') && (3.5)\\ &= (a^{b}a)^{c}(a^{b}a') && (2.1)\\ &= a^{c}b && (3.2)(3.5). \end{aligned}$$

(b)

$$\begin{aligned} a^{b}c &= a^{b}(b^{c}b') && (3.5)\\ &= (a^{b}b)^{c}(a^{b}b') && (2.1)\\ &= b^{c}a && (2.2)(2.3). \end{aligned}$$

(c) $a^{b}c=c^{b}a$. This follows from the propositions (a) and (b).

We shall call a ternary operation in which each pair of elements may be interchanged without changing its value completely commutative. If one pair of elements may be interchanged, but the others may not, then we have partial commutativity.

THEOREM 3.7. *For any x of K,*

$$a^{b}c = (a^{x}b)^{x'}(b^{x}c)^{x'}(c^{x}a).$$

PROOF.

$$
\begin{aligned}
a^b c &= a^b(x^c x') && (3.5)\\
&= (a^b x)^c(a^b x') && (2.1)\\
&= (a^x b)^c(a^{x'} b) && (3.6)\\
&= [(a^x b)^c(a^x x')]^{x'}[(a^x b)^c(x'^x b)] && (2.1)(2.3)\\
&= [a^x(b^c x')]^{x'}[(a^c x')^x b] && (2.1)\\
&= [(a^x b)^{x'}(a^x c)]^{x'}[(a^x b)^{x'}(c^x b)] && (3.6)(2.1)\\
&= (a^x b)^{x'}(b^x c)^{x'}(c^x a) && (3.2)(3.6).
\end{aligned}
$$

4. **Associated Boolean algebras.** Let p be a fixed element of K. Define

$$a \cap b = a^p b, \qquad a \cup b = a^{p'} b \tag{4.1}$$

and refer to the system formed by the elements of K and the operations $\cap$ and $\cup$ as $B(p)$. We may prove:

THEOREM I. *The system $B(p)$ forms a Boolean algebra with p as its universe element and p' as its null element.*

The operations $\cap$ and $\cup$ are commutative by (3.6), associative by (3.3), and distributive by (2.1). That p and p' are respectively the identities of multiplication and addition is easily verified by (2.3), and complementary elements satisfy the relations (3.5). Thus the system $B(p)$ has the properties of a Boolean algebra.

Theorem I enables us to translate theorems of Boolean algebra into theorems in K. For if we have a theorem in Boolean algebra, for example, deMorgan's theorem:

$$(a \cap b)' = a' \cup b', \tag{4.2}$$

it must hold in $B(p)$ since the latter is a Boolean algebra. But by (4.1) this becomes

$$(a^p b)' = a'^{p'} b', \tag{4.3}$$

for any p of K, and so we may state the following theorem.

THEOREM 4.3. *K possesses the deMorgan property* (4.3).

It will be noted that deMorgan's theorem takes a symmetric form in K and expresses the fact that the unitary operation $'$ is distributive over the ternary operation $a^b c$. In order to find the complement of an expression in K it is necessary only to put primes on the unprimed letters and unprime the primed ones.

Theorem I allows a further observation. For a fixed b, the ternary product $a^b c$ is a binary operation which has the properties of a Boolean binary operation. Since for each b there is such an operation, K may also be considered as a binary system in which there are as many Boolean operations as elements, and every pair of these operations satisfy the Boolean distributive law.

5. **Automorphisms of the ternary Boolean algebra.** We stated that K is homogeneous in its elements, meaning that every element of K has the same properties as any other. This fact will now be expressed in a concrete form in terms of automorphisms.

A function $f(x)$ is an automorphism of K if $f(x)$ is a one-to-one transformation of K on to itself leaving invariant the ternary operation and the operation of complementation. If the former is preserved, so is the latter; for if $f(x)$ preserves the ternary operation,

$$(5.1) \qquad f(x) = f(x^y y') = f(x)^{f(y)} f(y')$$

for all x and y. If x, y, and y' are distinct, so are $f(x)$, $f(y)$, and $f(y')$. By (2.3) and (3.1) it follows therefore that $[f(y)]' = f(y')$.

The function below will be shown by a sequence of lemmas to express an automorphism of K transforming a into b,

$$(5.2) \qquad f(x) = (b'^a x)^{a'}(b^a x').$$

LEMMA 1. $f(a) = b$.

This follows immediately by substitution.

LEMMA 2. $f(x)^a f(y) = (b'^a x^a y)^{a'}(b^a x'^a y')$.

This follows from the definition of $f(x)$ and the distributive law.

LEMMA 3. $f(x^y z) = f(x)^{f(y)} f(z)$.

$$\begin{aligned} f(x^y z) &= [b'^a(x^y z)]^{a'}[b^a(x'^{y'} z')] && (5.2)(4.3) \\ &= [b'^a\{(x^a y)^{a'}(x^a z)^{a'}(y^a z)\}]^{a'}\ [b^a\{(x'^a y')^{a'}(x'^a z')^{a'}(y'^a z')\}] && (3.7). \end{aligned}$$

By the results of §§2–3, simplification, and use of Lemma 2 we obtain

$$\begin{aligned} f(x^y z) &= [f(x)^a f(y)]^{a'}[(f(y)^a f(z)]^{a'}[f(z)^a f(x)] \\ &= f(x)^{f(y)} f(z) && (3.7). \end{aligned}$$

LEMMA 4. $f[f(x)] = x$.

This follows by direct substitution and simplification.

THEOREM II. *For given a and b, there exists an automorphism of K transforming a into b.*

PROOF. By Lemma 1, the function (5.2) carries a into b, by Lemma 3, it leaves invariant the ternary operation and hence complementation, and by Lemma 4, it has a single-valued inverse, namely, itself.

COROLLARY. *If p and q are any two elements of K, the Boolean algebras $B(p)$ and $B(q)$ are isomorphic.*

It is necessary only to consider the automorphism of K that transforms p into q. This transforms the operations of $B(p)$ into the corresponding ones of $B(q)$.

6. Realizations of the ternary algebra.

We have pointed out that the function $(a\cap b)\cup(b\cap c)\cup(c\cap a)$ in Boolean algebra satisfies the postulates for the ternary algebra. It will now be shown that this is the only realization of the ternary product a^bc in an associated Boolean algebra $B(p)$. For Theorem 3.7 states an identity in x; the value of a^bc is independent of the value of x and we may assign the latter at pleasure. Let $x=p$. Then in $B(p)$ we must have

$$a^bc = (a\cap b)\cup(b\cap c)\cup(c\cap a). \tag{6.1}$$

Hence the theorem:

THEOREM III. *The only realisation of the ternary operation in a Boolean algebra is*

$$a^bc = (a\cap b)\cup(b\cap c)\cup(c\cap a).$$

7. Relationship of the ternary and binary algebras.

We see from the preceding two paragraphs that there exists a one-to-one correspondence between a ternary Boolean algebra and an abstract Boolean algebra. Nonisomorphic Boolean algebras give rise to nonisomorphic ternary Boolean algebras and conversely.

It follows that Boolean algebra is fully characterized by the postulates (2.1)–(2.3) and the definitions (4.1).

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CONCERNING AUTOMORPHISMS OF NON-ASSOCIATIVE ALGEBRAS

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In their studies of non-associative algebras A. A. Albert and N. Jacobson have made much use of the relationships which exist between an arbitrary non-associative algebra $\mathfrak{A}$ and its associative transformation algebra $T(\mathfrak{A})$. In this paper we are interested in the automorphism group $\mathfrak{G}$ of $\mathfrak{A}$, and we sharpen the results of Jacobson [3, §4][1] and Albert [2, §9] in the sense that we prove $\mathfrak{G}$ isomorphic to a *well-defined* subgroup of the automorphism group of each of three associative algebras (§§2, 3).

Incidental to our proofs is the reconstruction (in the sense of equivalence) of an arbitrary non-associative algebra $\mathfrak{A}$ with unity element 1 from $T(\mathfrak{A})$ and from either of the enveloping algebras $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$ of respectively the right or left multiplications of $\mathfrak{A}$. This paper has been expanded in accordance with suggestions of the referee to include a more detailed study of the right ideals used in this reconstruction process (§5).

1. **Preliminaries.** Our notations are chiefly those of Albert as given in [1]. We regard a non-associative algebra $\mathfrak{A}$ of order n over a field $\mathfrak{F}$ as consisting of a linear space $\mathfrak{L}$ of order n over $\mathfrak{F}$, a linear space $R(\mathfrak{A})$ of linear transformations R_x on $\mathfrak{L}$ of order $m \leq n$ over $\mathfrak{F}$, and a linear mapping of $\mathfrak{L}$ on $R(\mathfrak{A})$,

$$x \rightarrow R_x. \tag{1}$$

The elements R_x of $R(\mathfrak{A})$ are called *right multiplications*, and $R(\mathfrak{A})$ the *right multiplication space* of $\mathfrak{A}$. Multiplication in $\mathfrak{A}$ is defined by

$$a \cdot x = aR_x. \tag{2}$$

The linearity of the right multiplications and of (1) insures distributivity in $\mathfrak{A}$ as well as the usual laws of scalar multiplication. We shall use the fact that, in case $\mathfrak{A}$ contains no absolute right divisor of zero (an element x such that $a \cdot x = 0$ for all a in $\mathfrak{A}$), the mapping (1) is nonsingular and the order of $R(\mathfrak{A})$ over $\mathfrak{F}$ is n.

The linear transformations L_x defined by

$$a \rightarrow x \cdot a = aL_x \tag{3}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

are called *left multiplications* of $\mathfrak{A}$ and form the *left multiplication space* $L(\mathfrak{A})$ of $\mathfrak{A}$. The algebra $\mathfrak{A}$ may equally well be regarded as consisting of $\mathfrak{L}$, $L(\mathfrak{A})$, and the linear mapping

$$x \to L_x \tag{4}$$

of $\mathfrak{L}$ on $L(\mathfrak{A})$. Both $R(\mathfrak{A})$ and $L(\mathfrak{A})$ are linear subspaces of the total matric algebra $(\mathfrak{F})_n$ of all linear transformations on $\mathfrak{L}$.

If $\mathfrak{M}$ is a subset of $(\mathfrak{F})_n$, the algebra of all polynomials in the transformations in $\mathfrak{M}$ with coefficients in $\mathfrak{F}$ is called the *enveloping algebra* of $\mathfrak{M}$, and is denoted by $E(\mathfrak{M})$. We are particularly concerned with the enveloping algebras $E(R(\mathfrak{A}))$ and $E(L(\mathfrak{A}))$ of respectively the right and left multiplications of $\mathfrak{A}$, and with the *transformation algebra* $T(\mathfrak{A}) = E(I, R(\mathfrak{A}), L(\mathfrak{A}))$ which is the algebra of all polynomials with coefficients in $\mathfrak{F}$ in the right and left multiplications of $\mathfrak{A}$ and the identity transformation I in $(\mathfrak{F})_n$. We shall have occasion to write an arbitrary element T of each of these algebras as follows:

$$T = f(R_x, R_y, \cdots) \qquad \text{for } T \text{ in } E(R(\mathfrak{A})), \tag{5}$$

$$T = f(L_x, L_y, \cdots) \qquad \text{for } T \text{ in } E(L(\mathfrak{A})), \tag{6}$$

$$T = f(I, R_x, L_x, R_y, \cdots) \qquad \text{for } T \text{ in } T(\mathfrak{A}), \tag{7}$$

where $x, y, \cdots$ are elements of $\mathfrak{A}$. In case $\mathfrak{A}$ contains a unity element 1, then $R(\mathfrak{A})$ contains I, and we may write

$$T = f(R_x, L_x, R_y, \cdots) \qquad \text{for } T \text{ in } T(\mathfrak{A}), \tag{8}$$

$x, y, \cdots$ in $\mathfrak{A}$.

If $\mathfrak{B}$ is a linear subspace of $\mathfrak{A}$, the set of all R_b for b in $\mathfrak{B}$ is a linear subspace $R(\mathfrak{B}, \mathfrak{A})$ of $R(\mathfrak{A})$, and the set of all L_b is a linear subspace $L(\mathfrak{B}, \mathfrak{A})$ of $L(\mathfrak{A})$.

An *automorphism* S of an algebra $\mathfrak{A}$ is a nonsingular linear transformation $x \to xS$ of $\mathfrak{A}$ on itself such that

$$(a \cdot x)S = aS \cdot xS \tag{9}$$

for all a, x in $\mathfrak{A}$. In terms of right and left multiplications, (9) may be written equivalently as

$$R_x S = S R_{xS} \tag{10}$$

or

$$L_x S = S L_{xS} \tag{11}$$

for all x in $\mathfrak{A}$. We shall use the facts that, if S is an automorphism of $\mathfrak{A}$, then S^{-1} is also, and if $\mathfrak{A}$ has a unity element 1, then $1S = 1$.

Inasmuch as the elements T of subalgebras of $(\mathfrak{F})_n$ are themselves linear transformations, we shall denote linear transformations on subalgebras of $(\mathfrak{F})_n$—such as $T(\mathfrak{A})$, $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$—by Greek capitals, so that if Σ is a linear transformation on $T(\mathfrak{A})$, say, we may write (without confusion) the image of T under Σ as $T\Sigma$.

An automorphism S of $\mathfrak{A}$ determines an automorphism Σ of $T(\mathfrak{A})$ as follows: let T in $T(\mathfrak{A})$ be written in the form (7); then Σ is the mapping

$$T \rightarrow T\,\Sigma = f(I, R_{xS}, L_{xS}, R_{yS}, \cdots) = S^{-1}TS. \tag{12}$$

For if S is an automorphism of $\mathfrak{A}$, then $R_x = SR_{xS}S^{-1}$, $L_x = SL_{xS}S^{-1}$ by (10), (11), and $T = f(I, R_x, L_x, R_y, \cdots) = f(I, SR_{xS}S^{-1}, SL_{xS}S^{-1}, SR_{yS}S^{-1}, \cdots) = Sf(I, R_{xS}, L_{xS}, R_{yS}, \cdots)S^{-1} = S(T\Sigma)S^{-1}$, or $T\Sigma = S^{-1}TS$. The mapping (12) is obviously an automorphism of $T(\mathfrak{A})$. Moreover, Σ induces automorphisms (which we do not distinguish notationally from Σ) on the subalgebras $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$ of $T(\mathfrak{A})$:

$$T \rightarrow T\,\Sigma = f(R_{xS}, R_{yS}, \cdots), \qquad T \text{ in } E(R(\mathfrak{A})) \text{ as in (5)}, \tag{13}$$

$$T \rightarrow T\,\Sigma = f(L_{xS}, L_{yS}, \cdots), \qquad T \text{ in } E(L(\mathfrak{A})) \text{ as in (6)}. \tag{14}$$

If S determines Σ as in (12), then

$$R(\mathfrak{A})\,\Sigma = R(\mathfrak{A}), \qquad L(\mathfrak{A})\,\Sigma = L(\mathfrak{A}), \tag{15}$$

since $R_x\Sigma = R_{xS}$ in $R(\mathfrak{A})$ while $L_x\Sigma = L_{xS}$ in $L(\mathfrak{A})$, and the nonsingularity of Σ eliminates the possibility of proper inclusion.

2. **Automorphisms of an algebra with unity element.** Let $\mathfrak{A}$ be a non-associative algebra of order n over $\mathfrak{F}$ with unity element 1. We consider the elements of $\mathfrak{A}$ as comprising a linear space $\mathfrak{L}$ of order n over $\mathfrak{F}$. Let $\mathfrak{B}$ be any (associative) algebra of linear transformations on $\mathfrak{L}$ which contains either $R(\mathfrak{A})$ or $L(\mathfrak{A})$. We intend to reconstruct $\mathfrak{A}$ (in the sense of equivalence) as an algebra of residue classes of $\mathfrak{B}$.

Denote by $\mathfrak{N}$ the set of all transformations N in $\mathfrak{B}$ which annihilate 1, that is, for which $1N = 0$. Then $\mathfrak{N}$ is a right ideal of $\mathfrak{B}$. For if N, N_1 are in $\mathfrak{N}$, then $1(\alpha N + \beta N_1) = \alpha 1N + \beta 1N_1 = 0$ for α, β in $\mathfrak{F}$, while $1NT = 0T = 0$ for any transformation T in $\mathfrak{B}$. Denote by $\mathfrak{D}$ whichever set $R(\mathfrak{A})$ or $L(\mathfrak{A})$ is assumed to be contained in $\mathfrak{B}$, and by D_x correspondingly the transformation R_x or L_x. Then $\mathfrak{B}$ is the supplementary sum $\mathfrak{B} = \mathfrak{D} + \mathfrak{N}$. For T in $\mathfrak{B}$ may be written uniquely in the form $T = D_t + N$, $1T = t$, N in $\mathfrak{N}$.

Since $1TN = tN$ which is not necessarily zero, $\mathfrak{N}$ is not in general a two-sided ideal of $\mathfrak{B}$ and we are not able to form the difference algebra $\mathfrak{B} - \mathfrak{N}$ when we take residue classes $[T]$ modulo $\mathfrak{N}$. Instead we form the difference group $\mathfrak{B}-\mathfrak{N}$ of residue classes $[T]$ modulo $\mathfrak{N}$ and have as usual a linear set over $\mathfrak{F}$ with respect to the operations $[T]+[U] = [T+U]$, $\lambda[T] = [\lambda T]$ of addition and scalar multiplication. Define multiplication in this linear set as follows:

$$[T][U] = [X], \qquad \text{where } 1X = 1T\cdot 1U, \tag{16}$$

where the multiplication on the right is that in $\mathfrak{A}$. To see that for any T, U in $\mathfrak{B}$ such an X exists, we need only to note that, if $X = D_x$ for $x = 1T\cdot 1U$ in $\mathfrak{A}$, then $1X = 1T\cdot 1U$. This definition of multiplication is independent of the representatives T, U since if $[T] = [T_1]$, $[U] = [U_1]$, then there exist N, N_1 in $\mathfrak{N}$ such that $T_1 = T+N$, $U_1 = U+N_1$, and $1T_1\cdot 1U_1 = 1(T+N)\cdot 1(U+N_1) = 1T\cdot 1U$. With this multiplication the distributive laws hold in $\mathfrak{B}-\mathfrak{N}$. Hence $\mathfrak{B}-\mathfrak{N}$ is a non-associative algebra over $\mathfrak{F}$. Since there are no difference algebras used in this paper, there should be no confusion in the use of the notation $\mathfrak{B}-\mathfrak{N}$ for this algebra with multiplication defined by (16).

THEOREM 1. *Let $\mathfrak{A}$ be a non-associative algebra over $\mathfrak{F}$ with unity element* 1, *and $\mathfrak{B}$ be any* (*associative*) *algebra of linear transformations on $\mathfrak{A}$ containing either $R(\mathfrak{A})$ or $L(\mathfrak{A})$. If $\mathfrak{N}$ is the right ideal of transformations in $\mathfrak{B}$ annihilating* 1, *then the non-associative algebra $\mathfrak{B}-\mathfrak{N}$ with multiplication defined by* (16) *is equivalent to $\mathfrak{A}$.*

For in each residue class $[T]$ there is a unique transformation D_t in $\mathfrak{D}$ ($=R(\mathfrak{A})$ or $L(\mathfrak{A})$) such that $1T = 1D_t = t$. Then, since $\mathfrak{A}$ contains neither absolute right nor absolute left divisors of zero, the (obviously linear) mapping

$$x \to D_x \to [D_x] \tag{17}$$

is one-to-one on $\mathfrak{A}$ to $\mathfrak{B}-\mathfrak{N}$. But

$$[D_x][D_y] = [D_{xy}], \qquad x, y \text{ in } \mathfrak{A}, \tag{18}$$

since $xy = 1D_x\cdot 1D_y = 1D_{xy}$. Then (17) is an equivalence of $\mathfrak{A}$ and $\mathfrak{B}-\mathfrak{N}$ since $xy \to D_{xy} \to [D_{xy}] = [D_x][D_y]$ under (17).

Now $T(\mathfrak{A})$, $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$ are among the algebras of linear transformations on the vector space $\mathfrak{L}$ underlying $\mathfrak{A}$ which contain either $R(\mathfrak{A})$ or $L(\mathfrak{A})$—or both, as in the case of $T(\mathfrak{A})$—and may be used as the algebra $\mathfrak{B}$ in Theorem 1. We denote by $\mathfrak{N}_T$ the set of all N in $T(\mathfrak{A})$ annihilating 1 and write $\mathfrak{N}_R = \mathfrak{N}_T \cap E(R(\mathfrak{A}))$, $\mathfrak{N}_L = \mathfrak{N}_T \cap E(L(\mathfrak{A}))$. Then Theorem 1 implies that if multiplication in the respective algebras of residue classes is defined by (16) we have $\mathfrak{A} \cong T(\mathfrak{A})-\mathfrak{N}_T \cong E(R(\mathfrak{A}))-\mathfrak{N}_R \cong E(L(\mathfrak{A}))-\mathfrak{N}_L$.

In the proof of the next theorem we must distinguish between the cases $\mathfrak{D}=R(\mathfrak{A})$ and $\mathfrak{D}=L(\mathfrak{A})$, and we use the following equations:

$$[R_x][R_y] = [R_{xy}] = [R_xR_y], \qquad x, y \text{ in } \mathfrak{A}, \tag{19}$$

$$[L_x][L_y] = [L_{xy}] = [L_yL_x], \qquad x, y \text{ in } \mathfrak{A}, \tag{20}$$

verification of which is similar to that of (18).

THEOREM 2. *Let* $\mathfrak{A}$, $\mathfrak{B}$, *and* $\mathfrak{N}$ *be as in Theorem* 1, *and* $\mathfrak{D}$ *be* $R(\mathfrak{A})$ *or* $L(\mathfrak{A})$, *whichever is assumed to be in* $\mathfrak{B}$. *If* Σ *is an automorphism of* $\mathfrak{B}$ *such that* $\mathfrak{N}\Sigma=\mathfrak{N}$ *and* $\mathfrak{D}\Sigma=\mathfrak{D}$, *then* Σ *determines an automorphism* S_Σ *of* $\mathfrak{A}$ *as follows:*

$$S_\Sigma:\quad x \rightarrow [D_x] \rightarrow [D_x\Sigma] = [D_{x'}] \rightarrow x' = xS_\Sigma, \tag{21}$$

for x, x' *in* $\mathfrak{A}$, *where the* $[D_x]$ *are elements of* $\mathfrak{B}-\mathfrak{N}\cong\mathfrak{A}$.

Note first that the mapping

$$[T] \rightarrow [T\Sigma], \tag{22}$$

of $\mathfrak{B}-\mathfrak{N}$ on itself is well-defined, since if $[T]=[T_1]$ then $T=T_1+N$ for N in $\mathfrak{N}$, and $T\Sigma=(T_1+N)\Sigma=T_1\Sigma+N\Sigma=T_1\Sigma+N_1$ with N_1 in $\mathfrak{N}$ since $\mathfrak{N}\Sigma=\mathfrak{N}$. Hence $[T\Sigma]=[T_1\Sigma]$. Inasmuch as the correspondences $x\rightarrow[D_x]$ and $[D_{x'}]\rightarrow x'$ are equivalences between $\mathfrak{A}$ and $\mathfrak{B}-\mathfrak{N}$, we need only to show that (22) is an automorphism of $\mathfrak{B}-\mathfrak{N}$ in order to show that (21) is an automorphism of $\mathfrak{A}$. Now (22) is linear since $\alpha[T]+\beta[U]=[\alpha T+\beta U] \rightarrow [(\alpha T+\beta U)\Sigma]=[\alpha T\Sigma+\beta U\Sigma]=\alpha[T\Sigma]+\beta[U\Sigma]$, and is nonsingular since $[T]\rightarrow[T\Sigma]=[0]$ implies $T\Sigma=N$ in $\mathfrak{N}$, $T=N\Sigma^{-1}=N_1$ in $\mathfrak{N}$, $[T]=[0]$. Since $\mathfrak{D}\Sigma=\mathfrak{D}$, there exists an element x_1 of $\mathfrak{A}$ such that $D_x\Sigma=D_{x_1}$. But then $x_1=x'$ since there is a unique transformation in $\mathfrak{D}$ in each residue class of $\mathfrak{B}$ modulo $\mathfrak{N}$. We may write $x'=xS_\Sigma$ and

$$D_x\Sigma = D_{xS_\Sigma}. \tag{23}$$

We distinguish now between the cases $\mathfrak{D}=R(\mathfrak{A})$ and $\mathfrak{D}=L(\mathfrak{A})$. Let $\mathfrak{D}=R(\mathfrak{A})$ so that (19) holds. Then, since Σ is an automorphism of $\mathfrak{B}$, we have $[R_x][R_y]=[R_xR_y]\rightarrow[(R_xR_y)\Sigma]=[(R_x\Sigma)(R_y\Sigma)]=[R_{xS_\Sigma}R_{yS_\Sigma}]=[R_{xS_\Sigma}][R_{yS_\Sigma}]=[R_x\Sigma][R_y\Sigma]$ under (22) which is an automorphism of $\mathfrak{B}-\mathfrak{N}$ as desired. In case $\mathfrak{D}=L(\mathfrak{A})$ it follows from (20) that $[L_x][L_y]=[L_yL_x]\rightarrow[(L_yL_x)\Sigma]=[(L_y\Sigma)(L_x\Sigma)]=[L_{yS_\Sigma}L_{xS_\Sigma}]=[L_{xS_\Sigma}][L_{yS_\Sigma}]=[L_x\Sigma][L_y\Sigma]$ under (22), completing the proof of the theorem.

We shall have occasion in the proof of the next theorem to use the fact that if $\mathfrak{B}$ contains both $R(\mathfrak{A})$ and $L(\mathfrak{A})$, and if both $R(\mathfrak{A})$ and $L(\mathfrak{A})$—as well of course as $\mathfrak{N}$—are their own images under an automorphism Σ of $\mathfrak{B}$, then

$$(24) \qquad R_x\Sigma = R_{xS_\Sigma}, \qquad L_x\Sigma = L_{xS_\Sigma}$$

for S_Σ defined by (21).

THEOREM 3. *Let $\mathfrak{A}$ be a non-associative algebra with unity element* 1 *and automorphism group $\mathfrak{G}$. Let $\mathfrak{H}_T$ be the group of automorphisms Σ of $T(\mathfrak{A})$ such that $\mathfrak{N}_T\Sigma = \mathfrak{N}_T$, $R(\mathfrak{A})\Sigma = R(\mathfrak{A})$, $L(\mathfrak{A})\Sigma = L(\mathfrak{A})$. Then the correspondence $S \rightarrow \Sigma$ of* (12) *is an isomorphism of $\mathfrak{G}$ onto $\mathfrak{H}_T$.*

If S is in $\mathfrak{G}$ and $S\rightarrow\Sigma$ under (12), then $1(N\Sigma) = 1S^{-1}NS = 1NS = 0S = 0$ for N in $\mathfrak{N}_T$. The nonsingularity of Σ gives $\mathfrak{N}_T\Sigma = \mathfrak{N}_T$. By (15) we have Σ in $\mathfrak{H}_T$. By Theorem 2 this Σ determines an automorphism S_Σ of $\mathfrak{A}$:

$$S_\Sigma\colon \quad x \rightarrow [R_x] \rightarrow [R_x\Sigma] = [R_{xS}] \rightarrow xS = xS_\Sigma$$

for all x in $\mathfrak{A}$, or $S = S_\Sigma$. Conversely, let Σ be in $\mathfrak{H}_T$. Then Σ determines an automorphism S_Σ of $\mathfrak{A}$ which in turn determines an automorphism

$$(25) \qquad \Sigma_*\colon \quad T \rightarrow T\Sigma_* = S_\Sigma^{-1}TS_\Sigma, \qquad T \text{ in } T(\mathfrak{A}),$$

of $T(\mathfrak{A})$ by (12). Write T in the form (8). Then $T\Sigma_* = f(R_{xS_\Sigma}, L_{xS_\Sigma}, R_{yS_\Sigma}, \cdots) = f(R_x\Sigma, L_x\Sigma, R_y\Sigma, \cdots) = \{f(R_x, L_x, R_y, \cdots)\}\Sigma = T\Sigma$ by (12), (24), and the fact that Σ is an automorphism of $T(\mathfrak{A})$. That is, $\Sigma_* = \Sigma$. It is clear then that $S\rightarrow\Sigma$ is a one-to-one mapping of $\mathfrak{G}$ onto $\mathfrak{H}_T$. To see that $S\rightarrow\Sigma$ is an isomorphism we note only that if S_1, S_2 are in $\mathfrak{G}$, $S_1\rightarrow\Sigma_1$, $S_2\rightarrow\Sigma_2$, then for T in $T(\mathfrak{A})$ we have $T\Sigma_1 = S_1^{-1}TS_1$, $T\Sigma_1\Sigma_2 = S_2^{-1}(S_1^{-1}TS_1)S_2 = (S_1S_2)^{-1}T(S_1S_2)$, or $S_1S_2\rightarrow\Sigma_1\Sigma_2$.

Variations in the proof of the following theorem from the proof above are trivial, consisting only of changes due to the fact that elements of $E(R(\mathfrak{A}))$ or $E(L(\mathfrak{A}))$ are generated by right or left multiplications alone.

THEOREM 4. *The correspondences $S\rightarrow\Sigma$ of* (13) *and* (14) *are isomorphisms of $\mathfrak{G}$ onto $\mathfrak{H}_R$ and $\mathfrak{H}_L$ respectively, where $\mathfrak{H}_R$ is the group of automorphisms Σ of $E(R(\mathfrak{A}))$ such that $\mathfrak{N}_R\Sigma = \mathfrak{N}_R$, $R(\mathfrak{A})\Sigma = R(\mathfrak{A})$, and $\mathfrak{H}_L$ is the group of automorphisms Σ of $E(L(\mathfrak{A}))$ such that $\mathfrak{N}_L\Sigma = \mathfrak{N}_L$, $L(\mathfrak{A})\Sigma = L(\mathfrak{A})$.*

3. **Automorphisms of an algebra without unity element.** In case we are concerned with an algebra $\mathfrak{A}_0$ of order $(n-1)$ over $\mathfrak{F}$ without a unity element, we can easily modify the results of §2 to include $\mathfrak{A}_0$. For we adjoin a unity element 1 to $\mathfrak{A}_0$ in the usual fashion to obtain an algebra $\mathfrak{A}$ of order n over $\mathfrak{F}$ containing $\mathfrak{A}_0$ (in the sense of equivalence) as an ideal. Every element x of $\mathfrak{A}$ may be written uniquely in the form

$$x = \xi 1 + x_0, \qquad \xi \text{ in } \mathfrak{F},\ x_0 \text{ in } \mathfrak{A}_0, \tag{26}$$

and if $y = \eta 1 + y_0$, then $x + y = (\xi + \eta)1 + (x_0 + y_0)$, $\delta x = (\delta\xi)1 + (\delta x_0)$ for δ in $\mathfrak{F}$, $xy = (\xi\eta)1 + (\eta x_0 + \xi y_0 + x_0 y_0)$. We shall write $\mathfrak{A} = 1\mathfrak{F} + \mathfrak{A}_0$ for the algebra so defined. Any automorphism S_0 of $\mathfrak{A}_0$ may be extended in a unique fashion to an automorphism S of $\mathfrak{A}$ by defining

$$S\colon\ x \to xS = \xi 1 + x_0 S_0, \tag{27}$$

x as in (26). Note that S induces the automorphism S_0 within $\mathfrak{A}_0$.

It is apparent that an automorphism S_0 of $\mathfrak{A}_0$ determines a unique automorphism Σ of $T(\mathfrak{A})$ as follows: $S_0 \to S$ by (27), $S \to \Sigma$ by (12). Moreover, the linear subspaces $R(\mathfrak{A}_0, \mathfrak{A})$ and $L(\mathfrak{A}_0, \mathfrak{A})$ of $T(\mathfrak{A})$ are their own images under Σ. For if x_0 is in $\mathfrak{A}_0$, then $R_{x_0}\Sigma = R_{x_0 S} = R_{x_0 S_0}$ is in $R(\mathfrak{A}_0, \mathfrak{A})$ and $L_{x_0}\Sigma = L_{x_0 S} = L_{x_0 S_0}$ is in $L(\mathfrak{A}_0, \mathfrak{A})$.

If $\mathfrak{B} - \mathfrak{N}$ is the non-associative algebra equivalent to $\mathfrak{A}$ which was defined in §2, then $\mathfrak{A}_0$ is equivalent to the ideal $\mathfrak{C}_0$ of $\mathfrak{B} - \mathfrak{N}$ consisting of residue classes $[D_{x_0}]$ for x_0 in $\mathfrak{A}_0$, that is, for D_{x_0} in $\mathfrak{D}_0 = R(\mathfrak{A}_0, \mathfrak{A})$ or $L(\mathfrak{A}_0, \mathfrak{A})$ according as $\mathfrak{D} = R(\mathfrak{A})$ or $L(\mathfrak{A})$. For, by Theorem 1, $\mathfrak{A}$ is isomorphic to $\mathfrak{B} - \mathfrak{N}$ under the mapping (17). Since $\mathfrak{A}_0$ is an ideal of $\mathfrak{A}$, the mapping

$$x_0 \to [D_{x_0}], \qquad x_0 \text{ in } \mathfrak{A}_0, \tag{28}$$

determines an ideal $\mathfrak{C}_0$ of $\mathfrak{B} - \mathfrak{N}$, and $\mathfrak{C}_0 \cong \mathfrak{A}_0$.

Let Σ be an automorphism of $\mathfrak{B}$ such that $\mathfrak{N}\Sigma = \mathfrak{N}$ and $\mathfrak{D}_0\Sigma = \mathfrak{D}_0$. Then Σ determines an automorphism $S_{0\Sigma}$ of $\mathfrak{A}_0$ as follows:

$$S_{0\Sigma}\colon\ x_0 \to [D_{x_0}] \to [D_{x_0}\Sigma] = [D_{x_0'}] \to x_0' = x_0 S_{0\Sigma}, \tag{29}$$

for x_0, x_0' in $\mathfrak{A}_0$. For $\mathfrak{D} = I\mathfrak{F} + \mathfrak{D}_0$, and any automorphism of $\mathfrak{B}$ leaves invariant the subspace $I\mathfrak{F}$ of order 1, so that $\mathfrak{D}\Sigma = \mathfrak{D}$. Then by Theorem 2, Σ determines an automorphism S_Σ of $\mathfrak{A}$. But S_Σ induces on $\mathfrak{A}_0$ the automorphism (29) since $D_{x_0}\Sigma = D_{x_0'}$ in $\mathfrak{D}_0$ implies x_0' is in $\mathfrak{A}_0$. Thus $x_0 \to [D_{x_0}] \to [D_{x_0}\Sigma] = [D_{x_0'}] \to x_0' = x_0 S_\Sigma$ is in $\mathfrak{A}_0$, or S_Σ induces $S_{0\Sigma}$ on $\mathfrak{A}_0$.

THEOREM 5. *Let $\mathfrak{A}_0$ be a non-associative algebra without unity element, and let $\mathfrak{A} = 1\mathfrak{F} + \mathfrak{A}_0$. Let $\mathfrak{H}_T^0$ be the group of automorphisms Σ of $T(\mathfrak{A})$ such that $R(\mathfrak{A}_0, \mathfrak{A})\Sigma = R(\mathfrak{A}_0, \mathfrak{A})$, $L(\mathfrak{A}_0, \mathfrak{A})\Sigma = L(\mathfrak{A}_0, \mathfrak{A})$, $\mathfrak{N}_T\Sigma = \mathfrak{N}_T$. Then the correspondence $S_0 \to S \to \Sigma$ of* (27) *and* (12) *is an isomorphism of the automorphism group $\mathfrak{G}_0$ of $\mathfrak{A}_0$ onto $\mathfrak{H}_T^0$.*

For if S_0 is in $\mathfrak{G}_0$, then $S_0 \to S \to \Sigma$ in $\mathfrak{H}_T^0$ and $\Sigma \to S_\Sigma = S$ by Theorem 3. But then S induces the automorphism $S_{0\Sigma}$ within $\mathfrak{A}_0$. That is, $S_{0\Sigma} = S_0$. Conversely, if Σ is in $\mathfrak{H}_T^0$, then $\Sigma \to S_{0\Sigma}$ in $\mathfrak{G}_0$ by (29). But $S_{0\Sigma} \to S_\Sigma \to \Sigma_*$ by (27) and (12) and $\Sigma_* = \Sigma$ by Theorem 3. Hence the mapping $S_0 \rightarrow S \rightarrow \Sigma$ of $\mathfrak{G}_0$ on $\mathfrak{H}_T^0$ is one-to-one, and is by Theorem 3 an isomorphism.

The results analogous to Theorem 4 for algebras $\mathfrak{A}_0$ without unity quantity may be stated as follows: let $\mathfrak{H}_R^0$ be the group of automorphisms Σ of $E(R(\mathfrak{A}))$ such that $R(\mathfrak{A}_0, \mathfrak{A})\Sigma = R(\mathfrak{A}_0, \mathfrak{A})$, $\mathfrak{N}_R\Sigma = \mathfrak{N}_R$. Then the correspondence $S_0 \rightarrow S \rightarrow \Sigma$ of (27) and (13) is an isomorphism of the automorphism group $\mathfrak{G}_0$ of $\mathfrak{A}_0$ onto $\mathfrak{H}_R^0$. Let $\mathfrak{H}_L^0$ be the group of automorphisms Σ of $E(L(\mathfrak{A}))$ such that $L(\mathfrak{A}_0, \mathfrak{A})\Sigma = L(\mathfrak{A}_0, \mathfrak{A})$, $\mathfrak{N}_L\Sigma = \mathfrak{N}_L$. Then the correspondence $S_0 \rightarrow S \rightarrow \Sigma$ of (27) and (14) is an isomorphism of $\mathfrak{G}_0$ onto $\mathfrak{H}_L^0$.

4. Inner automorphisms Σ of $T(\mathfrak{A})$.

An automorphism Σ of the associative algebra $T(\mathfrak{A})$ is called ***inner*** in case $T \rightarrow T\Sigma = K^{-1}TK$ for some nonsingular element K of $T(\mathfrak{A})$. We are concerned in this section with automorphisms S of $\mathfrak{A}$ which determine inner automorphisms Σ of $T(\mathfrak{A})$ under (12).

The group $\mathfrak{K}$ of all inner automorphisms of $T(\mathfrak{A})$ is an invariant subgroup of the automorphism group of $T(\mathfrak{A})$. If $\mathfrak{H}_T$ is the group of automorphisms of $T(\mathfrak{A})$ described in Theorem 3, then the intersection $\mathfrak{H}_T \cap \mathfrak{K}$ is an invariant subgroup of $\mathfrak{H}_T$. But then there is an invariant subgroup $\mathfrak{J}$ of the automorphism group $\mathfrak{G}$ of $\mathfrak{A}$ such that $\mathfrak{J} \cong \mathfrak{H}_T \cap \mathfrak{K}$ under the correspondence $S \rightarrow \Sigma$ of (12). The elements of $\mathfrak{J}$ are characterized as those automorphisms of $\mathfrak{A}$ which are themselves elements of $T(\mathfrak{A})$ by

THEOREM 6. *Let $\mathfrak{A}$ be a non-associative algebra over $\mathfrak{F}$ with unity element* 1 *and automorphism S determining an automorphism Σ of $T(\mathfrak{A})$ by* (12). *Then Σ is inner if and only if S is in $T(\mathfrak{A})$.*

If S is in $T(\mathfrak{A})$, then $T \rightarrow T\Sigma = S^{-1}TS$ is an inner automorphism of $T(\mathfrak{A})$. Conversely, if Σ is inner, there exists a nonsingular element K of $T(\mathfrak{A})$ such that $T\Sigma = K^{-1}TK$ for all T in $T(\mathfrak{A})$. In particular, $R_{xS} = R_x\Sigma = K^{-1}R_xK$. Let $1K = k$ so that $xSL_k = k \cdot xS = kR_{xS} = 1KK^{-1}R_xK = xK$ for all x in $\mathfrak{A}$, or $SL_k = K$. Since S and K are nonsingular, L_k^{-1} exists. Moreover, L_k^{-1} is in $T(\mathfrak{A})$, and $S = KL_k^{-1}$ is in $T(\mathfrak{A})$.

Perhaps it should be pointed out that Theorem 6 yields nothing in the case of central simple algebras (that is, algebras which are simple for all scalar extensions). For although it is true that, if $\mathfrak{A}$ is central simple, then $T(\mathfrak{A})$ is also and—by a well known theorem concerning associative algebras—every automorphism Σ of $T(\mathfrak{A})$ is inner, so that Theorem 6 implies that every automorphism S of $\mathfrak{A}$ is in $T(\mathfrak{A})$, it is also true [1, §8] that in this case $T(\mathfrak{A}) = (\mathfrak{F})_n$, the algebra of all linear transformations on $\mathfrak{A}$. Of course it is vacuous then to say that S is in $T(\mathfrak{A})$.

5. **The right ideals $\mathfrak{N}_T$, $\mathfrak{N}_E$, $\mathfrak{N}_L$.** We now make a more thorough analysis of the right ideals $\mathfrak{N}_T$, $\mathfrak{N}_E$, $\mathfrak{N}_L$ of $T(\mathfrak{A})$, $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$, respectively, and arrive in particular at criteria for the (right, left) simplicity of an algebra $\mathfrak{A}$ with unity quantity.

THEOREM 7. *An algebra $\mathfrak{A}$ with unity quantity is both commutative and associative if and only if $\mathfrak{N}_T=0$.*

For $\mathfrak{N}_T=0$ implies that $L_x-R_x=R_xR_y-R_{xy}=0$ for all x, y in $\mathfrak{A}$. That is,

$$R_x = L_x, \qquad R_xR_y = R_{xy}, \tag{30}$$

$\mathfrak{A}$ is both commutative and associative. Conversely, if (30) holds for all x, y in $\mathfrak{A}$, then T in $T(\mathfrak{A})$ has the form $T=f(R_x, L_x, R_y, \cdots) = g(R_x, R_y, \cdots) = R_{g(x,y,\cdots)}$. Then $1T=0$ implies $g(x, y, \cdots)=0$ or $T=0$. Hence $\mathfrak{N}_T=0$.

The *center* $\mathfrak{Z}$ of $\mathfrak{A}$ consists of all elements c in $\mathfrak{A}$ such that

$$xc = cx, \qquad c(xy) = (cx)y = x(cy), \tag{31}$$

or equivalently

$$cL_x = cR_x, \qquad cR_{xy} = cR_xR_y = cR_yL_x \tag{32}$$

for all x, y in $\mathfrak{A}$.

THEOREM 8. *An element c is in the center $\mathfrak{Z}$ of an algebra $\mathfrak{A}$ with unity quantity if and only if $c\mathfrak{N}_T=0$.*

Certainly L_x-R_x, $R_{xy}-R_xR_y$, $R_{xy}-R_yL_x$ are in $\mathfrak{N}_T$ for all x, y in $\mathfrak{A}$. Hence if $c\mathfrak{N}_T=0$, it follows that $c(L_x-R_x)=c(R_{xy}-R_xR_y)=c(R_{xy}-R_yL_x)=0$ or (32) holds, c is in the center of $\mathfrak{A}$. Conversely, if c is in the center of $\mathfrak{A}$, and if we write T in $T(\mathfrak{A})$ as in (8), it is seen by repeated application of (32) that $cT=cf(R_x, L_x, R_y, \cdots) = cR_{g(x,y,\cdots)}$ where the non-associative polynomial $g(x, y, \cdots) = 1f(R_x, L_x, R_y, \cdots)=1T$. But if T is in $\mathfrak{N}_T$, then $1T=0$ so that $g(x, y, \cdots)=0$ and $cT=0$, $c\mathfrak{N}_T=0$.

An algebra $\mathfrak{A}$, which is not the zero algebra of order 1, is called *simple* (*right simple*, *left simple*) in case the only ideals (right ideals, left ideals) of $\mathfrak{A}$ are 0 and $\mathfrak{A}$.

THEOREM 9. *A non-associative algebra $\mathfrak{A}$ with unity quantity is right simple if and only if $\mathfrak{N}_E$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$.*

If $\mathfrak{N}_E$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$, then the only right ideal of $E(R(\mathfrak{A}))$ containing $\mathfrak{N}_R$ properly is $E(R(\mathfrak{A}))$ itself. We assume that $\mathfrak{A}$ is not right simple, so that $\mathfrak{A}$ has a right ideal $\mathfrak{Q}\neq 0$, $\mathfrak{A}$. Let $\mathfrak{P}$ be the linear set $\mathfrak{P}=R(\mathfrak{Q}, \mathfrak{A})+\mathfrak{N}_R$. Then P in $\mathfrak{P}$ has the form $P=R_q+N$, q in $\mathfrak{Q}$, N in $\mathfrak{N}_R$, and any element T of $E(R(\mathfrak{A}))$ may be written as $T=R_t+N_1$, t in $\mathfrak{A}$, N_1 in $\mathfrak{N}_R$, so that $PT =(R_q+N)(R_t+N_1)=R_qR_t+R_qN_1+NT=R_{qt}+(R_qR_t-R_{qt})+R_qN_1 +NT$. Now $R_qN_1=R_a+N_2$ for a in $\mathfrak{A}$, N_2 in $\mathfrak{N}_R$, and $1R_qN_1=1R_a +1N_2$ or $a=qN_1$. Since $N_1=f(R_x, R_y, \cdots)$ while $\mathfrak{Q}$ is a right ideal of $\mathfrak{A}$, it follows that $a=qN_1=qf(R_x, R_y, \cdots)$ is in $\mathfrak{Q}$. Hence $PT=R_{qt+qN_1}+(R_qR_t-R_{qt}+N_2+NT)$ is in $\mathfrak{P}$ since $qt+qN_1$ is in $\mathfrak{Q}$ while $R_qR_t-R_{qt}+N_2+NT$ is in $\mathfrak{N}_R$. Hence $\mathfrak{P}$ is a right ideal of $E(R(\mathfrak{A}))$ containing $\mathfrak{N}_R$. Since $\mathfrak{Q}\neq 0$, $\mathfrak{A}$, it follows that $R(\mathfrak{Q}, \mathfrak{A})$, being of the same dimension over $\mathfrak{F}$ as $\mathfrak{Q}$, is neither 0 nor $R(\mathfrak{A})$, and then $\mathfrak{P}\neq\mathfrak{N}_R$, $E(R(\mathfrak{A}))$, a contradiction. Hence $\mathfrak{A}$ is right simple.

Conversely, let $\mathfrak{P}$ be any proper right ideal of $E(R(\mathfrak{A}))$ which contains $\mathfrak{N}_R$. Consider the set $\mathfrak{Q}$ of residue classes $[P]$ modulo $\mathfrak{N}_R$ for P in $\mathfrak{P}$. Then $\mathfrak{Q}$ is a linear subset of $E(R(\mathfrak{A}))-\mathfrak{N}_R\cong\mathfrak{A}$. Moreover, if $[P]$ is any element of $\mathfrak{Q}$, we write $P=R_p+N$ for p in $\mathfrak{A}$, N in $\mathfrak{N}_R$. Let $[R_t]$ be any element of $E(R(\mathfrak{A}))-\mathfrak{N}_R$. Then $PR_t=R_pR_t+NR_t =P_1$ in $\mathfrak{P}$ since $\mathfrak{P}$ is a right ideal of $E(R(\mathfrak{A}))$. Then

$$[P][R_t] = [R_p][R_t] = [R_pR_t] = [P_1] \tag{33}$$

in $\mathfrak{Q}$ by (19), and $\mathfrak{Q}$ is a right ideal of $E(R(\mathfrak{A}))-\mathfrak{N}_R\cong\mathfrak{A}$. If $\mathfrak{A}$ is right simple, then either $\mathfrak{Q}=[0]$ or $\mathfrak{Q}=E(R(\mathfrak{A}))-\mathfrak{N}_R$. In the latter case, $\mathfrak{Q}$ contains $[I]$, $\mathfrak{P}$ contains $I+N_1$ for some N_1 in $\mathfrak{N}_R$. Since $\mathfrak{P}$ also contains N_1, it follows that I is in $\mathfrak{P}$, whence $\mathfrak{P}=E(R(\mathfrak{A}))$, a contradiction. Hence $\mathfrak{Q}=[0]$, $\mathfrak{P}=\mathfrak{N}_R$, and $\mathfrak{N}_R$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$.

An exactly symmetrical argument, involving left multiplications instead of right multiplications, suffices to prove

THEOREM 10. *A non-associative algebra $\mathfrak{A}$ with unity quantity is left simple if and only if $\mathfrak{N}_L$ is a maximal proper right ideal of* $E(L(\mathfrak{A}))$.

Only obvious variations on the proof above are required in the proof of

THEOREM 11. *A non-associative algebra $\mathfrak{A}$ with unity quantity is simple if and only if $\mathfrak{N}_T$ is a maximal proper right ideal of* $T(\mathfrak{A})$.

For example, to prove the converse part of the theorem, we let $\mathfrak{P}$ be any proper right ideal of $T(\mathfrak{A})$ which contains $\mathfrak{N}_T$, and let $\mathfrak{Q}$ be the linear space of residue classes $[P]$ modulo $\mathfrak{N}_T$ for P in $\mathfrak{P}$. We may write $P=R_p+N=L_p+N_0$ for N, N_0 in $\mathfrak{N}_T$, and let $[R_t]=[L_t]$ be any element of $T(\mathfrak{A})-\mathfrak{N}_T$. Then we have (33) as before, where now the quantities involved are residue classes of $T(\mathfrak{A})$ modulo $\mathfrak{N}_T$, but also we have $PL_t = L_y L_t + N_0 L_t = P_2$ in $\mathfrak{P}$ so that $[L_t][P] = [L_t][L_y] = [L_y L_t] = [P_2]$ in $\mathfrak{Q}$ by (20), and $\mathfrak{Q}$ is an ideal of $T(\mathfrak{A})-\mathfrak{N}_T \cong \mathfrak{A}$. The remainder of the proof is as before.

We conclude with an analysis of the structure of the right ideal $\mathfrak{N}_T$ of $T(\mathfrak{A})$ in case $\mathfrak{A}$ of order n over $\mathfrak{F}$ (with unity quantity) is simple. In this case $T(\mathfrak{A}) = (\mathfrak{Z})_s$ where the center $\mathfrak{Z}$ of $\mathfrak{A}$ is a field of degree t over $\mathfrak{F}$, and $n = st$ (see [1, §§8, 19]).

THEOREM 12. *Let $\mathfrak{A}$ be a simple non-associative algebra of order $n = st$ over $\mathfrak{F}$ with unity quantity and with center $\mathfrak{Z}$ of degree t over $\mathfrak{F}$. Then $\mathfrak{N}_T = \mathfrak{R} + (\mathfrak{Z})_{s-1}$, where the radical $\mathfrak{R}$ of $\mathfrak{N}_T$ has order $(s-1)$ over $\mathfrak{Z}$ and the semi-simple component of $\mathfrak{N}_T$ is the total matric algebra $(\mathfrak{Z})_{s-1}$ of degree $(s-1)$ over $\mathfrak{Z}$.*

For $\mathfrak{A}$ is central simple over $\mathfrak{Z}$. Let $(1, u_2, \cdots, u_s)$ be a fixed basis of $\mathfrak{A}$ over $\mathfrak{Z}$. Then, since $T(\mathfrak{A}) = (\mathfrak{Z})_s$, it follows from Theorem 8 that $\mathfrak{N}_T$ (over $\mathfrak{Z}$) consists of all s-by-s matrices with first row zero. But the structure of this algebra of matrices, with principal idempotent

$$E = \begin{pmatrix} 0 & 0 \\ 0 & I_{s-1} \end{pmatrix},$$

is easily determined. Its radical $\mathfrak{R}$ consists of all matrices (with elements in $\mathfrak{Z}$) of the form

$$\begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}$$

where U is any $(s-1)$-by-1 matrix. Its semi-simple component consists of all matrices (with elements in $\mathfrak{Z}$) of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix}$$

where V is any $(s-1)$-rowed square matrix. This is a total matric algebra $(\mathfrak{Z})_{s-1}$.

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THE INSTITUTE FOR ADVANCED STUDY

ON THE REDUCTION OF THE CONJUGATING REPRESENTATION OF A FINITE GROUP

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1. **Introduction.** By the conjugating representation P of a finite group G of order $g>1$, with elements γ_i, is meant the representation of G by permutation matrices $P(\gamma_i)$ such that

$$\gamma_i^{-1}\gamma\gamma_i = P(\gamma_i)\gamma. \tag{1}$$

Here we define the group vector γ to be a $g\times 1$ column vector whose entries are the elements of G, arranged so that the identity element γ_1 is first, and so that the h_r elements of a class C_r of conjugate elements are listed consecutively, forming a class vector γ_r which is a subvector of the group vector γ.

From a study of two different partial decompositions of the linear group P and its subsequent complete reduction into irreducible components Γ_ρ, the principal theorem is obtained, which relates the multiplicities of the irreducible components of the direct products $\Gamma_\rho\times\Gamma_\rho$ with those of certain transitive constituents of P. Furthermore, a matrix T is described which completely and simultaneously reduces the right and left regular representations as well as the conjugating representation.

2. **The transitive constituents of the conjugating representation.** The $g\times g$ permutation matrices of the right and left regular representations, respectively, are defined by right or left multiplication of the group vector γ by a group element γ_i, thus:

$$\gamma\gamma_i = R(\gamma_i)\gamma, \qquad \gamma_i\gamma = L(\gamma_i)\gamma. \tag{2}$$

They form transitive groups of permutation matrices, one isomorphic and the other anti-isomorphic with G. The matrix $R(C_r)$, obtained by summing the matrices $R(\gamma_i)$ over a class C_r, is identical with the corresponding matrix $L(C_r)$. Each matrix $R(\gamma_i)$ is permutable with every matrix $L(\gamma_j)$.

A group of permutation matrices, which we call the *conjugating representation* P of G, is defined by assigning to the group element γ_i the matrix $P(\gamma_i)$, where

$$P(\gamma_i) = L(\gamma_i^{-1})R(\gamma_i), \qquad P(\gamma_i)P(\gamma_j) = P(\gamma_i\gamma_j). \tag{3}$$

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The matrices $P(\gamma_i)$ may also be defined directly by the equation (1).

From equation (1) it is apparent that the permutation group P is intransitive (for $g>1$), having one transitive constituent P corresponding to each of the r distinct classes C_r. In terms of the class vector γ_r we define the transitive "class representation" P_r as follows:

$$\gamma_i^{-1}\gamma_r\gamma_i = P_r(\gamma_i)\gamma_r. \tag{4}$$

It is well known[1] that in any transitive permutation group of degree n which is homomorphic with a given group G, there is a subgroup leaving a specified symbol fixed, to which corresponds a subgroup H of index n in G, and that the given permutation group is equivalent to the permutation group G_H on the right cosets $H\gamma_\alpha$ of G. To γ_i in G corresponds the permutation $H\gamma_\alpha \rightarrow H\gamma_\alpha\gamma_i$ in G_H. If K is the largest subgroup of H which is invariant in G, then the permutation group G_H is isomorphic with the factor group G/K.

For the transitive class representation P_r, the subgroup H is the normalizer N_r of a chosen element of the class C_r. Since each N_r contains the center C of G, each group P_r is a representation of the factor group G/C. It is never a faithful (isomorphic) representation of G when the center contains more than one element.

Each of the groups P_r, considered as a linear group, may be completely reduced by a change of basis into the direct sum of irreducible linear groups. Let μ_{rp} be the multiplicity in P_r of the irreducible component Γ_p and let Γ_1 be the identity representation. Then since P_r is transitive, $\mu_{r1}=1$. In the complete reduction of the conjugating representation P, if μ_p is the multiplicity of the component Γ_p,

$$P \cong \sum_p \mu_p\Gamma_p, \qquad \text{where } \mu_p = \sum_{r=1}^{r} \mu_{rp}. \tag{5}$$

In particular, if the center of G contains more than one element, the coefficient μ_p is zero for every faithful representation Γ_p of G, and for all other representations which do not represent every invariant element by the unit matrix.

3. **Reduction by idempotents of the group algebra.** The right and left regular representations of the group G induce corresponding representations of the group algebra $\mathfrak{A}$ whose typical element $a=\sum a_i\gamma_i$ is a linear combination of group elements with coefficients from a specified field such as the field of complex numbers. It is known[2] that matrices T exist which transform the group vector γ into some new basis

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[1] A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, 3d ed., Berlin, 1937 (Dover, New York, 1945) p. 113.

[2] A. Speiser, loc. cit. p. 178.

vector $T^{-1}\gamma$, such that both the right and left regular representations are thereby completely and simultaneously reduced, and such that equivalent irreducible components of the right representation are actually identical with each other, and are merely transposes of the corresponding components in the anti-isomorphic left representation. The component matrices $\Gamma_\rho(\gamma_i)$ may further be assumed to be in unitary form, and we have

$$(6) \qquad \Gamma'_\rho(\gamma_i^{-1}) = \bar{\Gamma}_\rho(\gamma_i).$$

In both of the transformed representations the class C_σ is represented by the same diagonal matrix $T^{-1}R(C_\sigma)T = T^{-1}L(C_\sigma)T$. The r classes C_σ of G are linearly independent in both regular representations, so these diagonal matrices must be linear combinations of r^* idempotent matrices I_ρ, where $r^* \geqq r$. By means of the primitive diagonal idempotents I_ρ we define the blocks R_ρ and L_ρ of which the transformed right and left representations are respectively the direct sums:

$$(7) \qquad R_\rho(a) = I_\rho T^{-1}R(a)TI_\rho; \qquad L_\rho(a) = I_\rho T^{-1}L(a)TI_\rho.$$

Since two nonequivalent irreducible components must differ in their representations of at least one class C_σ, the irreducible components of the block $R_\rho(a)$ are all equivalent, and may each be written as the same $\Gamma_\rho(a)$ by suitable choice of T. Equivalent to these, but written in the transposed form $\Gamma'_\rho(a)$ to produce the required anti-isomorphism, are the components of the left block $L_\rho(a)$. Because the matrices of R_ρ and L_ρ commute with each other, the multiplicity of the component Γ_ρ in R_ρ must equal the degree of Γ'_ρ in L_ρ, namely n_ρ. Hence the diagonal idempotent I_ρ has n_ρ^2 1's and defines a subspace of the g-dimensional vector space in which $R_\rho(a)$ and $L_\rho(a)$ may be written as direct products (in opposite orders) of a unit matrix and a representation matrix each of degree n_ρ:

$$(8) \qquad R_\rho(a) = \Gamma'_\rho(1) \times \Gamma_\rho(a); \qquad L_\rho(a) = \Gamma'_\rho(a) \times \Gamma_\rho(1).$$

Using the same suitably chosen T, whose coefficients we shall describe later, we transform the conjugating representation, defining

$$(9) \qquad Q(\gamma_i) = T^{-1}P(\gamma_i)T.$$

From equations (9), (3), (7), (8), and (6), it then follows that the block Q_ρ of Q defined by the idempotent I_ρ has the form

$$(10) \qquad Q_\rho(\gamma_i) = I_\rho Q(\gamma_i)I_\rho = \bar{\Gamma}_\rho(\gamma_i) \times \Gamma_\rho(\gamma_i),$$

and is thus the direct product of the two conjugate imaginary irreducible representations $\bar{\Gamma}_\rho$ and Γ_ρ of the group G. Consequently the multiplicity of any irreducible representation as a component of the conjugating representation P is the sum of its multiplicities in the r^* direct products $\bar{\Gamma}_\rho \times \Gamma_\rho$.

4. **The balancing of multiplicities.** Combining the result of §§2 and 3 we obtain our principal theorem:

THEOREM 1. *The sum of the multiplicities of a given irreducible representation* Γ_ρ *as a component in the direct products* $\bar{\Gamma}_\tau \times \Gamma_\tau$ *is equal to the sum of its multiplicities in the transitive class representations* P_σ *(permutations on cosets with respect to a normaliser).*

Applying Theorem 1 to the component Γ_1 which occurs just once in each $\bar{\Gamma}_\tau \times \Gamma_\tau$ and P_σ, we obtain the following well known result.

COROLLARY 1. *The number of nonequivalent irreducible representations of a finite group is equal to the number of its classes.*

To illustrate Theorem 1, we give below the two decompositions of the conjugating representation of the symmetric group of order 24. This group has five nonequivalent irreducible representations, of degrees $n_\rho = 1, 1, 3, 3, 2$, respectively, and five class representations P_σ, of degrees $h_\sigma = 1, 3, 8, 6, 6$, respectively. The decomposition of each of the five direct products $\bar{\Gamma}_\tau \times \Gamma_\tau$ is given in one of the left-hand columns, that of the conjugating representation P in the center column, and those of the class representations P_σ in the right-hand columns.

	$\bar{\Gamma}_\tau\times\Gamma_\tau$					P		P_σ				
τ:	1	2	3	4	5		σ:	1	2	3	4	5
n_τ^2:	1	1	9	9	4	$g=24$	h_σ:	1	3	8	6	6
Γ_1:	1	1	1	1	1	$\mu_1=5$		1	1	1	1	1
Γ_2:					1	$\mu_2=1$				1		
Γ_3:			1	1		$\mu_3=2$				1	1	
Γ_4:			1	1		$\mu_4=2$				1		1
Γ_5:			1	1	1	$\mu_5=3$			1		1	1

Table of multiplicities

From Theorem 1 and the discussion of §2, we obtain also the following theorem.

THEOREM 2. *If* Γ *is an irreducible representation of a finite group* G *whose center* C *contains more than one element, then each irreducible* *component of* $\bar{\Gamma}\times\Gamma$ *is a representation of* G/C, *and is not a faithful (isomorphic) representation of* G.

One illustration of Theorem 2 is afforded by considering a representation Γ of the quaternion group which is irreducible over the field of complex numbers. Since the center C contains two elements, no component of $\bar{\Gamma}\times\Gamma$ is a faithful representation of the quaternion group. Similarly let us consider a group G which is represented faithfully by a group Γ of unitary symplectic matrices, irreducible over the field of complex numbers. Weyl[8] applies the term *symplectic* to matrices of degree $n=2\nu$ having an alternating bilinear invariant. In terms of real ν-dimensional matrices A, B, C, D and the corresponding unit matrix I such unitary symplectic matrices M_n and their invariant j may be written in the form

$$M_n=\begin{pmatrix} A+Bi & C+Di \\ -C+Di & A-Bi \end{pmatrix},\qquad j=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{11}$$

This is equivalent to a set of ν-dimensional matrices $M_\nu=A+Bi+Cj+Dij$ having quaternion coefficients, and such that the inverse matrix is the transposed quaternion conjugate $A'-B'i-C'j-D'ij$. The group Γ is equivalent to $\bar{\Gamma}$. The existence in G of an invariant element of order 2 implies by Theorem 2 that $\bar{\Gamma}\times\Gamma$ is not a faithful representation of G.

5. The reducing transformation. It is known that a linear transformation with matrix T exists which completely reduces the right and left regular representations.[2] It is possible to find all such reducing matrices T in a fairly straightforward manner by making use of the conjugating representation.

We first define a $g\times g$ "entry matrix" $Z=\|Z_{ij}\|$ by assigning to the ith row in a specified order the $g=\sum n_p^2$ linearly independent entries (or coefficients) for the group element γ_i in a complete set of r non-equivalent irreducible unitary representations Γ_p. It is best to order these g entries first by representations Γ_p (Γ_1 leading), then by rows within the particular n_p-dimensional matrix of Γ_p, and finally by columns in Γ_p. The ordering of rows in Z shall be that of the group vector γ.

Then the matrix $P(\gamma_i)Z$ is a matrix similar to Z but with each group element which defines a row in Z replaced by its transform under γ_i, so that the rows of the matrix Z are permuted by $P(\gamma_i)$. The same reordering of the coefficients of Z could have been obtained by postmultiplying Z by the matrix $Q(\gamma_i)$ of (9), which is partially decomposed according to the idempotents I_ρ. For ZI_ρ is a rectangular matrix of g rows and n_ρ^2 columns, whose coefficients are transformed by a direct product matrix $\bar{\Gamma}_\rho(\gamma_i)\times\Gamma_\rho(\gamma_i)$ applied on the right just as if each individual group element were transformed by γ_i. Hence using (9), we have

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[8] H. Weyl, *The classical groups*, Princeton, 1939, p. 165 ff.

$$P(\gamma_i)Z = ZQ(\gamma_i) = ZT^{-1}P(\gamma_i)T. \tag{12}$$

A similar argument shows that $R(\gamma_i)ZI_\rho$ and $L(\gamma_i)ZI_\rho$ are matrices like ZI_ρ in which the coefficients corresponding to the group element γ_j are replaced by those of $\gamma_j\gamma_i$ or of $\gamma_i\gamma_j$ respectively. This however is the same as either $ZI_\rho R_\rho(\gamma_i)$ or $ZI_\rho L_\rho(\gamma_i)$, where R_ρ and L_ρ are defined in (7). Summing over ρ we have

$$R(\gamma_i)Z = ZT^{-1}R(\gamma_i)T; \qquad L(\gamma_i)Z = ZT^{-1}L(\gamma_i)T. \tag{13}$$

Writing $T = ZV$ in (12) and (13) we see that the nonsingular matrix V permutes with each of the matrices $T^{-1}R(\gamma_i)T$ and $T^{-1}L(\gamma_i)T$ and their product $Q(\gamma_i)$. Thus,

$$\begin{aligned} V[T^{-1}R(\gamma_i)T] &= V(ZV)^{-1}R(\gamma_i)ZV = V(ZV)^{-1}ZT^{-1}R(\gamma_i)TV \\ &= [T^{-1}R(\gamma_i)T]V, \\ V[T^{-1}L(\gamma_i)T] &= T^{-1}L(\gamma_i)TV, \\ VQ(\gamma_i) &= Q(\gamma_i)V. \end{aligned} \tag{14}$$

These relations (14) are possible for all γ_i if and only if V lies in the intersection of the commutators of $T^{-1}R(a)T$ and $T^{-1}L(a)T$. Hence V is a nonsingular linear combination of the idempotents I_ρ.

Now the well known orthogonality relations for the coefficients in the irreducible group representations imply that

$$Z'Z = \sum_\rho (g/n_\rho)I_\rho. \tag{15}$$

It follows that the matrix $T = ZV$ will be unitary if and only if

$$V = \sum_\rho \omega_\rho(n_\rho/g)^{1/2}I_\rho, \qquad \text{where } \bar{\omega}_\rho\omega_\rho = 1. \tag{16}$$

A convenient choice is to take $\omega_\rho = 1$.

THEOREM 3. *A unitary matrix T which completely reduces the right and left representations and the conjugating representation may be formed by multiplying each element of the entry matrix Z described above by the appropriate factor $(n_\rho/g)^{1/2}$, where n_ρ is the degree of the irreducible representation associated with the particular column of Z.*

MICHIGAN STATE COLLEGE

A NOTE ON FINITE ABELIAN GROUPS[1]

L. J. PAIGE

1. **Introduction.** R. H. Bruck[2] has pointed out that every finite group of odd order is isotopic to an idempotent quasigroup. It can be shown that a necessary and sufficient condition that a group G be isotopic to an idempotent quasigroup is that there exist one-to-one mappings θ and η of G upon G satisfying the relationship $\eta(x)=x\cdot\theta(x)$, for all x of G. The same condition is sufficient to prove the existence of a loop M whose automorphism group contains G as a subgroup. We shall not attempt to show either of these applications; but, since there may be others, the present paper is concerned with the existence of suitable θ and η for any finite abelian group G. For this we have a complete answer. Our methods are constructive, but (unfortunately from the standpoint of generalization) they make considerable use of the commutative law.

2. **Notation.** We shall consider a finite abelian group G of order $n=n(G)$.

The product of the n distinct elements of G will be designated by $p=p(G)$.

Let $x\rightarrow\theta(x)$ be any one-to-one mapping (*not* necessarily an automorphism) of G upon G. Consider the derived mapping $x\rightarrow\eta(x)=x\theta(x)$. The *order* of η, denoted by $O(\eta)$, is the number of distinct elements $\eta(x)$, for x in G.

It is our purpose to prove the following theorem:

THEOREM 1. *There exists a θ for which $O(\eta)=n(G)$ unless G possesses exactly one element of order 2. In the latter case there exists a θ for which $O(\eta)=n(G)-1$.*

3. **Evaluation of p.**

LEMMA 1. *$p(G)=1$ unless G possesses exactly one element of order 2. In the latter case, $p(G)$ is the unique element of order 2.*

PROOF. The set H consisting of the identity and all elements of G of order 2 is a uniquely defined subgroup of G. If $a\in G$ is of order

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[1] The author wishes to thank the referee for his suggestions which added considerably to the clarity of the paper.

[2] R. H. Bruck, *Some results in the theory of quasigroups*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 19–52, especially pp. 35, 36.

greater than 2, $a\neq a^{-1}$; thus both a and a^{-1} appear in $p(G)$ and hence $p(G)=p(H)$.

If H has order 1, $p(H)=1$. If H has order 2, elements 1, g, then $p(H)=1\cdot g=g$, and $p(H)$ is the unique element of H (and hence of G) of order 2.

Now suppose H has order greater than 2; so that H has order 2^k, $k>1$. Then H has k generators $g_1, \cdots, g_k$ and every element of H has a unique representation in the form $g_1^{n_1}g_2^{n_2}\cdots g_k^{n_k}$ where n_i is 0 or 1. Hence $p(H)=\prod(g_1^{n_1}g_2^{n_2}\cdots g_k^{n_k})$, where the product is over the distinct ordered sets $(n_1, \cdots, n_k)$ with n_i taking the values 0 or 1. By symmetry $p(H)=(g_1g_2\cdots g_k)^m$ where $m=2^{k-1}$ and since $k>1$ we have $p(H)=1$.

4. **A necessary condition.** It is easily shown that there are abelian groups for which a suitable θ does not exist.

LEMMA 2. *A necessary condition that $O(\eta)=n(G)$ is that $p(G)=1$.*

COROLLARY. *If $p(G)\neq 1$, $O(\eta)<n(G)$ for all θ.*

PROOF. Suppose there exists a θ for which $O(\eta)=n(G)$. Then if we denote the elements of G by x_i $(i=1, 2, \cdots, n)$,

$$\prod_{i=1}^{n}[x_i\theta(x_i)] = \prod_{i=1}^{n}\eta(x_i),$$

and since G is abelian, θ and η one-to-one mappings of G upon G, we have $p^2=p$ or $p=1$. The corollary should be obvious.

5. **The main theorem.** In order to avoid complexity, we prove the following lemma before proceeding with the proof of Theorem 1.

LEMMA 3. *If for θ, $O(\eta)\leq n-2$, where $n=n(G)$, there exists a θ' such that $O(\eta')>O(\eta)$.*

COROLLARY. *There exists a θ for which $O(\eta)=n(G)-1$.*

PROOF. Let θ be a mapping for which $O(\eta)=r\leq n-2$. Denoting the elements of G by x_i $(i=1, \cdots, n)$, let $\eta(x_i)$ $(i=1, \cdots, r)$ be the r distinct elements of $\eta(x)$, for x in G. If there exist integers h, $k>r$ such that $x_k\theta(x_h)\neq\eta(x_i)$ $(i\leq r)$, the problem is solved by setting $\theta'(x_k)=\theta(x_h)$, $\theta'(x_h)=\theta(x_k)$ and $\theta'(x)=\theta(x)$ for the remaining elements of G. Hence, assume that this is not the case. Since $\eta(x_{r+1})=\eta(x_i)$ for some $i\leq r$, there is no loss in generality in assuming that $\eta(x_{r+1})=\eta(x_1)$. If $x_1\theta(x_{r+2})\neq\eta(x_i)$ $(i\leq r)$, we can set $\theta'(x_1)=\theta(x_{r+2})$, $\theta'(x_{r+2})=\theta(x_1)$ leaving $\theta'(x)=\theta(x)$ for the remaining elements of G

and thus construct a θ' with $O(\eta')>r$. But if $x_1\theta(x_{r+2})=\eta(x_i)$ for some $i\leqq r$, we note that $x_1\theta(x_{r+2})\neq\eta(x_1)$. Hence we may assume without loss of generality that $x_1\theta(x_{r+2})=\eta(x_2)$.

Now $x_2\theta(x_1)\neq\eta(x_1), \eta(x_2)$. If $x_2\theta(x_1)\neq\eta(x_i)$ $(i\leqq r)$, we could change θ by setting $\theta'(x_1)=\theta(x_{r+2})$, $\theta'(x_2)=\theta(x_1)$, $\theta'(x_{r+2})=\theta(x_2)$ and thus construct θ' with $O(\eta')>r$. Otherwise we may assume without loss of generality that $x_2\theta(x_1)=\eta(x_3)$.

Continue in this manner and suppose we have reached the point where

$$(1) \qquad x_1\theta(x_{r+2}) = \eta(x_2), \qquad x_{i+1}\theta(x_i) = \eta(x_{i+2}) \qquad (i = 1, 2, \cdots, k).$$

From (1) we derive the equations

$$(2) \qquad \eta(x_1)\theta(x_{r+2}) = \eta(x_{i+1})\theta(x_i) \qquad (i = 1, 2, \cdots, k+1).$$

In fact $\eta(x_1)\theta(x_{r+2})=x_1\theta(x_1)\theta(x_{r+2})=x_1\theta(x_{r+2})\theta(x_1)=\eta(x_2)\theta(x_1)$; so assume $\eta(x_1)\theta(x_{r+2})=\eta(x_{j+1})\theta(x_j)$ for some j, with $1\leqq j\leqq k$. Then $\eta(x_{j+1})\theta(x_j)=x_{j+1}\theta(x_j)\theta(x_{j+1})=\eta(x_{j+2})\theta(x_{j+1})$; and the result follows by induction.

Now $x_{k+2}\theta(x_{k+1})\neq\eta(x_i)$ $(i\leqq k+2)$, for using (2) this would imply $\eta(x_i)\theta(x_{k+2}) = x_{k+2}\theta(x_{k+1})\theta(x_{k+2}) = \eta(x_{k+2})\theta(x_{k+1}) = \eta(x_i)\theta(x_{i-1})$, or $\theta(x_{k+2})=\theta(x_{i-1})$, which is impossible since $i\leqq k+2$. If $x_{k+2}\theta(x_{k+1})\neq\eta(x_i)$ $(i\leqq r)$, we could change θ by setting $\theta'(x_1)=\theta(x_{r+2})$, $\theta'(x_{i+1})=\theta(x_i)$ $(i=1, 2, \cdots, k+1)$, $\theta'(x_{r+2})=\theta(x_{k+2})$ and thus construct a θ' with $O(\eta')>r$. If $x_{k+2}\theta(x_{k+1})=\eta(x_i)$ for some $i\leqq r$ we may assume without loss of generality that $i=k+3$ and add to (1) the equation $x_{k+2}\theta(x_{k+1})=\eta(x_{k+3})$. However, since $O(\eta)$ is finite, we must reach a product $x_j\theta(x_{j-1})\neq\eta(x_i)$ $(i\leqq r)$. This completes the proof of Lemma 3. The corollary is obvious.

In order to prove Theorem 1 we may assume, by the corollary of Lemma 3, a θ for which $O(\eta)=n(G)-1$. Hence, let $\eta(x_i)$ $(i=1, \cdots, n-1)$ be the $n-1$ distinct elements of $\eta(x)$, for x in G; z the unique element of G not equal to some $\eta(x_i)$. Then since

$$\prod_{i=1}^{n-1} [x_i\theta(x_i)] = \prod_{i=1}^{n-1} \eta(x_i)$$

we have $px_n^{-1}p\theta(x_n)^{-1}=pz^{-1}$, where $p=p(G)$ as defined in §2. Thus $p^{-1}x_n\theta(x_n)=z$ or $p^{-1}\eta(x_n)=z$. Hence if $p(G)=1$, we see that $O(\eta)=n(G)$. But if $p(G)\neq 1$ we know by Lemma 2 that $O(\eta)<n(G)$ for all θ. This completes the proof.

Although there exist groups G for which a θ, such that $O(\eta)=n(G)$, is easily represented explicitly (for example, if G is of odd order let $\theta(x)=x$), the author found it necessary to use repeated applications of Lemma 3 to obtain suitable θ's for groups of the form $Z_1 \times Z_2 \times Z_3$ where Z_i are cyclic of order 2^{n_i}. However, it should be noted that if $G \cong G_1 \times G_2$, a one-to-one mapping θ of G upon G may be defined by

$$\theta[(x, y)] = [\theta_1(x), \theta_2(y)]$$

where θ_1 and θ_2 are one-to-one mappings of G_1 upon G_1 and G_2 upon G_2 respectively. Moreover θ satisfies the relationship $O(\eta) \gtrsim O(\eta_1) \cdot O(\eta_2)$. Thus if $O(\eta_1) = n(G_1)$, $O(\eta_2) = n(G_2)$ we would have $O(\eta) = n(G_1 \times G_2)$ and θ is represented explicitly in terms of θ_1 and θ_2.

University of Wisconsin

ON RINGS WHOSE ASSOCIATED LIE RINGS ARE NILPOTENT

S. A. JENNINGS

1. Introduction.

With any ring R we may associate a Lie ring $(R)_l$ by combining the elements of R under addition and commutation, where the commutator $x \circ y$ of two elements $x, y \in R$ is defined by

$$x \circ y = xy - yx.$$

We call $(R)_l$ the Lie ring associated with R, and denote it by $\mathfrak{R}$. The question of how far the properties of $\mathfrak{R}$ determine those of R is of considerable interest, and has been studied extensively for the case when R is an algebra, but little is known of the situation in general. In an earlier paper the author investigated the effect of the nilpotency of $\mathfrak{R}$ upon the structure of R if R contains a nilpotent ideal N such that R/N is commutative.[1] In the present note we prove that, for an arbitrary ring R, the nilpotency of $\mathfrak{R}$ implies that the commutators of R of the form $x \circ y$ generate a nil-ideal, while the commutators of R of the form $(x \circ y) \circ z$ generate a nilpotent ideal (cf. §3). If R is finitely generated, and $\mathfrak{R}$ is nilpotent then the ideal generated by the commutators $x \circ y$ is also nilpotent (cf. §4).

2. A lemma on L-nilpotent rings.

We recall that the Lie ring $\mathfrak{R}$ is said to be nilpotent of class γ if we have

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[1] *Central chains of ideals in an associative ring*, Duke Math. J. vol. 9 (1942) pp. 341–355, Theorem 6.5.

$$(1)\qquad \mathfrak{R} = \mathfrak{R}_1 \supset \mathfrak{R}_2 \supset \cdots \supset \mathfrak{R}_\gamma \supset \mathfrak{R}_{\gamma+1} = 0$$

where $\mathfrak{R}_k = [\mathfrak{R}_{k-1}, \mathfrak{R}]$ is the Lie ideal of $\mathfrak{R}$ generated by all elements of the form $x \circ y$ with $x \in \mathfrak{R}$ and $y \in \mathfrak{R}_{k-1}$. If R is a ring whose associated Lie ring is nilpotent of class γ then we shall say that R is *L-nilpotent of class* γ. It is well known that the lower central chain (1) has the property $[\mathfrak{R}_\lambda, \mathfrak{R}_\mu] \subseteq \mathfrak{R}_{\lambda+\mu}$ and hence in particular

$$(2)\qquad [\mathfrak{R}_\lambda, \mathfrak{R}_\lambda] = 0 \qquad \text{if } 2\lambda > \gamma.$$

We prove the following lemma.

LEMMA 1. *Let R be an L-nilpotent ring of class γ. If $c \in \mathfrak{R}_{\gamma-1}$ and if x, y are arbitrary elements of R then*

$$(c \circ x)(c \circ y) = 0,$$

and in particular

$$(c \circ x)^2 = 0.$$

If $c_1, c_2 \in \mathfrak{R}_{\gamma-1}$ and $c_1 \circ c_2 = 0$ then for arbitrary $x, y \in R$

$$(c_1 \circ x)(c_2 \circ y) = 0.$$

PROOF. Consider the identity

$$(\overline{a \circ b y} \circ x) = (\overline{a \circ b} \circ x)y + (a \circ b)(y \circ x) + b(\overline{a \circ y} \circ x) + (b \circ x)(a \circ y).$$

Setting $a = b = c$ we have, since $[\mathfrak{R}_{\gamma-1}, \mathfrak{R}, \mathfrak{R}] = 0$,

$$0 = (c \circ x)(c \circ y)$$

and, if $x = y$,

$$0 = (c \circ x)^2,$$

while if $a = c_2$, $b = c_1$ and $c_1 \circ c_2 = 0$

$$0 = (c_1 \circ x)(c_2 \circ y),$$

which proves the lemma.

3. Ideals generated by the lower central chain of $\mathfrak{R}$. In what follows, R will be an L-nilpotent ring, and we denote the lower central chain of $\mathfrak{R}$ as in (1). Let R_k, $k = 1, 2, \cdots, \gamma$, be the subring of R generated by the elements of $\mathfrak{R}_k$, and let $\bar{R}_k$ be the ideal of R generated by R_k. It is known[2] that every element of $\bar{R}_k$ may be written in the form $u_k + v_k$, where $u_k \in R_k$ and $v_k \in RR_k$, and since R_γ is in the centre of R, $\bar{R}_\gamma$ is a nilpotent or nil-ring whenever R_γ is.

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[2] Ibid. Lemma 5.3.

Let $R^* = R/\bar{R}_\gamma$; then the natural homomorphism of R upon R^* induces a homomorphism of $\mathfrak{R}$ upon $\mathfrak{R}^*$, where $\mathfrak{R}^*$ is the Lie ring associated with R^*, such that $\bar{R}_k \to \bar{R}_k^*$. Hence in particular $\mathfrak{R}_\gamma^* = 0$ and R^* is an L-nilpotent ring of class not greater than $\gamma - 1$.

Our principal theorem is the following:

THEOREM 1. *If R is an L-nilpotent ring, then the commutators of R generate a nil-ideal of R, that is, $\bar{R}_2$ is a nil-ideal. The elements of R of the form $(x \circ y) \circ z$ generate a nilpotent ideal of R, that is, $\bar{R}_3$ is nilpotent.*

PROOF. Consider first R_γ: every element of R_γ can be written as a finite sum of finite products of elements of $\mathfrak{R}_\gamma$ and since $\mathfrak{R}_\gamma = [\mathfrak{R}_{\gamma-1}, \mathfrak{R}]$, every element of $\mathfrak{R}_\gamma$ can be written as a finite sum of elements of the form $c \circ x$, where $c \in \mathfrak{R}_{\gamma-1}$ and $x \in R$. Hence every element of R_γ is a sum of products of elements of the form $c \circ x$. Now by Lemma 1 the square of every element of the form $c \circ x$ is zero, and these elements are all in the centre of R. Hence if

$$y = p_1 + p_2 + \cdots + p_n$$

is an element of R_γ, where the p_k are products of elements of the form $c \circ x$, we have $p_k^2 = 0$ and therefore, since these products p_k are all in the centre of R, we have $y^{n+1} = 0$, which proves that R_γ, and hence $\bar{R}_\gamma$, is a nil-ring. Now if $\gamma > 2$ we have, from (2)

$$[\mathfrak{R}_{\gamma-1}, \mathfrak{R}_{\gamma-1}] = 0$$

and hence $c_1 \circ c_2 = 0$ for all $c_1, c_2 \in \mathfrak{R}_{\gamma-1}$. From Lemma 1 it follows that $p_i p_j = 0$ in the representation of y above, and hence

$$\bar{R}_\gamma^2 = 0, \qquad \gamma > 2.$$

The proof of the theorem now proceeds easily by induction upon γ, since by the above it is true when $\gamma = 2$, that is whenever $\bar{R}_2 = \bar{R}_\gamma$, $\bar{R}_3 = 0$. We suppose, therefore, that the theorem holds for rings of class less than γ, and hence in particular for $R^* = R/\bar{R}_\gamma$. Then if $c \in \bar{R}_2$ and $c \to c^*$ in the homomorphism of R upon R^* we have

$$c^{*\sigma'} = 0, \qquad \sigma' \text{ some integer,}$$

by our induction, and hence

$$c^{\sigma'} \in \bar{R}_\gamma \qquad \text{for all } c \in \bar{R}_2.$$

Since $\bar{R}_\gamma^2 = 0$ whenever $\gamma > 2$ we have

$$c^\sigma = 0, \qquad \text{where } \sigma = 2\sigma',$$

and it follows that $\bar{R}_2$ is a nil-ring. Further, since $\bar{R}_2^*$ is nilpotent by our induction,

$$\bar{R}_2^{*\tau'} = 0 \qquad \text{for some integer } \tau'$$

and hence

$$\bar{R}_2^{\tau'} \subseteq \bar{R}_\gamma$$

and therefore

$$\bar{R}_2^{\tau} = 0, \qquad \text{where } \tau = 2\tau',$$

which proves that $\bar{R}_2$ is nilpotent, as required.

4. **Finitely generated *L*-nilpotent rings.** If R satisfies the maximal or minimal condition for one-sided ideals, so does $\bar{R}_2$ and hence $\bar{R}_2$ must be nilpotent.[3] We prove the following stronger result:

THEOREM 2. *If R is a finitely generated L-nilpotent ring, then the commutators of R generate a nilpotent ideal, that is, $\bar{R}_2$ is nilpotent.*

PROOF. If R is finitely generated, say by $x_1, x_2, \cdots, x_d$, then every element x of R can be written in the form $x = p_1 + p_2 + \cdots + p_n$ where the p_k are products of the $x_1, \cdots, x_d$ in some order. It is clearly sufficient to consider the case $\gamma = 2$, since if we show in general that $\bar{R}_2/\bar{R}_3$ is nilpotent, it will follow from Theorem 1 that $\bar{R}_2$ has this property. Because of the identity

$$(ab) \circ c = a(b \circ c) + (a \circ c)b$$

every element of $\bar{R}_2$ can be written as a sum of products of the form

$$\pi_r = a(x_{i_1} \circ x_{j_1})(x_{i_2} \circ x_{j_2}) \cdots (x_{i_r} \circ x_{j_r}), \qquad a \in R.$$

Now there are at most $d(d-1)/2$ nonzero commutators of the type $x_i \circ x_j$, and since by Lemma 1 we have

$$(x_i \circ x_j)(x_i \circ x_k) = 0$$

it follows that if the number of factors in any product π_r is greater than $d(d-1)/2$ this product vanishes. Hence

$$\bar{R}_2^{\tau} = 0, \qquad \tau = d(d-1)/2 + 1$$

and the theorem is established.

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[3] Cf. C. Hopkins, *Nilrings with minimum condition for admissible left ideals*, Duke Math. J. vol. 4 (1938) pp. 664–667; J. Levitzki, *Solution of a problem of G. Köthe*, Amer. J. Math. vol. 67 (1945) pp. 437–442.

In connection with Theorem 2 it would be of interest to know if there exist L-nilpotent rings for which R_2 is not nilpotent. It would be enough to exhibit a ring R for which

$$(x \circ y) \circ z = 0 \qquad \text{for all } x,\ y,\ z \in R$$

and such that the subring generated by elements of the form $(x \circ y)$ is not nilpotent. The author has been unable to construct such a ring but it seems fairly safe to conjecture that such a one exists, and indeed with a countable generating set.

Since $R/\overline{R}_2$ is commutative and $\overline{R}_2$ is nilpotent we have at once from an earlier result of the author:[1]

THEOREM 3. *A finitely generated L-nilpotent ring is of finite class.*

Finally, it is clear that we have the following criterion for the nilpotency of a finitely generated nil-ring:

THEOREM 4. *A finitely generated nil-ring is nilpotent if and only if its associated Lie ring is nilpotent.*

This last theorem may be compared with Kaplansky's result on finitely generated nil-algebras,[4] which states that, provided the ground field has enough elements, such an algebra is nilpotent if and only if there exists a fixed integer ρ such that $x^{\rho} = 0$ for all elements x in the algebra. Our theorem shows that this condition may be replaced by the requirement that all commutators of a fixed weight vanish.

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[4] I. Kaplansky, *On a problem of Kurosch and Jacobson*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 496–500. *Added in proof.* In a recent paper (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1033–1035) J. Levitzki has proved a more general theorem to the effect that every finitely generated nil-ring of bounded index is nilpotent.

A SUM CONNECTED WITH THE PARTITION FUNCTION

ALBERT LEON WHITEMAN

1. **Introduction.** In the formula for the number $p(n)$ of unrestricted partitions of an integer n there appears the sum [1][1]

$$A_k(n) = \sum_{h \bmod k}{}' \omega_{h,k} \exp(-2\pi i h n/k), \tag{1}$$

where the dash ′ beside the summation symbol indicates here and in the sequel that the letter of summation runs only through a reduced residue system with respect to the modulus. The symbol $\omega_{h,k}$ denotes certain $24k$th roots of unity given by

$$\omega_{h,k} = \exp(\pi i s(h, k)),$$

where $s(h, k)$ is a Dedekind sum [2] defined by

$$s(h, k) = \sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{\mu h}{k}\right)\right).$$

The symbol $((x))$, in turn, is defined as follows:

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{for } x \text{ not an integer,} \\ 0 & \text{for } x \text{ an integer,} \end{cases}$$

where $[x]$ denotes, as usual, the greatest integer not exceeding x. D. H. Lehmer [3] has investigated these sums on the basis of a different expression for the roots of unity involved. In the first place he factored the $A_k(n)$ according to the prime powers contained in k. Secondly, by reducing them to sums studied by H. D. Kloosterman [4] and H. Salié [5], he evaluated the $A_k(n)$ explicitly in the case in which k is a prime or a power of a prime. Both results together provide a method for calculating the $A_k(n)$. Alternate proofs of Lehmer's factorization theorems have been given in [2]. In the present note a new approach to the second of Lehmer's results is presented. The method given here is simpler than Lehmer's method, especially in the treatment of the case $k = 2^\lambda$.

2. **Some lemmas.** The proofs are based in part on three lemmas, which occur as Theorems 17, 18, 19 in [2].

LEMMA 1. *Let* $\theta = \theta(k)$ *denote* 1 *for* $3 \nmid k$ *and* 3 *for* $3 | k$ *so that* θk *is*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

either prime to 3 *or divisible by* 3^2. *For* $(h, k) = 1$ *we have*

$$12hks(h, k) \equiv h^2 + 1 \pmod{\theta k}.$$

Moreover

$$12ks(h, k) \equiv 0 \pmod{3}$$

if and only if $3 \nmid k$.

LEMMA 2. *For odd* k *we have* $12ks(h, k) \equiv k + 1 - 2(h|k) \pmod{8}$, *where* $(h|k)$ *denotes the Legendre-Jacobi symbol.*

LEMMA 3. *If* k *is equal to* $2^\lambda j$, $\lambda \geqq 0$, *and* j *and* h *are odd integers, then*

$$12hks(h, k) \equiv h^2 + k^2 + 3k + 1 + 2k(k \mid h) \pmod{2^{\lambda+3}}.$$

The following lemmas concerning generalized Kloosterman sums are also needed.

LEMMA 4. *Let* $k = p^\alpha$, *where* p *is an odd prime. Put*

$$S_k(n) = \sum_{h \bmod k}{}' (h \mid k) \exp\left(2\pi i(nh + \bar{h})/k\right),$$

where $\bar{h}$ *is defined as any solution of the congruence* $h\bar{h} \equiv 1 \pmod{k}$. *Then*

$$S_k(n) = \begin{cases} 0 & \textit{if } n \textit{ is a non-residue of } k \textit{ prime to } k, \\ 2i^{((k-1)/2)^2} k^{1/2} \cos 4\pi m/k & \textit{if } m^2 \equiv n \pmod{k},\ n \textit{ prime to } k, \\ 0 & \textit{if } n \textit{ is divisible by } p \textit{ and } \alpha > 1, \\ i^{((k-1)/2)^2} k^{1/2} & \textit{if } n \textit{ is divisible by } p \textit{ and } \alpha = 1. \end{cases}$$

LEMMA 5. *Let* $k = p^\beta$, $\beta > 1$, *and* $n \equiv 1 \pmod{p}$, *where* p *denotes an odd prime congruent to* $3 \pmod{4}$. *Then*

$$\sum_{h \bmod k}{}' (h \mid p^{\beta-1}) \exp\left(2\pi i(nh + \bar{h})/k\right) = 2i^{((k-1)/2)^2+1} (m \mid p) k^{1/2} \sin(4\pi m/k),$$

where m *is an integer such that* $m^2 \equiv n \pmod{k}$.

Lemmas 4 and 5 may be proved by employing a method due to the author [6]. The method carries over step by step with only slight modifications and it does not seem necessary to present it again here.

LEMMA 6. *If* k *is equal to* 2^λ, $\lambda \geqq 9$, *and* n *is an odd integer, then*

$$\sum_{h \bmod k}{}' (2k \mid h) \exp\left(2\pi i n(h + \bar{h})/k\right) = 2(k \mid n)(2k)^{1/2} \cos\left(\pi/4 + (-1)^{(n-1)/2} 4\pi n/k\right).$$

Lemma 6 may be established easily if use is made of Salié's [5] discussion (§3) of the corresponding sum without a quadratic character. In fact, the introduction of the character has no influence on the argument. It is therefore not necessary to reproduce the proof here.

3. **Proof of Lehmer's theorems.** In order to apply the lemmas of the preceding section to the evaluation of $A_k(n)$ for k a power of an odd prime p, we divide the discussion into two cases according as $p>3$ or $p=3$. For $p>3$ we employ the congruence

$$6k[-((k-1)/2)^2-1+(2h\mid k)]+24^2ah+24\cdot\overline{24}\bar{h} \equiv 12ks(h,k)-24hn \pmod{24k}, \tag{2}$$

where a and $\bar{h}$ are defined by means of the congruences

$$24^2a \equiv 1-24n \pmod{k}, \qquad h\bar{h} \equiv 1 \pmod{k}. \tag{3}$$

Divisibility with respect to k and with respect to 3 in congruence (2) follows at once from Lemma 1. Divisibility with respect to 8 follows from Lemma 2 and the well known relation $(2\mid k)=(-1)^{(k^2-1)/8}$. Hence, from (1) and (2), we get

$$\begin{aligned} A_k(n) &= \sum_{h \bmod k}{}' \exp\,[2\pi i(ks(h,k)/2-hn)/k] \\ &= \sum_{h \bmod k}{}' \exp\,\{2\pi i[k(-((k-1)/2)^2-1+(2h\mid k))/4 \\ &\qquad\qquad + 24ah+\overline{24}\bar{h}]/k\} \\ &= (-i)^{((k-1)/2)^2}(2\mid k)\sum_{h \bmod k}{}'(h\mid k)\exp\,[2\pi i(24ah+\overline{24}\bar{h})/k] \\ &= (-i)^{((k-1)/2)^2}(3\mid k)\sum_{h \bmod k}{}'(h\mid k)\exp\,[2\pi i(ah+\bar{h})/k]. \end{aligned}$$

Applying Lemma 4, and using (3), we obtain the following theorem.

THEOREM 1. *If $k=p^\alpha$, $\alpha\geqq 1$, and $v=1-24n$, where p is a prime greater than 3, then*

$$A_k(n)=\begin{cases} 0 & \text{if } v \text{ is a non-residue}^2 \text{ of } k, \text{ prime to } k, \\ 2(3\mid k)k^{1/2}\cos(4\pi m/k) & \text{if } v\equiv(24m)^2 \pmod{k}, \text{ prime to } k, \\ 0 & \text{if } v\equiv 0 \pmod{p} \text{ and } \alpha>1, \\ (3\mid k)k^{1/2} & \text{if } v\equiv 0 \pmod{p} \text{ and } \alpha=1. \end{cases}$$

We turn next to the case which arises when $k=p^\alpha$ and $p=3$. This

---

[2] This condition should not be confused with $(v\mid k)=-1$. We mean that no solution exists of the congruence $x^2\equiv v \pmod{k}$.

time we use the congruence

$$
\begin{aligned}
(4)\qquad 6k[-((k-1)/2)^2 - 1 + (2h \mid k)] + 8^2ah + 8\cdot\bar{8}\bar{h} \\
\equiv 12ks(h, k) - 24hn \pmod{24k},
\end{aligned}
$$

where a and $\bar{h}$ are defined by the congruences

$$
(5)\qquad 8^2a \equiv 1 - 24n \pmod{3k}, \qquad h\bar{h} \equiv 1 \pmod{3k}.
$$

Divisibility with respect to $3k$ in congruence (4) follows immediately from Lemma 1. Divisibility with respect to 8 follows from Lemma 2. Hence, by (1) and (4), we have

$$
\begin{aligned}
A_k(n) &= \sum_{h \bmod k}{}' \exp\,[2\pi i(3ks(h, k)/2 - 3hn)/3k] \\
&= \sum_{h \bmod k}{}' \exp\,\{2\pi i[3k(-((k-1)/2)^2 - 1 + (2h \mid k))/4 \\
&\qquad\qquad + 8ah + \bar{8}\bar{h}]/3k\} \\
&= (-i)^{((k-1)/2)^2}(2 \mid k) \sum_{h \bmod k}{}' (h \mid k) \exp\,[2\pi i(8ah + \bar{8}\bar{h})/3k].
\end{aligned}
$$

As h runs through a reduced residue system mod k, so does $h+k$. Replace $\bar{h}$ by $\bar{h}-k\bar{h}^2$, and observe that (5) implies that

$$
8a(h + k) + \bar{8}(\bar{h} - k\bar{h}^2) \equiv 8ah + \bar{8}\bar{h} \qquad \pmod{3k}.
$$

If h now runs over a reduced residue system mod $3k$, instead of mod k, we obtain $3A_k(n)$ instead of $A_k(n)$. Therefore,

$$
\begin{aligned}
A_k(n) &= \frac{1}{3}(-i)^{((k-1)/2)^2}(2 \mid k) \sum_{h \bmod 3k}{}' (h \mid k) \exp\,[2\pi i(8ah + \bar{8}\bar{h})/3k] \\
&= \frac{1}{3}(-i)^{((k-1)/2)^2} \sum_{h \bmod 3k}{}' (h \mid k) \exp\,[2\pi i(ah + \bar{h})/3k].
\end{aligned}
$$

Since $a \equiv 1 \pmod 3$, we may apply Lemma 5 with β replaced by $\alpha+1$. Thus we get the following theorem.

THEOREM 2. *If* $k=3^\beta$, *then*

$$
A_k(n) = 2(-1)^{\beta+1}(m \mid 3)(k/3)^{1/2} \sin\,(4\pi m/3k),
$$

where m *is an integer such that* $(8m)^2 \equiv 1-24n \pmod{3k}$.

Finally, we consider the case $k=2^\lambda$. For this purpose we use the congruence

$$
(6)\quad 3\bar{h}k - 6\bar{h}k(k \mid h) + 3^2ah + 3\cdot 3\bar{h} \equiv 12ks(h, k) - 24hn \pmod{24k},
$$

where a and $\bar{h}$ are defined by the congruences

$$(7) \qquad 3^2a \equiv 1 - 24n \pmod{8k}, \qquad h\bar{h} \equiv 1 \pmod{8k},$$

and where we have assumed that $\lambda \geqq 3$.

Divisibility with respect to 3 in congruence (6) follows from the second part of Lemma 1. Divisibility with respect to $8k$ follows from Lemma 3. From (1) and (6) we now obtain

$$\begin{aligned} A_k(n) &= \sum_{h \bmod k}{}' \exp\,[2\pi i(4ks(h, k) - 8hn)/8k] \\ &= \sum_{h \bmod k}{}' \exp\,[2\pi i(\bar{h}k - 2\bar{h}k(k \mid h) + 3ah + 3\bar{h})/8k] \\ &= \frac{1}{8} \sum_{h \bmod 8k}{}' \exp\,[2\pi i(k - 2h(k \mid h))/8] \exp[2\pi i(3ah + 3\bar{h})/8k]. \end{aligned}$$

It is easy to verify that

$$\exp\,[2\pi i(k - 2h(k \mid h))/8] = (-1)^{\lambda+1}(h \mid 3h) \exp\,[2\pi i(3h)/8].$$

Therefore,

$$(8) \quad A_k(n) = \frac{1}{8}(-1)^{\lambda+1} \sum_{h \bmod 8k}{}' (k \mid 3h) \exp\,[2\pi i(3ah + 3\bar{h}(1 + k))/8k].$$

Now, for $\lambda \geqq 5$, we have $(1+k/2)^2 \equiv 1+k \pmod{8k}$. Furthermore, it follows from (7) that a is a quadratic residue of $2^{\lambda+3}$ since $3^2a \equiv 1 \pmod 8$. Let $m^2 \equiv a \pmod{8k}$. Replacing h by $\bar{3}\bar{m}(1+k/2)h$, and applying Lemma 6, we obtain for $\lambda \geqq 6$,

$$\begin{aligned} A_k(n) &= \frac{1}{8}(-1)^{\lambda+1}(k \mid m) \sum_{h \bmod 8k}{}' (k \mid h) \exp\,[2\pi i m(1 + k/2)(h + \bar{h})/8k] \\ &= (-1)^{\lambda+1}(2 \mid m) k^{1/2} \cos\left\{\pi/4 + (-1)^{(m-1)/2} 4\pi[m(1 + k/2)]/8k\right). \end{aligned}$$

The last equation may be simplified by employing the relation $(2 \mid m) = (-1)^{(m^2-1)/8}$. We obtain thus the following theorem.

THEOREM 3. *If* $k = 2^\lambda$, $\lambda \geqq 0$, *then*

$$A_k(n) = (-1)^\lambda(-1 \mid m) k^{1/2} \sin\,(4\pi m/8k),$$

where m *is an integer such that* $(3m)^2 \equiv 1 - 24n \pmod{8k}$.

Actually we have established Theorem 3 only for $\lambda \geqq 6$. The verification of this theorem for $\lambda < 6$ is left as an exercise for the reader. For $0 \leqq \lambda \leqq 3$, use the definition of $A_k(n)$ given by (1). For $\lambda = 4, 5$ use the formula for $A_k(n)$ given by (8).

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PURDUE UNIVERSITY

A PRIME-REPRESENTING FUNCTION

W. H. MILLS

A function $f(x)$ is said to be a prime-representing function if $f(x)$ is a prime number for all positive integral values of x. It will be shown that there exists a real number A such that $[A^{3^x}]$ is a prime-representing function, where $[R]$ denotes the greatest integer less than or equal to R.

Let p_n denote the nth prime number. A. E. Ingham[1] has shown that

$$p_{n+1} - p_n < Kp_n^{5/8} \tag{1}$$

where K is a fixed positive integer.

LEMMA. *If N is an integer greater than K^8 there exists a prime p such that $N^3 < p < (N+1)^3 - 1$.*

PROOF. Let p_n be the greatest prime less than N^3. Then

$$\begin{aligned} N^3 < p_{n+1} < p_n + Kp_n^{5/8} < N^3 + KN^{15/8} < N^3 + N^2 \\ < (N+1)^3 - 1. \end{aligned} \tag{2}$$

Let P_0 be a prime greater than K^8. Then by the lemma we can construct an infinite sequence of primes, $P_0, P_1, P_2, \cdots$, such that $P_n^3 < P_{n+1} < (P_n+1)^3 - 1$. Let

$$u_n = P_n^{3^{-n}}, \qquad v_n = (P_n + 1)^{3^{-n}}. \tag{3}$$

Then

$$v_n > u_n, \qquad u_{n+1} = P_{n+1}^{3^{-n-1}} > P_n^{3^{-n}} = u_n, \tag{4}$$

$$v_{n+1} = (P_{n+1} + 1)^{3^{-n-1}} < (P_n + 1)^{3^{-n}} = v_n. \tag{5}$$

It follows at once that the u_n form a bounded monotone increasing sequence. Let $A = \lim_{n \to \infty} u_n$.

THEOREM. *$[A^{3^x}]$ is a prime-representing function.*

PROOF. From (4) and (5) it follows that $u_n < A < v_n$, or $P_n < A^{3^n} < P_n + 1$.

Therefore $[A^{3^n}] = P_n$ and $[A^{3^x}]$ is a prime-representing function.

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[1] A. E. Ingham, *On the difference between consecutive primes*, Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 255–266.

ON THE CHARACTERISTIC EQUATIONS OF CERTAIN MATRICES

ALFRED BRAUER

In a paper to be published soon in the Annals of Mathematical Statistics, R. v. Mises obtains the following theorem on matrices from results in the theory of probability.

THEOREM. *Let* $A=(a_{\kappa\lambda})$, $B=(b_{\kappa\lambda})$, *and* $C=(c_{\kappa\lambda})$ *be square matrices of order* n. *If the elements of* A *and* C *satisfy the conditions*

$$r_\kappa = \sum_{\nu=1}^{n} a_{\kappa\nu} = 0 \qquad (\kappa = 1, 2, \cdots, n), \tag{1}$$

$$s_\lambda = \sum_{\nu=1}^{n} a_{\nu\lambda} = 0 \qquad (\lambda = 1, 2, \cdots, n), \tag{2}$$

$$c_{\kappa\lambda} = c_\kappa + c_\lambda \qquad (\kappa, \lambda = 1, 2, \cdots, n) \tag{3}$$

where $c_1, c_2, \cdots, c_n$ *are arbitrary numbers, then the matrices* AB *and* $A(B+C)$ *have the same characteristic equation.*

In the following a purely algebraic proof of this theorem will be given.

PROOF. We set

$$\sum_{\nu=1}^{n} a_{\kappa\nu}c_\nu = q_\kappa \qquad (\kappa = 1, 2, \cdots, n).$$

Then we have by (1) and (3)

$$\begin{aligned} AC &= \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{pmatrix} \begin{pmatrix} c_1+c_1 & c_1+c_2 \cdots c_1+c_n \\ c_2+c_1 & c_2+c_2 \cdots c_2+c_n \\ \cdot & \cdot \quad \cdot \quad \cdot \\ c_n+c_1 & c_n+c_2 \cdots c_n+c_n \end{pmatrix} \\ &= \begin{pmatrix} q_1+c_1r_1 & q_1+c_2r_1 \cdots q_1+c_nr_1 \\ q_2+c_1r_2 & q_2+c_2r_2 \cdots q_2+c_nr_2 \\ \cdot & \cdot \quad \cdot \quad \cdot \\ q_n+c_1r_n & q_n+c_2r_n \cdots q_n+c_nr_n \end{pmatrix} = \begin{pmatrix} q_1 & q_1 \cdots q_1 \\ q_2 & q_2 \cdots q_2 \\ \cdot & \cdot \quad \cdot \\ q_n & q_n \cdots q_n \end{pmatrix}. \end{aligned} \tag{4}$$

Let P be the triangular matrix

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$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \cdots 0 \\ 1 & 1 & 0 & 0 \cdots 0 \\ 1 & 1 & 1 & 0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \cdot \cdot \cdot \\ 1 & 1 & 1 & 1 \cdots 1 \end{pmatrix}; \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ -1 & 1 & 0 \cdots 0 \\ 0 & -1 & 1 \cdots 0 \\ \cdot & \cdot & \cdot \cdot \cdot \cdot \\ 0 & 0 \cdots 0 & -1 \quad 1 \end{pmatrix}.$$

We have by (4)

$$\begin{aligned} PAC &= \begin{pmatrix} 1 & 0 \cdots & & 0 \\ 1 & 1 \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 \cdots & 1 & 0 \\ 1 & 1 \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 & q_1 & \cdots & q_1 \\ q_2 & q_2 & \cdots & q_2 \\ \cdot & \cdot & \cdot & \cdot \\ q_{n-1} & q_{n-1} & \cdots & q_{n-1} \\ q_n & q_n & \cdots & q_n \end{pmatrix} \\ (5) \qquad &= \begin{pmatrix} q_1 & q_1 & \cdots & q_1 \\ q_1+q_2 & q_1+q_2 & \cdots & q_1+q_2 \\ \cdot & \cdot & \cdot & \cdot \\ q_1+q_2+\cdots+q_{n-1} & q_1+q_2+\cdots+q_{n-1} & \cdots & q_1+q_2+\cdots+q_{n-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \end{aligned}$$

since by (2)

$$\sum_{\nu=1}^{n} q_\nu = \sum_{\nu=1}^{n}\sum_{\lambda=1}^{n} a_{\nu\lambda}c_\lambda = \sum_{\lambda=1}^{n} c_\lambda \sum_{\nu=1}^{n} a_{\nu\lambda} = \sum_{\lambda=1}^{n} c_\lambda s_\lambda = 0.$$

Hence

$$\begin{aligned} PACP^{-1} &= \begin{pmatrix} q_1 & q_1 & \cdots & q_1 \\ q_1+q_2 & q_1+q_2 & \cdots & q_1+q_2 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \cdots & 0 & 0 \\ -1 & 1 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 \cdots & -1 & 1 \end{pmatrix} \\ (6) \qquad &= \begin{pmatrix} 0 & 0 \cdots 0 & q_1 \\ 0 & 0 \cdots 0 & q_1+q_2 \\ \cdot & \cdot \cdot \cdot & \cdot \\ 0 & 0 \cdots 0 & q_1+q_2+\cdots+q_{n-1} \\ 0 & 0 \cdots 0 & 0 \end{pmatrix}. \end{aligned}$$

On the other hand, it follows from (2), similarly as in (5), that PA, and therefore also PAB and $PABP^{-1}$, are matrices in which all the elements of the last row are equal to 0. Hence $PABP^{-1}$ has the form

$$PABP^{-1} = \left(\begin{array}{cccc|c} & & & & t_1 \\ & & & & t_2 \\ & D_{nn} & & & \cdot \\ & & & & \cdot \\ & & & & t_{n-1} \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right) \tag{7}$$

where D_{nn} is a square matrix of order $n-1$ and $t_1, t_2, \cdots, t_{n-1}$ are certain elements. It follows from (7) and (6) that

$$PA(B+C)P^{-1} = \left(\begin{array}{cccc|c} & & & & t_1 + q_1 \\ & & & & t_2 + q_1 + q_2 \\ & D_{nn} & & & \cdot \\ & & & & t_{n-1} + q_1 + q_2 + \cdots + q_{n-1} \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right). \tag{8}$$

If we denote the characteristic polynomial of the matrix D_{nn} by $f(x)$, then it follows from (7) and (8) that $PABP^{-1}$ and $PA(B+C)P^{-1}$ both have the characteristic equation

$$xf(x) = 0. \tag{9}$$

Since similar matrices have the same characteristic equation, (9) is also the characteristic equation of AB and $A(B+C)$.

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NOTE ON HADAMARD'S DETERMINANT THEOREM

JOHN WILLIAMSON

Introduction. We shall call a square matrix A of order n an Hadamard matrix or for brevity an H-matrix, if each element of A has the value ± 1 and if the determinant of A has the maximum possible value $n^{n/2}$. It is known that such a matrix A is an H-matrix [1][1] if, and only if, $AA'=nE_n$ where A' is the transpose of A and E_n is the unit matrix of order n. It is also known that, if an H-matrix of order $n>1$ exists, n must have the value 2 or be divisible by 4. The existence of an H-matrix of order n has been proved [2, 3] only for the following values of $n>1$: (a) $n=2$, (b) $n=p^k+1\equiv 0 \bmod 4$, p a prime, (c) $n=m(p^k+1)$ where $m\geqq 2$ is the order of an H-matrix and p is a prime, (d) $n=q(q-1)$ where q is a product of factors of types (a) and (b), (e) $n=172$ and for n a product of any number of factors of types (a), (b), (c), (d) and (e).

In this note we shall show that an H-matrix of order n also exists when (f) $n=q(q+3)$ where q and $q+4$ are both products of factors of types (a) and (b), (g) $n=n_1n_2(p^k+1)p^k$, where $n_1>1$ and $n_2>1$ are orders of H-matrices and p is an odd prime, and (h) $n=n_1n_2m(m+3)$ where $n_1>1$ and $n_2>1$ are orders of H-matrices and m and $m+4$ are both of the form p^k+1, p an odd prime.

It is interesting to note the presence of the factors n_1 and n_2 in the types (g) and (h) and their absence in the types (d) and (f). Thus, if p is a prime and $p^k+1\equiv 0 \bmod 4$, an H-matrix of order $p^k(p^k+1)$ exists but, if $p^k+1\equiv 2 \bmod 4$, we can only be sure of the existence of an H-matrix of order $n_1n_2p^k(p^k+1)$ where $n_1>1$ and $n_2>1$ are orders of H-matrices. This is analogous to the simpler result that, if $p^k+1\equiv 0 \bmod 4$ an H-matrix of order p^k+1 exists but, if $p^k+1\equiv 2 \bmod 4$, we can only be sure of the existence of an H-matrix of order $n(p^k+1)$ where $n>1$ is the order of an H-matrix.

We shall denote the direct product of two matrices A and B by $A\cdot B$ and the unit matrix of order n by E_n.

Theorems on the existence of H-matrices. If a symmetric H-matrix of order $m>1$ exists, there exists an H-matrix H of order m with the form

$$H=\begin{pmatrix}1 & e\\ e' & D\end{pmatrix},$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

where e is the row vector $(1, 1, \cdots, 1)$ of dimension $m-1$ and e' the column vector which is the transpose of e. Since H is a symmetric H-matrix

$$H^2 = HH' = mE_m$$

and accordingly

$$\begin{pmatrix} m & e + eD \\ e' + De' & e'e + D^2 \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & mE_{m-1} \end{pmatrix}.$$

Therefore

$$eD = -e, \qquad De' = -e' \tag{1}$$

and

$$D^2 = mE_{m-1} - R, \tag{2}$$

where

$$R = e'e \tag{3}$$

and R is the square matrix of order $m-1$ each element of which has the value 1. It follows easily that

$$R^2 = (m-1)R \tag{4}$$

and by (1) and (3) that

$$RD = -R = DR. \tag{5}$$

If $F = 2E_{m-1} - R$, F is a symmetric matrix each element of which has the value ±1. Further

$$FD = DF \tag{6}$$

by (5) and

$$F^2 = 4E_{m-1} + (m-5)R \tag{7}$$

by (4). If n is a product of factors of types (a) and (b) there exists [3, p. 67] an H-matrix of order n with the form $E_n + S$ where S is skew-symmetric so that

$$S^2 = -(n-1)E_n. \tag{8}$$

If

$$W = F \cdot E_n + D \cdot S,$$

each element of W has the value ±1 and

$$
\begin{aligned}
WW' &= (F\cdot E_n + D\cdot S)(F\cdot E_n - D\cdot S) \\
&= F^2\cdot E_n - D^2\cdot S^2 && \text{(by (6))} \\
&= [4E_{m-1} + (m-5)R]\cdot E_n + (mE_{m-1} - R)\cdot(n-1)E_n \\
& && \text{(by (7), (2) and (8))} \\
&= [(4 + mn - m)E_{m-1}]\cdot E_n + (m - n - 4)R\cdot E_n.
\end{aligned}
$$

Therefore, if $n = m-4$,

$$WW' = (m-1)(m-4)E_{m-1}\cdot E_{m-4} = n(n+3)E_n\cdot E_{n+3}$$

and W is an H-matrix.

Since a symmetric H-matrix of order 2 exists and a symmetric H-matrix of order $p^k+1 \equiv 0 \bmod 4$, where p is a prime, exists [3, p. 67], there exists a symmetric H-matrix of order n where n is a product of factors of types (a) and (b). We have therefore proved the theorem:

THEOREM 1. *If n and $n+4$ are both products of factors of types* (a) *and* (b) *there exists an H-matrix of order $n(n+3)$.*

As a particular case of this theorem we have the corollary:

COROLLARY 1. *If $n-1$ and $n+3$ are both powers of primes and are congruent to* 3 *modulo* 4, *there exists an H-matrix of order $n(n+3)$.*

If $m = p^k+1 \equiv 2 \bmod 4$, where p is a prime, there exists [3, p. 66] a symmetric matrix T of order m, each diagonal element of which has the value 0 and each other element the value ± 1 and such that

$$T = \begin{pmatrix} 0 & e \\ e' & U \end{pmatrix}$$

and

(9) $$T^2 = (m-1)E_m.$$

It follows therefore that

(10) $$UU' = U^2 = (m-1)E_{m-1} - R,$$

where R is defined by (3). Let A_1 and B_1 be two H-matrices of order n_1 such that [3, p. 66]

(11) $$A_1B_1' = -B_1A_1'$$

and let $K = A_1\cdot E_{m-1} + B_1\cdot U$. Then each element of K has the value ± 1 and

$$(12)\qquad \begin{aligned} \mathrm{K}\mathrm{K}' &= A_1A_1'\cdot E_{m-1} + B_1B_1'\cdot U^2 && \text{(by (11))}\\ &= n_1E_{n_1}\cdot(mE_{m-1} - R) && \text{(by (10)).}\end{aligned}$$

Since $eU = 0 = Ue'$,

$$(13)\qquad RU = UR = 0.$$

Hence, if $\Gamma = A_1\cdot R$,

$$(14)\qquad \Gamma\Gamma' = n_1E_{n_1}\cdot(m-1)R \qquad \text{(by (4))}$$

and

$$(15)\qquad \Gamma\mathrm{K}' = A_1A_1'\cdot R = \mathrm{K}\Gamma' \qquad \text{(by (13)).}$$

Finally, if A_2 and B_2 are two H-matrices of order n_2 satisfying

$$(16)\qquad A_2B_2' = -B_2A_2',$$

and

$$\begin{aligned} W &= A_2\cdot\Gamma\cdot E_m + B_2\cdot\mathrm{K}\cdot T,\\ WW' &= A_2A_2'\cdot\Gamma\Gamma'\cdot E_m + B_2B_2'\cdot\mathrm{K}\mathrm{K}'\cdot T^2 && \text{(by (15) and (16))}\\ &= n_2E_{n_2}\cdot n_1E_{n_1}\cdot\left[(m-1)R + (mE_{m-1} - R)(m-1)\right]\cdot E_m\\ & && \text{(by (9), (12) and (14))}\\ &= rE_r && (r = n_1n_2m(m-1)).\end{aligned}$$

Therefore W is an H-matrix and we have proved the theorem:

THEOREM 2. *If H-matrices of orders n_1 and n_2 exist, $n_1 > 1$, $n_2 > 1$, and p is a prime such that $p^k+1 \equiv 2 \bmod 4$, there exists an H-matrix of order $n_1n_2p^k(p^k+1)$.*

Since, if p is a prime such that $p^k+1 \equiv 0 \bmod 4$, there exists an H-matrix of order $p^k(p^k+1)$, we have the corollary:

COROLLARY 1. *If H-matrices exist of orders $n_1 > 1$ and $n_2 > 1$, there exists an H-matrix of order $n_1n_2(p^k+1)p^k$ where p is an odd prime.*

Since an H-matrix of order 2 exists we have the corollary:

COROLLARY 2. *If p is an odd prime an H-matrix exists of order $4p^k(p^k+1)$.*

In the proof of the final theorem we require the following lemma:

LEMMA 1. *If there exists an H-matrix A of order $n > 1$, there exist two H-matrices B and C of order n such that $AB' = -BA'$, $AC' = CA'$, $BC' = CB'$.*

In fact the matrices $B = XA$ and $C = YA$, where X is the diagonal block matrix

$$\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$$

and Y is the diagonal block matrix

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$

satisfy the conditions of the lemma. For $BB' = CC' = nE_n$, $AB' = nX' = -nX = -BA'$, $CA' = nY' = nY = CA'$ and $BC' = nXY' = nYX' = CB'$.

Let $\mathrm{M} = C_1\cdot(2E_{m-1} - R)$ and $\mathrm{N} = A_1\cdot E_{m-1} + B_1\cdot U$, where R is defined by (14), U by (10) and A_1, B_1 and C_1 are matrices of order n_1 with the properties of Lemma 1. Then each element of the matrices M and N has the value ± 1. Further

$$\begin{aligned} \mathrm{MM}' &= n_1E_{n_1}\cdot(4E_{m-1} - 4R + R^2) \\ &= n_1E_{n_1}\cdot[4E_{m-1} + (m-5)R] \qquad \text{(by (4))}, \end{aligned} \tag{17}$$

$$\mathrm{NN}' = n_1E_{n_1}\cdot(E_{m-1} + U^2) = n_1E_{n_1}\cdot(mE_{m-1} - R) \qquad \text{(by (10))} \tag{18}$$

and

$$\begin{aligned} \mathrm{MN}' &= C_1A_1'\cdot(2E_{m-1} - R) + C_1B_1'\cdot(2E_{m-1} - R)U \\ &= C_1A_1'\cdot(2E_{m-1} - R) + C_1B_1'\cdot 2U \qquad \text{(by (13))}. \end{aligned}$$

Therefore by Lemma 1

$$\mathrm{MN}' = \mathrm{NM}'. \tag{19}$$

Let A_2 and B_2 be two H-matrices of order $n_2 > 1$ satisfying (16) and let $n = p^k + 1 \equiv 2 \bmod 4$ where p is a prime. Then there exists a matrix G of order n and of the same form as T in (9) and satisfying

$$G^2 = (n-1)E_n. \tag{20}$$

If finally $W = A_2\cdot\mathrm{M}\cdot E_n + B_2\cdot\mathrm{N}\cdot G$, each element of W has the value ± 1 and

$$\begin{aligned} WW' &= n_2E_{n_2}\cdot(\mathrm{MM}'\cdot E_n + \mathrm{NN}'\cdot G^2) \qquad \text{(by (16) and (19))} \\ &= n_1n_2E_{n_1n_2}\cdot[4E_{m-1} + (m-5)R]\cdot E_n \\ &\quad + (mE_{m-1} - R)\cdot(n-1)E_n \qquad \text{(by (17), (18) and (20))} \\ &= n_1n_2E_{n_1n_2}\cdot[(4 + mn - m)E_{m-1} + (m - 4 - n)R]\cdot E_n. \end{aligned}$$

Hence, if $m=n+4$ and $r=n_1n_2n(n+3)=n_1n_2(m-1)(m-4)$,

$$WW'=rE_r$$

and W is an H-matrix. We have therefore proved the theorem:

THEOREM 3. *If n and $n+4$ are both of the form $p^k+1\equiv 2 \bmod 4$ where p is a prime and if H-matrices of orders $n_1>1$ and $n_2>1$ both exist, there exists an H-matrix of order $n_1n_2n(n+3)$.*

As a consequence of Theorem 1 we have the corollary:

COROLLARY 1. *If n and $n+4$ are both of the form p^k+1 where p is an odd prime and if H-matrices of orders $n_1>1$ and $n_2>1$ both exist, there exists an H-matrix of order $n_1n_2n(n+3)$.*

Since an H-matrix of order 2 exists we also have the corollary:

COROLLARY 2. *If n and $n+4$ are both of the form p^k+1, where p is an odd prime, there exists an H-matrix of order $4n(n+3)$.*

Particular examples. That the above theorems do actually increase the values of n as orders of H-matrices which are known to exist is shown by the following examples.

By Theorem 1 an H-matrix of order (56)(59) exists. For $56=2(3^3+1)$ and 59 is prime. Further no one of (56)(59), (28)(59), (14)(59), 4(59) or 2(59) is of the form p^k+1. Therefore (56)(59) is not a product of factors of types (a), (b) or (c). By Theorem 2 an H-matrix of order 4(73)(74) exists and by Theorem 3 an H-matrix of order 4(230)(233) exists. Neither of the numbers 4(73)(74) nor 4(230)(233) is a product of factors of types (a), (b), (c) and (d).

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QUEENS COLLEGE

A GEOMETRIC THEOREM AND ITS APPLICATION TO BIORTHOGONAL SYSTEMS

ANGUS E. TAYLOR

1. **The geometric theorem.** Let E_n denote an n-dimensional Euclidean space. A set of points in E_n may constitute an E_{n-1}; if so, it will be called a plane in E_n. We shall sometimes denote planes by $p_1, p_2, \cdots$. Also, if $x_1, \cdots, x_{n-1}$ are linearly independent vectors emanating from a common point O in E_n, the plane through O and containing these vectors will be denoted by $E(x_1, \cdots, x_{n-1})$. A plane divides E_n into two closed half-spaces each having the plane as boundary, but otherwise having no points in common. If S is a point set in E_n, a plane p is called a supporting plane of S if (1) S lies in one of the closed half-spaces determined by p, and (2) the distance between S and p is zero.

THEOREM 1. *Let S be a bounded and closed point set in E_n. Let O be a point of E_n, and suppose that the set consisting of O and S does not lie in any plane. Then there exist n linearly independent vectors $x_1, \cdots, x_n$ emanating from O, with terminal points $P_1, \cdots, P_n$ in S, and n planes $p_1, \cdots, p_n$ satisfying the following conditions: For each i* (a) *p_i contains P_i*; (b) *p_i is parallel to* $E(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$; (c) *$p_i$ is a supporting plane of the set consisting of O and S.*

PROOF. Consider any n points $P_1, \cdots, P_n$ in S. Let x_i be the vector from O to P_i. Form the parallelepiped determined by the vectors $x_1, \cdots, x_n$. As a figure in E_n this parallelepiped has a certain content, which is a function of $P_1, \cdots, P_n$, say $V(P_1, \cdots, P_n)$. Because of the hypothesis concerning O and S, there is at least one choice of $P_1, \cdots, P_n$ for which the parallelepiped is nondegenerate; V therefore assumes a positive value. Now V is a continuous function of the variables $P_1, \cdots, P_n$, each of which ranges over the compact set S. It follows that V assumes a positive absolute maximum value. Throughout the remainder of the proof we shall use $P_1, \cdots, P_n$ to denote a set of points in S at which V attains its absolute maximum. The fact that the maximum is positive then implies that the vectors $x_1, \cdots, x_n$ are linearly independent.

Let p_i be the plane through P_i parallel to $E(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. The parallelepiped of maximum content lies between these planes, each of a pair of opposite faces lying in one of the planes.

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We have to prove that p_i is a supporting plane of the set consisting of O and S, that is (since O is not on p_i), that O and any point of S both lie in the same one of the two closed half-spaces determined by p_i. Suppose that for some point Q of S this were not the case. Consider the parallelepiped determined by the vectors $x_1, \cdots, x_{i-1}, OQ, x_{i+1}, \cdots, x_n$. Its content would evidently be greater than that of the parallelepiped determined by $x_1, \cdots, x_n$, in contradiction to the maximal property of the latter. For, the n-dimensional content of a parallelepiped is equal to the $(n-1)$-dimensional content of one of its faces multiplied by the distance between the plane of this face and that of the face opposite. The two parallelepipeds which we are comparing have a face through O in common, but the distance from O to p_i is less than the distance from O to the parallel plane through Q. The proof of Theorem 1 is now complete.[1]

2. **Normed linear spaces.** Let X denote a real normed linear space.[2] In this section we shall prepare the way for the proof of the theorem in the next section by discussing certain matters pertaining to n-dimensional linear subspaces of X. Let Y_n denote such a subspace, and let $y_1, \cdots, y_n$ be n linearly independent elements of Y_n, so that any y in Y_n is uniquely representable in the form

$$y = e_1y_1 + \cdots + e_ny_n. \tag{2.1}$$

If in an n-dimensional Euclidean space E_n we introduce a rectangular coördinate system, we may set up a correspondence between the element y of Y_n, as given by (2.1), and the point with coördinates $(e_1, \cdots, e_n)$ in E_n. This correspondence is one-to-one and bicontinuous. We state the following lemmas, leaving the proof of the first one to the reader.

LEMMA 2.1. *The points of E_n which correspond to the elements y of Y_n for which $\|y\| \leq 1$ form a bounded and closed convex set S which is symmetric about the origin. If $\|y\| < 1$, the corresponding point of E_n is an interior point of S.*

LEMMA 2.2. *Let S be the set referred to in Lemma 2.1. Let P_0 be a point on the boundary of S, and p_0 a plane of support of S through P_0. Denote by p the plane parallel to p_0 and through the origin of E_n. Let y_0 be the element of Y_n corresponding to P_0, and denote by M the set of*

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[1] The referee brought to my attention the fact that a theorem similar to Theorem 1, for the special case in which S is a convex body with the point O as a center of symmetry, was announced by M. M. Day, in Bull. Amer. Math. Soc. Abstract 51-11-237.

[2] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 53.

elements of Y_n *corresponding to points of* p. *Then there exists a linear functional* f_0 *defined on* X *with the properties* (1) $\|f_0\| = 1$, (2) $f_0(y_0) = 1$, (3) $f_0(y) = 0$ *for each* y *in* M.

Proof. Since p_0 does not pass through the origin, it may be described by an equation of the form $a_1e_1 + \cdots + a_ne_n = 1$. The points of p will then satisfy the equation $a_1e_1 + \cdots + a_ne_n = 0$. By means of the correspondence (2.1) we define a linear functional ϕ_0 on Y_n by the equation $\phi_0(y) = a_1e_1 + \cdots + a_ne_n$. It is easily proved, by using the last sentence of Lemma 2.1 and the fact that p_0 is a supporting plane of S, that $|\phi_0(y)| \leq 1$ if $\|y\| \leq 1$. Note that $\phi_0(y_0) = 1$. Consequently, by the Hahn-Banach theorem,[3] there exists a linear functional f_0 defined on X, having $\|f_0\| = 1$, and coinciding with ϕ_0 on Y_n. This functional evidently meets the requirements of Lemma 2.2.

3. **Biorthogonal systems.** Consider a real normed linear space X, and the conjugate space X^* of linear functionals defined on X. A pair of ordered sets $\{x_1, x_2, \cdots, x_n\} \subset X$ and $\{f_1, \cdots, f_n\} \subset X^*$ is called a biorthogonal system if $f_i(x_j) = 1$ when $i = j$ and $f_i(x_j) = 0$ when $i \neq j$ $(i, j = 1, 2, \cdots, n)$. If in addition $\|x_i\| = \|f_i\| = 1$ $(i = 1, 2, \cdots, n)$, we shall call the pair of sets a biorthonormal system.

Theorem 2. *Let* Y_n *be an n-dimensional linear subspace of* X. *Then there exists a biorthonormal system* $\{x_1, \cdots, x_n\}$, $\{f_1, \cdots, f_n\}$ *with* $x_1, \cdots, x_n$ *in* Y_n.

Proof. Introduce the mapping of Y_n on E_n, and the set $S \subset E_n$ as defined in §2, Lemma 2.1. With O the origin in E_n, let $P_1, \cdots, P_n$ and $p_1, \cdots, p_n$ be points and planes related to S as described in §1, Theorem 1 (here O and S do not lie in any plane). Let $x_1, \cdots, x_n$ be the elements of Y_n corresponding to $P_1, \cdots, P_n$. The points P_j, $j \neq i$, are in the plane through O parallel to p_i. By Lemma 2.2 it follows that there exists a linear functional f_i such that $\|f_i\| = 1$, $f_i(x_i) = 1$, $f_i(x_j) = 0$, $j \neq i$. This completes the proof of the theorem.

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[3] Banach, loc. cit. Théorème 2, p. 55.

LATTICES OF CONTINUOUS FUNCTIONS

IRVING KAPLANSKY

1. **Introduction.** Let X be a compact ($=$bicompact) Hausdorff space and $C(X)$ the set of real continuous functions on X. By defining addition and multiplication pointwise, we convert $C(X)$ into a ring. With the norm $\|f\| = \sup |f(x)|$, $C(X)$ becomes a Banach space. Finally, we may introduce an ordering by defining $f \geqq g$ to mean $f(x) \geqq g(x)$ for all x; this makes $C(X)$ a lattice.

Gelfand and Kolmogoroff [6][1] showed that, as a ring alone, $C(X)$ characterizes X. More precisely, if $C(X)$ and $C(Y)$ are isomorphic rings, then X and Y are homeomorphic. Banach [3, p. 170] proved that $C(X)$ as a Banach space characterizes X, if X is compact metric. Stone [5, p. 469] generalized this to any compact Hausdorff space, and Eilenberg [5] and Arens and Kelley [2] have since given other proofs. Finally, Stone [9] has shown that as a lattice-ordered group, $C(X)$ characterizes X. A negative result is that $C(X)$ as a topological linear space fails to characterize X [3, p. 184].

In this paper we shall prove the following result: as a *lattice alone* $C(X)$ characterizes X. This theorem is shown in §5 to subsume all the earlier results cited above. Moreover in this context we can replace the reals by an arbitrary chain, granted a suitable separation axiom. In §4 it is shown that the connectedness of X is equivalent to the indecomposability of $C(X)$ as a lattice.

I am greatly indebted to Professor A. N. Milgram for suggestions which led to a substantial simplification of my proof of Theorem 1.

2. **Main theorem.** Let R be a chain (simply ordered set). Until §6 it will be assumed that R has neither a minimal nor maximal element. There is a natural way of topologizing R [4, p. 27] which can be described as follows: for any $\alpha \in R$ let $U(\alpha)$ be the set of all $\beta \in R$ with $\beta > \alpha$, $L(\alpha)$ the set of all β with $\beta < \alpha$; then the U's and L's form a subbase of the open sets.

LEMMA 1. *If $\alpha, \beta \in R$ and $\alpha > \beta$, then there exist neighborhoods M, N of α, β such that $\gamma > \delta$ for all $\gamma \in M$, $\delta \in N$.*

PROOF. If there exists ξ with $\alpha > \xi > \beta$ we take $M = U(\xi)$, $N = L(\xi)$. If not, we take $M = U(\beta)$, $N = L(\alpha)$.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Let X be a topological space and $C=C(X)$ the set of all continuous functions from X to R. We order C by defining $f \geqq g$ to mean $f(x) \geqq g(x)$ for all $x \in X$. Then C becomes a lattice, and in fact a distributive lattice.

DEFINITION. X is *R-separated* if for any $x, y \in X$ $(x \neq y)$ and $\alpha, \beta \in R$ there exists a continuous function f with $f(x)=\alpha, f(y)=\beta$. X is *R-normal*, if for any disjoint closed sets F, $G \subseteq X$ and any α, $\beta \in R$ there exists a continuous function f equal to α on F and β on G.

Any R-separated space is necessarily a Hausdorff space. If R is the real number system, it is known that conversely any compact Hausdorff space is R-normal, and this is likewise true if R is the set of reals with $\pm\infty$ adjoined (the latter is the same as a bounded closed interval on the real line). If R is disconnected, it is clear that X must be totally disconnected if it is to be R-separated, and conversely a totally disconnected space is R-separated for every R. In the extreme case where R consists of two elements, $C(X)$ is a Boolean algebra in its natural ordering, and the results in this paper are subsumed in Stone's theory of Boolean spaces and rings [7].

For any R we have the following result, which can be proved in exactly the same way as the lemma on p. 487 of [1].

LEMMA 2. *Suppose R has neither a maximal nor minimal element. Then if X is compact and R-separated, it is R-normal.*

By a *prime ideal* P in C [4, p. 78] we mean the set of antecedents of 0 in some lattice homomorphism of C onto the two-element lattice $(0, I)$. A prime ideal P is a sublattice containing with any element all smaller ones, and its complement $C-P$ has the dual property.

We can construct prime ideals in C as follows. Let Z be the lower half of a Dedekind cut in R, and for a fixed point $x \in X$, let P consist of all f with $f(x) \in Z$. Such prime ideals can be characterized as follows: $f \in P$ and $g(x) \leqq f(x)$ imply $g \in P$. We shall see below (§3) that these do not in general exhaust all prime ideals. However, we can prove a certain weakened version of this property.

DEFINITION. A prime ideal P in C is *associated* with a point $x \in X$ if $f \in P$ and $g(x) < f(x)$ imply $g \in P$.

LEMMA 3. *If X is compact, then any prime ideal P in C is associated with some point. If X is R-separated, the point is unique.*

PROOF. Suppose P is associated with no point of X. Then for every $x \in X$ we have functions f, g with $g(x) < f(x)$ and $f \in P$, $g \in Q=C-P$. By Lemma 1 the inequality $g < f$ extends to a neighborhood of x. A finite number of these neighborhoods cover X. Let $f_1, \cdots, f_n$ and $g_1, \cdots, g_n$ be the corresponding functions, and define $h = f_1 \cup \cdots \cup f_n$, $k = g_1 \cap \cdots \cap g_n$. Then $h > k$, but $h \in P$, $k \in Q$, a contradiction.

Suppose now that X is R-separated and that P is associated with both x and x'. Let m, n be any functions in P, Q respectively. There exists a function r with $r(x) < m(x)$, $r(x') > n(x')$. But these inequalities require r to be in both P and Q.

LEMMA 4. *If two prime ideals in C are associated with the same point of X, then their intersection contains a prime ideal.*

PROOF. Suppose P and P' are both associated with x. Let $f \in P$, $g \in P'$ and take any $\alpha \in R$ smaller than both $f(x)$ and $g(x)$. Let P'' be the set of all h with $h(x) \leqq \alpha$; then $P'' \subset P \cap P'$.

LEMMA 5. *Suppose X is compact and R-separated. Let P, P', P'' be prime ideals in C with $P'' \subset P \cap P'$. Then P and P' are associated with the same point.*

PROOF. Suppose on the contrary that P and P' are associated with distinct points x and y. Let P'' be associated with z, where z is for definiteness different from x. Choose any f in P'', g in $C - P$, and then h with $h(z) < f(z)$, $h(x) > g(x)$. Then h is in P'' but not in P, a contradiction.

For use in §4 we insert at this point the following result.

LEMMA 6. *Suppose X is compact. If P is a prime ideal associated with x, $f \in P$, and $g \leqq f$ on an open and closed set U containing x, then $g \in P$.*

PROOF. Suppose $g \in Q = C - P$. We note that P cannot be associated with a point y in the complement V of U, for we can construct a function m with $m(x) < f(x)$, $m(y) > g(y)$ and obtain a contradiction. Hence at every $y \in V$ we have functions h, k with $h > k$ in a neighborhood of y and $h \in P$, $k \in Q$. A finite number of neighborhoods cover the compact space V; if $h_1, \cdots, h_n$, $k_1, \cdots, k_n$ are the corresponding functions we have $f \cup h_1 \cup \cdots \cup h_n$ in P but exceeding $g \cap k_1 \cap \cdots \cap k_n$ in Q.

LEMMA 7. *Suppose X is compact and R-separated. Let f_0 be a fixed function in C, and for any subset S of X define $A(S)$ to be the intersection of all prime ideals containing f_0 which are associated with points of S. Then a point x is in the closure of S if and only if $A(S)$ is contained in a prime ideal associated with x.*

PROOF. Suppose x is in the closure of S. Choose $\alpha > f_0(x)$ and let P

consist of all f with $f(x) \leqq \alpha$. Now $g \in A(S)$ implies $g \leqq f_0$ at all points of S and hence $g(x) \leqq f_0(x)$. It follows that $A(S) \subset P$. (For later use we remark that this half of the proof did not use R-separation.)

Suppose x is not in the closure of S. Then $A(S)$ cannot be contained in any prime ideal P associated with x. For let f be any function in $C-P$ and let $\alpha \in R$ be chosen with $f_0 > \alpha$ on S. By Lemma 2, a continuous function g exists which equals α on S and exceeds f at x. Then g is in $A(S)$ but not in P, a contradiction.

We can now prove our principal theorem.

THEOREM 1. *Let R be a chain with neither a minimal nor maximal element, and let it be endowed with its order topology. Let X be a compact R-separated space, and C the set of continuous functions from X to R. Then as a lattice, C characterizes X.*

PROOF. We say that two prime ideals in C are equivalent if their intersection contains a third prime ideal. Lemmas 3, 4, and 5 show that there is a one-one correspondence between points of X and classes of equivalent prime ideals. Lemma 7 shows that the topology of X can be expressed in terms of inclusion relations among the prime ideals, and this completes the proof.

3. An example.

We shall give an example to show that not all prime ideals in C are simply based on a point in X and a Dedekind cut in R.

Let R be the reals, X the unit interval $[0, 1]$. Let A be the set of functions f for which $f(x) \leqq -x$ in a neighborhood of 0 (the neighborhood depending on f). A is an ideal in C, that is, a sublattice containing with any element all smaller ones. Similarly let B be the set of all g with $g(x) \geqq x$ in a neighborhood of 0; B is a dual ideal disjoint from A. By [8, Theorem 6] we can expand A, B to a prime ideal P and its complementary dual prime ideal Q. All functions f with $f(0) < 0$ go into P, all with $f(0) > 0$ go into Q, but those vanishing at 0 are split between P and Q.

Examples of this type can be constructed provided neither X nor R is discrete. They apparently require the axiom of choice.

4. Connectedness and direct products.

If the space X splits into two open and closed sets X_1 and X_2, it is evident that the lattice $C = C(X)$ is the direct product [4, p. 13] of the lattices $C(X_1)$ and $C(X_2)$. We shall now prove the less trivial converse.

THEOREM 2. *Let R be a chain with neither a maximal nor minimal element, and let it be endowed with its order topology. Let X be a compact space and $C = C(X)$ the lattice of all continuous functions from X to R.*

Suppose C is a direct product of lattices, $C \cong C_1 \times C_2$. Then X splits into two open and closed sets X_1, X_2 in such a fashion that $C_i \cong C(X_i)$.

PROOF. We first remark that any prime ideal P in C is of the form $P_1 \times C_2$ or $C_1 \times P_2$, where P_i is a prime ideal in C_i. This follows readily from Theorem 10 of [8], or it can be proved by repeating the argument of Lemma 3, if we replace X by a two-point space and R by C_1 at one point and C_2 at the other. Let us write Z_1 for the class of prime ideals of the form $P_1 \times C_2$, Z_2 for those of the form $C_1 \times P_2$.

Now take any point x in X and any prime ideal P associated with it. We define X_i to be the set of those x for which $P \in Z_i$. It follows from Lemma 4 that this definition makes sense, since any two prime ideals associated with the same point must fall into the same class.

Next we show that X_1 is closed. If not, there is a point $y \in X_2$ which is in the closure of X_1. By the first half of Lemma 7, $A(X_1)$ is contained in a prime ideal P associated with y. But P is of the form $C_1 \times P_2$ while $A(X_1)$ is of the form $B \times C_2$, so this is impossible. Similarly, X_2 is closed, and each is thus open and closed.

We shall now set up a lattice isomorphism between C_1 and $C(X_1)$. Given $c_1 \in C_1$, we pair it with any c_2 in C_2, obtaining a function $f \in C$, whose specialization to X_1 we shall call f_1. We prove that the correspondence is order-preserving, which incidentally shows it to be well defined. Suppose we have $d_1 \in C_1$ with $d_1 \leq c_1$ and that d_1 has given rise to the function g on X specializing to g_1 on X_1. If $g_1(x) > f_1(x)$ at some point $x \in X_1$, we can build a prime ideal P associated with x such that $f \in P$, $g \in C - P$. But P is of the form $P_1 \times C_2$ and hence $f \in P$ implies $g \in P$, since the C_1-component of f exceeds that of g. This contradiction shows that $f_1 \geq g_1$, as desired.

To set up the correspondence the other way we take a function f_1 on X_1, pair it with any f_2 on X_2, to obtain a function f whose C_1-component is, say, c_1. Suppose $g_1 \leq f_1$ and that g_1 gives rise to $d_1 \in C_1$. We assert that $d_1 \leq c_1$. For if not, we note that C_1 is a distributive lattice, whence [4, Theorem 5.8] there exists a prime ideal $P_1 \subset C_1$ with $c_1 \in P_1$, $d_1 \in C_1 - P_1$. For the corresponding prime ideal $P = P_1 \times C_2$ we have $f \in P$, $g \in C - P$. But P is associated with a point in X_1; by Lemma 6 we have a contradiction.

There remains the trivial observation that the two correspondences we have defined are inverse to each other, and we have completed the proof of Theorem 2.

Remark. It is to be observed that no separation axiom is required for Theorem 2. In case R is the reals, we can even drop the assumption of compactness by making use of the Stone-Čech compactification

$\beta(X)$. (Cf. [5, p. 578].) But for the general case no such device is available, and the validity of Theorem 2 for non-compact spaces remains undecided.

5. **Rings and Banach spaces.** Suppose we are given $C(X)$ as the *ring* of real continuous functions. Now $f \geqq g$ is equivalent to the statement that $f-g$ is a square. Hence we are also given the lattice; the lattice characterization subsumes the ring characterization.

Suppose we are given $C(X)$ as a *Banach space*, with X a compact Hausdorff space. Let e be an extreme point on the unit sphere, that is, a point which is not an interior point of a segment lying in the unit sphere. It is easy to see that e can assume only the values 1 and -1; suppose it is 1 on Y_1 and -1 on Y_2, where Y_1, Y_2 are open and closed disjoint sets whose sum is X. Now a function $f \neq 0$ which satisfies $\| f/\|f\| - e\| \leqq 1$ must be non-negative on Y_1 and non-positive on Y_2. Hence if we write $f \geqq 0$ whenever it fulfills this condition, we preserve order on Y_1, and invert order on Y_2, as compared with the natural order in $C(X)$. This gives us a lattice which is at any rate isomorphic to $C(X)$. Hence $C(X)$ as a Banach space determines $C(X)$ as a lattice, and the lattice characterization subsumes the Banach space characterization.

Finally we remark that in our Theorem 2 we have likewise subsumed the analogous ring and Banach space theorems, in the following strong sense: if we have $C(X)$ expressed as a direct sum of rings or Banach spaces, then that very decomposition expresses the lattice as a direct product (it being understood in the Banach space case that the lattice being decomposed is the one obtained after inversion of order on Y_2). We omit the straightforward proof of this fact.

6. **Chains with 0 and I.** If R is a chain which has a minimal or maximal element or both, Theorems 1 and 2 remain valid, with the proviso that R-separation in the hypothesis of Theorem 1 is replaced by R-normality. In several of the proofs, however, there occur technical complications. We have preferred to avoid these complications, which add nothing to the fundamental idea. We shall merely mention the two main modifications that are necessary: (1) The definition of "equivalence" of prime ideals is revised to read: either their intersection contains a prime ideal, or the intersection of their complements contains a dual prime ideal. (2) In the characterization of the topology, x is in the closure of S if and only if $A(S)$ is contained in a prime ideal, or the object dual to $A(S)$ is contained in a dual prime ideal.

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LINEAR TOPOLOGICAL DIVISION ALGEBRAS

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1. Introduction. We present a generalization of the familiar theorem of Frobenius that any finite-dimensional linear division algebra A over the real number field is isomorphic to the real, the complex, or the quaternion number system. The generalization consists in replacing the hypothesis of finite dimensionality of A by the weaker hypothesis that A be a complete linear space with a topology in which multiplication is continuous and which is based on a countable system of convex open sets.

Previous generalizations of Frobenius' theorem have been indicated or proved by S. Mazur [6],[1] G. Silov [8], and I. Gelfand [4]. These writers have generally assumed that A have a norm; and only that case has been adequately considered in which scalar multiplication by complex numbers is assumed. We shall give the proof of the general case without limiting ourselves to the commutative case or to complex scalars.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

In the presence of a norm, the operation of inversion, that is, the passage from x to x^{-1} in A, is easily seen to be analytic (in a sense defined in [4]) and an application of the Liouville theorem establishes that A is the complex number system. In connection with nonnormed algebras one is hampered by the fact that inversion need not even be continuous on the set of nonsingular elements. In other words, that part of a topological ring which forms a multiplicative group need not be a topological group in its relative topology: a discussion of this follows in §4. However, in complete separable metric *division* rings, inversion is nevertheless continuous on the set of nonzero elements, as a result of a theorem on topological groups due to D. Montgomery [7]. Because of this, an adaptation of Gelfand's method can be made which establishes our Theorem 3.

In a section devoted to integral domains we generalize slightly a theorem of Silov [8] without being able to avoid hypothesizing a norm. We justify this limitation by exhibiting the peculiar properties of the ring A of functions holomorphic on an open set. This is a complete metric convex linear algebra and is an integral domain, even in a strong topological sense. Its existence with these properties shows that Frobenius' theorem formulated in terms of integral domains does not generalize as readily as the statement above in terms of the existence of inverses. The most remarkable property of this ring of holomorphic functions is probably the fact that the set of singular elements is *open* and not closed as is the case in complete *normed* rings.

2. **Topological division algebras.** A *topological algebra* A is a topological linear space in which not only addition and scalar multiplication by real numbers, but also multiplication of elements of A is defined, and in such a manner as to be continuous. If scalar multiplication by complex numbers is defined as well, we speak of a *complex* topological algebra. We call A a *convex* topological algebra if it is convex as a topological linear space (see [10]). If the topology of A can be given by a metric structure, we call it a *metric* linear space; and we say A is *complete* if it is a complete metric space. An important special case of convex complete metric algebras is presented by the so-called *normed rings* [4], otherwise known as Banach algebras. Theorem 2 below is known for commutative complex Banach algebras [4, p. 8]. Many devices introduced by Gelfand will be recognized in our proofs.

By a topological *division* algebra we mean one which possesses a unit 1 and a two-sided inverse x^{-1} for each nonzero element x in it.

The following theorem is a convenient starting point.

THEOREM 1. *Let A be a convex complex topological division algebra with continuous inversion. Then A consists only of scalar multiples of the unit element and is thus isomorphic with the complex number field.*

PROOF. If possible, select an element x from A which is not a scalar multiple of $1 \in A$. Since A is convex, there exists a continuous, linear, real-valued functional f defined on A and such that $f(x^{-1}) \neq 0$ [10, p. 162]. Define $g(z) = f(z) - if(iz)$ for any $z \in A$. Then $g(\lambda z) = \lambda(g(z))$ for both real or complex λ. The function $h(\lambda) = g[(x-\lambda)^{-1}]$ is defined and continuous for any complex number λ because x is such that $x - \lambda \neq 0$ for any λ. From the relations

$$\begin{aligned} h(\mu) - h(\lambda) &= g[(x-\mu)^{-1} - (x-\lambda)^{-1}] \\ &= g[(x-\mu)^{-1}(x - \lambda - x + \mu)(x-\lambda)^{-1}] \\ &= (\mu - \lambda) g[(x-\mu)^{-1}(x-\lambda)^{-1}], \end{aligned}$$

we can deduce that $h(\lambda)$ has a derivative $h'(\lambda) = g[(x-\lambda)^{-2}]$. Therefore h is an entire function. Moreover, as $\lambda \to \infty$, we evidently have $h(\lambda) \to 0$, whence $h(\lambda) \equiv 0$. This contradicts $f(x^{-1}) \neq 0$. Therefore each x in A is a scalar multiple of the unit of A.

If the division algebra A arises as the quotient algebra of a complex Banach algebra, one can easily verify the hypothesis of Theorem 1.

The following considerations will enable us to drop the hypothesis that the topological algebra A admit complex scalars.

Let us call A *formally real* if A is commutative and $x^2 = -y^2$ in A implies $x, y = 0$. Let $A(i)$ be the class of all ordered pairs; and denote the pair (0, 1) by the symbol "i." Then each element of $A(i)$ can be written in the form $x + iy$; and algebraic operations can be defined in $A(i)$ in an obvious fashion such that $i^2 = -1$. It is clear that complex scalars can be applied to $A(i)$. Let $\mathcal{V}$ be a system of neighborhoods U of 0 in A; then let us take the totality of sets $U + iU$, $U \in \mathcal{V}$, as a system of neighborhoods for 0 in $A(i)$. One can readily verify that $A(i)$ is a complex topological algebra; that $A(i)$ is convex or complete if and only if A is; that it is an integral domain or a field if and only if A is; and that inversion is continuous in $A(i)$ if it is in A.

These concepts having been introduced, we proceed to the case in which only real scalars are required.

THEOREM 2. *Let A be a convex topological division algebra with continuous inversion. The A is isomorphic with either the real, the complex, or the quaternion field.*

PROOF. *Commutative case*: If A is formally real, then the construction of $A(i)$ can be carried out, yielding a convex, complex, topological algebra, with continuous inversion. By Theorem 1, $A(i)$ consists only of scalar pairs; therefore A consists only of real multiples of the identity.

If A is not formally real, let $x^2=-y^2$ with $x, y\neq 0$. Define $j=xy^{-1}$. Next, scalar multiplication by complex numbers $a+bi$ can be defined, by $(a+bi)z=az+b(jz)$, for $z\in A$. Theorem 1 now tells us that A is the complex number field.

General case: The center Z of A comes under the previous case, and is therefore the real or complex field. Let x be selected outside of Z; and let X be the division sub-algebra generated by x over Z. Since this is still commutative, we may conclude that x satisfies some polynomial with coefficients from Z (and, in fact, of degree not greater than 2). Hence A is a division algebra algebraic over the real number field. The argument is now completed by an appeal to the following lemma, whose proof can be conducted in the same way as that of Frobenius' theorem for algebras of finite order (dimension); or one may refer to a recent result [5, Theorem 16]:

LEMMA. *An algebraic division algebra over the real number field is either the real, the complex, or the quaternion field.*

The requirement that inversion be continuous in A is troublesome, although of course it is necessary. The most convenient way of insuring it is to suppose that A can be given a norm having the property

$$\|xy\| \leq \|x\| \, \|y\|$$

as well as the usual normed linear space properties. In the case of such an A we can establish the continuity of inversion independently of completeness, as follows: let x have an inverse y, and suppose $x+h$ has an inverse $y+k$. Then $(y+k)(x+h)=1$, $(y+k)(1+hy)=y$, $yhy+k+khy=0$, $k=-khy-yhy$, whence

$$\|k\| \leq \frac{\|yhy\|}{1-\|hy\|}$$

if $\|h\| \, \|y\|<1$. Therefore we have the hypothesis of Theorem 2, with the following result:

COROLLARY 1 (Mazur [6, Theorem I]). *Let A be a normed division algebra whose norm has the property*

$$\|xy\| \leq \|x\| \, \|y\|.$$

Then A is the real, the complex, or the quaternion field.

From this one can deduce the following corollary.

COROLLARY 2 ([6, Theorem II]). *Let A be a normed algebra in which $\|xy\| = \|x\| \, \|y\|$. Then A is the real, the complex, or the quaternion field.*

PROOF. In the commutative case we can use a well known device [9, p. 246] to imbed A in a field to which Corollary 1 or even Ostrowski's theorem [9, p. 257] may be applied. The noncommutative case is reduced to the commutative case by an application of the lemma as in Theorem 2.

A less trivial result on integral domains will be considered in the next section.

The other case in which inversion is continuous is that in which A is a complete separable metric algebra, for then the set of nonzero elements forms a separable metric locally complete set which is a group with continuous multiplication, and, by [7, p. 881], inversion is continuous. Combining this with Theorem 2, we obtain the following.

THEOREM 3. *Let A be a convex, separable, metric and complete division algebra. Then A is isomorphic with either the real, the complex, or the quaternion field.*

I have been informed that the complex commutative case of this theorem was obtained recently by Mr. B. Yood. Mr. Yood's proof also makes use of Montgomery's theorem.

3. **Integral domains.** An element z of a topological ring A is called a *topological divisor* of zero if there exists a closed set $F \subset A$ not containing 0 but such that 0 is a limit point of zF or Fz. We shall call an element *singular* if it has no inverse in a ring with unit and *nonsingular*, otherwise.

Silov [8] has proved that if x is an element of a Banach algebra A and if λ_0 is a frontier point of the set of scalars λ for which $x-\lambda$ is singular, then $x-\lambda_0$ is a topological divisor of zero. Silov concludes from this the following:

Any complex Banach algebra satisfying the norm condition $k\|x\| \, \|y\| \leq \|xy\| \leq \|x\| \, \|y\|$ with some fixed positive k is the complex number field.

The argument of [8] applied only to complex Banach algebras because conceivably the set of λ for which $x-\lambda$ is singular might be void for all nonscalar x in a Banach algebra. But if this were so, then each $x \neq 0$ in A would be nonsingular, or A would be a division algebra and its structure would be given by Theorem 3. In somewhat greater detail: One can adapt the argument of [8] to show that if the set S of singular elements has a frontier point s in a normed algebra, then s is a topological zero divisor. In a Banach algebra the set S is always closed. Now the condition $k\|x\| \|y\| \leqq \|xy\|$ certainly precludes the existence of nonzero topological zero-divisors, hence S consists of 0 alone and A is a division algebra. We formulate this result as a theorem.

THEOREM 4. *Let A be a normed algebra in which $k\|x\| \|y\| \leqq \|xy\| \leqq \|x\| \|y\|$ with a fixed positive k. Then A is the real, the complex, or the quaternion number system.*

It is possible to omit the condition that A be complete, since A can be completed without losing the norm condition.

The reader will observe that we have confined ourselves to normed integral domains in this section. This seems to be necessary as we can exhibit a complete convex complex algebra which has no topological divisors of zero, and is not a division algebra:

Let A be the class of functions holomorphic on an open subset G of the complex plane; and let this function space be given the k-topology.[3] *Then this complex linear algebra is* (1) *convex,* (2) *metrisable,* (3) *complete,* (4) *not normable,* (5) *it contains no topological zero divisors, and* (6) *the set of singular elements is open.*

PROOF. (1) is obvious. For (2), see [1, Theorems 7 and 12]. For (3) observe [1, Theorem 10] and the fact that a sequence of holomorphic functions can converge uniformly on an open set only to a holomorphic function. Now (4) will follow from (6) and the well known properties of Banach algebras to which we have alluded in Theorem 4. As regards (5), let f, $g_n \in A$ with $f \neq 0$ and $fg_n \to 0$ in the k-topology, of A. Let a closed bounded set $K \subset G$ be given. We can cover K with a finite number of regions $G_1, \cdots, G_n$ whose boundaries lie in K and such that $|f(s)|$ has a positive minimum m on the point set C formed by these boundaries. For $s \in C$ we evidently have

$$| g_n(s) | \leqq m^{-1} \sup_{t \in G} | f(t) g_n(t) |.$$

Since the right-hand side tends to zero as n tends to infinity we have $g_n \to 0$ in the k-topology, using the principle of the maximum modulus. Hence f is not a topological zero-divisor. To prove (6) suppose that f

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[3] We have proposed this name for the topology usually introduced into such function spaces. For its definition, see [1, p. 481]. In it, convergence simply means uniform convergence on the bounded closed subsets of G.

has no inverse in A. This means simply that $f(z_0)=0$ for some $z_0 \in G$. About z_0 we can describe a small circle C lying within G, and on which f has a positive minimum m. The set U of all $h \in A$ for which $\sup_{z \in C} |h(z)| < m$ constitutes a neighborhood of 0 in A, in the k-topology. But by Rouché's theorem, if $h \in U$ then f and $f+h$ have the same number of zeros within C. Thus $f+h$ has no inverse in A. This proves our assertions.

4. The continuity of ring operations.

We have not required multiplication to be continuous in both factors simultaneously, as is customary in our topological algebras, because we did not need this property. Nevertheless, complete metric algebras have it. We append this section to consider such matters. For greater generality, define an *m-ring* as a ring whose elements form a metrizable topological group under addition, and whose product is continuous in each factor, under the topology.

THEOREM 5. *Let A be a complete m-ring. Then multiplication is continuous in both factors simultaneously. If the set of nonsingular elements is a separable G_δ and A has a unit, then the process of inversion is continuous in this set.*

PROOF. It will suffice to show that if $x_n \to 0$ and a neighborhood U of 0 be given, then there is a neighborhood V of 0 and an integer n such that $x_m V \subset U$ for $m \geqq n$.

Since A is a topological group under addition, we can find a neighborhood W of 0 such that $W^- - W \subset U$. Let A_n contain those $y \in A$ for which $x_m y \in W^-$ if $m \geqq n$. Since right multiplication is continuous, the set A_n is closed. Each $y \in A$ is contained in some A_n because $x_n y \to 0$ and so $x_m y \in W$ for m not less than some suitable n. Hence some A_n contains an open set $V+y$ where V can be selected as a neighborhood of 0. If $V+y \subset A_n$ then $x_m V + x_m y \subset W^-$ for $m \geqq n$. One may select n greater if necessary so that $x_m y \in W$ if $m \geqq n$. Then $x_m V \subset W^- - W$ for $m \geqq n$, as desired.

If the set R of nonsingular elements is a G_δ it is obviously of the second category. If R is also separable then the elements of R form a separable space of the second category and also a group in which multiplication is continuous. By [7, Theorem 2], R is a topological group and inversion is continuous on R.

In view of Theorem 5 it becomes natural to ask whether or not the nonsingular elements of a complete m-ring always form a topological group. We can give an example that this is not so. So far as we know, this point has never been mentioned before.

The ring we exhibit for this purpose is the intersection L^ω of all L^p-spaces based on the interval $[0, 1]$, with $p \geqq 1$. We say $f_n \to 0$ in L^ω if and only if $f_n \to 0$ in each L^p, $p \geqq 1$. Thus topologized, L^ω is a complete convex m-algebra [2]. Now consider the following sequence of nonsingular elements of L^ω:

$$f_n(t) = \begin{cases} 1/n & \text{for } 0 \leqq t \leqq 1/n, \\ 1 & \text{for } 1/n < t \leqq 1, \end{cases}$$

$n = 1, 2, \cdots$. This sequence approaches the unit function 1, $1(t) = 1$, of L^ω. But the sequence of inverses $f_1^{-1}, f_2^{-1}, \cdots$ does not approach 1 since

$$\int_0^1 (f_n^{-1}(t) - 1(t))^p dt = \int_0^{1/n} (n - 1)^p dt = \frac{(n-1)^p}{n}$$

tends to infinity as $n \to \infty$ and $p > 1$. Hence the nonsingular elements do not form a topological group. In other words, inversion is not continuous on the set of nonsingular elements.

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INSTITUTE FOR ADVANCED STUDY

ON THE TOPOLOGICAL PRODUCT OF PARACOMPACT SPACES

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A space S is defined by Jean Dieudonné[1] to be *paracompact* provided every covering of S by open sets has a neighborhood-finite refinement[2] which covers it. Dieudonné proves that every metric separable space is paracompact, that every paracompact Hausdorff space is normal, and that if S_1 is compact and S_2 is paracompact, then $S_1 \times S_2$ is paracompact. He leaves open the question as to whether the topological product of two paracompact spaces is paracompact. It is the purpose of this note to answer this question in the negative. The example given provides also a simple instance of a normal Hausdorff space S such that $S \times S$ is not normal.

The space S. The points of the space S are the non-negative real numbers, and a neighborhood of a point is any semi-open interval of the form $a \leqq x < b$ which contains it. It is easily seen that S is a regular Hausdorff space which is separable but not perfectly separable.

To show that S is paracompact, let α be any covering of S by open sets. It is evident that there is a refinement β of α whose elements are neighborhoods. Let E be the set of all points e of S which satisfy the relation $a < e < b$ for no neighborhood $a \leqq x < b$ of β. The set E is closed with respect to the natural topology of the real numbers. (For if z is a point in the complement of E, there is a neighborhood $a \leqq x < b$ of β such that $a < z < b$, and since the open interval $a < x < b$ contains no point of E, it follows that z is not a limit point of E.) The complement of E, therefore, is the sum of a countable number of disjoint open intervals $I_1, I_2, \cdots$. For each n the left end point of I_n is a point of E, and each point of E is the left end point of some interval I_n. Hence if I_n is the interval $e_n < x < f_n$, it follows that $E = \sum e_n$ and hence that $S = \sum (I_n + e_n)$. For each n there exists a neighborhood $e_n \leqq x < q_n$ of β; denote it by D_n. Let γ_n be the collection of all open intervals $a < x < b$ such that $a \leqq x < b$ belongs to β and is a subset of $e_n \leqq x < f_n$. Then γ_n covers $q_n \leqq x < f_n$. Since under the natural topology of the real

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[1] *Une généralisation des espaces compacts*, J. Math. Pures Appl. (9) vol. 23 (1944) pp. 65–76.

[2] The covering ω is a *refinement* of a covering α if every element of ω is a subset of some element of α; ω is *neighborhood-finite* if every point of the space has a neighborhood which intersects only a finite number of elements of ω.

numbers $q_n \leqq x < f_n$ is paracompact, it follows that there is a neighborhood-finite refinement δ_n of γ_n which covers it. Let ϵ_n be the collection of intersections of elements of δ_n with $q_n \leqq x < f_n$. The elements of ϵ_n are open sets in the topology of S, and each of them is a subset of an element of α. Hence if η_n is the collection obtained by adding to ϵ_n the open set D_n, η_n is a neighborhood-finite covering of $I_n + e_n$ whose elements are subsets of elements of α; therefore $\eta = \sum \eta_n$ is a neighborhood-finite refinement of α which covers S.

Hence S is paracompact and therefore normal.

The space $S \times S$. The points of $S \times S$ are, of course, sensed pairs (x, y) of non-negative real numbers, and neighborhoods are semi-open rectangles of the form $a \leqq x < b$, $c \leqq y < d$.

To show that $S \times S$ is not paracompact it will be sufficient to show that it is not normal. To this end let H be the set of all points (x, y) of $S \times S$ such that $x+y=1$ and $[(x-1)^2+y^2]^{1/2}$ is rational, and let K be the set of all points (x, y) of $S \times S$ such that $x+y=1$ and $[(x-1)^2+y^2]^{1/2}$ is irrational. It is readily seen that H and K are disjoint closed sets in $S \times S$. Let V be any open set containing K. For each point (k, k') of K there is a neighborhood $N(k, k')$ of the form $k \leqq x < k+d$, $k' \leqq y < k'+d$ which lies in V. For each positive integer n let K_n be the set of all points (k, k') of K such that the diameter of $N(k, k')$ is greater than $1/n$. Then $K = \sum K_n$, and since K is of the second category with respect to the natural topology of the line interval $x+y=1$, it follows that there exists an integer m such that K_m is dense in an interval $(s, 1-s)$, $(t, 1-t)$ of $x+y=1$. It follows easily that every point of the rectangle R bounded by $x+y=1$, $x+y=1+1/m$, $x-y=2s-1$, $x-y=2t-1$ belongs to some $N(k, k')$. Hence R is a subset of V. But every point of H in the interval $(s, 1-s)$, $(t, 1-t)$ of $x+y=1$ is a limit point of R and hence of V. It follows that H is intersected by the closure of every open set containing K.

Hence $S \times S$ is not normal and therefore not paracompact.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES

ON RESTORING FORCES WHICH ADMIT FORCING TERMS OF NON-CRITICAL AMPLITUDE

STEPHEN H. CRANDALL

Introduction. The forced vibrations of a conservative system with one degree of freedom can be represented by the equation

$$d^2x/dt^2 + g(x) = f(t). \tag{1}$$

In a mechanical system x is the displacement and t is the time. We call $g(x)$ a restoring force and $f(t)$ a forcing term. Throughout this paper $f(t)$ will be periodic and when we speak of a periodic solution of (1) we always mean a solution having the same period as $f(t)$.

If $g(x)$ is simply a constant multiplied by x, equation (1) represents a linear oscillator. In this case the amplitude of the forced oscillation is a constant multiple of the amplitude of the forcing function provided this latter is non-resonant. Here, a change in the amplitude of the forcing term merely changes the amplitude of the forced vibration. In the case of a non-linear restoring force, on the other hand, we usually expect [1][1] that changing the amplitude of the forcing term will alter the essential form of the periodic solution.

However, as we shall show, there is a class of non-linear restoring forces, for which forcing terms exist with the property that as the amplitude of the forcing function is varied, the periodic solutions resulting all have the same form but vary only in amplitude. A general method is given for determining such a forcing function whenever it exists. As an example, the Duffing equations with $g(x) = \pm x + bx^3$ are discussed, and the forcing terms mentioned are found to exist and are given explicitly in (16).

Definitions. A function $f(t)$ with period $4T$ is called a forcing term of non-critical amplitude for the system (1) if there is a solution to (1), $x(t)$, having period $4T$ and if a factor $h(k)$ exists so that $y = hx(t)$ is a solution of

$$d^2y/dt^2 + g(y) = kf(t) \tag{2}$$

for a continuous range of k which includes $k = 1$. In the following, we limit our discussion to odd restoring forces and for brevity will only define them for positive x, remembering always that $g(-x) = -g(x)$ and $g(0) = 0$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

THEOREM I. *The only restoring functions, $g(x)$, out of the class of functions possessing a derivative for $x > 0$, which permit forcing terms of non-critical amplitude have the form*

$$g(x) = ax + bx^{\alpha} \tag{3a}$$

or

$$g(x) = ax + bx \log x \tag{3b}$$

where a, b and α are constants.

Assume that in (1), $f(t)$ is in fact a restoring force of non-critical amplitude, and that $x(t)$ is a solution of period $4T$, then $h(k)x(t)$ will be a solution of (2). Substituting for $f(t)$ in (2) its value in (1) gives

$$d^2x/dt^2 + [g(hx) - kg(x)]/(h - k) = 0. \tag{4}$$

Since $x(t)$ is independent of k, we must have

$$[g(hx) - kg(x)]/(h - k) = G(x), \tag{5}$$

say, independent of k, that is, $h(k)$ must be taken so that G is a function of x only. The possibility of doing this is limited by the form of $g(x)$ as we shall see. Equation (5) can be rearranged to read:

$$G(x) - g(x) = [g(hx) - hg(x)]/(h - k). \tag{6}$$

Now the term on the right of (6) is also a function of x alone. Since the denominator is a function of k alone, the numerator must be separable into the product of a function of x with a function of k. Then the logarithmic derivative of the numerator with respect to x will be a function of x alone. Finally if we set $x = 1$ in this logarithmic derivative, we obtain

$$\frac{hg'(h) - hg'(1)}{g(h) - hg(1)} = \text{a constant,}$$

α, say. Solving this differential equation we obtain

$$g(h) = \{[g'(1) - \alpha g(1)]h + [g(1) - g'(1)]h^{\alpha}\}/(1 - \alpha)$$

for $\alpha \neq 1$. If $\alpha = 1$, we find $g(h) = [g'(1) - g(1)]h \log h + g(1)h$. These results are of the form stated in the theorem. It is an easy matter to verify for these forms that $h(k)$ can in fact be taken so that (5) becomes a function of x alone. A practical condition, namely the requirement that the velocity dx/dt be finite, limits the range of allowable values of α to $\alpha > -1$.

We shall now construct the non-critical forcing terms for the restoring forces of (3). We first choose $h(k)$ so that (5) is independent of k. There is a degree of freedom in this choice for, as we have seen, the right side of (6) must be the product of a function of x with a function of k and in choosing $h(k)$ what we do is constrain the function of k to be a constant. The magnitude of this constant is at our disposal. Call it C. Then (6) becomes

$$G(x) - g(x) = CG_1(x), \tag{7}$$

say, and (4) becomes

$$d^2x/dt^2 + g(x) + CG_1(x) = 0. \tag{8}$$

It is convenient to note at this point that $x(t)$, the solution of (1) when $f(t)$ is a forcing term of non-critical amplitude, is also according to (8) the free vibration solution of a conservative system with an odd restoring force. This gives us considerable information regarding $x(t)$; for example, it is completely symmetric, that is, its properties are entirely determined by its behavior during the quarter period interval, T, between the maximum amplitude position and the zero amplitude position. This is a not surprising consequence of our initial restriction that $g(x)$ be odd. To insure that (1) actually possesses a forcing term of non-critical amplitude with a certain given period $4T$, we must verify that (8) has a solution $x_0(t)$ with period $4T$. If $dx/dt=0$ when $x=A$, the quarter period time, τ, is given by [2]

$$\tau = \int_0^A dx \Big/ \left(\int_x^A [g(\xi) + CG_1(\xi)]d\xi \right)^{1/2}. \tag{9}$$

If, by suitably choosing A and C, τ can be made equal to T, call the corresponding solution of (8) $x_0(t)$. From this the required forcing term of non-critical amplitude is constructed as

$$f(t) = d^2x_0/dt^2 + g(x_0). \tag{10}$$

Note that this has the same symmetry as $x_0(t)$. When (10) is placed in (1), it is clear that $x_0(t)$ is a solution of (1). Moreover if this $f(t)$ is multiplied by k, we find by virtue of (4) and (8) that $hx_0(t)$ is indeed a solution. We have thus proved the following:

THEOREM II. *A sufficient condition for* (1) *to possess a forcing term of non-critical amplitude with period* $4T$ *when the restoring force has the form of* (3) *is that numbers* A *and* C *can be found which make* τ *of* (9) *equal to* T.

For any given $g(x)$ of the form (3) a study of (9) will reveal the range of periods for which forcing terms of non-critical amplitude exist. For example, if $g(x) = ax + bx^{\alpha}$ with a and b positive and $-1 < \alpha$, a little study shows that

$$0 < \tau < \pi/2a^{1/2}.$$

Example. As an illustration, we find the non-critical forcing terms for the Duffing equations [1]

$$d^2x/dt^2 \pm x + bx^3 = f(t), \qquad b > 0. \tag{11}$$

Equation (8) for these cases becomes

$$d^2x/dt^2 \pm x + Cx^3 = 0 \tag{12}$$

which can be solved by quadratures. As the integrals involved are elliptic, it is somewhat easier to try an elliptic function with undetermined parameters. In this way it is found that

$$x = A\,cn(Kt/T,\ k), \tag{13}$$

where k is the modulus and K is the real quarter period of the Jacobi elliptic functions, is a solution of (12) providing

$$T^2 = \pm K^2(1 - 2k^2), \tag{14a}$$

$$A^2 = 2K^2k^2/CT^2. \tag{14b}$$

Thus non-critical forcing terms for (11) exist, with periods $4T$, when, with the upper sign

$$0 < k < 2^{1/2}/2, \qquad 0 < T < \pi/2 \tag{15a}$$

and with the lower sign

$$2^{1/2}/2 < k < 1, \qquad 0 < T < \infty. \tag{15b}$$

The forcing terms of non-critical amplitude are found to be, from (10),

$$f(t) = A\left[bA^2 \pm 2k^2/(2k^2 - 1)\right]cn^3(Kt/T,\ k). \tag{16}$$

For a given T, k and $K(k)$ are determined from (14a). The relationship between the amplitude of the forcing term and the amplitude of the solution is clearly shown in (16) and (13).

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

THE AMERICAN MATHEMATICAL SOCIETY AND APPLIED MATHEMATICS

Since the beginning of World War II, there has been a growing emphasis on Applied Mathematics, a field in which American mathematicians had not been so active in the past decades. Many programs for Applied Mathematics were set up and mathematicians made great contributions to the solution of problems connected with the country's war effort. During the war the Society established a special Committee on Addresses in Applied Mathematics, under the chairmanship of Professor Richard Courant; this committee arranged symposia and invited addresses on topics of interest in Applied Mathematics. There was a widespread feeling that this activity should be continued and expanded.

At the time of the 1946 Summer Meeting at Cornell University, the Council of the Society discussed possible steps which might be taken toward the stimulation of the proper development in applied mathematics and the need for a program within the Society which would serve the needs of the applied mathematicians. As a result of this discussion, the President was authorized to appoint a committee to study these problems and the following Special Committee on Applied Mathematics was subsequently appointed: Professors J. L. Synge (Chairman), R. V. Churchill, Richard Courant, G. C. Evans, W. T. Martin, John von Neumann, and J. W. Tukey.

This Special Committee on Applied Mathematics presented recommendations to the Council at the 1946 Annual Meeting, providing for: (1) a study of financial support by the Society to the Journal of Mathematics and Physics and the Quarterly of Applied Mathematics by the appropriate committee of the Society; (2) the appointment of a Committee on Applied Mathematics which would organize programs in applied mathematics and make recommendations to the Council in all matters pertaining to the interests of applied mathematics; (3) the establishment of an annual symposium on applied mathematics, the proceedings of which might be published in book form. These recommendations were adopted by the Council.

The Special Committee on Applied Mathematics presented its final report to the Council in April, 1947, and at that time certain specific problems which this committee had had under discussion but on which it had come to no decision were referred to the Committee on Applied Mathematics. It further recommended to the Council that a statement entitled *Instruction and Research in Applied Mathematics* be published in the BULLETIN. The Council approved this statement which is published in this issue of the BULLETIN.

The Committee on Applied Mathematics has made plans for the first annual Symposium, on the topic of *Non-linear problems in mechanics of continua*, which will be held at Brown University on August 2–4, 1947.

The Committee on Applied Mathematics (Professors J. L. Synge, Chairman, Richard Courant, G. C. Evans, John von Neumann, William Prager, and Dr. Warren Weaver) invites suggestions from members of the Society regarding problems which should be studied by this committee and welcomes comments as to the manner in which the American Mathematical Society may aid in the development of Applied Mathematics in this country. Please communicate comments and suggestions to the Chairman of the Committee, Professor J. L. Synge, Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania.

J. R. KLINE,
Secretary

May 8, 1947.

SECOND SYMPOSIUM ON APPLIED MATHEMATICS

The American Mathematical Society will hold its second Symposium on Applied Mathematics in the summer of 1948, and seeks invitations from institutions willing to act as hosts.

The subject of the Symposium will be chosen by the Committee on Applied Mathematics in consultation with the host institution, and institutions offering themselves as hosts are requested to indicate the subject preferred. The subject might be defined in terms of recognized divisions of science or engineering, or it might be defined in terms of mathematical methods which cut across these divisions. The Committee will welcome suggestions for subjects not only from inviting institutions but also from members of the Society and others.

It is proposed that the Symposium will last two or three days, with four one-hour invited lectures at the Symposium, also a number of twenty-minute contributions (invited or uninvited), with adequate time for discussion.

Invitations from institutions willing to act as host should be sent, before September 30, 1947, to the Chairman of the Committee on Applied Mathematics, Professor J. L. Synge, Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania.

INSTRUCTION AND RESEARCH IN APPLIED MATHEMATICS[1]

The history of mathematics reveals a continuous interplay of ideas between the mathematician and the experimental or practical scientist. In the last half-century this interplay has decreased, to the detriment, it may be believed, of both parties. The war caused a lively cooperation, but unless this continues under peace-time conditions, the prospect for the future is serious and warrants earnest consideration.

The widening gap between the mathematician and other scientists may be traced to several causes. The most obvious of these is the great activity in all branches of science, which has necessitated a high degree of specialization in each branch, and consequently less opportunity for roaming interests. However, side by side with the expansion of science, increased understanding of basic theory acts as a simplifying compensation. Such simplification is achieved largely through mathematical methods, of increasing depth and generality, and it would be indeed unfortunate if mathematicians should not be in a position to contribute towards this simplification.

As our educational system is at present organized, there are relatively few opportunities for a student of mathematics, undergraduate or graduate, to become acquainted with the mathematical structures of the theories underlying other sciences. His needs in this connection are not identical with those of students pursuing other branches of science as their major interest; many practical details, of importance to them, are not essential for his purpose, and he can go further and faster by concentration on the mathematical aspects of the subject. It is suggested that the study of some branches of other sciences from the mathematical standpoint (and that is a meaning commonly attached to the words "applied mathematics") should be regarded, wherever feasible, as an essential part of the training of mathematics majors, undergraduate or graduate. Apart from the general advantage to science which we might hope to see as a result of this procedure, it seems only fair to offer a young mathematician the opportunity of pursuing a career in applied mathematics if his natural inclinations are so directed. Unless he has the opportunity of viewing the field of applied mathematics from the mathematical standpoint, it is unlikely that he will be attracted to it if his interests are basically those of a mathematician.

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[1] Prepared by the Special Committee on Applied Mathematics and approved by the Council of the American Mathematical Society on April 26, 1947.

Consequently it is suggested that departments of mathematics throughout the country should consider the feasibility of enlarging their offerings in the direction of applied mathematics, both on the undergraduate and graduate levels, insofar as the departments are qualified to offer such instruction. Many branches of applied mathematics have not only well established axiomatic mathematical structures, but are also fruitful fields of research, involving mathematical ideas and techniques of the highest order. Attention may in particular be directed to mathematical statistics, theoretical mechanics (including elasticity and fluid dynamics), statistical mechanics and thermodynamics, heat conduction, electromagnetic theory, relativity, quantum mechanics, genetics, and the theory of high polymers.

In some institutions the problem is already solved in part by the activities of other departments, containing members well qualified in mathematics. Nevertheless, since each of the subjects listed above has not only its mathematical side, but also an even greater experimental or practical side, it appears likely that full justice can be done to the mathematical theories only by a specialization on that aspect. Needless to say, the task of maintaining and increasing an interplay between mathematics and the other sciences is one that can be dealt with only by full cooperation between all the parties concerned. It is to be expected that scientists in other fields would welcome the desire of young mathematicians to increase their knowledge of the mathematical structures of those fields, for the sake of a wider mathematical perspective and, in the case of those qualified to pursue research, with a view to solving outstanding problems and simplifying basic theory by mathematical methods of increased generality and power.

A similar gap in the background of our students lies in the field of computation. Even if we carefully distinguish mathematical computation and computation engineering, there is much of importance concerning numerical computation and its organization for hand computer, IBM equipment, relay calculator or electronic computer, which will be of value to the student whether he is to become a topologist, a flutter analyst, or a mathematical logician. Of equal importance is the field of algebraic and analytic computation and its possibilities of mechanization. With the exception of a few institutions where the engineers have begun to work on computation engineering, this whole field is at the disposal of the departments of mathematics.

Consequently, it is suggested that departments of mathematics throughout the country should consider the feasibility of enlarging their offerings in computation, algebraic, analytic, or numerical, at both the undergraduate and graduate levels.

THE TRANSFER OF RADIATION IN STELLAR ATMOSPHERES

S. CHANDRASEKHAR

1. **Introductory remarks.** The advances in the various branches of theoretical physics have often resulted in the creation of new mathematical disciplines: disciplines which, in their way, are as characteristic of the subjects as the physical phenomena they are devoted to. That several of the earlier Gibbs lectures should illustrate this point is not surprising since the example provided by the scientific writings of Gibbs is indeed the most striking in this connection. With your permission, I would like to illustrate the same thing this evening, on a more modest level, by considering the recent advances in theoretical astrophysics. More specifically, I should like to show how astrophysical studies relating to the transfer of radiation in stellar atmospheres have led to some characteristic developments in the theory of integro-differential and functional equations.

2. **The physical problem.** In a general way, the preoccupation of the astrophysicist with the transport of radiant energy in an atmosphere which absorbs, emits and scatters radiation is not difficult to understand. For, it is in the characteristics of the radiation emergent from a star—in its variation over the stellar disc, and in its distribution with wavelength through the spectrum—that he has to seek for information concerning the constitution and structure of the stellar atmosphere.

3. **Definitions. Intensity.** Now, the basic equation which governs the transfer of radiation in any medium is the so-called *equation of transfer*. This is the equation which governs the variation of radiant intensity in terms of the local properties of the medium. But, first, I should explain that for most purposes it is sufficient to characterize the radiation field by the specific intensity I; in terms of this quantity the amount of radiant energy crossing an element of area $d\sigma$, in a direction inclined at an angle ϑ to the normal, and confined to an element of solid angle $d\omega$, and in time dt can be expressed in the form

$$I \cos \vartheta d\sigma d\omega dt. \tag{1}$$

Accordingly, to characterize a radiation field completely, the specific

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The twentieth Josiah Willard Gibbs lecture delivered at Swarthmore, Pennsylvania, December 26, 1946, under the auspices of the American Mathematical Society; received by the editors January 13, 1947.

intensity will have to be defined at every point and for every direction through a point.

4. The equation of transfer. To establish the equation which the intensity must satisfy in a stationary radiation field (that is, in a radiation field which is constant with time) we consider the equilibrium with respect to the transport of energy of a small cylindrical element of cross section $d\sigma$ and height ds in the medium. From the definition of intensity, it follows that the difference in the radiant energy crossing the two faces normally, in a time dt and confined to an element of solid angle $d\omega$, is

$$\frac{dI}{ds}\,dsd\sigma d\omega dt. \tag{2}$$

Here I denotes the specific intensity in the direction of s. This difference (2) must arise from the excess of emission over absorption of radiant energy in time dt and the element of solid angle $d\omega$ considered. Now the absorption of radiation by an element of mass is expressed in terms of the *mass absorption coefficient* denoted by κ, such that, of the energy $Id\sigma d\omega dt$ normally incident on the face $d\sigma$, the amount absorbed in time dt is

$$\kappa\rho ds \times Id\sigma d\omega dt, \tag{3}$$

where ρ denotes the density of the material. Similarly, the emission coefficient j is defined in such a way that the element of mass $\rho d\sigma ds$ emits in directions confined to $d\omega$ and in time dt an amount of radiant energy given by

$$j\rho d\sigma ds d\omega dt. \tag{4}$$

It should be noted that, according to our definition, the emission coefficient j can very well depend on the direction at each point, differing in this respect from the absorption coefficient which is a function of position only.

From our earlier remarks and the expressions (3) and (4) for absorption and emission, we have the equation of transfer

$$\frac{dI}{\rho ds} = -\kappa I + j, \tag{5}$$

expressing the conservation of radiant energy.

5. The equation of transfer for plane-parallel atmospheres. The

source function. In our further discussion we shall restrict ourselves to transfer problems in a semi-infinite plane-parallel atmosphere (Fig. 1). Let z measure distances normal to the plane of stratification and ϑ and φ the corresponding polar angles. The specific intensity I in such an atmosphere will, in general, depend on all three variables, z, ϑ and φ. For such an atmosphere the equation of transfer can be written in the form

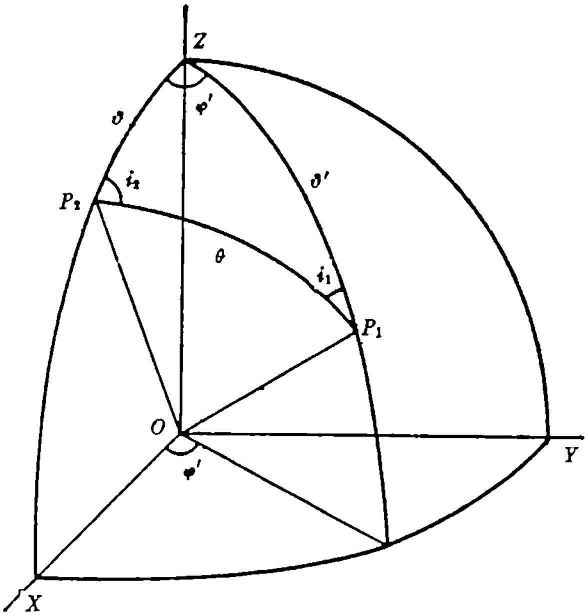


FIG. 1.

$$\cos\vartheta \frac{dI}{\rho dz} = -\kappa I + j. \tag{6}$$

It is convenient at this stage to introduce the *optical thickness*

$$\tau = \int_z^\infty \kappa\rho dz, \tag{7}$$

measured from the boundary *inward* as our variable instead of z. In terms of τ, the equation of transfer becomes

$$\cos\vartheta \frac{dI}{d\tau} = I(\tau, \vartheta, \varphi) - \mathfrak{J}(\tau, \vartheta, \varphi), \tag{8}$$

where we have written

$$\mathfrak{J} = \frac{j}{\kappa}. \tag{9}$$

This is the *source function.*

6. The source function for a scattering atmosphere. The phase function. Different physical problems lead to different *functional* dependences of the source function on I. To illustrate the nature of such functional dependences we shall consider the case of a *scattering atmosphere.* In such an atmosphere, absorption arises simply on account of the fact that when a pencil of radiation is incident on an element of gas, a determinate part of the radiant energy is scattered in other directions. In the same way, the radiation in any given direction can be reinforced by the scattering of radiation from other directions into the particular direction considered. It is now evident that, to formulate quantitatively the concept of scattering, we must specify the angular distribution of the scattered radiation when a pencil of radiation is incident on an element of gas. This angular distribution is generally given by a *phase function* $p(\cos\Theta)$ in such a way that

$$p(\cos\Theta)\frac{d\omega}{4\pi} \tag{10}$$

governs the probability that radiation will be scattered in a direction inclined at an angle Θ with the direction of incidence. If, as we have assumed, scattering is the only process by which radiation and matter interact with each other, then it is evident that

$$\int p(\cos\Theta)\frac{d\omega}{4\pi} = 1 \tag{11}$$

when the integration is extended over the complete sphere. The corresponding source function is

$$\begin{aligned}\mathfrak{J}(\tau, \vartheta, \varphi) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau, \vartheta', \varphi')p(\cos\vartheta\cos\vartheta' \\ + \sin\vartheta\sin\vartheta'\cos[\varphi' - \varphi])\sin\vartheta' d\vartheta' d\varphi'.\end{aligned} \tag{12}$$

When the scattering is *isotropic*

$$p(\cos\Theta) = 1 \tag{13}$$

and the source function reduces to

$$\mathfrak{J}(\tau) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau, \vartheta', \varphi') \sin \vartheta' d\vartheta' d\varphi'. \tag{14}$$

Next to this isotropic case, greatest interest is attached to *Rayleigh's phase function*

$$p(\cos \Theta) = \frac{3}{4}(1 + \cos^2 \Theta). \tag{15}$$

In general, we may, however, suppose that the phase function can be expanded as a series in Legendre polynomials of the form

$$p(\cos \Theta) = \sum \varpi_l P_l(\cos \Theta) \tag{16}$$

where $\varpi_0 = 1$ and ϖ_l $(l \neq 0)$ are some constants. (The condition $\varpi_0 = 1$ follows from equation (11).)

7. **Phase-functions with absorption.** Our foregoing remarks strictly apply to the case of perfect scattering; that is, to cases in which the interaction between matter and radiation results only in the transformation of radiation in one direction into radiation in other directions. However, cases are known in which, in the process of scattering, a certain amount of the radiation is consumptively transformed into other forms of energy. In such cases, we can still describe the scattered radiation by a phase function $p(\cos \Theta)$; but the "normalizing condition" (11) will no longer be satisfied and we shall have instead

$$\int p(\cos \Theta) \frac{d\omega}{4\pi} = \lambda \qquad (\lambda \leqq 1). \tag{17}$$

The quantity λ defined in this way is called the *albedo*. Formally, this does not introduce anything very essential, as the source function will continue to be given by equation (12).

The case

$$p(\cos \Theta) = \lambda(1 + x \cos \Theta) \qquad (-1 < x < 1) \tag{18}$$

is of particular interest for the analysis of planetary illumination.

8. **The problem of a semi-infinite plane-parallel atmosphere with a constant net flux.** Returning to the case of a perfectly scattering atmosphere, we observe that the equation of transfer

$$\cos \vartheta \frac{dI(\tau, \vartheta, \varphi)}{d\tau} = I(\tau, \vartheta, \varphi) - \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau, \vartheta', \varphi') p(\cos \Theta) d\omega', \tag{19}$$

where

$$\cos \Theta = \cos \vartheta \cos \vartheta' + \sin \vartheta \sin \vartheta' \cos (\varphi' - \varphi), \tag{20}$$

admits the first integral

$$\frac{d}{d\tau}\int_0^{\pi}\int_0^{2\pi} I(\tau, \vartheta, \varphi)\cos\vartheta\sin\vartheta\, d\vartheta\, d\varphi = 0, \tag{21}$$

or

$$\pi F = -\int_0^{\pi}\int_0^{2\pi} I(\tau, \vartheta, \varphi)\cos\vartheta\sin\vartheta\, d\vartheta\, d\varphi = \text{constant.} \tag{22}$$

The integral representing F is proportional to the net flux of radiant energy crossing unit area in the plane of stratification; and this is a constant for a perfectly scattering atmosphere. Because of this constancy of the net flux, a type of transfer problem which arises in connection with such perfectly scattering atmsopheres is that of a plane-parallel semi-infinite atmosphere with no incident radiation, and a constant net flux πF through the atmosphere normal to the plane of stratification. It is evident that for these problems, solutions of the appropriate equations of transfer must be sought which depend only on τ and ϑ, that is, solutions which exhibit rotational symmetry about the z-axis. Further, in these problems, the greatest interest is attached to the angular distribution of the *emergent radiation* $I(0, \mu)$ at $\tau = 0$ and for $\mu > 0$.

Finally, we may note that for the cases of isotropic scattering and scattering in accordance with Rayleigh's phase function, the equations of transfer have explicitly the forms

$$\text{I} \qquad \boxed{\mu\frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2}\int_{-1}^{+1} I(\tau, \mu')d\mu',}$$

and

$$\text{II} \qquad \boxed{\begin{aligned}\mu\frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{3}{16}\Big[(3 - \mu^2)&\int_{-1}^{+1} I(\tau, \mu')d\mu' \\ &+ (3\mu^2 - 1)\int_{-1}^{+1} I(\tau, \mu')\mu'^2 d\mu'\Big].\end{aligned}}$$

In the foregoing equations we have written μ for $\cos\vartheta$.

We require solutions of the integro-differential equations (I) and (II) which satisfy the boundary condition

$$I(0, \mu) = 0 \qquad \text{for } -1 < \mu < 0. \tag{23}$$

9. **The problem of diffuse reflection.** Another type of transfer problem which arises quite generally (that is, also in cases when λ is different from unity) is that relating to the phenomenon of *diffuse reflection*. In these contexts we are principally interested in the angular distribution $I(\vartheta, \varphi; \vartheta_0, \varphi_0)$ of the radiation diffusely reflected when a parallel beam of radiation of net flux πF per unit area normal to itself is incident on the atmosphere in some specified direction $(\pi - \vartheta_0, \varphi_0)$.

One general remark relating to problems of diffuse reflection may be made here. It is that, at any level, we may distinguish between the reduced incident radiation $\pi F e^{-\tau \sec \vartheta_0}$ which penetrates to the depth τ without suffering any scattering or absorbing mechanism, and the *diffuse radiation field* which arises in consequence of multiple scattering. We shall characterize this diffuse radiation field by $I(\tau, \mu, \varphi)$. Making this distinction between these two fields of radiation, we may write the equation of transfer for problems of diffuse reflection in the form

$$\begin{aligned}\cos\vartheta \frac{dI(\tau, \vartheta, \varphi)}{d\tau} = I(\tau, \vartheta, \varphi) &- \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau, \vartheta', \varphi') \\ &\times p(\cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos[\varphi' - \varphi]) \sin\vartheta' d\vartheta' d\varphi' \\ &- \frac{1}{4} F e^{-\tau \sec \vartheta_0} p(-\cos\vartheta \cos\vartheta_0 + \sin\vartheta \sin\vartheta_0 \cos[\varphi_0 - \varphi]).\end{aligned} \tag{24}$$

For the cases of isotropic scattering, and scattering in accordance with the phase functions (15) and (18), we have the equations

$$\boxed{\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2}\int_{-1}^{+1} I(\tau, \mu') d\mu' - \frac{1}{4} F e^{-\tau/\mu_0},} \tag{III}$$

$$\boxed{\begin{aligned}\mu \frac{dI(\tau, \vartheta, \varphi)}{d\tau} &= I(\tau, \vartheta, \varphi) \\ &- \frac{\lambda}{4\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau, \vartheta', \varphi')[1 + x(\cos\vartheta \cos\vartheta' \\ &\qquad + \sin\vartheta \sin\vartheta' \cos[\varphi' - \varphi])] \sin\vartheta' d\vartheta' d\varphi' \\ &- \frac{1}{4}\lambda F e^{-\tau \sec \vartheta_0}[1 + x(-\cos\vartheta \cos\vartheta_0 + \sin\vartheta \sin\vartheta_0 \cos\varphi)]\end{aligned}} \tag{IV}$$

and

$$\boxed{\begin{aligned}
\cos\vartheta\,\frac{dI(\tau,\vartheta,\varphi)}{d\tau} &= I(\tau,\vartheta,\varphi)\\
&\quad-\frac{3}{16\pi}\int_0^{\pi}\int_0^{2\pi} I(\tau,\vartheta',\varphi')\Big[1+\cos^2\vartheta\cos^2\vartheta'\\
&\quad+\frac{1}{2}\sin^2\vartheta\sin^2\vartheta' + 2\cos\vartheta\sin\vartheta\cos\vartheta'\sin\vartheta'\cos(\varphi'-\varphi)\\
&\quad+\frac{1}{2}\sin^2\vartheta\sin^2\vartheta'\cos 2(\varphi'-\varphi)\Big]\sin\vartheta' d\vartheta' d\varphi'\\
&\quad-\frac{3}{16}Fe^{-\tau\sec\vartheta_0}\Big[1+\cos^2\vartheta\cos^2\vartheta_0\\
&\quad+\frac{1}{2}\sin^2\vartheta\sin^2\vartheta_0 - 2\cos\vartheta\sin\vartheta\cos\vartheta_0\sin\vartheta_0\cos\varphi\\
&\quad+\frac{1}{2}\sin^2\vartheta\sin^2\vartheta_0\cos 2\varphi\Big].
\end{aligned}} \tag{V}$$

And again we require solutions of these equations which satisfy the boundary condition (23).

It appears from the foregoing equations that for problems of diffuse reflection which arise in practice, we can expand $I(\tau, \mu, \varphi)$ in a finite Fourier series of the form

$$I(\tau, \mu, \varphi) = \sum_{m=0}^{N} I^{(m)}(\tau, \mu) \cos m\varphi. \tag{25}$$

And finally, it should be remarked that in the problem of diffuse reflection by a semi-infinite atmosphere, greatest interest is attached to the *scattering function* $S(\mu, \varphi; \mu_0, \varphi_0)$ which is related to the reflected intensity $I(0, \mu, \varphi)$ according to the equation

$$I(0, \mu, \varphi) = \frac{1}{4\mu} FS(\mu, \varphi; \mu_0, \varphi_0). \tag{26}$$

Defined in this manner $S(\mu, \varphi; \mu_0, \varphi_0)$ *must be symmetrical in the pair of variables* (μ, φ) *and* (μ_0, φ_0). This is required by a *principle of reciprocity* due to Helmholtz.

10. **The equations of transfer incorporating the polarization of the scattered radiation in accordance with Rayleigh's law. The axially symmetric case.** The transfer problems which we have described in

the preceding sections do not exhaust all the physical possibilities that may arise. For example, an important factor which we have not included in our discussion so far concerns the *state of polarisation* of the radiation field. For many problems, this must be taken into account before we may be said to have an adequate physical description of the phenomenon under consideration, since, on scattering, light in general gets polarized. On Rayleigh's classical laws for instance, when an initially unpolarized beam is scattered in a direction inclined at an angle Θ with the direction of incidence, it becomes partially plane polarized with a ratio of intensities $1:\cos^2\Theta$ in directions perpendicular, respectively, parallel to the *plane of scattering*. (This is the plane which contains the directions of the incident and the scattered light.) It is, therefore, apparent that the diffuse radiation field in a scattering atmosphere must be partially polarized. The question immediately arises as to how best we can characterize the radiation field under these conditions in order that the relevant equations of transfer may be most conveniently formulated. This is a fundamental question: on its answer will depend the solution of a variety of problems, including the important one of the illumination and the polarization of the sunlit sky—a problem which was in fact first considered in a general way by Rayleigh in 1871. It is, therefore, somewhat surprising to find that even the basic equations of the problem should not have been written down before the problem arose in an astrophysical context earlier this year. Perhaps I may briefly explain the particular astrophysical problem which gave the final impetus for formulating and solving transfer problems in which the polarization characteristics of the radiation field are incorporated and allowance for the polarization of the scattered radiation is made:

It is now believed that in the atmospheres of stars with surface temperatures exceeding 15,000°K, the transfer of radiation must be predominantly controlled by the scattering by free electrons. And, according to J. J. Thomson's laws for this process, scattering by free electrons must result in the partial polarization of the scattered radiation. Thomson's laws agree, in fact, with Rayleigh's as regards the angular distribution and the state of polarization of the scattered radiation.

Now, if our belief in the important role played by the Thomson scattering in the atmospheres of hot stars is correct, we should expect that the radiation emergent from the atmospheres of such stars should be partially polarized. The question of the *degree of polarization* to be expected under such conditions therefore becomes one of exceptional astrophysical interest. But, before we can answer this question, we must formulate the relevant equations of transfer, distinguishing the different states of polarization, and solve them! One circumstance, however, simplifies this particular problem. It is, that in a plane-parallel atmosphere with no incident radiation, the radiation field at any point must exhibit rotational symmetry about the z-axis; accordingly, the plane of polarization must be along the principal meridian (or, at right angles to it). In other words, under the conditions of our present problem, the specific intensities I_l and I_r, referring respectively to the two states of polarization in which the electric vector vibrates in the meridian plane and at right angles to it, are sufficient to characterize the radiation field completely. And the relevant equations of transfer appropriate for this problem are found to be:

$$\tag{VI} \boxed{\begin{aligned} \mu \frac{dI_l}{d\tau} &= I_l - \frac{3}{8}\left\{\int_{-1}^{+1} I_l(\tau, \mu')\left[2(1-\mu'^2) + \mu^2(3\mu'^2 - 2)\right]d\mu' \right.\\ &\qquad \left. + \mu^2 \int_{-1}^{+1} I_r(\tau, \mu')d\mu'\right\},\\ \mu \frac{dI_r}{d\tau} &= I_r - \frac{3}{8}\left\{\int_{-1}^{+1} I_l(\tau, \mu')\mu'^2 d\mu' + \int_{-1}^{+1} I_r(\tau, \mu')d\mu'\right\}. \end{aligned}}$$

In other words, we now have a pair of simultaneous integro-differential equations to solve with the boundary conditions

$$\tag{27} I_l(0, \mu) = I_r(0, \mu) = 0 \qquad (-1 < \mu < 0).$$

11. **The parametric representation of partially plane-polarized light. The vector form of the equation of transfer.** Returning to the general problem of a partially polarized radiation field, we at once realize that the intensities I_l and I_r are not sufficient to characterize the radiation field completely. A third quantity is needed which will determine the plane of polarization. But the inclination of the plane of polarization itself to the meridian plane (for example) would be a most unsuitable parameter to choose: for, a proper parametric representation of partially polarized light should not only provide a complete set of variables to describe the radiation field, but also enable a convenient formulation of the rule of composition of a mixture of *independent* polarized streams in a given direction into a single partially polarized stream. On consideration it appears that the most convenient third variable to choose is a quantity U also of the dimensions of intensity and in terms of which the inclination, χ, of the plane of polarization to the direction to which I_l refers is given by

$$\chi = \frac{1}{2}\tan^{-1}\frac{U}{I_l - I_r}. \tag{28}$$

The rule of composition of a number of independent plane-polarized streams can be formulated as follows: If we have a number of independent plane-polarized streams of intensities $I^{(n)}$ (say) in a given direction, the resulting mixture will be partially plane-polarized and will be described by the parameters

$$I_l = \sum I^{(n)} \cos^2 \chi_n; \qquad I_r = \sum I^{(n)} \sin^2 \chi_n, \tag{29}$$

and

$$U = \sum I^{(n)} \sin 2\chi_n \tag{29'}$$

where χ_n denotes the inclination of the plane of polarization of the stream $I^{(n)}$ to the direction (in the transverse plane of the electric and the magnetic vectors) to which I_l refers.

With the rule of composition, as we have just stated, the equations of transfer for I_l, I_r and U can be formulated in accordance, for example, with Rayleigh's laws. For the problem of diffuse reflection of a partially plane-polarized parallel beam of radiation (characterized by the parameters F_l, F_r and $U^{(0)}$) incident on a plane-parallel atmosphere in the direction $(-\mu_0, \varphi_0)$, the three equations of transfer for the intensities I_l, I_r and U of the diffuse radiation field can be combined into a single vector equation of the form

$$\boxed{\begin{aligned}\mu\frac{dI}{d\tau} &= I(\tau, \mu, \varphi)\\ &\quad - \frac{3}{16\pi}Q\int_{-1}^{+1}\int_0^{2\pi} J(\mu, \varphi; -\mu', \varphi')I(\tau, \mu', \varphi')d\mu' d\varphi'\\ &\quad - \frac{3}{16}QJ(\mu, \varphi; \mu_0, \varphi_0)Fe^{-\tau/\mu_0},\end{aligned}} \tag{VII}$$

where

$$I = (I_l, I_r, U), \qquad F = (F_l, F_r, U^{(0)}), \tag{30}$$

$$Q = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}, \tag{31}$$

and

$$J(\mu, \varphi; \mu_0, \varphi_0)$$

$$(32) = \begin{pmatrix} 2(1-\mu^2)(1-\mu_0^2)+\mu^2\mu_0^2 & \mu^2 & \\ -4\mu\mu_0(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\cos(\varphi-\varphi_0) & & -2\mu(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\sin(\varphi-\varphi_0) \\ +\mu^2\mu_0^2\cos 2(\varphi-\varphi_0) & -\mu^2\cos 2(\varphi-\varphi_0) & +\mu^2\mu_0\sin 2(\varphi-\varphi_0) \\ \mu_0^2-\mu_0^2\cos 2(\varphi-\varphi_0) & 1+\cos 2(\varphi-\varphi_0) & -\mu_0\sin 2(\varphi-\varphi_0) \\ -2\mu_0(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\sin(\varphi-\varphi_0) & & (1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}\cos(\varphi-\varphi_0) \\ +\mu\mu_0^2\sin 2(\varphi-\varphi_0) & -\mu\sin 2(\varphi-\varphi_0) & -\mu\mu_0\cos 2(\varphi-\varphi_0) \end{pmatrix}.$$

The matrix QJ therefore plays the same role for this problem as the phase function $p(\cos\Theta)$ does in the simpler problems in which polarization is not allowed for.

We require to solve (VII) which satisfies the boundary condition

$$(33) \qquad I(0, \mu, \varphi) = 0 \qquad (-1 < \mu < 0).$$

Since the equation of transfer is linear in F, it is clear that the intensity $I(\mu, \varphi; \mu_0, \varphi_0)$ diffusely reflected by the atmosphere in the outward direction (μ, φ) must be expressible in the form

$$(34) \qquad I(\mu, \varphi; \mu_0, \varphi_0) = \frac{3}{16\mu} QS(\mu, \varphi; \mu_0, \varphi_0)F.$$

And our principal interest in this problem is in the determination of this *scattering matrix* S.

12. **The Schwarzschild-Milne integral equation.** In the preceding sections we have formulated several typical problems in the theory of radiative transfer and we have seen how all of them lead to integro-differential equations of varying degrees of complexity. The question now arises as to whether there are any general methods for solving such equations. Until recently, the only example of such integro-differential equations which had been considered to any extent is the simplest we have written down, namely equation (I). And here the method used was one of reducing the integro-differential equation to an integral equation. This reduction to an integral equation was achieved in the following way:

Writing equation (I) in the form

$$(35) \qquad \mu \frac{dI}{d\tau} = I - J(\tau),$$

where

$$(36) \qquad J(\tau) = \frac{1}{2} \int_{-1}^{+1} I(\tau, \mu) d\mu,$$

we observe that its formal solution is

$$(37)\qquad \begin{aligned} I(\tau, \mu) &= \int_{\tau}^{\infty} e^{-(t-\tau)/\mu} J(t)\, \frac{dt}{\mu} && (0 < \mu \leqq 1) \\ &= -\int_{0}^{\tau} e^{-(t-\tau)/\mu} J(t)\, \frac{dt}{\mu} && (-1 \leqq \mu < 0). \end{aligned}$$

Substituting this solution for $I(\tau, \mu)$ back into the equation defining $J(\tau)$ (equation (36)) we find, after some elementary reductions, that

$$(38)\qquad J(\tau) = \frac{1}{2} \int_{0}^{\infty} E_1(|t - \tau|) J(t) dt,$$

where

$$(39)\qquad E_1(x) = \int_{x}^{\infty} e^{-y} \frac{dy}{y},$$

is the "first exponential integral." This is the *Schwarzschild-Milne integral equation* for $J(\tau)$.

13. **The method of Hopf and Wiener.** The integral equation (38) for $J(\tau)$ is seen to be homogeneous with a symmetric kernel. However, the kernel has a logarithmic singularity at $t = \tau$. Integral equations of this general structure have therefore a definite mathematical interest and have, in fact, been considered by a number of mathematicians including E. Hopf, J. Brönstein and N. Wiener. But this method based on the reduction of the equation of transfer to a linear integral equation for the source function has not so far been extended successfully to the solution of the other more interesting equations of transfer we have formulated. I shall therefore pass on to a different method of attack on these equations which has been developed during the past three years and which allows the solution of practically all the different types of transfer problems which arise in practice, by a systematic method of approximation. Also, for the particular problems enumerated, *exact solutions* for the angular distribution of the emergent (or the diffusely reflected) radiation can be found explicitly by passing to the limit of "infinite" approximation.

14. **The method of replacing the integro-differential equations by systems of linear equations.** The fundamental idea in this new method of solving an integro-differential equation is to replace it in a certain approximation by a system of ordinary linear equations. This reduction to an equivalent linear system is made by approximating the integrals over μ which occur in the equation by a weighted

average of the values which the various integrands take at a suitably selected set of points in the interval $(-1, +1)$. It is evident that in making the necessary division of the interval $(-1, +1)$, we must be guided by reasons of "economy" in the sense that with a given number of divisions we must try to achieve the maximum accuracy in the evaluation of the integrals. The problem which we encounter here is therefore the same as that considered by Gauss in 1814 in deriving his formula for numerical quadratures. In this formula of Gauss, the interval $(-1, +1)$ is divided according to the zeros of a Legendre polynomial, $P_m(\mu)$ (say), and the integral of a function $f(\mu)$ over the interval $(-1, +1)$ is expressed as a sum in the form

$$\int_{-1}^{+1} f(\mu)d\mu \simeq \sum a_j f(\mu_j) \tag{40}$$

where the *weights* a_j are given by

$$a_j = \frac{1}{P_m'(\mu_j)} \int_{-1}^{+1} \frac{P_m(\mu)}{\mu - \mu_j} d\mu. \tag{41}$$

The reason Gauss's formula is superior to other formulae for quadratures in the interval $(-1, +1)$ is that for a given m it evalues the integrals *exactly* for *all* polynomial of degree less than, or equal to, $2m-1$; in other words, it is almost "twice as accurate" as we should expect a formula which uses only m values of the function in the interval to be.

For certain formal reasons, we shall not enter into here, in our further work we shall use divisions of the interval $(-1, +1)$ only according to the zeros of the even Legendre polynomials. Further, when the division according to the zeros of $P_{2n}(\mu)$ is selected, we shall say that we are working in the nth *approximation*. It is, therefore, apparent that when an equation of transfer is reduced to an equivalent linear system in the nth approximation, we are effectively using a polynomial representation of order $4n-1$ (in μ) for the various intensities which occur in our problem.

Finally, it may be observed that, when working in the nth approximation, we may use the following relations between the points of the Gaussian division, μ_j, and the Gaussian weights a_j:

$$a_j = a_{-j}; \qquad \mu_j = -\mu_{-j} \qquad (j = \pm 1, \cdots, \pm n), \tag{42}$$

and

$$\sum_{j=1}^{n} a_j \mu_j^{2m} = \frac{1}{2m+1} \qquad (m = 1, \cdots, 2n-1). \tag{43}$$

The identity (43) arises from the fact that in the nth approximation Gauss's formula evaluates exactly integrals of all polynomials of degree less than or equal to $4n-1$.

15. **The solution of equation (I) by the new method.** We shall illustrate the method of solving linear integro-differential equations by reducing them to equivalent linear systems in the various approximations by considering equation I.

In our nth approximation we replace equation (I) by the system of $2n$ equations

$$\mu_i \frac{dI_i}{d\tau} = I_i - \frac{1}{2} \sum a_j I_j \qquad (i = \pm 1, \cdots, \pm n), \tag{44}$$

where, for the sake of brevity, we have written $I_i(\tau)$ for $I(\tau, \mu_i)$.

In solving the system of equations represented by equation (44), we shall first obtain the different linearly independent solutions and later, by combining them, obtain the general solution.

As our first trial, we seek a solution of the form

$$I_i = g_i e^{-k\tau} \qquad (i = \pm 1, \cdots, \pm n), \tag{45}$$

where the g_i's and k are constants, for the present unspecified. Introducing equation (45) in equation (44), we obtain the relation

$$g_i(1 + \mu_i k) = \frac{1}{2} \sum a_j g_j. \tag{46}$$

Hence,

$$g_i = \frac{\text{constant}}{1 + \mu_i k} \qquad (i = \pm 1, \cdots, \pm n), \tag{47}$$

where the "constant" is independent of i. Substituting the foregoing form for g_i in equation (46), we obtain the *characteristic equation* for k:

$$1 = \frac{1}{2} \sum \frac{a_j}{1 + \mu_j k}. \tag{48}$$

Remembering that $a_{-j} = a_j$ and $\mu_{-j} = -\mu_j$, we can rewrite the characteristic equation in the form

$$1 = \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}. \tag{49}$$

It is, therefore, seen that k^2 must satisfy an algebraic equation of order n. However, since

$$\sum_{j=1}^{n} a_j = 1, \tag{50}$$

$k^2 = 0$ is a root of equation (49). Equation (48) accordingly admits only $2n-2$ distinct nonzero roots which must occur in pairs as

$$\pm k_\alpha \qquad (\alpha = 1, \cdots, n-1). \tag{51}$$

And corresponding to these $2n-2$ roots, we have $2n-2$ independent solutions of equation (44). The solution is completed by observing that

$$I_i = b(\tau + \mu_i + Q) \qquad (i = \pm 1, \cdots, \pm n), \tag{52}$$

where b and Q are two arbitrary constants, also satisfies equation (44). The general solution of equation (44) can therefore be written in the form

$$I_i = b\left\{ \sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} + \sum_{\alpha=1}^{n-1} \frac{L_{-\alpha} e^{+k_\alpha \tau}}{1 - \mu_i k_\alpha} + \tau + \mu_i + Q \right\} \qquad (i = \pm 1, \cdots, \pm n), \tag{53}$$

where b, $L_{\pm\alpha}$ $(\alpha = 1, \cdots, n-1)$ and Q are the $2n$ arbitrary constants of integration.

For the astrophysical problem on hand, the boundary conditions are that none of the I_i's increase exponentially as $\tau \to \infty$ and that, further, there is no radiation incident on $\tau = 0$. The first of these conditions requires that in the general solution (53) we omit all terms in $\exp(+k_\alpha \tau)$, thus leaving

$$I_i = b\left\{ \sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} + \tau + \mu_i + Q \right\} \qquad (i = \pm 1, \cdots, \pm n). \tag{54}$$

Next, the absence of any radiation in the directions $-1 < \mu < 0$ at $\tau = 0$ implies that in our approximation we must require

$$I_{-i} = 0 \qquad \text{at } \tau = 0 \text{ for } i = 1, \cdots, n. \tag{55}$$

Hence, according to equation (54),

$$\sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - \mu_i k_\alpha} - \mu_i + Q = 0 \qquad (i = 1, \cdots, n). \tag{56}$$

These are the n equations which determine the n constants of integration L_α $(\alpha = 1, \cdots, n-1)$ and Q. The constant b is left arbitrary, and this, as we shall presently see, is related to the assigned constant net flux of radiation through the atmosphere.

16. Some elementary identities. The further discussion of the solution obtained in §15 requires certain relations which we shall now establish.

Let

$$(57)\quad D_m(x) = \sum_i \frac{a_i\mu_i^m}{1+\mu_i x} = (-1)^m \sum_i \frac{a_i\mu_i^m}{1-\mu_i x} \qquad (m = 0, \cdots, 4n).$$

There is a simple recursion formula which $D_m(x)$ defined in this manner satisfies. We have:

$$(58)\qquad D_m(x) = \frac{1}{x}\sum_i a_i\mu_i^{m-1}\left(1 - \frac{1}{1+\mu_i x}\right),$$

or, using equations (42) and (43),

$$(59)\qquad D_m(x) = \frac{1}{x}\left[\frac{2}{m}\epsilon_{m,\mathrm{odd}} - D_{m-1}(x)\right],$$

where

$$(60)\qquad \epsilon_{m,\mathrm{odd}} = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

For odd, respectively, even values of m, equation (59) takes the form

$$(61)\qquad D_{2j-1}(x) = \frac{1}{x}\left[\frac{2}{2j-1} - D_{2j-2}(x)\right],$$

and

$$(62)\qquad D_{2j}(x) = -\frac{1}{x}D_{2j-1}(x).$$

Combining these relations, we have

$$(63)\quad D_{2j-1}(x) = \frac{1}{x}\left[\frac{2}{2j-1} + \frac{1}{x}D_{2j-3}(x)\right] = -xD_{2j}(x).$$

From this formula, we readily deduce that

$$(64)\qquad \begin{aligned} D_{2j-1}(x) = {} & \frac{2}{(2j-1)x} + \frac{2}{(2j-3)x^3} + \cdots \\ & + \frac{2}{3x^{2j-3}} + \frac{2}{3x^{2j-1}}\,[2 - D_0(x)], \end{aligned}$$

and

$$D_{2j}(x) = -\frac{2}{(2j-1)x^2} - \frac{2}{(2j-3)x^4} - \cdots - \frac{2}{3x^{2j-2}} - \frac{2}{3x^{2j}}\,[2 - D_0(x)]. \tag{65}$$

If we now let x be a root k of the characteristic equation (48),

$$D_0(k) = 2, \tag{66}$$

and we find from equations (64) and (65) that

$$D_1(k) = D_2(k) = 0, \tag{67}$$

$$D_{2j}(k) = -\frac{2}{(2j-1)k^2} - \frac{2}{(2j-3)k^4} - \cdots - \frac{2}{3k^{2j-2}} \qquad (j = 2, \cdots, n), \tag{68}$$

$$D_{2j-1}(k) = \frac{2}{(2j-1)k} + \frac{2}{(2j-3)k^3} + \cdots + \frac{2}{3k^{2j-3}} \qquad (j = 2, \cdots, n). \tag{69}$$

17. **A relation between the characteristic roots and the zeros of the Legendre polynomial.** In terms of the $D_{2j}(k)$'s introduced in §16, we can express the characteristic equation for k in a form which does not explicitly involve the Gaussian weights and divisions: Let p_{2j} be the coefficient of μ^{2j} in the polynomial representation of the Legendre polynomial $P_{2n}(\mu)$, so that

$$P_{2n}(\mu) = \sum_{j=0}^{n} p_{2j}\mu^{2j}. \tag{70}$$

Now consider

$$\sum_{j=0}^{n} p_{2j}D_{2j}(k) = \sum_{i} \frac{a_i}{1 + \mu_i k}\left(\sum_{j=0}^{n} p_{2j}\mu_i^{2j}\right). \tag{71}$$

Since the μ_i's are the zeros of $P_{2n}(\mu)$

$$\sum_{j=0}^{n} p_{2j}\mu_i^{2j} = 0 \tag{72}$$

and the characteristic equation can be expressed in the form

$$\sum_{j=0}^{n} p_{2j}D_{2j}(k) = 0, \tag{73}$$

where the D_{2j}'s are defined as in equation (68). Substituting in particular for D_{2n} and D_0, we have

$$\frac{2}{3}\frac{p_{2n}}{k^{2n-2}} + \cdots + 2p_0 = 0. \tag{74}$$

From this equation it follows that

$$\frac{1}{(k_1 \cdots k_{n-1})^2} = (-1)^{n-1}\frac{3p_0}{p_{2n}} = 3\mu_1^2 \cdots \mu_n^2, \tag{75}$$

or

$$k_1 \cdots k_{n-1}\mu_1 \cdots \mu_n = 1/3^{1/2}. \tag{76}$$

18. **The law of darkening.** Returning to the solution (54) of the transfer problem considered in §15, we evaluate the net flux πF according to the formula

$$F = 2\int_{-1}^{+1} I\mu d\mu. \tag{77}$$

In our present scheme of approximation, we can write

$$F = 2\sum a_i I_i \mu_i. \tag{78}$$

Evaluating this sum with the I_i's given by equation (54), we obtain

$$F = 2b\left\{\sum_{\alpha=1}^{n-1} L_\alpha e^{-k_\alpha \tau} D_1(k_\alpha) + \sum_i a_i\mu_i^2\right\}. \tag{79}$$

Using equations (43) and (67), we have

$$F = 4b/3. \tag{80}$$

We can now rewrite the solution (54) in the form

$$I_i = \frac{3}{4}F\left\{\sum_{\alpha=1}^{n-1}\frac{L_\alpha e^{-k_\alpha \tau}}{1+\mu_i k_\alpha} + \tau + \mu_i + Q\right\} \quad (i = \pm 1, \cdots, \pm n). \tag{81}$$

In terms of this solution, the source function for the problem under consideration can be obtained in the following manner. We have

$$\begin{aligned} J &= \frac{1}{2}\int_{-1}^{+1} I d\mu \simeq \frac{1}{2}\sum a_j I_j \\ &= \frac{3}{8}F\left\{\sum_{\alpha=1}^{n-1} L_\alpha e^{-k_\alpha \tau} D_0(k_\alpha) + 2(\tau + Q)\right\}, \end{aligned} \tag{82}$$

or, using equation (66),

$$J = \frac{3}{4} F \left\{ \sum_{\alpha=1}^{n-1} L_\alpha e^{-k_\alpha \tau} + \tau + Q \right\}. \tag{83}$$

The angular distribution $I(0, \mu)$ of the emergent radiation can be found from the source function (83) in accordance with the formula (cf. equation (37))

$$I(0, \mu) = \int_0^\infty J(\tau) e^{-\tau/\mu} \frac{d\tau}{\mu}. \tag{84}$$

We find

$$I(0, \mu) = \frac{3}{4} F \left\{ \sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 + \mu k_\alpha} + \mu + Q \right\}. \tag{85}$$

It is to be particularly noted that the foregoing expression for $I(0, \mu)$ is in agreement with the solution (81) for $\tau = 0$ and at the points of the Gaussian division $\mu = \mu_i$.

Comparing equation (85) with equation (56) which determines the constants L_α and Q, we observe that the angular distribution of the emergent radiation defined for the interval $0 \leq \mu \leq 1$ is determined in terms of a function which has zeros assigned in the complementary interval $-1 < \mu < 0$. Thus, letting

$$S(\mu) = \sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - \mu k_\alpha} - \mu + Q, \tag{86}$$

the boundary conditions require that

$$S(\mu_i) = 0 \qquad (i = 1, \cdots, n), \tag{87}$$

while the angular distribution of the emergent radiation is given by

$$I(0, \mu) = \frac{3}{4} F S(-\mu). \tag{88}$$

19. The elimination of the constants and the expression of the solution in closed form. We shall now show how an explicit formula for $S(\mu)$ can be found without solving explicitly for the constants L_α and Q:

Consider the function

$$\prod_{\alpha=1}^{n-1} (1 - \mu k_\alpha) S(\mu). \tag{89}$$

This is a polynomial of degree n in μ which vanishes for $\mu = \mu_i$, $i = 1, \cdots, n$. Consequently there must exist a proportionality of the form

$$\prod_{\alpha=1}^{n-1} (1 - k_\alpha \mu) S(\mu) \propto \prod_{i=1}^{n-1} (\mu - \mu_i). \tag{90}$$

The constant of proportionality can be found from a comparison of the coefficients of the highest powers of μ on either side. In this manner we find that

$$S(\mu) = (-1)^n k_1 \cdots k_{n-1} \frac{\prod_{i=1}^{n} (\mu - \mu_i)}{\prod_{\alpha=1}^{n-1} (1 - k_\alpha \mu)} \cdot \tag{91}$$

This is the required formula.

According to equation (91)

$$S(-\mu) = k_1 \cdots k_{n-1} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha=1}^{n-1} (1 + k_\alpha \mu)}, \tag{92}$$

or, using the result (76), we can write

$$S(-\mu) = \frac{1}{3^{1/2}} H(\mu), \tag{93}$$

where

$$H(\mu) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha=1}^{n-1} (1 + k_\alpha \mu)} \cdot \tag{94}$$

In terms of the function $H(\mu)$ defined in this manner, the angular distribution of the emergent radiation can be expressed in the form

$$I(0, \mu) = \frac{3^{1/2}}{4} F H(\mu). \tag{95}$$

20. **The solution of equation (III) by the new method. A particular integral.** As a further illustration of our method of solving equations

of transfer, we shall next consider equation III appropriate for the problem of diffuse reflection by an isotropically scattering atmosphere. The equivalent system of linear equations in the nth approximation is

$$(96)\qquad \mu_i \frac{dI_i}{d\tau} = I_i - \frac{1}{2}\sum a_j I_j - \frac{1}{4} F e^{-\tau/\mu_0} \qquad (i = \pm 1, \cdots, \pm n).$$

It is seen that the homogeneous system associated with equation (96) is the same as equation (44). Accordingly, the complimentary function for the solution of equation (96) is the same as the general solution (53) of the homogeneous system. To complete the solution we require only a particular integral. This can be found in the following manner: Setting

$$(97)\qquad I_i = \frac{1}{4} F h_i e^{-\tau/\mu_0} \qquad (i = \pm 1, \cdots, \pm n)$$

in equation (96) (the h_i's are constants unspecified for the present) we verify that we must have

$$(98)\qquad h_i(1 + \mu_i/\mu_0) = \frac{1}{2}\sum a_j h_j + 1.$$

Equation (98) implies that the constants h_i must be expressible in the form

$$(99)\qquad h_i = \frac{\gamma}{1 + \mu_i/\mu_0} \qquad (i = \pm 1, \cdots, \pm n),$$

where the constant γ has to be determined from the condition (cf. equation (98))

$$(100)\qquad \gamma = \frac{1}{2}\gamma \sum \frac{a_i}{1 + \mu_i/\mu_0} + 1.$$

In other words

$$(101)\qquad I_i = \frac{1}{4} F \frac{\gamma e^{-\tau/\mu_0}}{1 + \mu_i/\mu_0} \qquad (i = \pm 1, \cdots, \pm n),$$

with

$$(102)\qquad \gamma = \frac{1}{1 - \sum_{j=1}^{n} \dfrac{a_j}{1 - \mu_j^2/\mu_0^2}},$$

represents the required particular integral of equation (96).

The constant γ defined as in equation (102) can be expressed in terms of the H-function introduced in §19 (equation (94)). For this purpose consider the function

$$T(x) = 1 - \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 x}. \tag{103}$$

This clearly vanishes for

$$x = 0 \quad \text{and} \quad x = k_\alpha^2 \qquad (\alpha = 1, \cdots, n-1). \tag{104}$$

Accordingly

$$T(x) \prod_{j=1}^{n} (1 - \mu_j^2 x) \tag{105}$$

cannot differ from

$$x \prod_{\alpha=1}^{n-1} (x - k_\alpha^2) \tag{106}$$

by more than a constant factor since (105) represents a polynomial of degree n in x. The constant of proportionality can be determined by comparing the coefficients of the highest powers. In this manner we find that

$$T(x) = (-1)^n \mu_1^2 \cdots \mu_n^2 \frac{x \prod_{\alpha=1}^{n-1} (x - k_\alpha^2)}{\prod_{j=1}^{n} (1 - \mu_j^2 x)}. \tag{107}$$

But

$$\gamma = \frac{1}{T(\mu_0^{-2})}. \tag{108}$$

Hence

$$\gamma = (-1)^n \frac{1}{\mu_1^2 \cdots \mu_n^2} \frac{\prod_{i=1}^{n} (\mu_0^2 - \mu_i^2)}{\prod_{\alpha=1}^{n-1} (1 - k_\alpha^2 \mu_0^2)} \tag{109}$$

or according to our definition of $H(\mu)$

$$\gamma = H(\mu_0)H(-\mu_0). \tag{110}$$

21. **The law of diffuse reflection for the case of isotropic scattering.** Now, adding to the particular integral (101) the general solution of the homogeneous system compatible with our present requirement of boundedness of the solution for $\tau \rightarrow \infty$, we have

$$I_i = \frac{1}{4} F \left\{ \sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} + Q + \frac{H(\mu_0)H(-\mu_0)}{1 + \mu_i/\mu_0} e^{-\tau/\mu_0} \right\} \tag{111}$$

$$(i = 1, \cdots, n),$$

where the constants L_α $(\alpha = 1, \cdots, n-1)$ and Q are to be determined by the boundary conditions at $\tau = 0$.

At $\tau = 0$ we have no diffuse radiation directed inward. The conditions which determine the constants of integration are therefore

$$\sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - \mu_i k_\alpha} + Q + \frac{H(\mu_0)H(-\mu_0)}{1 - \mu_i/\mu_0} = 0 \quad (i = \pm 1, \cdots, \pm n). \tag{112}$$

The angular distribution of the diffusely reflected radiation can be found in terms of the source function

$$\mathfrak{J}(\tau) = \frac{1}{2} \sum a_i I_i + \frac{1}{4} F e^{-\tau/\mu_0}, \tag{113}$$

according to the formula

$$I(0, \mu) = \int_0^\infty \mathfrak{J}(\tau) e^{-\tau/\mu} \frac{d\tau}{\mu}. \tag{114}$$

We find

$$I(0, \mu; \mu_0) = \frac{1}{4} F \left\{ \sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 + \mu k_\alpha} + Q + \frac{H(\mu_0)H(-\mu_0)}{1 + \mu/\mu_0} \right\}. \tag{115}$$

As in §18, we again observe that the angular distribution of the diffusely reflected radiation defined in the interval $0 \leqq \mu \leqq 1$ is described in terms of a function which has zeros assigned in the interval $0 > \mu > -1$. Thus, letting

$$S(\mu) = \sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - \mu k_\alpha} + Q + \frac{H(\mu_0)H(-\mu_0)}{1 - \mu/\mu_0}, \tag{116}$$

we have

$$S(\mu_i) = 0 \qquad (i = 1, \cdots, n), \tag{117}$$

and

$$I(0, \mu; \mu_0) = \frac{1}{4} FS(-\mu). \tag{118}$$

And, again as in the earlier problem, we can find an explicit formula for $S(\mu)$ without having to solve explicitly for the constants L_α and Q. Thus, considering the function

$$(1 - \mu/\mu_0)\prod_{\alpha=1}^{n-1} (1 - k_\alpha\mu)S(\mu) \tag{119}$$

we observe that it is a polynomial of degree n in μ which vanishes for $\mu = \mu_i$, $i = 1, \cdots, n$. There must, therefore, exist a relation of the form

$$S(\mu) = (-1)^n \frac{X}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1}^{n} (\mu - \mu_i)}{\prod_{\alpha=1}^{n-1} (1 - k_\alpha\mu)} \frac{1}{(1 - \mu/\mu_0)}, \tag{120}$$

where X is a constant. In terms of $H(\mu)$ we can rewrite the foregoing equation as

$$S(\mu) = X \frac{H(-\mu)}{1 - \mu/\mu_0}. \tag{121}$$

The constant X appearing in equation (121) can be determined from the relation (cf. equation (116))

$$\lim_{\mu \to \mu_0} (1 - \mu/\mu_0)S(\mu) = H(\mu_0)H(-\mu_0). \tag{122}$$

According to equation (121) the left-hand side of equation (122) is $XH(-\mu_0)$. Hence

$$X = H(\mu_0) \tag{123}$$

and

$$S(\mu) = H(\mu_0)H(-\mu) \frac{\mu_0}{\mu_0 - \mu}. \tag{124}$$

The expression (118) for the angular distribution of the reflected radiation therefore becomes

$$I(0, \mu; \mu_0) = \frac{1}{4} FH(\mu)H(\mu_0) \frac{\mu_0}{\mu + \mu_0}. \tag{125}$$

22. **Tabulation of the solutions of equations** (I)–(VII). The method of solution of transfer problems which we have illustrated in the preceding sections by considering the two simplest problems is, on examination, found to be sufficiently general for adaptation to the solution of the more difficult problems presented by the other equations of transfer formulated in §§8–11. While the details of the solution of these other equations (particularly equations VI and VII) are considerably more elaborate and complex, the analysis nevertheless shows similarities with the simple problems we have considered in the broad features. Thus, it is found that, in all cases, the angular distribution of the emergent (equivalently, reflected) radiation is described by a function for the argument in the range (0, 1) while the boundary conditions specify the zeros of the same function in the interval (0, −1). This remarkable *reciprocity*, which exists in all the problems, enables the *elimination of the constants* and allows the reduction of the solutions to *closed forms* in the general nth approximation. And finally, these solutions, apart from certain constants, involve only H-functions of the form

$$H(\mu) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha=1}^{n} (1 + k_\alpha \mu)}, \tag{126}$$

where the μ_i's are the zeros of the Legendre polynomial $P_{2n}(\mu)$ and the k_α's are the positive (or zero) roots of a characteristic equation of the form

$$1 = 2\sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{1 - k^2 \mu_j^2}, \tag{127}$$

where $\Psi(\mu)$ is an even polynomial in μ satisfying the condition

$$\int_0^1 \Psi(\mu) d\mu \leqq \frac{1}{2} \cdot \tag{128}$$

(This last condition is necessary for $H(\mu)$ to be real.)

The different physical problems naturally lead to different characteristic equations and therefore to different H-functions. However, as the H-functions differ from one another only through the characteristic equations which define the roots k_α, we may properly call $\Psi(\mu)$ the *characteristic function* in terms of which H is defined.

We tabulate below the solutions for the various transfer problems (equations I–VII) obtained in the nth approximation of our method of solutions.

SOLUTIONS FOR THE EMERGENT AND THE DIFFUSELY REFLECTED RADIATION FOR VARIOUS TRANSFER PROBLEMS

A. *Isotropic scattering*

Problem with constant net flux: the law of darkening:

$$I(\mu) = \frac{3^{1/2}}{4} F H(\mu).$$

Law of diffuse reflection:

$$I(\mu; \mu_0) = \frac{1}{4} F H(\mu) H(\mu_0) \frac{\mu_0}{\mu + \mu_0} .$$

The characteristic function in terms of which $H(\mu)$ is defined is

$$\Psi(\mu) = 1/2.$$

B. *Scattering in accordance with Rayleigh's phase function* $3(1+\cos^2\Theta)/4$

Problem with constant net flux: the law of darkening:

$$I(\mu) = \frac{3}{4} q F H^{(0)}(\mu) .$$

Law of diffuse reflection:

$$\begin{aligned} I(\mu, \varphi; \mu_0, \varphi_0) = \frac{3}{32} F \big\{ & H^{(0)}(\mu) H^{(0)}(\mu_0) [3 - (3 - 8q^2)^{1/2}(\mu + \mu_0) + \mu\mu_0] \\ & - 4\mu\mu_0(1 - \mu^2)^{1/2}(1 - \mu_0^2)^{1/2} H^{(1)}(\mu) H^{(1)}(\mu_0) \cos(\varphi - \varphi_0) \\ & + (1 - \mu^2)(1 - \mu_0^2) H^{(2)}(\mu) H^{(2)}(\mu_0) \cos 2(\varphi - \varphi_0) \big\} \frac{\mu_0}{\mu + \mu_0} . \end{aligned}$$

The characteristic functions in terms of which $H^{(0)}(\mu)$, $H^{(1)}(\mu)$ and $H^{(2)}(\mu)$ are defined are respectively

$$\Psi^{(0)}(\mu) = \frac{3}{16}(3 - \mu^2),$$

$$\Psi^{(1)}(\mu) = \frac{3}{8}\mu^2(1 - \mu^2),$$

$$\Psi^{(2)}(\mu) = \frac{3}{32}(1 - \mu^2)^2.$$

And finally,

$$q = \frac{2(3^{1/2})}{H^{(0)}(+3^{1/2}) - H^{(0)}(-3^{1/2})}.$$

C. *Scattering in accordance with the phase function* $\lambda(1+x\cos\Theta)$

Law of diffuse reflection:

$$I(\mu, \varphi; \mu_0, \varphi_0)$$
$$= \frac{1}{4}\lambda F\{H^{(0)}(\mu)H^{(0)}(\mu_0)[1 - c(\mu+\mu_0) - x(1-\lambda)\mu\mu_0]$$
$$+ x(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}H^{(1)}(\mu)H^{(1)}(\mu_0)\cos(\varphi-\varphi_0)\}\frac{\mu_0}{\mu+\mu_0}.$$

The characteristic functions in terms of which $H^{(0)}(\mu)$ and $H^{(1)}(\mu)$ are defined are respectively

$$\Psi^{(0)}(\mu) = \frac{1}{2}\lambda[1 + x(1-\lambda)\mu^2],$$

$$\Psi^{(1)}(\mu) = \frac{1}{4}x\lambda(1-\mu^2).$$

The constant c depends in a somewhat complicated manner on the characteristic roots defining $H^{(0)}(\mu)$ (cf. S. Chandrasekhar, Astrophysical Journal vol. 103 (1946) p. 165, equation [108]).

D. *Scattering in accordance with Rayleigh's law and allowing for the polarization of the scattered radiation*

Problem with constant net flux: the law of darkening in the two states of polarization:

$$I_l(\mu) = \frac{3}{8}F(1-c^2)^{1/2}H_l(\mu),$$

$$I_r(\mu) = \frac{3}{8}F\frac{1}{2^{1/2}}H_r(\mu)(\mu+c).$$

Problem of diffuse reflection: the scattering matrix:

$$I(\mu, \varphi; \mu_0, \varphi_0) = \frac{3}{16\mu}QS(\mu, \varphi; \mu_0, \varphi_0)F$$

where

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S(\mu,\varphi;\mu_0,\varphi_0)$$

$$=\left\{\begin{array}{lll}
\begin{array}{l} 2H_l(\mu)H_l(\mu_0)[1+\mu\mu_0-c(\mu+\mu_0) \\ -4\mu\mu_0(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}H^{(1)}(\mu)H^{(1)}(\mu_0)\cos(\varphi-\varphi_0) \\ +\mu^2\mu_0^2H^{(2)}(\mu)H^{(2)}(\mu_0)\cos 2(\varphi-\varphi_0) \end{array} &
\begin{array}{l} (2(1-c^2))^{1/2}H_r(\mu_0)H_l(\mu)(\mu+\mu_0) \\ \\ -\mu^2H^{(2)}(\mu)H^{(2)}(\mu_0)\cos 2(\varphi-\varphi_0) \end{array} &
\begin{array}{l} \\ -2\mu(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}H^{(1)}(\mu)H^{(1)}(\mu_0)\sin(\varphi-\varphi_0) \\ +\mu^2\mu_0H^{(2)}(\mu)H^{(2)}(\mu_0)\sin 2(\varphi-\varphi_0) \end{array} \\
\begin{array}{l} (2(1-c^2))^{1/2}H_r(\mu)H_l(\mu_0)(\mu+\mu_0) \\ -\mu_0^2H^{(2)}(\mu)H^{(2)}(\mu_0)\cos 2(\varphi-\varphi_0) \end{array} &
\begin{array}{l} H_r(\mu)H_r(\mu_0)[1+\mu\mu_0+c(\mu+\mu_0)] \\ +H^{(2)}(\mu)H^{(2)}(\mu_0)\cos 2(\varphi-\varphi_0) \end{array} &
\begin{array}{l} \\ -\mu_0H^{(2)}(\mu)H^{(2)}(\mu_0)\sin 2(\varphi-\varphi_0) \end{array} \\
\begin{array}{l} -2\mu_0(1-\mu_0^2)^{1/2}(1-\mu^2)^{1/2}H^{(1)}(\mu)H^{(1)}(\mu_0)\sin(\varphi-\varphi_0) \\ +\mu\mu_0^2H^{(2)}(\mu)H^{(2)}(\mu_0)\sin 2(\varphi-\varphi_0) \end{array} &
\begin{array}{l} \\ -\mu H^{(2)}(\mu)H^{(2)}(\mu_0)\sin 2(\varphi-\varphi_0) \end{array} &
\begin{array}{l} (1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}H^{(1)}(\mu)H^{(1)}(\mu_0)\cos(\varphi-\varphi_0) \\ -\mu\mu_0H^{(2)}(\mu)H^{(2)}(\mu_0)\cos 2(\varphi-\varphi_0) \end{array}
\end{array}\right\}.$$

The characteristic functions in terms of which $H_l(\mu)$, $H_r(\mu)$, $H^{(1)}(\mu)$ and $H^{(2)}(\mu)$ are defined are respectively

$$\Psi_l(\mu) = \frac{3}{4}(1-\mu^2);$$

$$\Psi^{(1)}(\mu) = \frac{3}{8}(1-\mu^2)(1+2\mu^2);$$

$$\Psi_r(\mu) = \frac{3}{8}(1-\mu^2),$$

$$\Psi^{(2)}(\mu) = \frac{3}{16}(1+\mu^2)^2.$$

And finally, the constant c is given by

$$c = \frac{H_l(+1)H_r(-1) + H_l(-1)H_r(+1)}{H_l(+1)H_r(-1) - H_l(-1)H_r(+1)}.$$

23. **Remarks on the tabulated solutions. The possibility of passage to the "infinite approximation."** An examination of the solutions of the various transfer problems given in the preceding section discloses remarkable relationships between the *laws of darkening* of the emergent radiation from a semi-infinite plane-parallel atmosphere with constant net flux (and no incident radiation) and the *laws of diffuse reflection* by the same atmosphere. The relationship is naturally the simplest for the case of an isotropically scattering atmosphere and can indeed be established directly from the equations of transfer. However, in the other cases, the relationship is of a more complex nature and has to be sought between the darkening function for the problem with a constant net flux and the *azimuth independent* term in the law of diffuse reflection. It is seen that both these functions involve the same H-functions and the same constants.

While the relationship between the two problems has been established only in a particular scheme of approximation, it is apparent that the *relationship itself* must be an *exact one* since it is present in every approximation and must consequently be also present in the *limit of infinite approximation* when the solutions will become the exact ones of the problem. A further result of this train of thought is the realization that if we can solve the problem of diffuse reflection exactly, we shall, at the same time, have also solved exactly the axially symmetric problem with a constant net flux and no incident radiation.

Having thus been led to conceive the passage to the limit of infinite approximation, we naturally ask ourselves: *Can we in fact perform this limiting process and thus obtain the exact solutions for the various problems?* The answer to this question must clearly depend on our ability to pass to the limit of the H-functions as we have defined them, as $n\to\infty$. We shall now indicate how this limiting process can be achieved in practice.

24. The equation satisfied by $H(\mu)$. For the purposes of passing to the limit of infinite approximations of the solutions of the various transfer problems, we shall first establish the following basic theorem relating to the H-functions.

THEOREM 1. *Let $\Psi(\mu)$ be an even polynomial of degree $2m$ in μ such that*

$$\int_0^1 \Psi(\mu)d\mu \leq \frac{1}{2}\cdot$$

Let μ_j ($j=\pm 1, \cdots, \pm n$) denote the division of the interval $(-1, +1)$ according to the zeros of the Legendre polynomial of order $2n$ ($>m$); further, let a_j ($=a_{-j}$) denote the corresponding Gaussian weights. Finally, let k_α ($\alpha=1, \cdots, n$) denote the distinct positive (or zero) roots of the characteristic equation

$$1 = \sum_j \frac{a_j\Psi(\mu_j)}{1+k\mu_j} = 2\sum_{j=1}^{n} \frac{a_j\Psi(\mu_j)}{1-k^2\mu_j^2}\cdot$$

Then the function

$$H(\mu) = \frac{1}{\mu_1\cdots\mu_n}\,\frac{\prod_{i=1}^{n}(\mu+\mu_i)}{\prod_{\alpha=1}^{n}(1+k_\alpha\mu)}$$

satisfies identically the equation

$$H(\mu) = 1 + \mu H(\mu)\sum_{j=1}^{n}\frac{a_jH(\mu_j)\Psi(\mu_j)}{\mu+\mu_j}\cdot$$

PROOF. We shall first consider the case

$$\int_0^1 \Psi(\mu)d\mu < \frac{1}{2}\cdot \tag{129}$$

In this case the characteristic equation admits n distinct nonvanishing positive roots and we consider the function

$$S(\mu) = \sum_{\alpha=1}^{n} \frac{L_\alpha}{1 - k_\alpha \mu} + 1, \tag{130}$$

where L_α $(\alpha = 1, \cdots, n)$ are certain constants to be determined from the equations

$$\sum_{\alpha=1}^{n} \frac{L_\alpha}{1 - k_\alpha \mu_i} + 1 = 0 \qquad (i = 1, \cdots, n), \tag{131}$$

or equivalently

$$S(\mu_i) = 0 \qquad (i = 1, \cdots, n). \tag{132}$$

By considerations of the type we are now familiar with, it can be shown that

$$S(\mu) = k_1 \cdots k_n \mu_1 \cdots \mu_n \frac{(-1)^n}{\mu_1 \cdots \mu_n} \frac{\prod_i (\mu - \mu_i)}{\prod_\alpha (1 - k_\alpha \mu)}. \tag{133}$$

In other words,

$$S(\mu) = k_1 \cdots k_n \mu_1 \cdots \mu_n H(-\mu). \tag{134}$$

Since $H(0) = 1$, we can rewrite equation (134) alternatively in the form

$$S(\mu) = S(0) H(-\mu), \tag{135}$$

where

$$S(0) = \sum_{\alpha=1}^{n} L_\alpha + 1 = k_1 \cdots k_n \mu_1 \cdots \mu_n. \tag{136}$$

Now, since $\Psi(\mu)$ is even in μ, we can rewrite the equation which a characteristic root satisfies in either of the forms

$$1 = \sum_j \frac{a_j \Psi(\mu_j)}{1 + k\mu_j} \tag{137}$$

or

$$1 = \sum_j \frac{a_j \Psi(\mu_j)}{1 - k\mu_j}. \tag{138}$$

Let k_α denote a particular characteristic root. Then, on account of equations (137) and (138), which are satisfied by any of the characteristic roots, we can clearly write

$$S(0) = \sum_{\beta=1}^{n} L_\beta + 1$$

$$(139)\qquad = \sum_{\beta=1}^{n} \frac{L_\beta}{k_\alpha + k_\beta}\left[\sum_j a_j\Psi(\mu_j)\left\{\frac{k_\alpha}{1+k_\alpha\mu_j} + \frac{k_\beta}{1-k_\beta\mu_j}\right\}\right] + \sum_j \frac{a_j\Psi(\mu_j)}{1+k_\alpha\mu_j}.$$

Simplifying the quantity in brackets in the foregoing equation, we have

$$(140)\qquad S(0) = \sum_{\beta=1}^{n} L_\beta\left[\sum_j \frac{a_j\Psi(\mu_j)}{(1+k_\alpha\mu_j)(1-k_\beta\mu_j)}\right] + \sum_j \frac{a_j\Psi(\mu_j)}{1+k_\alpha\mu_j}$$

or, inverting the order of the summation,

$$(141)\qquad S(0) = \sum \frac{a_j\Psi(\mu_j)}{1+k_\alpha\mu_j}\left[\sum_{\beta=1}^{n}\frac{L_\beta}{1-k_\beta\mu_j} + 1\right].$$

But the quantity in brackets in equation (141) is $S(\mu_j)$. Hence

$$(142)\qquad S(0) = \sum_j \frac{a_j S(\mu_j)\Psi(\mu_j)}{1+k_\alpha\mu_j}.$$

In equation (142) (as in equations (137)–(141)) the summation is, of course, extended over all values of j, positive and negative. However, since $S(+\mu_j) = 0$ (equation (132)), in equation (142) only the terms with negative j make a nonzero contribution. We can, therefore, write

$$(143)\qquad S(0) = \sum_{j=1}^{n} \frac{a_j S(-\mu_j)\Psi(\mu_j)}{1-k_\alpha\mu_j} \qquad (\alpha = 1, \cdots, n)$$

or, in view of equation (135),

$$(144)\qquad 1 = \sum_{j=1}^{n} \frac{a_j H(\mu_j)\Psi(\mu_j)}{1-k_\alpha\mu_j} \qquad (\alpha = 1, \cdots, n).$$

Now, consider the function

$$(145)\qquad 1 - \mu\sum_{j=1}^{n} \frac{a_j H(\mu_j)\Psi(\mu_j)}{\mu+\mu_j}.$$

According to equation (144), this vanishes for $\mu = -1/k_\alpha$ $(\alpha = 1, \cdots, n)$; for

$$(146)\qquad 1 + \frac{1}{k_\alpha}\sum_{j=1}^{n} \frac{a_j H(\mu_j)\Psi(\mu_j)}{(-1/k_\alpha) + \mu_j} = 1 - \sum_{j=1}^{n} \frac{a_j H(\mu_j)\Psi(\mu_j)}{1 - k_\alpha \mu_j} = 0.$$

Hence

$$(147)\qquad \prod_{j=1}^{n} (\mu + \mu_j) - \mu \sum_{j=1}^{n} a_j H(\mu_j)\Psi(\mu_j) \prod_{i \neq j} (\mu + \mu_i)$$

also vanishes for

$$(148)\qquad \mu = -1/k_\alpha \qquad (\alpha = 1, \cdots, n).$$

But the expression (147) is a polynomial of degree n in μ. It cannot, therefore, differ from

$$(149)\qquad \prod_{\alpha=1}^{n} (1 + k_\alpha \mu)$$

by more than a constant factor; and the constant of proportionality is seen to be

$$(150)\qquad \mu_1 \cdots \mu_n$$

from a comparison of the two functions at $\mu = 0$. It therefore follows that

$$(151)\qquad 1 - \mu \sum_{j=1}^{n} \frac{a_j H(\mu_j)\Psi(\mu_j)}{\mu + \mu_j} = \mu_1 \cdots \mu_n \frac{\prod_{\alpha=1}^{n} (1 + k_\alpha \mu)}{\prod_{j=1}^{n} (\mu + \mu_j)} = \frac{1}{H(\mu)}.$$

Hence

$$(152)\qquad H(\mu) = 1 + \mu H(\mu) \sum_{j=1}^{n} \frac{a_j H(\mu_j)\Psi(\mu_j)}{\mu + \mu_j}.$$

This proves the theorem for the case (129).

Turning now to the case

$$(153)\qquad \int_0^1 \Psi(\mu) d\mu = \frac{1}{2},$$

we observe that, in this case, $k = 0$ is a root of the characteristic equation, and we are left with only $(n-1)$ positive roots. We, therefore, consider in this case the function

$$S(\mu) = \sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - k_\alpha \mu} + L_0 - \tag{154}$$

in place of equation (130). However, the constants L_0 and L_α ($\alpha = 1, \cdots, n-1$) are again to be determined by the conditions

$$S(\mu_i) = 0 \qquad (i = 1, \cdots, n). \tag{155}$$

With this definition of $S(\mu)$, equation (135) continues to be valid and the rest of the proof follows on similar lines. The only essential point of departure that needs to be noted is that, at the stage of the proof corresponding to equation (140) and before inverting the order of the summation, we must add the extra term

$$-\sum_j \frac{a_j \Psi(\mu_j)\mu_j}{1 + k_\alpha \mu_j} \tag{156}$$

to the right-hand side of the equation. We can do this without altering anything, since the quantity we thus add is zero; for

$$\begin{aligned} \sum_j \frac{a_j \Psi(\mu_j)\mu_j}{1 + k_\alpha \mu_j} &= \frac{1}{k_\alpha}\left[\sum_j a_j \Psi(\mu_j) \left\{ 1 - \frac{1}{1 + k_\alpha \mu_j} \right\} \right] \\ &= \frac{1}{k_\alpha}\left[1 - \sum_j \frac{a_j \Psi(\mu_j)}{1 + k_\alpha \mu_j} \right] = 0. \end{aligned} \tag{157}$$

(Note that we are permitted to set $\sum a_j \Psi(\mu_j) = 1$, since the Gauss sum in the nth approximation evaluates the integrals exactly for all polynomials of degree less than or equal to $4n-1$; and we have assumed $2n > m$.)

This completes the proof of the theorem.

25. **The limit of $H(\mu)$ as $n \to \infty$. The basic functional equation.** The theorem proved in §24 suggests how the limit of the H-function as $n \to \infty$ can be obtained. We shall state this in the form of a theorem.

THEOREM 2. *The solution of the functional equation*

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{H(\mu')\Psi(\mu')}{\mu + \mu'} d\mu',$$

where $\Psi(\mu)$ *is an even polynomial satisfying the condition*

$$\int_0^1 \Psi(\mu) d\mu \leq \frac{1}{2},$$

is the limit function

$$\lim_{n\to\infty} \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha=1}^{n} (1 + k_\alpha \mu)}$$

where the μ_i*'s and the* k_α*'s have the same meanings as in Theorem* 1.

SKETCH OF PROOF. The theorem arises in the following way: It is known that the integral of a bounded function over the interval (0, 1) can be approximated by a Gauss sum with any desired degree of accuracy by choosing a division of the interval according to the zeros of a Legendre polynomial of a sufficiently high degree. The integral which occurs on the right-hand side of the functional equation for $H(\mu)$ can, therefore, be replaced by the Gauss sum

$$\sum_{j=1}^{n} \frac{a_j H(\mu_j) \Psi(\mu_j)}{\mu + \mu_j} \tag{158}$$

to any desired accuracy by choosing a sufficiently large n. But by Theorem 1, for a finite n, no matter how large, the unique solution of the equation

$$H(\mu) = 1 + \mu H(\mu) \sum_{j=1}^{n} \frac{a_j H(\mu_j) \Psi(\mu_j)}{\mu + \mu_j} \tag{159}$$

is

$$\frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha=1}^{n} (1 + k_\alpha \mu)}. \tag{160}$$

If we now let $n \to \infty$, equation (159) becomes

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{H(\mu') \Psi(\mu')}{\mu + \mu'} d\mu'. \tag{161}$$

The solution of this functional equation is, therefore, seen to be the limit of the function (160) as $n \to \infty$.

It is realized that the sketch of the proof of Theorem 2 we have just outlined does not meet the full demands of a rigorous mathematical demonstration. It is, moreover, probable that precisely the questions of uniqueness and existence which we have ignored will cause the principal difficulties in the constructions of a rigorous mathematical proof. However, as the equation arises in a physical context, and the physical situations are such as to leave no room for ambiguity, it is hardly to be doubted that the theorem is true. Indeed, as we shall presently show quite rigorously by an entirely different line of argument, the exact solutions for the various transfer problems are of the forms tabulated in §22 with the H-functions redefined in terms of functional equations of the form (161) instead of in terms of the Gaussian division and characteristic roots. Nevertheless, it may be of interest to pursue further the purely mathematical questions raised by Theorem 2.

26. **A practical method of determining the exact H-functions as solutions of the functional equations they satisfy.** Assuming for the present the indications of the preceding sections that the exact solutions for the various transfer problems are of the *forms* found in our method of solution and that the H-functions which occur in them have to be redefined as solutions of functional equations of the form (161), we may observe that Theorem 1 of §24 suggests a simple practical method for determining the exact H-functions numerically. For, starting with an approximate solution for $H(\mu)$ (in the third approximation, for example) we can determine the exact H-functions by a process of iteration using for this purpose the functional equation which it satisfies. In this manner the exact H-functions which occur in the solutions of the various transfer problems involving isotropic scattering with an albedo $\lambda \leq 1$, Rayleigh phase function and Rayleigh scattering (including the state of polarization of the scattered radiation) have all been numerically evaluated. We have therefore now available exact numerical solutions for the cases A, B, C (for $x=0$) and D tabulated in §22.

27. **The constants in the solution.** It is to be noted that the determination of the limit to which the H-functions which occur in the solutions for the emergent (or diffusely reflected) radiation in the nth approximation tend, as $n \rightarrow \infty$, still leaves open the question of the exact limiting values of the constants which occur in these solutions. It does not seem that any direct or simple limiting process can be applied to the formulae which define them in the nth approximation. Attention may be particularly drawn in this connection to the fact that the formulae which define these constants (in the nth approximation) often involve the values of H-functions (now defined as rational functions in terms of the Gaussian division and the characteristic

roots) *outside* the interval (0, 1) (sometimes even for complex values of the argument!), whereas it would seem that in the limit of infinite approximation $H(\mu)$ has a meaning only in the interval (0, 1). (See, for example, the definitions of the constants c and q under the headings B and D in the tabulation of §22.) However, it appears that these constants can be determined *indirectly* by appealing to certain other identical relations which the problems must satisfy.

Thus, considering the transfer problems involving the Rayleigh phase function, we have

$$I(\mu) = \frac{3}{4} qFH^{(0)}(\mu), \tag{162}$$

where $H^{(0)}(\mu)$ is defined as the solution of the functional equation

$$H^{(0)}(\mu) = 1 + \frac{3}{16}\mu H^{(0)}(\mu) \int_0^1 \frac{(3 - \mu'^2)H(\mu')}{\mu + \mu'} d\mu'. \tag{163}$$

Now, as equation (162) gives the angular distribution of the emergent radiation for the axially symmetric problem with constant net flux, it follows that the outward flux of the emergent radiation must also equal πF. In other words, we must have

$$F = 2\int_0^1 I(\mu)\mu d\mu = \frac{3}{2} qF\alpha_1, \tag{164}$$

where α_1 denotes the first moment of $H^{(0)}(\mu)$. Hence,

$$q = \frac{2}{3\alpha_1}. \tag{165}$$

Thus, once the solution of equation (163) has been determined (by iteration based on an approximate $H(\mu)$ as suggested in §26) the constant q can be determined directly in terms of its first moment. Since q is the only constant which occurs in the solutions, they become determinate in this fashion.

Similarly, in the transfer problems in which the polarization of the scattered radiation in accordance with Rayleigh's law has to be properly allowed for, the solutions again involve a constant c. It does not seem possible to pass directly to the limit of infinite approximation in the formula defining this constant in the nth approximation. But we can determine it by appealing to the flux condition in the problem with the constant net flux. Thus, with the solutions $I_l(\mu)$ and $I_r(\mu)$ for the emergent radiations in the two states of polarization as given in §22 (with the functions $H_l(\mu)$ and $H_r(\mu)$ now defined properly in terms of functional equations of the form (161)), we must have

$$(166)\qquad \begin{aligned} F &= 2\int_0^1 [I_l(\mu) + I_r(\mu)]\mu d\mu \\ &= \frac{3}{4} F\left[(1 - c^2)^{1/2}\alpha_1 + \frac{1}{2^{1/2}}(A_2 + cA_1)\right] \end{aligned}$$

where α_1 denotes the first moment of $H_l(\mu)$ and A_1 and A_2 are the first and the second moments, respectively, of $H_r(\mu)$. Hence, we can determine c from the equation

$$(167)\qquad (2(1 - c^2))^{1/2}\alpha_1 + A_2 + cA_1 = \frac{4}{3} 2^{1/2}.$$

28. **The functional equation for the problem of diffuse reflection.** The discussion in the preceding sections has shown how we can obtain in *practice* the exact solutions for the angular distribution of the emergent and the reflected radiations from a semi-infinite plane-parallel atmosphere for a wide variety of scattering laws. From a strictly mathematical point of view, the limiting process by which the passage to the limit of infinite approximation was achieved may not have been as rigorously justified as one might have wished. We shall, therefore, now show how the exact solutions obtained in the manner of the preceding sections (by redefining, for example, the H-functions which occur in the solutions in the general nth approximation, in terms of functional equations of the form (161)) can be justified by following an entirely different line of argument. The basic idea in the development we are now going to describe is due to the Armenian astrophysicist, V. A. Ambarzumian.

In the problem of diffuse reflection, we are interested in the solution of the relevant equations of transfer principally, only to the extent that we want to establish the law of diffuse reflection as specified by the $\sigma(\mu, \varphi; \mu_0, \varphi_0)$ which gives the intensity reflected in the direction (μ, φ) when a parallel beam of radiation of unit flux normal to itself is incident on the atmosphere in the direction $(-\mu_0, \varphi_0)$. Now, Ambarzumian starts with the almost trivial observation that the intensity $\sigma(\mu, \varphi; \mu_0, \varphi_0)$ *must be invariant to the addition* (*or removal*) *of layers of arbitrary optical thickness to* (*from*) *the atmosphere* and shows (and this is really the point of the observation) how this invariance can be used to derive a *functional equation* for the *scattering function* $\sigma(\mu, \varphi; \mu_0, \varphi_0)\mu$. Ambarzumian has explicitly derived the form of this functional equation for the law of diffuse reflection from an atmosphere scattering radiation in accordance with a general phase function, that is, the functional equation associated with the equation of transfer (24). However, when one proceeds to solve the resulting functional equation, one is soon led to simultaneous systems of nonlinear, nonhomogeneous functional equations of such a highly complex nature that one might almost despair of solving them! But, on examination, it soon appears that a knowledge of the *forms* of the solution obtained by the method described in the earlier parts of this lecture enables us, in all cases, to reduce the Ambarzumian type of functional equations to equations of the following standard form

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{H(\mu')\Psi(\mu')}{\mu + \mu'} d\mu', \tag{168}$$

and helps us, moreover, to confirm the results obtained by our method of passage to the limit of infinite approximation.

29. **The functional equation for the scattering matrix.** In this section we shall derive, following Ambarzumian's general ideas, the functional equation for the scattering matrix *S* introduced in §11 (equation (34)). This problem is, therefore, more advanced than the ones consered by Ambarzumian; but it serves to illustrate the power of his idea.

To obtain the functional equation for S, we first rewrite the equation of transfer VII in the form

$$\mu \frac{dI}{d\tau} = I(\tau, \mu, \varphi) - B(\tau, \mu, \varphi), \tag{169}$$

where

$$\begin{aligned} B(\tau, \mu, \varphi) = {} & \frac{3}{16\pi} Q \int_{-1}^{+1} \int_0^{2\pi} J(\mu, \varphi; -\mu', \varphi') I(\tau, \mu', \varphi') d\mu' d\varphi' \\ & + \frac{3}{16} Q J(\mu, \varphi; \mu_0, \varphi_0) F e^{-\tau/\mu_0}, \end{aligned} \tag{170}$$

and express the intensity $I(\mu, \varphi; \mu_0, \varphi_0)$ reflected in the direction (μ, φ), where radiation with a net flux πF is incident in the direction $(-\mu_0, \varphi_0)$, in the form

$$I(\mu, \varphi; \mu_0, \varphi_0) = \frac{3}{16\mu} Q S(\mu, \varphi; \mu_0, \varphi_0) F. \tag{171}$$

Now, consider a level at a depth $d\tau$ below the boundary of the atmosphere at $\tau = 0$. At this level the radiation field present can be decomposed into two parts: first, there is the reduced incident flux of amount

$$\pi F\left(1 - \frac{d\tau}{\mu_0}\right) \tag{172}$$

and, second, there is a diffuse radiation field. The amount of this diffuse radiation field which is directed inward can be inferred from the equation of transfer: for, since at $\tau = 0$ there is no inward intensity, at the level $d\tau$, we must have an inward intensity

$$I(d\tau, -\mu', \varphi') = B(0, -\mu', \varphi')\frac{d\tau}{\mu'} \tag{173}$$

in the direction $(-\mu', \varphi')$. Both of these radiation fields will be reflected by the atmosphere below $d\tau$ by the *same laws* as those by which the atmosphere below $\tau = 0$ reflects. *This invariance is due to the fact that the removal of a layer of arbitrary thickness from a semi-infinite atmosphere cannot alter its reflecting power.* This is Ambarzumian's basic idea. Accordingly, the reflection of the radiations (172) and (173) by the atmosphere below $d\tau$ will contribute to an *outward* intensity, in the direction (μ, φ), the amount

$$\begin{aligned} I(d\tau, \mu, \varphi) &= \frac{3}{16\mu}\left(1 - \frac{d\tau}{\mu_0}\right) QS(\mu, \varphi; \mu_0, \varphi_0)F \\ &\quad + \frac{3}{16\pi\mu} d\tau Q \int_0^1 \int_0^{2\pi} S(\mu, \varphi; \mu', \varphi')B(0, -\mu', \varphi')\frac{d\mu}{\mu'}\, d\varphi'. \end{aligned} \tag{174}$$

On the other hand, from the equation of transfer, we conclude that

$$\begin{aligned} I(d\tau, \mu, \varphi) &= I(0, \mu, \varphi) + \frac{d\tau}{\mu}\left[I(0, \mu, \varphi) - B(0, \mu, \varphi)\right] \\ &= \frac{3}{16\mu}\left(1 + \frac{d\tau}{\mu}\right) QS(\mu, \varphi; \mu_0, \varphi_0)F - \frac{d\tau}{\mu} B(0, \mu, \varphi). \end{aligned} \tag{175}$$

Combining equations (174) and (175) and passing to the limit $d\tau = 0$, we have

$$\begin{aligned} \frac{3}{16}\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) QS(\mu, \varphi; \mu_0, \varphi_0)F &= B(0, \mu, \varphi) \\ &\quad + \frac{3}{16\pi} Q \int_0^1 \int_0^{2\pi} S(\mu, \varphi; \mu', \varphi')B(0, -\mu', \varphi')\frac{d\mu'}{\mu'}\, d\varphi'. \end{aligned} \tag{176}$$

But, according to equations (170) and (171)

$$(177)\quad \begin{aligned} B(0, \mu, \varphi) = \frac{3}{16} Q \Bigg[& J(\mu, \varphi; \mu_0, \varphi_0) \\ & + \frac{3}{16\pi} \int_0^1 \int_0^{2\pi} J(\mu, \varphi; -\mu'', \varphi'') Q S(\mu'', \varphi''; \mu_0, \varphi_0) \frac{d\mu''}{\mu''} d\varphi'' \Bigg] F. \end{aligned}$$

Substituting for $B(0, \mu, \varphi)$ from the foregoing equation in equation (176) and remembering that F can be an arbitrary vector, we find, after some minor reductions, that

$$(178)\quad \begin{aligned} \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S(\mu, \varphi; \mu_0, \varphi_0) &= J(\mu, \varphi; \mu_0, \varphi_0) \\ & + \frac{3}{16\pi} \int_0^1 \int_0^{2\pi} J(\mu, \varphi; -\mu'', \varphi'') Q S(\mu'', \varphi''; \mu_0, \varphi_0) \frac{d\mu''}{\mu''} d\varphi'' \\ & + \frac{3}{16\pi} \int_0^1 \int_0^{2\pi} S(\mu, \varphi; \mu', \varphi') Q J(-\mu', \varphi'; \mu_0, \varphi_0) \frac{d\mu'}{\mu'} d\varphi' \\ & + \frac{9}{256\pi^2} \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} S(\mu, \varphi; \mu', \varphi') Q J(-\mu', \varphi'; -\mu'', \varphi'') Q \\ & \qquad \times S(\mu'', \varphi''; \mu_0, \varphi_0) \frac{d\mu'}{\mu'} d\varphi' \frac{d\mu''}{\mu''} d\varphi''. \end{aligned}$$

This is the required functional equation for the scattering matrix.

Now, the matrix J (see equation (32)) has the property

$$(179)\quad J_{ik}(\mu, \varphi; \mu_0, \varphi_0) = J_{ki}(\mu_0, \varphi; \mu, \varphi_0).$$

From this property of J for transposition, it follows from equation (178) that S has also the same property:

$$(180)\quad S_{ik}(\mu, \varphi; \mu_0, \varphi_0) = S_{ki}(\mu_0, \varphi; \mu, \varphi_0).$$

It can be verified that equation (180) is equivalent to *Helmholtz' principle of reciprocity* for the problem under consideration.

30. **The reduction of the functional equation for S.** We shall now indicate how the functional equation for S derived in the preceding section can be solved.

From the form of the equation for S and the manner of its relation to J, it is evident that in a Fourier analysis of the elements of S in $(\varphi - \varphi_0)$ we must have the same nonvanishing components as in the corresponding elements of J. We may accordingly assume without loss of generality that S has the form

$$S(\mu, \varphi; \mu_0, \varphi)$$

$$(181) \quad = \left\{ \begin{array}{lll} S_{11}^{(0)}(\mu, \mu_0) & S_{12}^{(0)}(\mu, \mu_0) & \\ -4\mu\mu_0(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}S_{11}^{(1)}(\mu, \mu_0)\cos(\varphi-\varphi_0) & & -2\mu(1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}S_{13}^{(1)}(\mu, \mu_0)\sin(\varphi-\varphi_0) \\ +\mu^2\mu_0^2S_{11}^{(2)}(\mu, \mu_0)\cos 2(\varphi-\varphi_0) & -\mu^2S_{12}^{(2)}(\mu, \mu_0)\cos 2(\varphi-\varphi_0) & +\mu^2\mu_0S_{13}^{(2)}(\mu, \mu_0)\sin 2(\varphi-\varphi_0) \\ & & \\ S_{21}^{(0)}(\mu, \mu_0) & S_{22}^{(0)}(\mu, \mu_0) & \\ -\mu_0^2S_{21}^{(2)}(\mu, \mu_0)\cos 2(\varphi-\varphi_0) & +S_{22}^{(2)}(\mu, \mu_0)\cos 2(\varphi-\varphi_0) & -\mu_0S_{23}^{(2)}(\mu, \mu_0)\sin 2(\varphi-\varphi_0) \\ & & \\ -2\mu_0(1-\mu_0^2)^{1/2}(1-\mu^2)^{1/2}S_{31}^{(1)}(\mu, \mu_0)\sin(\varphi-\varphi_0) & & (1-\mu^2)^{1/2}(1-\mu_0^2)^{1/2}S_{33}^{(1)}(\mu, \mu_0)\cos(\varphi-\varphi_0) \\ +\mu_0^2\mu S_{31}^{(2)}(\mu, \mu_0)\sin 2(\varphi-\varphi_0) & -\mu S_{32}^{(2)}(\mu, \mu_0)\sin 2(\varphi-\varphi_0) & -\mu\mu_0S_{33}^{(2)}(\mu, \mu_0)\cos 2(\varphi-\varphi_0) \end{array} \right]$$

where, as the notation indicates, $S_{11}^{(0)}$, and so on, are all functions of μ and μ_0 only. From the property (180) of S for transposition, we now conclude that

$$S_{jk}^{(i)}(\mu, \mu_0) = S_{kj}^{(i)}(\mu_0, \mu). \tag{182}$$

If we now substitute the form (181) for S in equation (178) and equate the different Fourier components of the various elements, we shall clearly obtain three systems of functional equations governing the functions of the different orders, distinguished by their superscripts. Of these systems, the first, involving the zero-order functions $S_{11}^{(0)}$, $S_{12}^{(0)}$, $S_{21}^{(0)}$ and $S_{22}^{(0)}$, is the most important and, at the same time, the most difficult. We shall accordingly consider this system briefly.

First, we may write down the equations which are found for $S_{11}^{(0)}$, $S_{12}^{(0)}$, $S_{21}^{(0)}$, and $S_{22}^{(0)}$. The equations are

$$\begin{aligned}
\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) & S_{11}^{(0)}(\mu, \mu_0) \\
&= \left\{\mu^2 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'} \left[\mu'^2 S_{11}^{(0)}(\mu, \mu') + S_{12}^{(0)}(\mu, \mu')\right]\right\} \\
&\quad \times \left\{\mu_0^2 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'} \left[\mu'^2 S_{11}^{(0)}(\mu', \mu_0) + S_{21}^{(0)}(\mu', \mu_0)\right]\right\} \\
&\quad + 2\left\{1 - \mu^2 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'} (1 - \mu'^2) S_{11}^{(0)}(\mu, \mu')\right\} \\
&\quad \times \left\{1 - \mu_0^2 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'} (1 - \mu'^2) S_{11}^{(0)}(\mu', \mu_0)\right\};
\end{aligned} \tag{183}$$

$$\begin{aligned}
\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) & S_{12}^{(0)}(\mu, \mu_0) \\
&= \left\{\mu^2 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'} \left[\mu'^2 S_{11}^{(0)}(\mu, \mu') + S_{12}^{(0)}(\mu, \mu')\right]\right\} \\
&\quad \times \left\{1 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'} \left[\mu'^2 S_{12}^{(0)}(\mu', \mu_0) + S_{22}^{(0)}(\mu', \mu_0)\right]\right\} \\
&\quad + 2\left\{1 - \mu^2 + \frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'} (1 - \mu'^2) S_{11}^{(0)}(\mu, \mu')\right\} \\
&\quad \times \left\{\frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'} (1 - \mu'^2) S_{12}^{(0)}(\mu', \mu_0)\right\};
\end{aligned} \tag{184}$$

$$
\begin{aligned}
&\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S_{21}^{(0)}(\mu,\mu_0)\\
&\quad=\left\{\mu_0^2+\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}\left[\mu'^2S_{11}^{(0)}(\mu',\mu_0)+S_{21}^{(0)}(\mu',\mu_0)\right]\right\}\\
&\qquad\times\left\{1+\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}\left[\mu'^2S_{21}^{(0)}(\mu,\mu')+S_{22}^{(0)}(\mu,\mu')\right]\right\}\\
&\qquad+2\left\{1-\mu_0^2+\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}(1-\mu'^2)S_{11}^{(0)}(\mu',\mu_0)\right\}\\
&\qquad\times\left\{\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}(1-\mu'^2)S_{21}^{(0)}(\mu,\mu')\right\};
\end{aligned}
\tag{185}
$$

$$
\begin{aligned}
&\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S_{22}^{(0)}(\mu,\mu_0)\\
&\quad=\left\{1+\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}\left[\mu'^2S_{12}^{(0)}(\mu',\mu_0)+S_{22}^{(0)}(\mu',\mu_0)\right]\right\}\\
&\qquad\times\left\{1+\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}\left[\mu'^2S_{21}^{(0)}(\mu,\mu')+S_{22}^{(0)}(\mu,\mu')\right]\right\}\\
&\qquad+\frac{9}{32}\int_0^1\frac{d\mu'}{\mu'}(1-\mu'^2)S_{21}^{(0)}(\mu,\mu')\\
&\qquad\times\int_0^1\frac{d\mu'}{\mu'}(1-\mu'^2)S_{12}^{(0)}(\mu',\mu_0).
\end{aligned}
\tag{186}
$$

An inspection of these equations shows what we have already seen from the functional equation for $\mathbf{S}$, that among these functions of zero-order the relation (182) must hold in particular. In other words

$$
\begin{gathered}
S_{11}^{(0)}(\mu,\mu_0)=S_{11}^{(0)}(\mu_0,\mu);\qquad S_{12}^{(0)}(\mu,\mu_0)=S_{21}^{(0)}(\mu_0,\mu);\\
S_{22}^{(0)}(\mu,\mu_0)=S_{22}^{(0)}(\mu_0,\mu).
\end{gathered}
\tag{187}
$$

In view of these relations, it follows from equations (183)–(186) that we can express the functions $S_{11}^{(0)}$, $S_{12}^{(0)}$, $S_{21}^{(0)}$ and $S_{22}^{(0)}$ in the forms

$$
\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S_{11}^{(0)}(\mu,\mu_0)=\psi(\mu)\psi(\mu_0)+2\phi(\mu)\phi(\mu_0),
\tag{188}
$$

$$
\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S_{12}^{(0)}(\mu,\mu_0)=\psi(\mu)\chi(\mu_0)+2\phi(\mu)\zeta(\mu_0),
\tag{189}
$$

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S_{21}^{(0)}(\mu,\mu_0)=\psi(\mu_0)\chi(\mu)+2\phi(\mu_0)\zeta(\mu),\tag{190}$$

and

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S_{22}^{(0)}(\mu,\mu_0)=\chi(\mu)\chi(\mu_0)+2\zeta(\mu)\zeta(\mu_0),\tag{191}$$

where

$$\psi(\mu)=\mu^2+\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}\left[\mu'^2S_{11}^{(0)}(\mu,\mu')+S_{12}^{(0)}(\mu,\mu')\right],\tag{192}$$

$$\phi(\mu)=1-\mu^2+\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}(1-\mu'^2)S_{11}^{(0)}(\mu,\mu'),\tag{193}$$

$$\chi(\mu)=1+\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}\left[\mu'^2S_{21}^{(0)}(\mu,\mu')+S_{22}^{(0)}(\mu,\mu')\right],\tag{194}$$

and

$$\zeta(\mu)=\frac{3}{8}\int_0^1\frac{d\mu'}{\mu'}(1-\mu'^2)S_{21}^{(0)}(\mu,\mu').\tag{195}$$

Substituting for $S_{11}^{(0)}$, and so on, from equations (188)–(191) back into equations (192)–(195) we obtain the functional equations for the problem in their normal forms. We have

$$\begin{aligned}\psi(\mu)&=\mu^2+\frac{3}{8}\mu\psi(\mu)\int_0^1\frac{d\mu'}{\mu+\mu'}\left[\mu'^2\psi(\mu')+\chi(\mu')\right]\\&\quad+\frac{3}{4}\mu\phi(\mu)\int_0^1\frac{d\mu'}{\mu+\mu'}\left[\mu'^2\phi(\mu')+\zeta(\mu')\right],\end{aligned}\tag{196}$$

$$\begin{aligned}\phi(\mu)&=1-\mu^2+\frac{3}{8}\mu\psi(\mu)\int_0^1\frac{d\mu'}{\mu+\mu'}(1-\mu'^2)\psi(\mu')\\&\quad+\frac{3}{4}\mu\phi(\mu)\int_0^1\frac{d\mu'}{\mu+\mu'}(1-\mu'^2)\phi(\mu'),\end{aligned}\tag{197}$$

$$\begin{aligned}\chi(\mu)&=1+\frac{3}{8}\mu\chi(\mu)\int_0^1\frac{d\mu'}{\mu+\mu'}\left[\mu'^2\psi(\mu')+\chi(\mu')\right]\\&\quad+\frac{3}{4}\mu\zeta(\mu)\int_0^1\frac{d\mu'}{\mu+\mu'}\left[\mu'^2\phi(\mu')+\zeta(\mu')\right],\end{aligned}\tag{198}$$

$$\zeta(\mu) = \frac{3}{8}\mu\chi(\mu)\int_0^1 \frac{d\mu'}{\mu+\mu'}(1-\mu'^2)\psi(\mu') + \frac{3}{4}\mu\zeta(\mu)\int_0^1 \frac{d\mu'}{\mu+\mu'}(1-\mu'^2)\phi(\mu'). \tag{199}$$

31. **The solution of the functional equations** (196)–(199). Equations (196)–(199) represent a nonlinear, nonhomogeneous system of four simultaneous functional equations which one might despair of even attempting to solve. However, with the guidance provided by the form of the solution obtained in our general nth approximation, it is possible to reduce the solution of equations (196)–(199) to two simple functional equations, each of form (168). Thus, it can be verified by direct substitution that the solution of the system of equations (196)–(199) is given by

$$\psi(\mu) = (2(1-c^2))^{1/2}\mu H_l(\mu), \tag{200}$$

$$\phi(\mu) = H_l(\mu)(1-c\mu), \tag{201}$$

$$\chi(\mu) = H_r(\mu)(1+c\mu), \tag{202}$$

and

$$\zeta(\mu) = \frac{(1-c^2)^{1/2}}{2^{1/2}}\mu H_r(\mu), \tag{203}$$

where $H_l(\mu)$ and $H_r(\mu)$ are defined in terms of the functional equations

$$H_l(\mu) = 1 + \frac{3}{4}\mu H_l(\mu)\int_0^1 \frac{H_l(\mu')}{\mu+\mu'}(1-\mu'^2)d\mu' \tag{204}$$

and

$$H_r(\mu) = 1 + \frac{3}{8}\mu H_r(\mu)\int_0^1 \frac{H_r(\mu')}{\mu+\mu'}(1-\mu'^2)d\mu', \tag{205}$$

and c is a constant related in a determinate way with the moments of $H_l(\mu)$ and $H_r(\mu)$. We find

$$c = \frac{8(A_1-\alpha_1)+3(2\alpha_1\alpha_0 - A_1A_0)}{3(A_1^2+2\alpha_1^2)}, \tag{206}$$

where α_0, A_0 and α_1, A_1 are the moments of order zero and one of $H_l(\mu)$ and $H_r(\mu)$, respectively.

With $\psi(\mu)$, $\phi(\mu)$, and so on, defined as in equations (200)–(203) we can verify that the solutions for $S_{11}^{(0)}$, and so on, are in entire agreement with our earlier results obtained by passing to the limit of infinite approximation.

32. **The completion of the solution for S.** The discussion of the other two systems for the functions of order one and two turns out to be very simple, as it appears that all the functions $S^{(1)}$ are equal to each other and similarly all the functions $S^{(2)}$ are equal to each other. Therefore, writing

$$S_{i,j}^{(1)}(\mu, \mu_0) = S^{(1)}(\mu, \mu_0), \tag{207}$$

and

$$S_{i,j}^{(2)}(\mu, \mu_0) = S^{(2)}(\mu, \mu_0), \tag{208}$$

it is found that the equations governing $S^{(1)}$ and $S^{(2)}$ are

$$\begin{aligned}\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)&S^{(1)}(\mu, \mu_0)\\ &= \left\{1+\frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}(1-\mu'^2)(1+2\mu'^2)S^{(1)}(\mu', \mu_0)\right\}\\ &\quad\times\left\{1+\frac{3}{8}\int_0^1 \frac{d\mu'}{\mu'}(1-\mu'^2)(1+2\mu'^2)S^{(1)}(\mu, \mu')\right\},\end{aligned} \tag{209}$$

and

$$\begin{aligned}\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(2)}(\mu, \mu_0) &= \left\{1+\frac{3}{16}\int_0^1 \frac{d\mu'}{\mu'}(1+\mu'^2)^2S^{(2)}(\mu', \mu_0)\right\}\\ &\quad\times\left\{1+\frac{3}{16}\int_0^1 \frac{d\mu'}{\mu'}(1+\mu'^2)^2S^{(2)}(\mu, \mu')\right\}.\end{aligned} \tag{210}$$

From these equations it follows that the functions $S^{(1)}(\mu, \mu_0)$ and $S^{(2)}(\mu, \mu_0)$ are symmetrical in the variables μ and μ_0 and that they are expressible in the forms

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(1)}(\mu, \mu_0) = H^{(1)}(\mu)H^{(1)}(\mu_0), \tag{211}$$

and

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S^{(2)}(\mu, \mu_0) = H^{(2)}(\mu)H^{(2)}(\mu_0) \tag{212}$$

where $H^{(1)}(\mu)$ and $H^{(2)}(\mu)$ are solutions of the functional equations

$$H^{(1)}(\mu) = 1 + \frac{3}{8}\mu H^{(1)}(\mu)\int_0^1 \frac{H^{(1)}(\mu')}{\mu+\mu'}(1-\mu'^2)(1+2\mu'^2)d\mu', \tag{213}$$

and

$$H^{(2)}(\mu) = 1 + \frac{3}{16}\mu H^{(2)}(\mu)\int_0^1 \frac{H^{(2)}(\mu')}{\mu+\mu'}(1+\mu'^2)^2 d\mu'. \tag{214}$$

With this, the solution of the functional equation for S is completed and it will be observed that the solution for S which we have now obtained is of exactly the same form as that given in §22 (under D) with the only difference that the H-functions which appear in the solution are now defined in terms of the exact functional equations which they satisfy; further, the constant c is shown to be related in a definite way with the moments of $H_l(\mu)$ and $H_r(\mu)$.

33. **A class of functional equations and their solution.** The solution of the functional equations for the problem of diffuse reflection for laws of scattering, other than the one we have described, can be carried out in an analogous manner. It is not our intention to go into the details of the solution of these other cases here, but it may be of some mathematical interest to see the type of functional equations which these problems lead us to consider.

The problem of diffuse reflection in accordance with Rayleigh's phase function leads to the following simultaneous pair of functional equations:

$$\begin{aligned}\psi(\mu) = 3 - \mu^2 &+ \frac{1}{16}\mu\psi(\mu)\int_0^1 \frac{\psi(\mu')}{\mu+\mu'}(3-\mu'^2)d\mu' \\ &+ \frac{1}{2}\mu\phi(\mu)\int_0^1 \frac{\phi(\mu')}{\mu+\mu'}(3-\mu'^2)d\mu'\end{aligned} \tag{215}$$

and

$$\begin{aligned}\phi(\mu) = \mu^2 &+ \frac{1}{16}\mu\psi(\mu)\int_0^1 \frac{\psi(\mu')}{\mu+\mu'}\mu'^2 d\mu' \\ &+ \frac{1}{2}\mu\phi(\mu)\int_0^1 \frac{\phi(\mu')}{\mu+\mu'}\mu'^2 d\mu'.\end{aligned} \tag{216}$$

Again, guided by the form of the solution obtained by the direct solution of the equation of transfer (see the tabulation in §22, under B), we are led to surmise that the solutions of equations (215) and

(216) must be of the forms

$$\psi(\mu) = H^{(0)}(\mu)(3 - c\mu) \tag{217}$$

and

$$\phi(\mu) = q\mu H^{(0)}(\mu), \tag{218}$$

where $H^{(0)}(\mu)$ satisfies the functional equation

$$H^{(0)}(\mu) = 1 + \frac{3}{16}\mu H^{(0)}(\mu)\int_0^1 \frac{H^{(0)}(\mu')}{\mu + \mu'}(3 - \mu'^2)d\mu', \tag{219}$$

and q and c are two constants related in the manner

$$8q^2 = 3 - c^2. \tag{220}$$

Direct substitution confirms that the solutions of equations (215) and (216) are indeed of the forms surmised and shows further that, in agreement with equation (165),

$$q = \frac{2}{3\alpha_1} \tag{221}$$

where α_1 is the first moment of $H^{(0)}(\mu)$.

Similarly, the problem of diffuse reflection in accordance with the phase function $\lambda(1+x \cos \Theta)$ leads to the following pair of functional equations:

$$\begin{aligned} \psi(\mu) = 1 &+ \frac{1}{2}\lambda\mu\psi(\mu)\int_0^1 \frac{\psi(\mu')}{\mu + \mu'}d\mu' \\ &- \frac{1}{2}x\lambda\mu\phi(\mu)\int_0^1 \frac{\phi(\mu')}{\mu + \mu'}d\mu', \end{aligned} \tag{222}$$

and

$$\begin{aligned} \phi(\mu) = \mu &- \frac{1}{2}\lambda\mu\psi(\mu)\int_0^1 \frac{\psi(\mu')}{\mu + \mu'}\mu'd\mu' \\ &+ \frac{1}{2}x\lambda\mu\phi(\mu)\int_0^1 \frac{\phi(\mu')}{\mu + \mu'}\mu'd\mu'. \end{aligned} \tag{223}$$

And, it is found by direct verification that the solution of these equations can be expressed in the form

$$\psi(\mu) = H^{(0)}(\mu)(1 - c\mu), \tag{224}$$

and

$$\phi(\mu) = q\mu H^{(0)}(\mu), \tag{225}$$

where $H^{(0)}(\mu)$ now satisfies the functional equation

$$H^{(0)}(\mu) = 1 + \frac{1}{2}\lambda\mu H^{(0)}(\mu)\int_0^1 \frac{H^{(0)}(\mu')}{\mu+\mu'}[1 + x(1-\lambda)\mu'^2]d\mu', \tag{226}$$

and q and c are two constants related in the manner

$$xq^2 = c^2 + x(1-\lambda), \tag{227}$$

and given explicitly by the formulae

$$q = \frac{2(1-\lambda)}{2-\lambda\alpha_0} \quad\text{and}\quad c = x\lambda(1-\lambda)\frac{\alpha_1}{2-\lambda\alpha_0}, \tag{228}$$

α_0 and α_1 being the moments of order zero and one of $H^{(0)}(\mu)$.

34. **Some general remarks.** In some ways it is remarkable that systems of functional equations as complex in appearance as equations (196)–(199), (215)–(216) and (222)–(223) are should be capable of being reduced to single functional equations of the form (168). There must clearly be something in the structure of these equations which makes this reduction possible. But, as to what it precisely is, is at present shrouded in mystery!

35. **Some integral properties of the functions $H(\mu)$.** The discussion in the preceding sections has disclosed the important role which functional equations of the form

$$H(\mu) = 1 + \mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu+\mu'}\Psi(\mu')d\mu' \tag{229}$$

play in the theory of radiative transfer. The investigation of the properties of these equations is, therefore, a matter of considerable interest. Of course, from the practical standpoint of solving such equations numerically, the most important property is that derived from Theorem 1 (§24), namely, that when we replace the integral on the right-hand side by a Gauss sum, the solution can be explicitly written down as a rational function involving the points of the Gaussian division and the roots of the associated characteristic equation

$$1 = 2\sum_{j=1}^{n} \frac{a_j\Psi(\mu_j)}{1-k^2\mu_j^2}. \tag{230}$$

However, in addition to this property, there are a number of integral theorems (of an essentially elementary kind) which can be proved for functions satisfying equations of the form (229). We shall give two examples.

THEOREM 3. $\int_0^1 H(\mu)\Psi(\mu)d\mu = 1 - [1 - 2\int_0^1 \Psi(\mu)d\mu]^{1/2}$.

PROOF. Multiplying the equation satisfied by $H(\mu)$ by $\Psi(\mu)$ and integrating over the range of μ, we have

$$\begin{aligned}\int_0^1 & H(\mu)\Psi(\mu)d\mu \\ &= \int_0^1 \Psi(\mu)d\mu + \int_0^1\int_0^1 \frac{\mu}{\mu+\mu'} H(\mu)\Psi(\mu)H(\mu')\Psi(\mu')d\mu d\mu'.\end{aligned} \tag{231}$$

Interchanging μ and μ' in the double integral in equation (231) and taking the average of the two equations, we obtain

$$\begin{aligned}\int_0^1 & H(\mu)\Psi(\mu)d\mu \\ &= \int_0^1 \Psi(\mu)d\mu + \frac{1}{2}\int_0^1\int_0^1 H(\mu)\Psi(\mu)H(\mu')\Psi(\mu')d\mu d\mu'\end{aligned} \tag{232}$$

or, alternatively,

$$\frac{1}{2}\left[\int_0^1 H(\mu)\Psi(\mu)d\mu\right]^2 - \int_0^1 H(\mu)\Psi(\mu)d\mu + \int_0^1 \Psi(\mu)d\mu = 0. \tag{233}$$

Solving this equation for the integral in question, we have

$$\int_0^1 H(\mu)\psi(\mu)d\mu = 1 \pm \left[1 - 2\int_0^1 \Psi(\mu)d\mu\right]^{1/2}. \tag{234}$$

The ambiguity in the sign in equation (234) can be removed by the consideration that the integral on the left-hand side must uniformly converge to zero when $\Psi(\mu)$ tends to zero uniformly in the interval (0, 1). This requires us to choose the negative sign in equation (234) and the result stated follows.

COROLLARY. *A necessary and sufficient condition that $H(\mu)$ be real is*

$$\int_0^1 \Psi(\mu)d\mu \leq \frac{1}{2}\cdot$$

This is, of course, an immediate consequence of the theorem.

The physical meaning of the limitation on $\Psi(\mu)$ implied by this corollary is interesting: it is really equivalent to the condition that, on each scattering, more radiation should not be emitted than was incident; further, the equality sign is admissible only in the case of perfect scattering in the sense of equation (11).

THEOREM 4. $[1-2\int_0^1\Psi(\mu)d\mu]^{1/2}\int_0^1 H(\mu)\Psi(\mu)\mu^2d\mu+[\int_0^1 H(\mu)\Psi(\mu)\mu d\mu]^2/2 = \int_0^1\Psi(\mu)\mu^2d\mu$.

PROOF. To prove this theorem, we multiply the equation defining $H(\mu)$ by $\Psi(\mu)\mu^2$ and integrate over the range of μ. We find

$$\begin{aligned}
&\int_0^1 H(\mu)\Psi(\mu)\mu^2d\mu \\
&\quad = \int_0^1 \Psi(\mu)\mu^2d\mu + \int_0^1\int_0^1 \frac{H(\mu)H(\mu')\Psi(\mu)\Psi(\mu')}{\mu+\mu'}\mu^2d\mu d\mu' \\
&\quad = \int_0^1 \Psi(\mu)\mu^2d\mu \\
&\qquad + \frac{1}{2}\int_0^1\int_0^1 \frac{H(\mu)H(\mu')\Psi(\mu)\Psi(\mu')}{\mu+\mu'}(\mu^3+\mu'^3)d\mu d\mu' \\
&\quad = \int_0^1 \Psi(\mu)\mu^2d\mu \\
&\qquad + \frac{1}{2}\int_0^1\int_0^1 H(\mu)H(\mu')\Psi(\mu)\Psi(\mu')(\mu^2-\mu\mu'+\mu'^2)d\mu d\mu' \\
&\quad = \int_0^1 \Psi(\mu)\mu^2d\mu + \left[\int_0^1 H(\mu)\Psi(\mu)\mu^2d\mu\right]\left[\int_0^1 H(\mu)\Psi(\mu)d\mu\right] \\
&\qquad - \frac{1}{2}\left[\int_0^1 H(\mu)\Psi(\mu)\mu d\mu\right]^2.
\end{aligned} \tag{235}$$

Using Theorem 3 we obtain, after some minor reductions,

$$\begin{aligned}
&\left[1-2\int_0^1\Psi(\mu)d\mu\right]^{1/2}\int_0^1 H(\mu)\Psi(\mu)\mu^2d\mu \\
&\qquad\qquad + \frac{1}{2}\left[\int_0^1 H(\mu)\Psi(\mu)\mu d\mu\right]^2 = \int_0^1\Psi(\mu)\mu^2d\mu,
\end{aligned} \tag{236}$$

which is the required result.

COROLLARY. *For the case of a perfectly scattering atmosphere when*

$$\int_0^1 \Psi(\mu)d\mu = 1/2,$$

we have the further integral

$$\int_0^1 H(\mu)\Psi(\mu)\mu d\mu = \left[2\int_0^1 \Psi(\mu)\mu^2 d\mu \right]^{1/2}.$$

The corollary we have just stated generalizes a classical result of Hopf and Bronstein for the case of an isotropically scattering atmosphere. For, in this latter case

$$\Psi(\mu) = \text{constant} = 1/2, \tag{237}$$

and, according to Theorem 3 and the corollary of Theorem 4, we have

$$\int_0^1 H(\mu)d\mu = 2, \tag{238}$$

$$\int_0^1 H(\mu)\mu d\mu = \frac{2}{3^{1/2}}. \tag{239}$$

Hence

$$\frac{J(0)}{F} = \frac{\int_0^1 H(\mu)d\mu}{4\int_0^1 H(\mu)\mu d\mu} = \frac{3^{1/2}}{4}. \tag{240}$$

This is the Hopf-Bronstein relation. It, therefore, follows that for all cases of perfect scattering we have an integral of the Hopf-Bronstein type which is essentially that given by the corollary of Theorem 4.

36. **The equation of transfer in spherical atmospheres, and its reduction for the case $\kappa\rho\alpha r^{-n}$.** So far we have restricted ourselves to transfer problems in plane parallel atmospheres. We shall now briefly indicate how the methods we have described can be extended to treat transfer problems in spherically symmetric atmospheres. In such cases, the intensity in the radiation field will be a function of the distance r from the center of symmetry and the angle ϑ measured from the positive direction of the radius vector; and the equation of transfer will take the form

$$\mu\frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = -\kappa\rho(I - \mathfrak{J}), \tag{241}$$

where $\mu = \cos\vartheta$ and $\mathfrak{J}$ denotes, as usual, the source function.

In outlining the manner in which equations of transfer of the form (241) can be solved, we shall restrict ourselves to an isotropically scattering atmosphere. In this case equation (241) becomes

$$(242)\qquad \mu\frac{\partial I}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu}=-\kappa\rho I+\frac{1}{2}\kappa\rho\int_{-1}^{+1}I(r,\mu')d\mu'.$$

According to the ideas developed in the earlier parts of this lecture, we shall replace the integral which occurs on the right-hand side of equation (242) by a sum according to Gauss's formula for numerical quadratures, and reduce the integrodifferential equation to the system

$$(243)\qquad \mu_i\frac{dI_i}{dr}+\frac{1-\mu_i^2}{r}\left(\frac{\partial I}{\partial \mu}\right)_{\mu=\mu_i}=-\kappa\rho I_i+\frac{1}{2}\kappa\rho\sum a_jI_j$$
$$(i=\pm 1,\cdots,\pm n)$$

of $2n$ ordinary linear equations in the nth approximation. It is at once seen that our present system of equations differs, in an essential way, from those which we have considered so far: equation (243) now involves $(\partial I/\partial\mu)_{\mu=\mu_i}$ and, before we can proceed any further, we must know the values which we are to assign to $\partial I/\partial\mu$ at the points of the Gaussian division in our present scheme of approximation. At first sight it might be supposed that the assignment of values to $\partial I/\partial\mu$ at $\mu=\mu_i$, $i=\pm 1,\cdots,\pm n$, is largely arbitrary, particularly when n is small. However, on consideration, it appears that this assignment can be done in a satisfactory manner in only one way and, indeed, according to the following device:

Define the polynomials $Q_m(\mu)$ according to the formula

$$(244)\qquad P_m(\mu)=-\frac{dQ_m}{d\mu}\qquad (m=1,\cdots,2n),$$

and adjust the constant of integration in Q_m by requiring that

$$(245)\qquad Q_m=0\qquad \text{for } |\mu|=1.$$

This can always be accomplished, since, when m is odd, Q_m is even and when m is even the indefinite integral of $P_m(\mu)$ already contains $(1-\mu^2)$ as a factor. The first few polynomials $Q_m(\mu)$ are given below:

m	$P_m(\mu)$	$Q_m(\mu)$	$\mathfrak{Q}_m(\mu)$
1	μ	$(1-\mu^2)/2$	$1/2$
2	$(3\mu^2-1)/2$	$\mu(1-\mu^2)/2$	$\mu/2$
3	$(5\mu^3-3\mu)/2$	$1/8(5\mu^2-1)(1-\mu^2)$	$1/8(5\mu^2-1)$
4	$1/8(35\mu^4-30\mu^2+3)$	$1/8\mu(7\mu^2-3)(1-\mu^2)$	$1/8\mu(7\mu^2-3)$
5	$1/8(63\mu^5-70\mu^3+15\mu)$	$1/16(21\mu^4-14\mu^2+1)(1-\mu^2)$	$1/16(21\mu^4-14\mu^2+1)$

Now, by an integration by parts, we arrive at the identity

$$\int_{-1}^{+1} Q_m(\mu)\frac{\partial I}{\partial\mu}\,d\mu = -\int_{-1}^{+1} I\frac{dQ_m}{d\mu}\,d\mu = \int_{-1}^{+1} IP_m(\mu)\,d\mu. \tag{246}$$

Expressing the first and the last integrals in equation (246) as sums according to Gauss's formula, we have, in the nth approximation,

$$\sum a_i Q_m(\mu_i)\left(\frac{\partial I}{\partial\mu}\right)_{\mu=\mu_i} = \sum a_i I_i P_m(\mu_i) \qquad (m = 1, \cdots, 2n). \tag{247}$$

Equation (247) provides us with exactly the right number of equations to express $(\partial I/\partial\mu)_{\mu=\mu_i}$ $(i=\pm 1, \cdots, \pm n)$ as linear combinations of I_i. Essentially what equation (247) allows is to determine in a "best possible way" the derivatives of a function in terms of its values at the points of the Gaussian division. This problem has apparently not been considered before.

Returning to equation (243) we now observe that this equation, together with equation (246), provides the required reduction of the equation of transfer (242) to an equivalent system of linear equations.

For purposes of practical solution it appears most convenient to combine equations (243) and (247) in the following manner.

Since we have arranged $Q_m(\mu)$ to be divisible by $(1-\mu^2)$, we can clearly write

$$Q_m(\mu) = \mathfrak{Q}_m(\mu)(1-\mu^2). \tag{248}$$

The first few of the polynomials $\mathfrak{Q}_m(\mu)$ are listed in the preceding tabulation.

Now, multiply equation (243) by $a_i\mathfrak{Q}_m(\mu_i)$ and sum over all i's. We obtain

$$\begin{aligned} &\frac{d}{dr}\sum a_i\mu_i\mathfrak{Q}_m(\mu_i)I_i + \frac{1}{r}\sum a_i(1-\mu_i^2)\mathfrak{Q}_m(\mu_i)\left(\frac{\partial I}{\partial\mu}\right)_{\mu=\mu_i} \\ &\qquad = -\kappa\rho\sum a_i\mathfrak{Q}_m(\mu_i)I_i + \frac{1}{2}\kappa\rho\left(\sum a_iI_i\right)\left[\sum a_i\mathfrak{Q}_m(\mu_i)\right] \\ &\qquad\qquad (m = 1, \cdots, 2n). \end{aligned} \tag{249}$$

But, according to equations (247) and (248)

$$\begin{aligned} \sum a_i(1-\mu_i^2)\mathfrak{Q}_m(\mu_i)\left(\frac{\partial I}{\partial\mu}\right)_{\mu=\mu_i} &= \sum a_iQ_m(\mu_i)\left(\frac{\partial I}{\partial\mu}\right)_{\mu=\mu_i} \\ &= \sum a_iP_m(\mu_i)I_i. \end{aligned} \tag{250}$$

Equation (249), therefore, reduces to

$$\frac{d}{dr}\sum a_i\mu_i\mathfrak{Q}_m(\mu_i)I_i+\frac{1}{r}\sum a_iP_m(\mu_i)I_i = -\kappa\rho\sum a_i\mathfrak{Q}_m(\mu_i)I_i+\frac{1}{2}\kappa\rho\left(\sum a_iI_i\right)\left[\sum a_i\mathfrak{Q}_m(\mu_i)\right] \tag{251}$$

$$(m = 1, \cdots, 2n).$$

This is the required system of linear equations in the nth approximation.

Equation (251) for the case $m=1$ admits of immediate integration. For, when $m=1$

$$P_1(\mu) = \mu \quad \text{and} \quad \mathfrak{Q}_1(\mu) = 1/2, \tag{252}$$

and equation (251) yields

$$\frac{1}{2}\frac{d}{dr}\sum a_i\mu_iI_i+\frac{1}{r}\sum a_i\mu_iI_i = 0 \tag{253}$$

or

$$\sum a_i\mu_iI_i = \frac{1}{2}\frac{F_0}{r^2}, \tag{254}$$

where F_0 is a constant. This is the equivalent, in our approximation, of the flux integral

$$F = 2\int_{-1}^{+1} I\mu d\mu = \frac{F_0}{r^2}, \tag{255}$$

which the equation of transfer (242) admits directly.

Again, since $\mathfrak{Q}_m(\mu)$ is odd when m is even, equation (251) reduces for even values of m to the form

$$\frac{d}{dr}\sum a_i\mu_i\mathfrak{Q}_m(\mu_i)I_i+\frac{1}{r}\sum a_iP_m(\mu_i)I_i = -\kappa\rho\sum a_i\mathfrak{Q}_m(\mu_i)I_i \tag{256}$$

$$(m = 2, 4, \cdots, 2n).$$

For $m=2n$, the foregoing equation further simplifies to

$$\frac{d}{dr}\sum a_i\mu_i\mathfrak{Q}_{2n}(\mu_i)I_i = -\kappa\rho\sum a_i\mathfrak{Q}_{2n}(\mu_i)I_i. \tag{257}$$

Finally, we may note the explicit forms of the equations in the second approximation. They are

$$\sum a_i\mu_i I_i = \frac{1}{2}\frac{F_0}{r^2},$$

$$\frac{d}{dr}\sum a_i\mu_i^2 I_i + \frac{1}{r}\sum a_i(3\mu_i^2 - 1)I_i = -\kappa\rho\sum a_i\mu_i I_i,$$

$$(258)\quad \frac{d}{dr}\sum a_i\mu_i(5\mu_i^2 - 1)I_i + \frac{4}{r}\sum a_i\mu_i(5\mu_i^2 - 3)I_i = -\frac{5}{3}\kappa\rho\sum a_i(3\mu_i^2 - 1)I_i,$$

$$\frac{d}{dr}\sum a_i\mu_i^2(7\mu_i^2 - 3)I_i = -\kappa\rho\sum a_i\mu_i(7\mu_i^2 - 3)I_i.$$

From the point of view of astrophysical applications, greatest interest is attached to the case when $\kappa\rho$ varies as some inverse of power. And when

$$(259)\qquad \kappa\rho = \frac{\text{constant}}{r^n} \qquad (n > 1),$$

the equations of the second approximation (258) can be solved explicitly and the various physical quantities expressed as integrals over the Bessel functions, I_ν and K_ν, of purely imaginary argument, and of order

$$(260)\qquad \nu = \frac{n + 5}{2(n - 1)}.$$

It does not, however, seem that the passage to the limit of infinite approximation can be achieved as simply as in the case of transfer problems in plane parallel atmospheres.

37. **The equation of transfer in a differentially moving atmosphere. The influence of Doppler effect.** Finally, we shall turn to a class of transfer problems which is of an altogether different character from the ones we have considered so far. The problems in question arise in connection with the study of the transfer of radiation in atmospheres in which the different parts are in relative motion. To be specific, consider a plane-parallel atmosphere in which the material at height z has a velocity $w(z_1)$ with respect to a stationary observer. The novelty of the situation arises on account of Doppler effect which makes the radiation scattered in different directions have different frequencies

as judged by a stationary observer. Consequently, the radiation field in the different frequencies will interact with each other in a manner which is not always easy to visualize. However, in the astrophysical contexts, two circumstances simplify the problem. First, the velocities which are involved are small compared to the velocity of light, c, and second, the only effects of consequence are those which arise from the sensitive dependence of the scattering coefficient $\sigma(\nu)$ on frequency. This last circumstance, in particular, allows us to ignore all effects such as aberration and so on, and concentrate only on the effects arising from the change of frequency on scattering. Under these conditions, the equation of transfer can be shown to take the form

$$\begin{aligned}\mu\frac{\partial I(\nu, z, \mu)}{\rho\partial z} = -\sigma\left(\nu - \nu_0\frac{w}{c}\mu\right)\Big\{&I(\nu, z, \mu) \\ &-\frac{1}{2}\int_{-1}^{+1} I\left(\nu - \nu_0\frac{w}{c}\mu + \nu_0\frac{w}{c}\mu', z, \mu'\right)d\mu'\Big\},\end{aligned} \tag{261}$$

where ν_0 denotes the frequency of the "center of the line."

With suitable simplifying assumptions concerning $\sigma(\nu)$ and $w(z)$, the discussion of the equation of transfer (261), in the first approximation in our scheme of replacing integrals by Gauss sums, leads to a variety of novel types of boundary value problems in hyperbolic equations. It may be of some interest to specify the nature of these boundary value problems and indicate the methods which have been developed for their solution.

38. **A new class of boundary value problems in hyperbolic equations.** As related to the equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + f = 0 \tag{262}$$

the boundary value problems which are of most frequent occurrence are of the following general type:

The value of f and its derivatives are assigned for

$$y = 0 \quad \text{and} \quad 0 \leq x \leq l_1, \tag{263}$$

that is, along AB in Fig. 2. Along AD ($x=0$ and $0 \leq y \leq l_2$) and BC ($x=l_1$, $0 \leq y \leq l_2$), we are further given that

$$\left(\frac{\partial f}{\partial x}\right)_{x=0} = \left(\frac{\partial f}{\partial y}\right)_{x=0} + \phi(y) \qquad (0 \leq y \leq l_2), \tag{264}$$

$$f(l_1, y) = \psi(y) \qquad (0 \leq y \leq l_2) \tag{265}$$

where $\phi(y)$ and $\psi(y)$ are two known functions. The problem is to solve equation (1) in the rectangular strip $ABCD$ satisfying the stated boundary conditions. For the particular boundary value problems which occur in the astrophysical contexts, the following "systematic method" of solution has been found convenient.

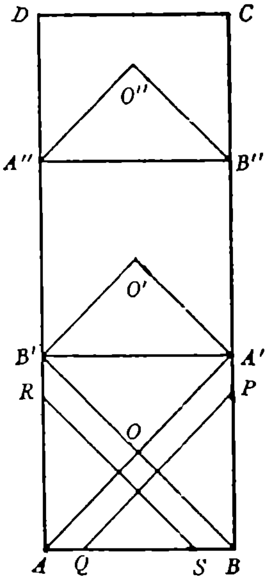


Fig. 2.

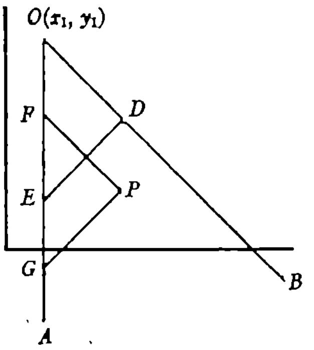


Fig. 3.

Let the characteristics $x = y$ and $l_1 - x = y$ through A and B intersect BC and AD at A' and B', respectively. Further, let AA' and BB' intersect at O. First, since the function and its derivatives are known along AB, we can find the solution inside and on the sides of the triangle OAB directly by Riemann's well known method. Next we use the requirement that the solution be unambiguously defined along AO and OB together with the boundary conditions specified along AD and BC to determine f along AB' and $\partial f/\partial x$ along BA' as solutions of certain integral equations of Volterra's type. With the knowledge of the function and its derivatives thus completed along the part $B'ABA'$ of the "supporting curve" $ABCD$, the solution inside the entire region $O'B'ABA'$ becomes determinate by Riemann's method. In particular, the function and its derivatives along $B'A'$ can be found and the continuation of the solution in the second square $B'A'B''A''$ follows along similar lines.

While the method outlined above shows how solutions satisfying the given boundary conditions can be found in principle, it suffers from the disadvantage that the method of solution depends on solving a succession of Volterra integral equations; and, unless the boundary conditions specified along $DABC$ are specially simple, we should not expect to go very far in the explicit carrying out of the solution by this method. It would, therefore, be useful if an alternative method of solution can be devised which will eliminate the need of solving integral equations and reduce the practical problem to one involving (at most!) only quadratures. It is remarkable that this can actually be accomplished by constructing suitable Green's functions and applying Green's theorem to contours, such as RAS and PQB.

39. **The Green's functions $C(x, y; x_1, y_1)$ and $\Gamma(x, y; x_1, y_1)$.** It is found that Green's functions which are appropriate for the solution of the boundary value problems of the type formulated in §38 are $C(x, y; x_1, y_1)$ and $\Gamma(x, y; x_1, y_1)$, defined as follows:

$C(x, y; x_1, y_1)$ is a solution of the hyperbolic equation (262) which satisfies the boundary conditions

$$C(x_1, y; x_1, y_1) = 1 \qquad (y \leq y_1) \tag{266}$$

and

$$C(x, x_1 + y_1 - x; x_1, y_1) = 1 \qquad (x \geq x_1). \tag{267}$$

In other words, if O represents the point (x_1, y_1), OB the characteristic $x+y=x_1+y_1$ through O and OA the line through O parallel to the y-axis, then the boundary conditions require that C take the value 1 along both OA and OB (see Fig. 3). A solution satisfying these boundary conditions can be found explicitly. It is given by

$$\begin{aligned} C(x, y; x_1, y_1) = \cos(x - x_1) + \frac{1}{2}(x - x_1)\int_0^\pi J_0([x - x_1]\sin\vartheta) \\ \times Ii_1(y_1 - y - [x - x_1]\cos\vartheta)\sin\vartheta d\vartheta, \end{aligned} \tag{268}$$

where J_0 denotes the Bessel function of order zero and $Ii_1(z)$ the "Bessel integral"

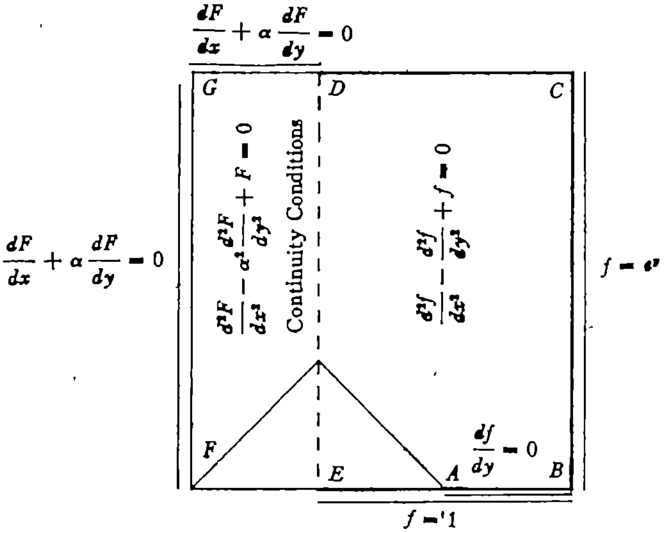
$$Ii_1(z) = \int_0^z \frac{I_1(t)}{t}dt. \tag{269}$$

($I_1(t)$ denotes the Bessel function of order 1 for a purely imaginary argument.)

The second function, $\Gamma(x, y; x_1, y_1)$, is defined in terms of

$C(x, y; x_1, y_1)$ according to the formula

$$\Gamma(x, y; x_1, y_1) = \frac{\partial C}{\partial x} - \frac{\partial C}{\partial y}. \tag{270}$$


$\frac{dF}{dx} + \alpha\frac{dF}{dy} = 0$
G
D
C
$\frac{d^2F}{dx^2} - \alpha^2\frac{d^2F}{dy^2} + F = 0$
Continuity Conditions
$\frac{d^2f}{dx^2} - \frac{d^2f}{dy^2} + f = 0$
$\frac{dF}{dx} + \alpha\frac{dF}{dy} = 0$
$f = e^y$
F
E
A
B
$\frac{df}{dy} = 0$
$f = 1$

Continuity Conditions:

$$e^{-y}f = e^{y/\alpha}F$$

$$e^{-y}\left(\frac{df}{dx} - \frac{df}{dy}\right) = e^{y/\alpha}\left(\frac{dF}{dr} + \alpha\frac{dF}{dy}\right)$$

FIG. 4.

40. **Some further boundary problems.** The boundary value problem formulated in §38 does not exhaust the type of problems which occur in theory of radiative transfer in moving atmospheres. However, no progress has so far been made in the solution of these other problems. It may, therefore, be of particular interest to describe here the nature of these more complex boundary value problems.

A typical problem is to solve (see Fig. 4)

$$\frac{\partial^2 F}{\partial x^2} - \alpha^2\frac{\partial^2 F}{\partial y^2} + F = 0 \qquad (\alpha = \text{constant}), \tag{271}$$

for F in the rectangular strip $FEDG$ $(0 \leqq x \leqq x_1,\ 0 \leqq y \leqq y_1)$, and

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + f = 0, \tag{272}$$

for f in the rectangular strip $EBCD$ ($x_2 \leqq x \leqq x_1$, $0 \leqq y \leqq y_1$) satisfying the following boundary conditions:

Along FG ($x=0$, $0 \leqq y \leqq y_1$) and GD ($y=y_1$, $0 \leqq x \leqq x_2$) relations of the form

$$\left(\frac{\partial F}{\partial x} + \alpha \frac{\partial F}{\partial y}\right)_{x=0} = \phi(y) \qquad (0 \leqq y \leqq y_1) \tag{273}$$

and

$$\left(\frac{\partial F}{\partial x} + \alpha \frac{\partial F}{\partial y}\right)_{y=y_1} = \psi(x) \qquad (0 \leqq x \leqq x_2) \tag{274}$$

are specified where $\phi(y)$ and $\psi(x)$ are two known functions. Along EB ($y=0$, $x_2 \leqq x \leqq x_1$) and BC ($x=x_1$, $0 \leqq y \leqq y_1$) f is given, while along the part AB ($x^* = x_2(1+\alpha) \leqq x \leqq x_1$, $y=0$) of the x-axis, the derivative $\partial f/\partial y$ is also given. And, finally, along ED ($x=x_2$, $0 \leqq y \leqq y_1$) certain "continuity conditions" of the type

$$f(x_2, y) = Q(y)F(x_2, y) \qquad (0 \leqq y \leqq y_1) \tag{275}$$

and

$$\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)_{x=x_2} = Q(y)\left(\frac{\partial F}{\partial x} + \alpha \frac{\partial F}{\partial y}\right)_{x=x_2} \qquad (0 \leqq y \leqq y_1) \tag{276}$$

are specified, where $Q(y)$ is a known function.

It would be of considerable interest to know how such boundary value problems can be solved. (In Fig. 4 the particular boundary conditions which occur in a specific problem are indicated.)

41. **Concluding remarks.** In concluding, I may recall what I said at the beginning, namely, that the advance of a branch of theoretical physics often leads to the creation of a new mathematical discipline. I think it may be conversely said, with almost equal truth, that the creation of a new mathematical discipline is often the sign that the particular branch of theoretical physics has reached maturity. I hope I have given you the impression that theoretical astrophysics has now come of age.

Since the lecture was given in December, the mathematical theory of radiative transfer has advanced along several directions and in this Addendum we shall briefly summarize the results of these newer investigations. The particular sections of the lecture to which these advances refer are indicated by the numbering of the paragraphs which follow; however, §42 breaks new ground not covered by the lecture.

11a. **Elliptically polarized radiation field.** In §11 we outlined how the equations of transfer for a partially plane-polarized radiation field can be formulated and gave the explicit form of these equations for the particular case of Rayleigh scattering. It is not difficult to extend this discussion to include the case of a general elliptic polarization of the radiation field. In this latter case, we must consider, in addition to the intensities I_l and I_r in two directions at right angles to each other in the plane of the electric and the magnetic vectors, the two further quantities

$$(277) \qquad U = (I_l - I_r) \tan 2\chi \text{ and } V = (I_l - I_r) \tan 2\beta \sec 2\chi$$

where χ denotes the inclination of the plane of polarization to the direction to which l refers and $-\pi/2 \leqq \beta \leqq +\pi/2$ is an angle the tangent of which is equal to the ratio of axes of the ellipse characterizing the state of polarization. (The sign of β depends on whether the polarization is right-handed (+) or left-handed (−).) And the rule of composition (due to Stokes) is that a mixture of several *independent* streams of polarized light is characterized by values of the parameters I_l, I_r, U and V which are the sums of the respective parameters of the individual streams. With this rule of composition, the equations of transfer for I_l, I_r, U and V can be formulated in terms of the basic laws of single scattering.

For the case of Rayleigh scattering, it is found that the equations of transfer for I_l, I_r and U are of exactly the same forms as when, only, plane polarization is contemplated; that is, in the case of Rayleigh scattering, the equations are *reducible*. Moreover, it is found that for Rayleigh scattering, V is simply scattered in accordance with a phase function $3/2 \cos \Theta$; and the exact solution of the equation for V therefore presents no difficulty.

23a. **The functional equation relating the law of darkening and the scattering function for semi-infinite plane-parallel atmospheres.** In §23 we have remarked on the remarkable relationships between the angular distributions of the emergent radiation in the problem with a constant net flux and the law of diffuse reflection. It has since been possible to trace the origin of these relationships: they arise simply in consequence of the invariance of the emergent radiation $I(0, \mu)$ (in the problem with a constant net flux) to the addition (or removal) of layers of arbitrary thickness to (or from) the atmosphere. The mathematical expression of this invariance is that the outward radiation $I(\tau, +\mu)$ $(0<\mu<1)$ at any level τ differs from the emergent radiation $I(0, \mu)$ only on account of the fact that at τ there is an inward directed radiation field specified by $I(\tau, -\mu')$ $(0<\mu'<1)$ which will be reflected by the atmosphere below τ by the law of diffuse reflection of a semi-infinite atmosphere. In other words, we must have

$$I(\tau, +\mu) = I(0, \mu) + \frac{1}{2\mu}\int_0^1 S^{(0)}(\mu, \mu')I(\tau, -\mu')d\mu', \tag{278}$$

where

$$S^{(0)}(\mu, \mu') = \frac{1}{2\pi}\int_0^1 S(\mu, \phi; \mu', \phi')d\phi' \tag{279}$$

is the azimuth independent term in the scattering function $S(\mu, \phi; \mu', \phi')$ (cf. equation (26)).

Differentiating equation (278) with respect to τ and passing to the limit $\tau=0$, we obtain

$$\left[\frac{dI(\tau, +\mu)}{d\tau}\right]_{\tau=0} = \frac{1}{2\mu}\int_0^1 S^{(0)}(\mu, \mu')\left[\frac{dI(\tau, -\mu')}{d\tau}\right]_{\tau=0} d\mu'. \tag{280}$$

On the other hand, from the equation of transfer

$$\mu\frac{dI}{d\tau} = I(\tau, \mu) - B(\tau, \mu) \tag{281}$$

where $B(\tau, \mu)$ is the source function appropriate for the problem (see equations I, II and VI) we conclude that

$$\left[\frac{dI(\tau, +\mu)}{d\tau}\right]_{\tau=0} = \frac{1}{\mu}\left[I(0, \mu) - B(0, \mu)\right], \tag{282}$$

and

$$\left[\frac{dI(\tau, -\mu')}{d\tau}\right]_{\tau=0} = \frac{1}{\mu'}B(0, -\mu'). \tag{283}$$

Now combining equations (280), (282) and (283), we obtain

$$I(0, \mu) = B(0, \mu) + \frac{1}{2}\int_0^1 S^{(0)}(\mu, \mu')B(0, -\mu')\frac{d\mu'}{\mu'}, \tag{284}$$

which is a functional equation relating $I(0, \mu)$ and $S^{(0)}(\mu, \mu')$; it can be shown that it is precisely in consequence of this equation that the relationship between $I(0, \mu)$ and $S^{(0)}(\mu, \mu')$ noticed in §23 arises.

35a. **Representation of $H(\mu)$ as a complex integral.** According to equations (102) and (110)

$$1-\sum_{j=1}^{n}\frac{a_j}{1-\mu_j^2/z^2}=\frac{1}{H(z)H(-z)}\tag{285}$$

where $H(z)$ is defined as usual in terms of the roots of the characteristic equation (see equation (49))

$$1=\sum_{j=1}^{n}\frac{a_j}{1-\mu_j^2k^2}.\tag{286}$$

The arguments leading to equation (285) (§20) are seen to be sufficiently general to establish the identity

$$\begin{aligned}1-2\sum_{j=1}^{n}\frac{a_j\Psi(\mu_j)}{1-\mu_j^2/z^2}&=\frac{1}{H(z)H(-z)}\\&=\frac{\prod_{\alpha}(1-k_\alpha^2z^2)}{\prod_{j}(1-z^2/\mu_j^2)},\end{aligned}\tag{287}$$

where $H(z)$ is now defined as in equation (126) in terms of the roots of the characteristic equation

$$1=2\sum_{j=1}^{n}\frac{a_j\Psi(\mu_j)}{1-k^2\mu_j^2},\tag{288}$$

and $\Psi(\mu)$ has the same meaning as in equation (127).

Now let

$$G(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log T(z)\,\frac{xdz}{z^2-x^2},\tag{289}$$

where

$$T(z)=1-2z^2\sum_{j=1}^{n}\frac{a_j\Psi(\mu_j)}{z^2-\mu_j^2}=\frac{\prod_{\alpha}(1-k_\alpha^2z^2)}{\prod_{j}(1-z^2/\mu_j^2)}.\tag{290}$$

It is seen that defined in this manner $G(x)$ is regular for $R(x)>0$.

By evaluating the residue at the pole on the right

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log\left(1+\frac{z}{a}\right)\frac{xdz}{z^2-x^2}=-\frac{1}{2}\log\left(1+\frac{x}{a}\right),\tag{291}$$

if $R(x)>0$ and $R(a)>0$. Similarly, by evaluating the residue at the pole on the left, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log\left(1-\frac{z}{a}\right)\frac{x\,dz}{z^2-x^2}=-\frac{1}{2}\log\left(1+\frac{x}{a}\right). \tag{292}$$

Hence,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log\left(1-\frac{z^2}{a^2}\right)\frac{x\,dz}{z^2-x^2}=-\log\left(1+\frac{x}{a}\right). \tag{293}$$

Accordingly

$$\begin{aligned} G(x) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left\{\sum_{\alpha=1}^{n}\log\left(1-k_\alpha^2 z^2\right)-\sum_{j=1}^{n}\log\left(1-\frac{z^2}{\mu_j^2}\right)\right\}\frac{x\,dz}{z^2-x^2} \\ &= -\sum_{\alpha=1}^{n}\log\left(1+xk_\alpha\right)+\sum_{j=1}^{n}\log\left(1+\frac{z}{\mu_j}\right) \\ &= \log H(z). \end{aligned} \tag{294}$$

We have thus shown that

$$\log H(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log T(z)\,\frac{x\,dz}{z^2-x^2}\,. \tag{295}$$

From the representation (295) of the H-function as a complex integral, it would appear that the solution of the *functional equation*

$$H(\mu)=1+\mu H(\mu)\int_0^1\frac{H(\mu')}{\mu+\mu'}\Psi(\mu')d\mu' \tag{296}$$

has the representation (cf. Theorem 2, §25)

$$\log H(\mu)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log T(z)\,\frac{\mu\,dz}{z^2-\mu^2}\,, \tag{297}$$

where, now (see equation (290)),

$$T(z)=1-2z^2\int_0^1\frac{\Psi(\mu)d\mu}{z^2-\mu^2}\,. \tag{298}$$

Our arguments do not of course establish rigorously the representation (297) of the solution of equation (296). However, Professor E. C. Titchmarsh, with whom I have corresponded, has kindly communicated to me a rigorous demonstration of the representation (297) by one of his colleagues, Mr. M. M. Crum.

42. The theory of radiative transfer in atmospheres of finite optical thicknesses. In the lecture attention was directed almost exclusively to transfer problems in semi-infinite plane-parallel atmospheres. The extension of this theory to the study of the transfer of radiation in plane-parallel atmospheres of finite optical thicknesses raises problems of a higher order of difficulties; these difficulties arise principally from the circumstance that boundary conditions have to be *explicitly* satisfied on both sides of the atmosphere. Thus, if we consider an atmosphere of optical thickness τ_1 $(< \infty)$, we are interested, for example, in solutions of the equations of transfer I–VII which satisfy the boundary conditions

$$I(0, -\mu) = 0 \qquad (0 < \mu < 1), \tag{299}$$

and

$$I(\tau_1, +\mu) = 0 \qquad (0 < \mu < 1). \tag{300}$$

However, in analogy with the theory of semi-infinite atmospheres, we may expect that in the case of finite atmospheres, also, the angular distributions of the emergent radiations can be expressed in terms of functions (like $H(\mu)$) which will be explicitly known in any finite approximation and which, in the limit of infinite approximation, will become solutions of functional equations of a standard form. It now appears that this reduction can in fact be achieved and that the pair of functional equations

$$X(\mu) = 1 + \mu \int_0^1 \frac{X(\mu)X(\mu') - Y(\mu)Y(\mu')}{\mu + \mu'} \Psi(\mu')d\mu', \tag{301}$$

and

$$Y(\mu) = e^{-\tau_1/\mu} + \mu \int_0^1 \frac{Y(\mu)X(\mu') - X(\mu)Y(\mu')}{\mu - \mu'} \Psi(\mu')d\mu', \tag{302}$$

where $\Psi(\mu)$ is an even polynomial in μ satisfying the condition

$$\int_0^1 \Psi(\mu)d\mu < \frac{1}{2}, \tag{303}$$

plays the same basic role in the theory of atmospheres of finite optical thicknesses as the functional equation

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{H(\mu')}{\mu + \mu'} \Psi(\mu')d\mu' \tag{304}$$

played in the theory of semi-infinite atmospheres. And just as

$$H(\mu) = \frac{1}{\mu_1 \cdots \mu_n} \frac{\prod_i (\mu + \mu_i)}{\prod_\alpha (1 + k_\alpha \mu)}, \tag{305}$$

where the k_α's are the positive roots of the characteristic equation

$$1 = 2 \sum_{j=1}^{n} \frac{a_j \Psi(\mu_j)}{1 - k^2 \mu_j^2}, \tag{306}$$

provides a rational approximation to the solution of equation (304), so also the functions $X(\mu)$ and $Y(\mu)$ defined in the manner of the following equations provides an approximation to the solution of equations (301) and (302):

$$X(\mu) = \frac{(-1)^n}{\mu_1 \cdots \mu_n} \frac{1}{[C_0^2(0) - C_1^2(0)]^{1/2}} \frac{1}{W(\mu)} \cdot [P(-\mu) C_0(-\mu) - e^{-\tau_1/\mu} P(\mu) C_1(\mu)] \tag{307}$$

$$Y(\mu) = \frac{(-1)^n}{\mu_1 \cdots \mu_n} \frac{1}{[C_0^2(0) - C_1^2(0)]^{1/2}} \frac{1}{W(\mu)} \cdot [e^{-\tau_1/\mu} P(\mu) C_0(\mu) - P(-\mu) C_1(-\mu)] \tag{308}$$

where

$$P(\mu) = \prod_{i=1}^{n} (\mu - \mu_i), \qquad W(\mu) = \prod_{\alpha=1}^{n} (1 - k_\alpha^2 \mu^2), \tag{309}$$

$$C_0(\mu) = \sum_{2^{n-1} \text{ terms}} \epsilon_{2l}^{(0)} \frac{\prod_{i=1}^{2l} \prod_{m=1}^{n-2l} (k_{r_i} + k_{s_m})}{\prod_{i=1}^{2l} \prod_{m=1}^{n-2l} (k_{r_i} - k_{s_m})} \times \prod_{i=1}^{2l} \xi_{r_i} (1 + k_{r_i} \mu) \prod_{m=1}^{n-2l} \eta_{s_m} (1 - k_{s_m} \mu), \tag{310}$$

$$C_1(\mu) = \sum_{2^{n-1} \text{ terms}} \epsilon_{2l+1}^{(1)} \frac{\prod_{i=1}^{2l+1} \prod_{m=1}^{n-2l-1} (k_{r_i} + k_{s_m})}{\prod_{i=1}^{2l+1} \prod_{m=1}^{n-2l-1} (k_{r_i} - k_{s_m})} \times \prod_{i=1}^{2l+1} \xi_{r_i} (1 + k_{r_i} \mu) \prod_{m=1}^{n-2l-1} \eta_{r_i} (1 - k_{s_m} \mu). \tag{311}$$

In equations (310) and (311), $(r_1, \cdots, r_j)$ and $(s_1, \cdots, s_{n-j})$ are j, respectively, $n-j$ *distinct* integers from the set $(1, 2, \cdots, n)$,

$$\epsilon_{2l}^{(0)} = \begin{cases} +1 \text{ for even integers of the form } 4l, \\ -1 \text{ for even integers of the form } 4l+2, \end{cases} \tag{312}$$

$$\epsilon_{2l+1}^{(1)} = \begin{cases} +1 \text{ for odd integers of the form } 4l+1, \\ -1 \text{ for odd integers of the form } 4l+3, \end{cases} \tag{313}$$

and

$$\xi_\alpha = e^{k_\alpha \tau_1/2} P(-1/k_\alpha) \quad \text{and} \quad \eta_\alpha = e^{-k_\alpha \tau_1/2} P(+1/k_\alpha) \qquad (\alpha = 1, \cdots, n). \tag{314}$$

Finally, it should be noted that the definitions of $C_0(\mu)$ and $C_1(\mu)$ according to equations (307) and (308) are valid only in even orders of approximation; in odd orders the role of C_0 and C_1 should be interchanged.

Moreover, there exist also functional equations for the scattering and the transmission functions for the problem of diffuse reflection and transmission by atmospheres of finite optical thicknesses. These equations arise from general invariances of the type considered in §§28, 29 and 23a and lead to a whole new class of systems of functional equations which can all be reduced to the solution of pairs of functional equations of the form (301) and (302). It is therefore apparent that the study of the transfer of radiation in atmospheres of finite optical thicknesses will lead to the development of a mathematical theory at least as extensive as the one described in the lecture in the context of semi-infinite atmospheres.

REFERENCES

The following papers and monographs substantially form the basis of the report given in the lecture. The list is not intended to be a complete bibliography of the subject of the transfer of radiation in stellar atmospheres.

V. A. AMBARZUMIAN

C. R. (Doklady) Acad. Sci. URSS. vol. 38 (1943) p. 257.

Russian Astronomical Journal vol. 19 (1942).

Journal of Physics of the Academy of Sciences of the U.S.S.R. vol. 8 (1944) p. 65.

S. CHANDRASEKHAR

Astrophysical Journal vol. 99 (1944) p. 180; vol. 100 (1944) pp. 76, 117, 355; vol. 101 (1945) pp. 95, 328, 348; vol. 102 (1945) p. 402; vol. 103 (1946) pp. 165, 351; vol. 104 (1946) pp. 110, 191; vol. 105 (1947) pp. 151, 164, 424, 435, 441, 461. (Further papers in this series are in course of publication in the same journal.)

Reviews of Modern Physics vol. 17 (1945) p. 138.
Proc. Chambridge Philos. Soc. vol. 42 (1946) p. 250.
E. Hopf
Mathematical problems of radiative equilibrium, Cambridge Mathematical Tracts, No. 31, Cambridge, England, 1934.
G. Münch
Astrophysical Journal vol. 104 (1946) p. 87.
Lord Rayleigh
Scientific papers of Lord Rayleigh, vol. 1, Cambridge, England, 1899, pp. 87, 104, 518.
Sir George Stokes
Trans. Cambridge Philos. Soc. vol. 9 (1852) p. 399.
M. Tuberg
Astrophysical Journal vol. 103 (1946) p. 145.
G. C. Wick
Zeitschrift für Physik vol. 120 (1943) p. 702.

Yerkes Observatory, University of Chicago

THE JANUARY MEETING ON FLUID DYNAMICS

The first program of the Committee on Fluid Dynamics of the American Physical Society was held at Columbia University on January 30, 31 and February 1, 1947. The American Mathematical Society acted as a coösponsor. These meetings of the Committee on Fluid Dynamics continue a series of closed meetings conducted during the war under the sponsorship of the Bureau of Ordnance of the U. S. Navy. The total attendance at the meeting was approximately three hundred. An audience survey on Thursday afternoon showed that the following 70 members of the American Mathematical Society were present:

Valentine Bargmann, Shepard Bartnoff, Stefan Bergman, Lipman Bers, Garrett Birkhoff, H. W. Bode, E. M. Boggs, T. E. Caywood, L. L. Cronvich, W. D. Duthie, Nat Edmonson, K. O. Friedrichs, W. J. Fry, Abe Gelbert, J. H. Giese, H. E. Goheen, N. A. Hall, Gerald Harrison, E. L. Haynes, A. D. Hestenes, J. H. Hett, Banesh Hoffmann, J. S. Isenberg, Aida Kalish, B. O. Koopman, P. A. Lagerstrom, Solomon Lefschetz, Joseph Lehner, B. A. Lengyel, Benjamin Lepson, C. C. Lin, D. P. Ling, Charles Loewner, E. R. Lorch, A. N. Lowan, J. K. L. MacDonald, A. K. Mitchell, Vladimir Morkovin, E. N. Nilson, G. H. Peebles, C. L. Pekeris, W. H. Pell, F. V. Pohle, Harry Polachek, Hillel Poritsky, William Prager, N. A. Renzetti, Daniel Resch, Moses Richardson, J. K. Riess, J. H. Rosenbloom, Arthur Sard, Hyman Serbin, Max Shiffman, S. S. Shu, Joseph Slepian, R. J. Slutz, T. L. Smith, George Springer, Benjamin Tannenbaum, A. H. Taub, J. E. Thompson, C. A. Truesdell, Y. W. Tschen, H. S. Tsien, J. W. Tukey, J. V. Wehausen, F. J. Weyl, H. A. Wood, A. W. Wundheiler.

Invited papers F1 to F6 were presented at the Thursday morning session, at which Professor H. W. Emmons presided. In the afternoon three invited addresses were given on aspects of *Recent developments in compressible flow* by Professors John von Neumann, G. F. J. Temple (of the Aeronautical Research Council of Great Britain) and Theodore von Kármán. Dr. H. L. Dryden presided.

Invited papers G1 to G6 were presented at the Friday morning session at which Dr. E. U. Condon presided. Contributed papers H1 to H13 were presented at the Saturday morning session at which Dr. R. J. Seeger presided.

Abstracts of all lettered papers except G4, G5 and G6 appeared in the Bulletin of the American Physical Society for January 30, 1947.

Of the 34 persons participating as speakers or chairmen, 9 were members of the American Mathematical Society.

F1. Rudolf Ladenburg: *Interferometry of faster-than-sound phenomena.*

F2. A. H. Taub: *Refraction of shocks in ideal gases.*

F3. H. W. Liepmann: *On the interaction between boundary layer and shock waves.*

F4. Arthur Kantrowitz: *The formation and stability of normal shock waves in channel flows.*

F5. Francis Clauser: *Notes on the thermodynamic propulsion of wind tunnels.*

F6. W. F. Hilton: *Variation of aerodynamic lift in the transonic speed range.*

G1. Harold Wayland: *Scale factors in water entry.*

G2. Garrett Birkhoff and T. E. Caywood: *Fluid flow patterns.*

G3. R. W. Wood: *The interaction of shock-waves, the spectra of cavity charge explosions, and the production of artificial meteors.*

G4. H. W. Emmons: *The solution of transonic fluid flow problems.*

G5. C. C. Lin: *Some recent investigations in hydrodynamic stability.*

G6. J. W. Tukey: *Linearization of solutions in supersonic flow.*

H1. H. A. Einstein: *On the hydraulic analogy between surface waves on a liquid and pressure waves in a compressible fluid.*

H2. Valentine Bargmann: *On nearly glancing reflection of shocks.*

H3. J. H. Giese: *Compressible flows with degenerate hodographs.*

H4. Stefan Bergman: *On an operator method in the theory of two-dimensional flows of a compressible fluid.*

H5. Zdenek Kopal: *Some remarks on the limitations of linearized theory of supersonic flow around cones.*

H6. F. L. Whipple: *Supersonic problems in astronomy.*

H7. R. N. Thomas: *Comments on the motion of a solid body at high velocity through a compressible medium.*

H8. H. K. Skramstad and G. B. Schubauer: *Laminar boundary-layer oscillations and stability of laminar flow.*

H9. Albert May and Jean C. Woodhull: *Drag coefficient of steel spheres entering water.*

H10. J. H. McMillan and E. N. Harvey: *Drag and cavity of fast spheres shot into water.*

H11. P. N. Metzelaar and D. E. Allmand: *Impact pressure produced by flat plate striking water surface.*

H12. B. L. Hicks: *On the characterization of fields of diabatic flow.*

H13. Y. H. Kuo: *On the initial motion of a strong spherical shock.*

J. W. TUKEY

THE APRIL MEETING IN NEW YORK

The four hundred twenty-fourth meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, April 25–26, 1947. The attendance was over 300, including the following 300 members of the Society:

C. R. Adams, C. F. Adler, R. P. Agnew, E. J. Akutowicz, Leonidas Alaoglu, T. W. Anderson, C. B. Allendoerfer, R. G. Archibald, L. A. Aroian, M. C. Ayer, R. A. Bari, Joshua Barlaz, P. T. Bateman, G. E. Bates, M. F. Becker, E. G. Begle, Richard Bellman, Stefan Bergman, Arthur Bernstein, Felix Bernstein, Lipman Bers, D. H. Blackwell, Gertrude Blanch, J. H. Blau, R. P. Boas, H. W. Bode, G. L. Bolton, Samuel Borofsky, J. M. Boyer, A. D. Bradley, H. W. Brinkmann, Paul Brock, A. B. Brown, G. W. Brown, R. H. Brown, E. F. Buck, R. C. Buck, L. H. Bunyan, R. S. Burington, J. C. Burkill, Herbert Busemann, K. A. Bush, Hobart Bushey, J. H. Bushey, K. E. Butcher, Albert Cahn, S. S. Cairns, W. R. Callahan, H. H. Campaigne, K. Chandrasekharan, J. O. Chellevold, Herman Chernoff, Claude Chevalley, Edmund Churchill, J. A. Clarkson, M. D. Clement, R. M. Cohn, I. S. Cohen, Nancy Cole, T. F. Cope, Richard Courant, M. J. Cox, F. G. Critchlow, H. B. Curry, J. H. Curtiss, M. D. Darkow, A. S. Day, S. P. Diliberto, J. L. Doob, C. H. Dowker, Y. N. Dowker, William H. Durfee, Churchill Eisenhart, Benjamin Epstein, Ky Fan, J. M. Feld, William Feller, F. G. Fender, A. D. Fialkow, L. A. Fine, C. D. Firestone, W. B. Fite, D. A. Flanders, M. C. Foster, F. H. Fowler, G. A. Foyle, W. C. G. Fraser, Gerald Freilich, Bernard Friedman, K. O. Friedrichs, C. S. Gardner, Irving Gerst, J. H. Giese, B. P. Gill, Leonard Gillman, V. D. Gokhale, H. E. Goheen, H. H. Goldstine, A. W. Goodman, R. O. Goodman, Saul Gorn, W. H. Gottschalk, F. G. Gravalos, C. H. Graves, H. J. Greenberg, Bernard Greenspan, Harriet Griffin, George Grossman, E. J. Gumbel, D. W. Hall, G. J. Haltiner, R. W. Hamming, Duncan Harkin, Gerald Harrison, Philip Hartman, K. E. Hazard, L. B. Hedge, G. A. Hedlund, A. E. Heins, M. H. Heins, Alex Heller, L. A. Henkin, T. H. Hildebrandt, Einar Hille. P. G. Hoel, Banesh Hoffmann, T. R. Hollcroft, E. M. Hull, Witold Hurewicz, L. C. Hutchinson, W. H. Ingram, R. C. James, S. A. Joffe, Fritz John, R. E. Johnson, A. W. Jones, L. G. Jones, Mark Kac, Aida Kalish, Edward Kasner, Stanley Katz, M. E. Kellar, H. S. Kieval, F. L. Kiokemeister, J. R. Kline, Morris Kline, E. G. Kogbetliantz, E. R. Kolchin, B. O. Koopman, Jack Laderman, M. E. Ladue, R. E. Langer, P. D. Lax, Solomon Lefschetz, D. H. Lehmer, Benjamin Lepson, Howard Levi, Norman Levinson, D. C. Lewis, Marie Litzinger, Charles Loewner, L. H. Loomis, E. R. Lorch, Lee Lorch, A. N. Lowan, C. I. Lubin, E. H. Luchins, Brockway McMillan, E. J. McShane, L. A. MacColl, R. E. MacKenzie, G. W. Mackey, Saunders MacLane, H. M. MacNeille, H. F. MacNeish, A. W. Manning, Murray Mannos, A. J. Maria, M. H. Maria, J. W. Marshall, M. H. Martin, Imanuel Marx, F. I. Mautner, S. Minakshisundaram, A. K. Mitchell, T. W. Moore, K. A. Morgan, I. R. Moses, G. D. Mostow, H. T. Muhly, F. J. Murray, D. S. Nathan, C. A. Nelson, E. D. Nering, E. N. Nilson, Louis Nirenberg, P. B. Norman, Lawrence Norwood, C. O. Oakley, E. J. Olson, A. F. O'Neill, F. W. Owens, J. C. Oxtoby, J. S. Oxtoby, G. W. Patterson, L. G. Peck, Anna Pell-Wheeler, C. R. Phelps, R. S. Phillips, Everett Pitcher, F. V. Pohle, William Prager, Walter Prenowitz, G. B. Price, M. H. Protter, Hans Rademacher, Arthur Radin, John Raleigh, C. E. Rickart, G. de B. Robinson, Mina Rees, Daniel Resch, J. S. Rhodes, Moses Richardson, R. G. D. Richardson,

F. D. Rigby, J. F. Ritt, M. S. Robertson, J. B. Robinson, R. M. Robinson, S. L. Robinson, Ira Rosenbaum, L. J. Rosenbaum, R. A. Rosenbaum, J. H. Rosenbloom, M. A. Rosenlicht, A. E. Ross, Charles Saltzer, Arthur Sard, Samuel Saslaw, S. A. Schaaf, A. T. Schafer, R. D. Schafer, I. J. Schoenberg, Lowell Schoenfeld, I. E. Segal, H. N. Shapiro, Max Shiffman, D. N. Silver, David Singer, P. A. Smith, Andrew Sobczyk, George Springer, E. P. Starke, Fritz Steinhardt, J. J. Stoker, C. N. Stokes, E. G. Straus, L. M. Straus, Walter Strodt, Fred Supnick, J. L. Synge, M. M. Sullivan, Otto Szász, W. C. Taylor, J. M. Thomas, Hing Tong, C. C. Torrance, J. W. Tukey, Annita Tuller, H. E. Vansant, H. S. Vandiver, G. L. Walker, A. D. Wallace, Henry Wallman, C. T. Wang, J. A. Ward, W. G. Warnock, Herbert Federer, Alan Wayne, J. V. Wehausen, Alexander Weinstein, F. P. Welch, Louis Weisner, G. W. Whitehead, P. M. Whitman, W. F. Whitmore, Albert Wilansky, C. S. Williams, John Williamson, Jacob Wolfowitz, A. W. Wundheiler, M. A. Wurster, J. W. Young, Oscar Zariski, H. J. Zimmerberg, Leo Zippin, O. J. Zobel, M. F. Zucker, Antoni Zygmund.

The meeting opened Friday afternoon at 2:00 with an address on *The small and the large in variational analysis* by Professor Max Shiffman of New York University. Vice President Saunders MacLane presided.

On Saturday morning there were two sections for contributed papers: one in Analysis in which Professor G. A. Hedlund presided and one in Algebra and Topology in which Professor A. D. Wallace presided. On Saturday afternoon there were two sections for contributed papers: one in Analysis, in which Professor I. J. Schoenberg presided, and one in Applied Mathematics, Geometry and Logic in which Professor H. A. Rademacher presided.

A 2:30 P.M. Saturday Professor Claude Chevalley of Princeton University gave an address on the *Theory of algebraic Lie groups*. Professor Solomon Lefschetz presided.

The Council met in the Men's Faculty Club at 12:15 P.M. on April 26, 1947.

The Secretary announced the election of the following seventy-five persons to ordinary membership in the Society:

Mr. Loren Wesley Akers, University of Kansas;
Dr. Binyamin A. Amirà, Hebrew University, Jerusalem, Palestine;
Professor German Ancochea, Universidad de Salamanca, Salamanca, Spain;
Mr. James I. Asnin, Colorado Fuel and Iron Corporation, Denver, Colo.;
Mr. Frederick Edward Atkins, State Teachers College, California, Pa.;
Mr. Howard Hayden Barnett, University of Kansas;
Mr. Frank Samuel Beckman, Pratt Institute;
Professor William Henry Bradford, John McNeese Junior College of Louisiana State University;
Miss Helen J. Bradley, University of Tennessee;
Mr. Roy P. Brady, Hunter College;
Miss Ruth Alice Brendel, University of Buffalo;
Professor Grover Preston Burns, Marshall College, Huntington, W. Va.;
Mr. Franklyn George Creese, San Leandro, Calif.;

Mr. Walter Melvin DeCew, McGraw-Hill Book Company;
Mr. Eliseo Mark Di Domenica, Drexel Institute of Technology;
Miss Genevieve Hazel Dixon, Buffalo, N. Y.;
Miss Mary Patricia Dolciani, Cornell University;
Professor Theodore Dreier, Black Mountain College, Black Mountain, N. C.;
Professor Leonard Philip Edwards, University of New Brunswick;
Professor Frank Mahony Ellis, Youngstown College;
Professor Herman Harvey Ferns, University of Saskatchewan;
Professor Aaron D. Fleshler, Sampson College, Sampson, N. Y.;
Dr. Curtis Maxwell Fulton, University of California at Los Angeles;
Mr. Lewis McLeod Fulton, Jr., Duke University;
Dr. Lars Jakob Gårding, Lund University;
Dr. Irving Gerst, Control Instrument Company, Brooklyn, N. Y.
Mr. Charles C. Goldman, University of Cincinnati;
Mr. Irwin Leonard Goldman, Columbia University;
Mr. Leon Goldstein, Naval Ordnance Plant Laboratory, Indianapolis, Ind.;
Mr. Max Goldstein, Los Alamos Scientific Laboratory, Los Alamos, N. Mex.;
Mr. Arthur Grad, Stanford University;
Deborah Tepper Haimo (Mrs. Franklin), Cambridge, Mass.;
Mr. George Charles Helme, Pratt Institute;
Mr. Donald Lincoln Herr, Control Instrument Company, Brooklyn, N. Y.;
Mr. Robert Eugene Horton, University of California at Los Angeles;
Mr. Hai-Tsin Hsü, Yale University;
Mr. Eugene Isaacson, New York University;
Mr. Hyman B. Kaitz, National Income Division, Department of Commerce, Washington, D. C.;
Mr. Stanley Katz, Hydrocarbon Research, Inc., New York, N. Y.;
Mr. J. Lee Kavanau, University of California at Los Angeles;
Professor M. Wiles Keller, Purdue University;
Mr. Samuel George Kneale, University of Kansas;
Mr. Hendrik S. Konijn, State Department, Washington, D. C.;
Mr. Thomas Phillip George Liverman, University of Delaware;
Mr. Jens Lloyd Lund, Oakland, Calif.;
Mr. Charles Edward Mack, Jr., Grumman Aircraft Engineering Corporation, Bethpage, N. Y.;
Mr. L. Kenton Meals, University of Missouri;
Mr. James William Mettler, State Teachers College, California, Pa.;
Mr. Harry Frederick Mist, University of Chicago;
Professor D. S. Mitrinovitch, University of Skopié, Skopié, Jugoslavia;
Professor Alexander McFarlane Mood, Iowa State College of Agriculture and Mechanical Arts;
Mr. Albert Harrison Moore, Pratt Institute;
Mr. Allen Thurmond Myers, University of Arizona;
Mr. Kenneth Neal Nickel, University of Kansas;
Mr. Henry Joseph Osner, University of California;
Dr. Morris Ostrofsky, Gulf Oil Corporation, Pittsburgh, Pa.;
Mr. Arthur Radin, College of the City of New York;
Mr. Hans Vilhelm Rådström, Matematiska Institutet, Stockholms Högskola, Stockholm, Sweden;
Mr. Andreas Owen Ridgway, University of Maryland;
Mr. James Daniel Riley, University of Kansas;

Mr. Saul Rosen, University of Delaware;
Mr. Maxwell Alexander Rosenlicht, Brooklyn, N. Y.;
Miss Joyce Rubin, Polytechnic Institute of Brooklyn and Seward Park High School, Brooklyn, N. Y.;
Mr. Harry Daniel Ruderman, Manhattan High School of Aviation Trades, New York, N. Y.
Mr. Arnold Bernard Schacknow, Polytechnic Institute of Brooklyn;
Sister Mary Charles Schaffer, Immaculate Heart College, Los Angeles, Calif.;
Dr. Menahem Max Schiffer, Institute of Theoretical Physics, Hebrew University, Jerusalem, Palestine;
Dr. Alfred Ernest Schild, Carnegie Institute of Technology;
Dr. Mohammad Raziuddin Siddiqi, Research Institute, Osmania University, Hyderabad-Deccan, India;
Mr. Daniel N. Silver, Newark College of Engineering;
Mr. Theodor Teichmann, Princeton University;
Professor Jose Joaquin Trejos-Fernández, University of Costa Rica;
Mr. Gordon Allen Tully, Comptroller's Office, State Capitol, Little Rock, Ark.;
Mr. John R. Van Andel, Aeronautical Instruments Laboratory, Naval Air Material Center, Philadelphia, Pa.;
Dr. Leo M. Wulf, Monmouth Junior College, Long Branch, N. J.

It was reported that the following had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Mr. Haskell Cohen;
University of Arizona: Mr. Joseph Foster Foster, Jr.;
University of Chicago: Messrs. Harley Flanders, Albert Furman, Daniel Orloff, Efrem Herbert Ostrow, Lyman C. Peck, Herman Rubin, and Clayton James Thomas;
College of the City of New York: Mr. Martin David Davis;
Columbia University: Mr. John Bartholomew Webster;
Harvard University: Messrs. Paul Olum and William Gurdon Saltonstall;
Institute for Advanced Study: Dr. Andrew Donald Booth, Miss Kathleen Hyldia Valerie Britten, Dr. John Charles Burkill, Mr. Komaravolu Chandrasekharan, Dr. Jean Frédéric Delsarte, Professor Beno Eckmann, Dr. Subbaramiah Minakshisundaram;
State University of Iowa: Miss Helen Emogene McCue;
Kenyon College: Mr. Thomas Forbes Smith;
University of Maryland: Miss Jean Marie Boyer, Mr. Stuart Troy Haywood;
Michigan State College: Mr. James Goodrich Renno, Jr.;
University of Missouri: Mr. Jerry William Gaddum;
New York University: Mr. Louis Nirenberg;
Oklahoma Agricultural and Mechanical College: Mr. R. R. Reynolds;
Pennsylvania State College: Messrs. Frank T. Kocher, Jr., and James Russel Ziegler;
Rice Institute: Mrs. Erna Herzog Pearson, Messrs. Howard Edward Taylor and Thomas Jefferson White;
University of Rochester: Mr. Erich W. Marchand;
Stanford University: Messrs. Wray Grayson Brady, Burnett Chandler Meyer, Albert Boris Novikoff, and Joseph Ullman;
University of Toronto: Messrs. Hubert Whitfield Ellis, Theodore Franklin Morris, Leo Moser, and Ralph Gordon Stanton;

University of Washington: Mr. George Edward Uhrich;
Wesleyan University: Mr. Richard Godfrey Billard;
University of Wisconsin: Messrs. Edmond Darrell Cashwell, Paul Lawrence Chessin, Robert Davies, Arie Gaalswyk, Carlton Wayne Larson, Harvey C. McKenzie, Merrill Roberts Moore, Donald Ross Morrison, Donald Alan Norton, Francis Harry Pott, Elmer George Schuld.

The Secretary announced that Major I. B. Perrott of the College of Technology, Leicester, England, had been admitted to the Society in accordance with the reciprocity agreement with the London Mathematical Society.

The Secretary announced the election by mail of the University of British Columbia and the University of Tennessee to institutional contributing membership in the Society. The Council voted to elect Southern Illinois Normal University to institutional membership.

The following appointments by President Einar Hille of representatives of the Society were reported: Professor E. J. Moulton at the Conference on Science as part of the Centennial Celebration of Rockford College on February 21–23, 1947; Professor H. J. Ettlinger at the inauguration of William Vermillion Houston as President of The Rice Institute on April 10, 1947; Professor H. A. Rademacher at the Fifty-first Annual Meeting of the American Academy of Political and Social Science on April 18–19, 1947; Professor F. E. Johnston at the inauguration of Leonard M. Elstad as President of Gallaudet College on April 26, 1947; Professor F. R. Bamforth at the Centennial Convocation of Otterbein College on April 26, 1947; Professor R. C. Huffer at the Centennial Ceremony of Rockford College on May 3, 1947; Professor W. L. Ayres at the inauguration of George Dinsmore Stoddard as President of the University of Illinois on May 16, 1947.

The following additional appointments by the President were reported: Professors A. W. Tucker (Chairman), Nelson Dunford, Samuel Eilenberg, W. T. Martin, and Gabor Szegö as a Committee on the Reorganization of the Society's Programs; Professor T. F. Cope and Dean A. E. Meder as auditors of the Society's accounts for 1947; Professor W. T. Reid as Chairman of the Committee on Places of Meetings for 1947 and Professor C. B. Allendoerfer as a member of the Committee on Places of Meetings for 1947–1949 (committee now consists of Professors W. T. Reid, Chairman, C. B. Allendoerfer, and G. B. Price); Professors Solomon Lefschetz (Chairman), Garrett Birkhoff and Nelson Dunford as a committee to study the question of cooperation with the Pan American Congress, to be held in Mexico City in November, 1947; Professors C. C. MacDuffee (Chairman), L. M. Graves, E. J. McShane, A. P. Morse, and A. W. Tucker as a

Committee on Nomination of Officers and Members of the Council for 1948; Professors F. D. Murnaghan (Chairman), W. T. Martin, and Gabor Szegö as a Committee to Select Gibbs Lecturers for 1948 and 1949; Professors J. R. Kline and M. H. Stone as representatives of the Society on the A.A.A.S. Inter-Society Committee on Science Foundation Legislation; Professor William Feller as liaison officer of the Quarterly of Applied Mathematics for the period 1947–1949; Professor C. O. Oakley as Chairman and Professors Garrett Birkhoff and R. M. Thrall as members of the Committee on Publicity, for the period 1947–1949 (committee now consists of Professors C. O. Oakley, Chairman, Garrett Birkhoff, J. B. Rosser, Dr. J. M. Thompson, Professors R. M. Thrall and R. J. Walker); Professors J. L. Synge (Chairman), Richard Courant, G. C. Evans, John von Neumann, William Prager, and Dr. Warren Weaver as a Committee on Applied Mathematics; Professors Saunders MacLane (Chairman), R. C. Archibald, J. L. Coolidge, G. C. Evans, Marston Morse, G. B. Price, Hermann Weyl, and J. W. T. Youngs as a Finance Committee in connection with the Birkhoff Memorial Project; Professors D. V. Widder (Chairman), C. R. Adams, R. E. Langer, Marston Morse, and M. H. Stone as an Editorial Committee in connection with the Birkhoff Memorial Project.

The Secretary reported the following actions taken by mail since the Christmas meeting of the Council: (1) the adoption of a report, setting up procedures in accordance with which a mathematical organization might become a sponsor of Mathematical Reviews; (2) the approval of a report of the Committee on Applied Mathematics, providing for a Symposium on *Non-linear problems in mechanics of continua*, to be held at Brown University on August 2–4, 1947; (3) approval of a recommendation of the Colloquium Editorial Committee that Professor Richard Brauer of the University of Toronto be invited to deliver a series of Colloquium Lectures at the 1948 Summer Meeting. The Secretary also reported Professor Brauer's acceptance of this invitation.

The Secretary reported that the following had been chosen as the voting representatives of editorial committees on the Council for the year 1947: Bulletin, Professor R. E. Langer; Transactions, Professor Oscar Zariski; Colloquium Publications, Professor J. F. Ritt; American Journal of Mathematics, Professor Richard Brauer; Mathematical Reviews, Professor O. E. Neugebauer; Mathematical Surveys, Professor J. L. Walsh. It was also reported that Professor J. F. Ritt had been selected as Chairman of the Colloquium Editorial Committee for the year 1947.

It was reported that certain changes in the By-Laws had been approved by the Society at the February, 1947, meeting. These changes create rotation in the terms of office of members of the Board of Trustees and provide for an increase in membership dues. The sections of the By-Laws, as revised, were published as part of the report of the February Meeting in New York in the May issue of this BULLETIN.

It was reported that the Society had cooperated with the Journal of Mathematics and Physics by distributing to members a circular, advertising the Journal and offering a reduced subscription rate to Society members.

Recommendations regarding meetings of the Society in the midwest were presented and adopted as follows: (1) That the Society continue the policy of holding one midwestern meeting a year at some institution away from Chicago; (2) That two meetings be held annually in the Chicago area; (3) That no regular meeting of the Society be held in the midwest at the time of the Annual Meeting at the University of Georgia; (4) That the November, 1947, meeting in the midwest be held at Washington University.

The Council voted unanimously to accept the invitation of Harvard University to hold the International Congress of Mathematicians at that institution in 1950.

A report from the Board of Trustees regarding the questions of the Society's library and adequate housing for the Society was presented and discussed. The Council authorized the President to appoint, in conjunction with the Board of Trustees, a committee to study the problems involved.

The resignation of Professor A. C. Schaeffer as Associate Secretary for the far west was accepted with regret.

Certain invitations to deliver addresses were announced: Professor C. B. Allendoerfer for the October, 1947, meeting in New York City; Professors Marshall Hall and P. R. Halmos for the November, 1947, meeting in the midwest; and Professor D. H. Lehmer for the November, 1947, meeting in the far west.

The Council voted to accept the recommendation of the Committee to Select Gibbs Lecturers that Professor Hermann Weyl be invited to deliver the twenty-second Josiah Willard Gibbs Lecture at the 1948 Annual Meeting. The Secretary reported the appointment of Professor Gabor Szegö as Chairman of this Committee, replacing Professor F. D. Murnaghan, and of Professor J. J. Stoker as a new member of the Committee.

The final report of the Special Committee on Applied Mathematics

(Professor J. L. Synge, Chairman) was presented. In accordance with this report, certain problems were referred by the Council to the Committee on Applied Mathematics, for study and recommendation to the Council. These included problems concerning the desirability of setting up a separate division for applied mathematics, the establishment of a Committee on the Organization of Known Results, and a Committee on Advanced Reference Texts in Applied Mathematics. A statement entitled *Instruction and Research in Applied Mathematics*, prepared by the Special Committee on Applied Mathematics and approved by the Council, appears elsewhere in this issue of the BULLETIN.

Professor Saunders MacLane, reporting for the Finance Committee of the Birkhoff Memorial Project, announced that the solicitation of particular groups was in progress and that subscriptions and contributions would be requested from members of the Society about June 1, 1947.

Titles and cross references to the abstracts of papers read at the meeting follow below.

Papers 1–8 were read in the Analysis section Saturday morning; papers 9–17 in the Algebra and Topology section Saturday morning; papers 18–22 in the Analysis section Saturday afternoon; papers 23–27 in the Applied Mathematics, Geometry and Logic section Saturday afternoon; papers 28–67 were presented by title. Paper number 8 was read by Dr. Szász, number 14 by Professor Hall, number 16 by Miss Boyer, and number 18 by Mr. Chandrasekharan. Dr. Toralballa was introduced by Professor J. J. Lynch, Mr. Shen by Professor Antoni Zygmund, and Professor Fekete by Professor Otto Szász.

1. L. V. Toralballa: *The integral in a normed division quasi-ring.* (Abstract 53-5-235.)

2. Yael N. Dowker: *Invariant measure and the ergodic theorems.* (Abstract 53-5-197.)

3. F. G. Gravalos: *A note on dynamical systems with two degrees of freedom.* (Abstract 53-5-203.)

4. D. C. Lewis: *Generalized orthonormality of polynomials.* (Abstract 53-5-215.)

5. Charles Loewner: *A topological characterization of a class of integral operators.* (Abstract 53-5-216.)

6. Albert Wilansky: *An application of Banach linear functionals to the theory of summability.* Preliminary report. (Abstract 53-5-240.)

7. Alexander Weinstein: *On Dirichlet's discontinuous factor and a class of many-valued functions.* (Abstract 53-5-238.)

8. K. Chandrasekharan and Otto Szász: *On Bessel summation.* (Abstract 53-5-196.)

9. R. D. Schafer: *The exceptional simple Jordan algebras.* (Abstract 53-5-180.)

10. Grace E. Bates: *Decompositions of a loop into characteristic free summands.* (Abstract 53-5-164.)

11. William H. Durfee: *Quadratic forms over fields with a valuation.* (Abstract 53-5-168.)

12. J. A. Ward: *Analytic functions in linear algebras.* (Abstract 53-5-185.)

13. G. de B. Robinson: *On a theorem of Nakayama concerning the prime factors of the degrees of the irreducible representations of the symmetric group.* (Abstract 53-5-179.)

14. D. W. Hall and D. C. Lewis: *On coloring six-rings.* (Abstract 53-5-273.)

15. C. H. Dowker: *Topologies of infinite complexes.* (Abstract 53-5-271.)

16. Jean M. Boyer and D. W. Hall: *On Peano spaces as continuous images of intervals.* (Abstract 53-5-270.)

17. A. E. Ross: *On continuous transformations of a Jordan curve.* (Abstract 53-3-162.)

18. K. Chandrasekharan and S. Minakshisundaram: *Some results on double Fourier series.* (Abstract 53-5-195.)

19. D. H. Lehmer: *Approximations to the surface area and electrostatic capacity of the ellipsoid.* (Abstract 53-5-214.)

20. E. R. Lorch: *On certain implications which characterise Hilbert space.* (Abstract 53-5-217.)

21. R. C. Buck: *Integral valued entire functions.* (Abstract 53-5-193.)

22. Mark Kac: *Distribution properties of certain gap sequences.* (Abstract 53-7-290.)

23. A. E. Heins: *Water waves over a channel with a dock.* (Abstract 53-5-248.)

24. R. M. Robinson: *On the decomposition of spheres.* (Abstract 53-5-262.)

25. Ira Rosenbaum: *A method of determining the nth q-ary truth-function in m-valued logic.* (Abstract 53-5-264.)

26. Felix Bernstein: *A lattice color problem.* Preliminary report. (Abstract 53-5-267.)

27. J. L. Synge: *Apsidal angles for symmetrical dynamical systems with two degrees of freedom.* (Abstract 53-7-296.)

28. P. T. Bateman: *Modular subgroup of finite index*. Preliminary report. (Abstract 53-3-109-*t*.)

29. Richard Bellman: *On the boundedness of solutions of nonlinear differential and difference equations*. (Abstract 53-3-120-*t*.)

30. Richard Bellman: *On the solutions of the heat equation*. Preliminary report. (Abstract 53-5-190-*t*.)

31. Richard Bellman: *On the stability of solutions of differential-difference equations*. Preliminary report. (Abstract 53-5-191-*t*.)

32. Stefan Bergman: *Functions satisfying linear partial differential equations and their properties*. (Abstract 53-5-121-*t*.)

33. Stefan Bergman and Menahem Schiffer: *On Green's and Neumann's functions in the theory of partial differential equations*. (Abstract 53-5-192-*t*.)

34. Richard Brauer: *Applications of induced characters*. (Abstract 53-3-110-*t*.)

35. Richard Brauer: *On Artin's L-series with general group characters*. (Abstract 53-3-111-*t*.)

36. Richard Brauer: *On the zeta-functions of algebraic number fields*. (Abstract 53-3-112-*t*.)

37. Leonard Carlitz and E. L. Cohen: *Divisor functions of polynomials in a Galois field*. (Abstract 53-3-113-*t*.)

38. R. P. Cesco: *On strong summability*. (Abstract 53-5-194-*t*.)

39. E. L. Cohen: *Sums of an even number of squares in* $GF[p^n, x]$. (Abstract 53-3-115-*t*.)

40. John DeCicco: *An extension of Euler's theorem on homogeneous functions*. (Abstract 53-3-150-*t*.)

41. John DeCicco: *Characterization of Halphen's theorem on central and parallel fields of force*. (Abstract 53-5-257-*t*.)

42. J. B. Díaz and H. J. Greenberg: *The determination of upper and lower bounds for the deflection of a clamped plate*. (Abstract 53-5-243-*t*.)

43. J. B. Díaz and Alexander Weinstein: *Schwarz' inequality and the methods of Rayleigh-Ritz and Trefftz*. (Abstract 53-5-244-*t*.)

44. C. H. Dowker: *An imbedding theorem for paracompact metric spaces*. (Abstract 53-3-157-*t*.)

45. Bernard Epstein: *Some inequalities relating to conformal mapping upon canonical slit-domains*. (Abstract 53-5-200-*t*.)

46. M. Fekete: *On generalized transfinite diameters*. (Abstract 53-7-288-*t*.)

47. K. O. Friedrichs: *On the non-occurrence of a limiting line in transonic flow*. (Abstract 53-7-292-*t*.)

48. K. O. Friedrichs: *Water waves on a shallow sloping beach*. (Abstract 53-7-293-*t*.)

49. K. O. Friedrichs and Hans Lewy: *Dock problem.* (Abstract 53-7-294-*t.*)

50. H. J. Greenberg and Witold Hurewicz: *The stability of periodic dynamical processes subject to discontinuous impulses.* (Abstract 53-5-247-*t.*)

51. Philip Hartman and Aurel Wintner: *Töplerian* (L^2)*-bases.* (Abstract 53-7-289-*t.*)

52. R. E. Johnson: *The modules of a ring.* Preliminary report. (Abstract 53-5-172-*t.*)

53. Mark Kac, Raphael Salem and Antoni Zygmund: *A gap theorem.* (Abstract 53-5-210-*t.*)

54. Edward Kasner and John DeCicco: *Differential geometric properties of the polar surfaces of a general algebraic surface.* (Abstract 53-5-260-*t.*)

55. C. N. Moore: *Generalized limits in general analysis.* III. (Abstract 53-5-220-*t.*)

56. John von Neumann and E. R. Lorch: *On the euclidean character of the perpendicularity relation.* (Abstract 53-5-221-*t.*)

57. Ira Rosenbaum: *A method of determining the number n correlated with the given truth table of an arbitrary q-ary function in m-valued logic.* (Abstract 53-5-265-*t.*)

58. I. M. Sheffer: *A limit theorem.* (Abstract 53-5-230-*t.*)

59. Y. C. Shen: *Interpolation to some classes of analytic functions by functions with pre-assigned poles.* (Abstract 53-5-231-*t.*)

60. Walter Strodt (National Research Fellow): *Note on quasi-harmonic functions.* Preliminary report. (Abstract 53-5-234-*t.*)

61. Otto Szász: *Quasi-monotone series.* (Abstract 53-7-291-*t.*)

62. L. V. Toralballa: *A generalization of the finite integral.* (Abstract 53-5-184-*t.*)

63. W. R. Utz: *On the decomposition of meromorphic functions.* (Abstract 53-5-236-*t.*)

64. J. L. Walsh: *The location of the critical points of simply and doubly periodic functions.* (Abstract 53-5-237-*t.*)

65. G. T. Whyburn: *On locally simple curves.* (Abstract 53-5-285-*t.*)

66. G. T. Whyburn: *On n-arc connectedness.* (Abstract 53-5-286-*t.*)

67. Bertram Yood: *Regular and singular elements in the ring of operators on a Banach space.* Preliminary report. (Abstract 53-3-139-*t.*)

T. R. HOLLCROFT,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The four hundred twenty-fifth meeting of the American Mathematical Society was held in Eckhart Hall at the University of Chicago, Chicago, Illinois, on Friday and Saturday, April 25–26, 1947. Thus the Society returned to its traditional Chicago meeting place after an absence of four years. This meeting was the largest April meeting in Chicago in the history of the Society. The previous maximum occurred in 1939, with an attendance of about 250; this time the attendance was about 300, including the following 259 members of the Society:

W. R. Allen, E. W. Anderson, R. V. Andree, M. L. Anthony, K. J. Arrow, Max Astrachan, W. F. Atchison, H. C. Ayres, W. L. Ayres, Reinhold Baer, R. W. Ball, R. H. Bardell, R. C. F. Bartels, H. J. Barten, J. H. Bell, P. O. Bell, Arthur Bernhart, Herman Betz, S. F. Bibb, R. H. Bing, K. E. Bisshopp, H. L. Black, L. M. Blumenthal, D. G. Bourgin, A. J. Brandt, Foster Brooks, F. L. Brown, R. H. Bruck, P. B. Burcham, I. W. Burr, L. E. Bush, Lee Byrne, R. H. Cameron, J. W. Calkin, B. G. Carlson, R. E. Carr, P. W. Carruth, Abraham Charnes, E. W. Chittenden, A. M. Christiansen, R. V. Churchill, H. M. Clark, Nathaniel Coburn, R. H. Cole, L. A. Colquitt, B. H. Colvin, E. G. H. Comfort, N. B. Conkwright, J. A. Cooley, J. J. Corliss, Byron Cosby, H. S. M. Coxeter, H. J. Curtis, M. M. Day, H. F. DeBaggis, Flora Dinkines, J. M. Dobbie, C. L. Dolph, M. D. Donsker, J. L. Dorroh, Charlotte Dragstedt, John Dyer-Bennett, W. F. Eberlein, Beno Eckmann, R. H. Edwards, Samuel Eilenberg, J. H. Engel, Paul Erdös, M. H. M. Esser, H. P. Evans, G. M. Ewing, J. V. Finch, Harley Flanders, H. M. Fliess, L. R. Ford, R. S. Fouch, J. S. Frame, Evelyn Frank, Marianne Freundlich, C. G. Fry, Albert Furman, R. E. Fullerton, J. W. Gaddum. Wallace Givens, Michael Golomb, S. H. Gould, A. A. Grau, L. M. Graves, L. J. Green, Edison Greer, L. W. Griffiths, V. G. Grove, Marshall Hall, P. R. Halmos, O. H. Hamilton, W. L. Hart, G. E. Hay, C. T. Hazard, E. D. Hellinger, R. G. Helsel, Fritz Herzog, M. R. Hestenes, J. F. Heyda, E. H. C. Hildebrandt, R. C. Hildner, D. L. Holl, C. C. Hsiung, S. P. Hughart, H. K. Hughes, Rufus Isaacs, T. J. Jaramillo, Madeline Johnsen, H. F. S. Jonah, L. H. Kanter, Wilfred Kaplan, Irving Kaplansky, William Karush, J. L. Kelley, L. M. Kelly, J. F. Kenney, D. E. Kibbey, Lois Kiefer, W. M. Kincaid, S. C. Kleene, J. C. Koken, M. S. Kramer, W. C. Krathwohl, M. Z. Krzywoblocki, Joseph Landin, E. P. Lane, R. E. Lane, E. H. Larguier, J. P. LaSalle, C. G. Latimer, W. G. Leavitt, Harry Levy, S. L. Levy, D. J. Lewis, A. O. Lindstrum, B. J. Lockhart, A. T. Lonseth, Eugene Lukacs, C. T. McCormick, S. W. McCuskey, J. C. C. McKinsey, G. R. Magee, Morris Marden, W. S. Massey, M. S. Matchett, C. W. Matthews, H. F. Mathis, A. E. May, J. R. Mayor, L. K. Meals, B. E. Meserve, H. L. Meyer, Herman Meyer, E. J. Mickle, H. F. Mist, A. C. Moeller, J. T. Moore, M. G. Moore, F. V. E. Morfoot, D. R. Morrison, M. E. Munroe, J. R. Musselman, S. B. Myers, August Newlander, Ivan Niven, E. A. Nordhaus, M. J. Norris, J. I. Northam, Rufus Oldenburger, J. M. H. Olmsted, Isaac Opatowski, Morris Ostrofsky, E. H. Ostrow, L. J. Paige, Gordon Pall, H. C. Parrish, R. S. Pate, P. M. Pepper, Sam Perlis, H. P. Pettit, George Piranian, J. C. Polley, D. H. Potts, J. E. Powell, A. L. Putnam, Tibor Rado, E. D. Rainville, W. C. Randels, M. K. Rapp, M. O. Reade, O. W. Rechard, P. V. Reichelderfer, W. T. Reid, J. G. Renno, Haim Reingold, M. M. Resnikoff,

P. R. Rider, R. F. Rinehart, D. H. Rock, G. F. Rose, E. H. Rothe, Herman Rubin, Hans Samelson, R. G. Sanger, L. J. Savage, Robert Schatten, O. F. G. Schilling, W. T. Scott, M. E. Shanks, A. S. Shapiro, L. W. Sheridan, Seymour Sherman, H. A. Simmons, D. M. Smiley, M. F. Smiley, A. H. Smith, F. C. Smith, G. W. Smith, W. S. Snyder, E. H. Spanier, R. D. Specht, Abraham Spitzbart, R. H. Stark, N. E. Steenrod, Rothwell Stephens, B. M. Stewart, W. M. Stone, E. B. Stouffer, L. W. Swanson, T. T. Tanimoto, H. P. Thielman, Leonard Tornheim, E. A. Trabant, E. F. Trombley, H. L. Turrittin, Henry Van Engen, J. I. Vass, H. E. Vaughan, Bernard Vinograde. R. D. Wagner, M. T. Wechsler, C. P. Wells, E. T. Welmers, George Whaples, N. A, Wiegmann, L. R. Wilcox, R. L. Wilder, J. E. Wilkins, C. O. Williamson, R. L. Wilson, R. S. Wolfe, G. S. Young, P. M. Young, J. W. T. Youngs, Daniel Zelinsky.

Professor N. E. Steenrod of the University of Michigan opened the sessions both mornings at 9:30 with the Symposium lectures, delivered under the title *Foundations of algebraic topology*. Professors Rado and Eilenberg presided at these lectures on Friday and Saturday respectively. On Friday at 1:45 P.M., with Professor Hestenes presiding, Professor Marcel Riesz of Lunds Universitets Matematiska Institution and the University of Chicago gave an hour address entitled *Lorentz transformations and Clifford numbers.*

Sessions for short research papers were held as follows (Presiding officers are indicated in parentheses): 10:45 A.M. Friday: Section I. Algebra (Professor Baer), Section II. Analysis (Professor Graves); 3:00 P.M. Friday: Section I. Topology (Professor Eilenberg), Section II. Applied Mathematics, Foundations and Analysis (Professor Kleene); 10:45 A.M. Saturday: Section I. Analysis (Professor Halmos), Section II. Algebra and Geometry (Professor Pall).

At the close of the Friday afternoon sessions afternoon tea was served in the Eckhart Hall Common Room and in the Library across the hall. Thanks are due to the Mathematics Department of the University of Chicago for their hospitality, and also to Dr. Irving Kaplansky who was in charge of the local arrangements.

An informal business meeting was held at 8:00 P.M. Friday, for the purpose of crystallizing opinion in regard to location of future midwestern Thanksgiving meetings. Although only thirty-six members attended, they represented the opinions of a much wider group, and the whole question of midwestern meetings was thoroughly discussed. Several resolutions were passed and these were telegraphed to Secretary Kline for consideration by the Council of the Society at its meeting in New York on Saturday, April 26.

Titles and cross references to the abstracts of papers read follow. Papers presented by title are indicated by the letter "*t*." Papers numbered 1 to 8 and 9 to 16 were presented Friday morning in Sections I and II respectively; papers 17 to 22 and 23 to 28, Friday afternoon; papers 29 to 36 and 37 to 44, Saturday morning. Paper 1 was presented by Dr. Kaplansky, paper 7 by Professor Whaples, paper 10 by Professor Mickle, paper 12 by Mr. Rechard, paper 15 by Dr. Piranian, paper 19 by Professor Helsel, paper 27 by Professor McKinsey, and paper 43 by Professor Niven. Mr. Bearman and Mr. Ostrom were introduced by Professor Cameron.

1. Irving Kaplansky and R. F. Arens: *Topological representation of algebras.* (Abstract 53-5-174.)

2. M. J. Norris: *Indeterminates over a universal algebra.* (Abstract 53-5-177.)

3. R. L. Wilson: *A finite method for the determination of the Galois group of an equation with an application to the problem of reducibility.* (Abstract 53-5-188.)

4. L. J. Paige: *Neofields.* Preliminary report. (Abstract 53-5-178.)

5. P. W. Carruth: *Generalised power series fields.* (Abstract 53-3-114.)

6. M. F. Smiley: *Alternative regular rings without nilpotent elements.* (Abstract 53-5-181.)

7. C. J. Everett and George Whaples: *Representation of classes of finite sets.* (Abstract 53-5-170.)

8. H. S. M. Coxeter: *Continuity in real projective geometry.* (Abstract 53-5-256.)

9. Tibor Rado: *On two-dimensional concepts of bounded variation and absolute continuity.* (Abstract 53-3-131.)

10. Tibor Rado and E. J. Mickle: *A new geometrical interpretation of the Lebesgue area of a surface.* (Abstract 53-5-223.)

11. P. V. Reichelderfer: *The effect of a lipschitzian transformation on the area of a continuous surface.* (Abstract 53-5-225.)

12. O. W. Rechard and P. V. Reichelderfer: *A new criterion for the extension of rectangle functions.* (Abstract 53-5-224.)

13. W. G. Leavitt: *A normal form for matrices whose elements are holomorphic functions.* (Abstract 53-5-213.)

14. L. H. Kanter: *On the roots of orthogonal polynomials and the related Christoffel numbers.* (Abstract 53-5-211.)

15. Paul Erdös and George Piranian: *Over-convergence on the circle of convergence.* (Abstract 53-5-201.)

16. Wilfred Kaplan: *The level curves of a harmonic function.* (Abstract 53-5-212.)

17. M. E. Shanks: *Decomposition homology groups.* Preliminary report. (Abstract 53-5-282.)

18. Beno Eckmann: *On infinite complexes with automorphisms.* Preliminary report. (Abstract 53-5-272.)

19. R. G. Helsel and Tibor Rado. *On the Cauchy area of a Fréchet surface.* (Abstract 53-5-207.)

20. O. H. Hamilton: *A fixed point theorem for upper semi-continuous transformations on n-cells for which the images of points are non-acyclic.* (Abstract 53-5-274.)

21. R. H. Bing: *A homogeneous indecomposable plane continuum.* (Abstract 53-5-268.)

22. R. L. Wilder: *Recognition of manifolds by accessibility properties.* (Abstract 53-5-287.)

23. Rufus Isaacs: *Recent progress in the theory of compressible fluids.* (Abstract 53-5-249.)

24. J. W. Calkin: *Incipient shock waves in one dimension.* (Abstract 53-5-242.)

25. M. Z. Krzywoblocki: *On the two-dimensional steady flow of a compressible viscous fluid far behind a solid symmetrical body.* (Abstract 53-5-251.)

26. Isaac Opatowski: *Application of a theorem of Jacobi to compressible and rotational flows.* (Abstract 53-5-252.)

27. J. C. C. McKinsey and Alfred Tarski: *Some theorems about the sentential calculi of Lewis and Heyting.* (Abstract 53-5-263.)

28. J. E. Wilkins: *Neumann series of Bessel functions.* (Abstract 53-5-241.)

29. G. M. Ewing: *Variation problems formulated in terms of the Weierstrass integral.* (Abstract 53-5-202.)

30. E. H. Rothe: *Extrema of functions in Banach spaces.* Preliminary report. (Abstract 53-5-226.)

31. J. E. Bearman: *Rotations in the product of two Wiener spaces.* Preliminary report. (Abstract 53-5-189.)

32. C. W. Mathews: *Cauchy type double integral representations for functions of a complex variable.* (Abstract 53-5-218.)

33. T. G. Ostrom: *The solution of linear integral equations by means of Wiener integrals.* (Abstract 53-5-222.)

34. R. H. Stark: *Some classes of monotone functions.* (Abstract 53-5-232.)

35. W. F. Eberlein: *A note on ergodic theory.* (Abstract 53-5-199.)

36. M. R. Hestenes: *Sufficient conditions for isoperimetric multiple integral problem in the calculus of variations.* (Abstract 53-5-208.)

37. Marshall Hall: *Cyclic projective planes.* (Abstract 53-3-152.)

38. B. M. Stewart: *Left associated matrices with elements in an algebraic domain.* II. (Abstract 53-5-183.)

39. V. G. Grove: *On the Darboux tangents.* (Abstract 53-5-258.)

40. R. D. Wagner: *The generalized Laplace equations in a function theory for commutative algebras.* (Abstract 53-3-119.)

41. C. C. Hsiung: *Differential geometry of a surface at a parabolic point.* (Abstract 53-5-259.)

42. B. J. Lockhart: *Covariants of a valence correspondence on an algebraic curve.* (Abstract 53-5-261.)

43. Paul Erdös and Ivan Niven: *On the roots of a polynomial and its derivative.* (Abstract 53-5-169.)

44. Arthur Bernhart: *Quantitative analysis of rings in minimal maps.* (Abstract 53-5-166.)

45. R. H. Bing: *Extending a metric.* (Abstract 53-5-269-*t*.)

46. W. F. Eberlein: *A nonlinear differential equation.* (Abstract 53-5-198-*t*.)

47. W. F. Eberlein: *A note on spheroidal wave functions.* (Abstract 53-5-245-*t*.)

48. Benjamin Epstein: *On the mathematical description of certain breakage mechanisms.* (Abstract 53-5-246-*t*.)

49. Rufus Isaacs: *Inverse iterates.* (Abstract 53-5-209-*t*.)

50. G. K. Kalisch: *On extensions of topological fields.* (Abstract 53-5-173-*t*.)

51. S. C. Kleene: *Analysis of lengthening of modulated repetitive pulses.* (Abstract 53-5-250-*t*.)

52. D. H. Lehmer: *On the Tarry-Escott problem.* (Abstract 53-5-175-*t*.)

53. E. J. Mickle: *Metric foundations of continuous transformations.* (Abstract 53-5-219-*t*.)

54. M. J. Norris: *A note on regular and completely regular topological spaces.* (Abstract 53-5-275-*t*.)

55. M. J. Norris: *On the representation of real numbers by sequences of integers.* (Abstract 53-5-276-*t*.)

56. A. E. Ross: *On continuous transformations of a continuous closed curve.* (Abstract 53-3-161-*t*.)

57. H. M. Schaerf: *Generalization of the theory of distance sets to topological groups.* Preliminary report. (Abstract 53-5-277-*t*.)

58. H. M. Schaerf: *On A. Weil's condition M.* Preliminary report. (Abstract 53-5-278-*t*.)

59. H. M. Schaerf: *On sets of the form* $B^{-1}A$ *in topological groups.* Preliminary report. (Abstract 53-5-279-*t*.)

60. H. M. Schaerf: *On the continuity of measurable mappings of neighborhood spaces.* (Abstract 53-5-280-*t*.)

61. H. M. Schaerf: *Properties of measures biinvariant under a composition law.* Preliminary report. (Abstract 53-5-228-*t*.)

62. H. M. Schaerf: *Two theorems on measure spaces with a composition law*. Preliminary report. (Abstract 53-5-229-*t*.)

63. G. E. Schweigert and G. S. Young. *Remarks concerning invariants for certain finite transformations*. II. (Abstract 53-5-281-*t*.)

64. M. F. Smiley: *Binary systems which are almost loops*. (Abstract 53-5-182-*t*.)

65. C. F. Stephens: *Concerning linear and nonlinear difference equations*. (Abstract 53-5-233-*t*.)

66. E. A. Trabant: *The Riemannian geometry of the symmetric top*. (Abstract 53-5-255-*t*.)

67. S. M. Ulam and John von Neumann: *On the group of homeomorphisms of the surface of the sphere*. (Abstract 53-5-283-*t*.)

68. N. A. Wiegmann: *Some theorems on normal matrices with analogs of the generalized principal axis transformation*. (Abstract 53-5-187-*t*.)

R. H. Bruck,
Associate Secretary

THE APRIL MEETING IN STANFORD UNIVERSITY

The four hundred twenty-sixth meeting of the American Mathematical Society was held at Stanford University, California, on April 26, 1947. The total attendance of approximately ninety included the following sixty-three members of the Society:

T. M. Apostol, H. M. Bacon, G. A. Baker, R. A. Beaumont, E. M. Beesley, B. A. Bernstein, Z. W. Birnbaum, H. F. Bohnenblust, A. H. Bowker, W. G. Brady, J. L. Brenner, A. C. Burdette, F. G. Creese, P. H. Daus, G. C. Evans, E. J. Farrell, E. A. Fay, F. G. Fisher, Evelyn Fix, A. L. Foster, J. W. Green, Leonard Greenstone, William Gustin, S. M. Hallam, Frank Harary, J. G. Herriot, J. L. Hodges, Alfred Horn, C. G. Jaeger, E. L. Lehmann, R. B. Leipnik, Hans Lewy, Rhoda Manning, B. C. Meyer, E. D. Miller, W. E. Milne, C. B. Morrey, F. R. Morris, Jerzy Neyman, C. D. Olds, Edmund Pinney, George Pólya, Robert Russel, A. C. Schaeffer, Henry Scheffé, Abraham Seidenberg, Ernst Snapper, Herbert Solomon, D. C. Spencer, Lee Swinford, J. D. Swift, Gabor Szegö, Alfred Tarski, A. H. Taub, A. E. Taylor, J. L. Ullman, F. A. Valentine, A. H. Van Tuyl, R. K. Wakerling, Morgan Ward, P. A. White, A. R. Williams, František Wolf.

The morning session was devoted to contributed papers, Professor C. B. Morrey presiding. During the afternoon session Dr. Abraham Seidenberg of the University of California delivered the invited hour address entitled *The local uniformization theorem*. Following this the remaining contributed papers were presented, Professor H. F. Bohnenblust presiding.

Titles and cross references to the abstracts of papers read at the meeting follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper 7 was read by Mrs. Lehmer.

1. P. A. White: *On the equivalence between avoidability and co-local connectedness*. (Abstract 53-3-163.)

2. A. E. Taylor: *The inverse of a polynomial function of a closed operator*. (Abstract 53-3-135.)

3. F. A. Valentine: *The determination of connected linear sections*. (Abstract 53-5-284.)

4. R. A. Beaumont: *Rings over a group*. (Abstract 53-5-165.)

5. Morgan Ward: *Elliptic divisibility sequences*. (Abstract 53-5-186.)

6. B. A. Bernstein: *Field in terms of multiplication and a unary operation*. (Abstract 53-5-167.)

7. D. H. Lehmer: *On the vanishing of Ramanujan's function*. (Abstract 53-5-176.)

8. Edmund Pinney: *Aerodynamically driven oscillations in suspension bridges*. (Abstract 53-5-253.)

9. Z. W. Birnbaum: *Probabilities of sample-means for bounded random variablies.* (Abstract 53-5-266.)

10. William Gustin: *A bilinear integral identity for harmonic functions.* (Abstract 53-5-205.)

11. A. H. Taub: *Orbits of charged particles in constant electromagnetic fields.* (Abstract 53-5-254.)

12. George Pólya: *On virtual masses.* (Abstract 53-7-295.)

13. Howard Eves: *Some consequences of a simple theorem on torque.* (Abstract 53-3-142-*t*.)

14. Howard Eves: *Systems of particles with a common centroid.* (Abstract 53-3-143-*t*.)

15. Leonard Greenstone: *Mapping by analytic functions.* Part II. *Pseudo-conformal distortion theorems.* Preliminary report. (Abstract 53-5-204-*t*.)

16. H. J. Hamilton: *Mertens' theorem and sequence transformations.* (Abstract 53-5-206-*t*.)

17. A. C. Schaeffer and D. C. Spencer: *A general class of problems in conformal mapping.* (Abstract 53-5-227-*t*.)

A. C. SCHAEFFER,
Associate Secretary

BOOK REVIEWS

Mathematical methods of statistics. By H. Cramér. (Princeton Mathematical Series, no. 9.) Uppsala, Almqvist and Wiksells, 1945; Princeton University Press, 1946. 16+575 pp. $6.00.

This book has long been needed, and its effect on the future development of mathematical statistics in both teaching and research will be sharp and lasting. The achievement of the author is to present the first treatise on statistics in which the mathematical developments are carried through according to standards of rigor comparable to those now customary in pure mathematics and excelling those in many fields of applied mathematics. He has accomplished this by integrating modern statistical concepts developed by the English and American schools of statisticians, with the mathematical methods of the modern work in probability of the French and Russians. While uniting these two streams he has added important contributions of his own.

The book is divided into three parts. Part I, called *Mathematical introduction*, consists mainly of an exposition of measure and integration, but contains also needed parts of matrix theory, theory of Fourier integrals, and some other topics. Part II, called *Random variables and probability distributions*, develops general concepts associated with random variables and applies them to some specific distributions such as the binomial, normal, chi-square, and so on. Part III, called *Statistical inference*, deals also with asymptotic and exact sampling distributions. Actually only a little over a fourth of the book deals specifically with statistical testing and estimation, and here the treatment is not exhaustive but limited to fundamentals. Much recent work of a fundamental character in statistical inference is omitted because the author's contact with foreign publications was curtailed by the war; for example, there is no reference to any of Wald's work. The main value of the book will be to supply mathematical background for those working in advanced mathematical statistics.

In his preface the author states that he has tried to make his book accessible to the reader with "a good working knowledge of the elements of the differential and integral calculus, algebra, and analytic geometry." However, a student not previously familiar with ϵ-δ analysis would find the book too difficult. On the other hand, the level of difficulty is much below that of the author's Cambridge Tract. The reviewer would guess that American students beginning graduate mathematics should be able to handle the book. At its chosen level of difficulty it is a masterwork of exposition. The student is led with skill and charm from the familiar to the new: For example, beginning on the real line, Lebesgue measure is introduced as a natural generalization of the notion of length of an interval. Using upper and lower Darboux sums, the Lebesgue integral of a bounded function on a set of finite measure is revealed as a generalization of the Riemann integral, the allowable subdivisions of the set in the Darboux sums being now less restricted. With a fixed function for the integrand and a variable set, the Lebesgue integral furnishes a concrete example of a completely additive set function, and this leads to the consideration of more general measures called "P-measures." Next a simple modification of the Lebesgue integral, in which Lebesgue measure is replaced by P-measure, yields the Lebesgue-Stieltjes integral. With this one-dimensional development behind him the student is then prepared for a k-dimensional development using fewer steps, namely: Lebesgue measure, P-measure, Lebesgue-Stieltjes integrals.

Among subjects usually developed in a mathematically questionable way but here presented with rigor we may mention (1) the asymptotic normality of large classes of statistics, (2) the asymptotic distribution of Karl Pearson's chi-square statistic for goodness of fit in cases where parameters of the theoretical distribution are estimated from the sample, (3) the optimum properties of R. A. Fisher's maximum likelihood estimates. All of these developments are enriched by original contributions of the author; especially exciting is the new small sample theory of efficiency of estimates created by him in relation to (3).

One hesitates to make any adverse criticisms of this book for fear of giving a wrong impression of the balance of good and bad: the weight of the mathematically praiseworthy is overwhelming, while the few features which are possibly objectionable are mainly so for nonmathematical and subjective reasons based on one's statistical taste and philosophy. Of the half dozen such features which troubled the reviewer two may be worth mentioning. A correct exposition is given of the meaning of Neyman's confidence intervals, in which the interval is a random interval such that the probability that it cover the unknown parameter value θ is a fixed preassigned number α, independent of the true value of θ. But after this the author gives a new twist to the method. A fixed interval (a, b) is preassigned. A "confidence coefficient" α for the statement $a \leqq \theta \leqq b$ is then calculated from the sample, so that α is a random variable. While there seem to be no definitely wrong statements made, nevertheless the result may be confusion of the reader as to the meaning of confidence intervals.

The simple frequency interpretation is lost: Thus with Neyman's confidence intervals if the statistical assumptions were satisfied in the applications, the user would know that of all statements he makes with confidence coefficients $\alpha \geqq \alpha_0$, the fraction of these statements which are correct will be not less than α_0 "in the long run," regardless of what the true values of the parameters happened to be in the situations encountered. The user of the Cramér "confidence intervals" has no such assurance: It becomes obvious that in a sequence of confidence statements *all* may be wrong, by thinking of a sequence of experiments in which θ is fixed and outside the interval (a, b).

Another criticism, which may be of a controversial nature, is that the book shows insufficient concern with the question whether data generated in a given way may safely be assumed to be results of trials on the same random variable, or in Shewhart's language, whether the underlying chance cause system is constant. This manifests itself in several ways: (1) Among the illustrations of "random experiments" are some, such as commodity prices, or quality characteristics of factory output, whose "randomness" is highly questionable. (2) In the first description given of statistical tests, the order of the observations is disregarded, that is, it is assumed that the critical region of the test is always a part of the sample space completely symmetrical in all coordinates. (3) No tests of randomness are mentioned.

Clearly an oversight is the statement that in using the analysis of variance to test the hypothesis of equality of a set of unknown means the appropriate critical region of Fisher's statistic z is $|z| > z_p$, and that the z-tables are calculated for such a two-tailed test.

The number of misprints is so amazingly low as to deserve comment.

Here are some suggestions to the author for additions to a future edition of this fine work, or to teachers for supplementing the present edition: (1) It would be useful in view of the recent development of sequential analysis to have some material on measure and integration in the space of infinite sequences $(x_1, x_2, \cdots)$. (2) It would be enlightening to relate the estimation and testing problems arising in the analysis of variance, the analysis of covariance, and linear regression, as examples of a single theory for "linear hypotheses." (3) Even though the treatment of statistical inference is not intended to be extensive, a brief mention at the end of the book of Wald's formulation of the general problem of statistical inference and his concept of statistical decision function might be justified. (4) The value of the book to most users would be increased by augmenting the 39 problems.

HENRY SCHEFFÉ

Moderna teoria delle funzioni di variabile reale. By G. Vitali and G. Sansone. (Monografie di matematica applicata per cura del Consiglio Nazionale delle Ricerche.) 2d ed., Bologna, Zanichelli, 1943–1946. Vol. 1, 9+194 pp., vol. 2, 7+511 pp.

The first edition of this excellent treatise was reviewed in vol. 43 (1936) p. 15 and vol. 45 (1939) p. 218 of this BULLETIN. The first volume has not been changed much in the revision; a discussion of the theorems of Egoroff-Severini and Lusin in chapter III seems to be the most important addition. On the other hand, the second volume has grown by more than 200 pages. A completely new chapter on approximation and interpolation has been added. About 80 pages are devoted to the theorem of Weierstrass (the Landau-de la Vallée-Poussin approach for the rational case and that of Dunham Jackson for the trigonometric one), best order of approximation, Chebychev polynomials, the interpolation polynomials of Lagrange and Hermite, Bernstein polynomials, and so on.

There are other important additions. Thus in chapter I a discussion of L_p spaces is added. In chapter II the theory of Fourier series has been enlarged somewhat, the main addition being devoted to (C, α) summability. In chapter III the formula of Mehler has been added as well as a discussion of the zeros of Legendre polynomials. The major addition to this chapter, however, is a discussion of the functions on the sphere as well as (C, α) and Poisson summability of the series of Laplace and of Legendre. Chapter IV on Hermite and Laguerre polynomials is practically unchanged. Finally, in the last chapter on Stieltjes integrals the discussion of characteristic functions (=Fourier-Stieltjes transforms of distribution functions) has been completed by the addition of the Bochner-P. Lévy inversion formula and some convergence theorems. The additions have done much to enhance the value of the treatise which has become a convenient and reliable handbook in classical analysis. The book can be strongly recommended. It is to be hoped for that the misspelling of the name of Walsh will be corrected in the third edition.

EINAR HILLE

Cours d'analyse mathématique. By G. Valiron. Paris, Masson, 1942–1945. Vol. 1, 2+522 pp., vol. 2, 2+605 pp.

The two volumes under review constitute a comprehensive account of the following topics: the differential and integral calculus, the general theory of analytic functions of a complex variable, algebraic functions and Abelian integrals, a brief introduction to analytic functions of several variables, the theory of ordinary differential equations in the real and complex domain, a brief introduction to the calculus of variations, differential geometry, partial differential equations of the first and second order. The primary object of these volumes is to give an exposition of the material of the "*certificat de calcul différentiel et intégral.*" However on many occasions the author amplifies the scope of his work and goes out of the immediate confines of his curriculum. In spirit, the treatment is classical and aims at giving a rigorous and at the same time succinct account of the topics treated. The author has attained his goal in a distinguished manner and has made available to the mathematical public a very useful pair of volumes on the subject of classical analysis.

One notable feature of the work is the extensive treatment given to the differential and integral calculus (some three hundred pages, about three-fifths of the first volume). In this way an unequivocally solid foundation is laid for the material that follows. This fact lends significance to the book on the American scene where just such material is basic in the cradle phase of graduate mathematical instruction.

We have remarked that the spirit of the work is classical; there is, however, constant reference to research in mathematical analysis of recent years that keeps these volumes in close contact with modern interest. The general pattern of each chapter is to start with a brief historical account of the topics treated and terminate with reference to recent monographs and special tracts for further study.

The first volume starts with a resumé of the fundamental properties of the real and complex number systems, point sets, sequences, and a sketch of the theory of continued fractions. The second chapter treats systematically infinite series and products. The author then turns to a study of functions of a single variable which includes the usual theorems on continuous and differentiable functions, monotone functions, convex functions, arc length, the Riemann integral and applications, including proofs of the transcendental character of e and π. There follows a brief treatment of the Lebesgue integral of a complementary nature.

The remaining chapters on functions of a single real variable are devoted to functions defined by series or integrals, Fourier series and orthogonal functions, the reduction and mechanical calculation of integrals.

A careful account is given of functions of several variables and multiple integral theory. These topics are treated in fitting detail. Particular mention should be made of the care used in discussing change of variable in double integrals. Of course these topics are treated under hypotheses which are restrictive enough to allow simple proofs but which are quite adequate for the usual applications.

The author then proceeds to his account of analytic functions of a single complex variable. The major topics treated are: the elementary functions; the Cauchy theory and applications including the calculus of residues; the Weierstrass theory, analytic continuation, and representation theorems for entire functions; conformal mapping including the fundamental mapping theorem, the results of Schwarz concerning polygonal domains, and as an application of the fundamental theorem a study of the elliptic modular function and theorems of the Picard type; elliptic functions; analytic functions defined by integrals and Hadamard's prime number theorem treated by the method of Landau.

The first chapters of Volume II are concerned with the theory of algebraic functions of a single variable and the associated theory of Abelian integrals. Applications are made to plane algebraic curves and the Jacobi inversion problem.

As preparation for the comprehensive treatment of the theory of ordinary differential equations which follows, a study is made of analytic functions of several variables, implicit function theorems, the method of majorants, and in addition a brief account is given of the theory of contact and envelopes.

The theory of ordinary differential equations is treated in both the real and complex domains. The ground covered is extensive and is well suited for a semester graduate course. Topics treated include the elementary theory; classical equations of the first order; the linear theory, and in particular, the problem of isolated singular points in the complex domain (studied by the methods developed by G. D. Birkhoff); nonlinear differential equations in the complex domain; the classical existence theorems; a brief account of the applications of Lie's theory; the theories of Poincaré and Bendixson concerning singularities in the real domain; Sturmian theorems.

The remaining chapters are concerned with the elements of the calculus of variations, the classical differential geometry of curves and surfaces in euclidean 3-space, and partial differential equations of the first and second order.

Even this brief account of the topics treated will indicate the vast labor involved in the preparation of these volumes. Apart from presenting a wealth of material in a concise rigorous manner, the book has the special merit of synthesizing the subject matter of a *Traité d'analyse* with the results of the research of the last decades.

MAURICE HEINS

Algèbre. Vol. 1. *Équivalences, opérations, groupes, anneaux, corps.* By Paul Dubreil. (Cahiers scientifiques, no. 20.) Paris, Gauthiers-Villars, 1946. 7+305 pp. 585 fr.

This book is the first volume of a series which is to cover the material necessary to a thorough introduction to modern algebra. The author presents a rather complete discussion of the basic facts of algebra such as the theory of groups, rings and fields. The material is elementary, that is, fundamental definitions and their consequences are discussed in a self-contained fashion. Thus, complications of proofs which usually arise when a mathematical system is specialized do not occur. No appeal to previous knowledge is made; at no place are facts stated without proof. The following list of contents illustrates the aim of the author to emphasize basic principles, constructions and techniques: Chapter I, sets and correspondences; chapter II, operations; chapter III, groups; chapter IV, regular equivalences; chapter V, fields; chapter VI, rings; chapter VII, algebraic equations.

Starting with elementary facts of set theory and emphasis on a single operation (defined by the use of abstract relations or equivalently by subsets of a cartesian product) additional postulates for the underlying mathematical system are gradually introduced. This method of presentation helps clarify the significance and implications of a given set of hypotheses, and the student becomes aware, at an early stage, of the interplay of ideas in a proof. An illustration of this approach may be found in the careful exposition of the general principles for the existence proof of complete fields; all details here are prepared carefully. In this connection it is to be pointed out that the "abstract" work of the initial chapters is by no means dry. The illustrations and exercises are chosen with thought; they are not constructions ad hoc. Rather, the reinterpretation of familiar results are used to enliven the general theory.

It is good to find in this book (p. 209) a clear statement of the distinction which should be made between the concepts of polynomial prevalent in algebra and analysis. The main body of the text is supplemented by four notes which are particularly useful since they emphasize the characterization of the system of natural numbers and the principles of induction.

O. F. G. SCHILLING

Funzioni analitiche. By Francesco Tricomi. 2d ed., Bologna, Zanichelli, 1946. 7+134 pp. 300 lire.

This is a brief sketch of the elementary theory of analytic functions of one complex variable. The author succeeds admirably in presenting a very readable account which makes plain the importance of the concepts, the power of the methods, and the broad outlines of the proofs.

More space than is usual in such a brief account is given to the connections of this theory with the theory of harmonic functions and of contour integration in vector analysis. Nine pages are devoted to simple applications to the theory of hydrodynamics. About six pages are devoted to well-illustrated discussions of the graphical representation of analytic functions by means of level lines of the real and imaginary parts. There are twenty-nine very good diagrams.

Little attention is given to questions of foundations. The reader is apparently assumed to have a working knowledge of real analysis from the point of view of a moderately rigorous upper-college course in the calculus, and the proofs are handled in the spirit of such a course.

The first chapter treats the definitions of continuity and analyticity, with some emphasis on the connection of the latter with the problem of conformal mapping. The second chapter discusses the Cauchy integral formula, with special attention to the relation of the Cauchy integral theorem to Green's theorem; the principle of the maximum is here derived for harmonic functions, and for the modulus of an analytic function. The third chapter considers Taylor's series and Laurent series, and the classification of singularities. The fourth chapter treats the general problem of analytic continuation, the Weierstrass concept of the complete analytic function, and Riemann surfaces; the special topics of Schwarzian reflection, exponential and trigonometric functions, Mittag-Leffler's theorem for meromorphic functions, and the Weierstrass product-theorem for entire functions are considered.

WALTER STRODT

NOTES

Professor P. A. Alexandroff of the University of Moscow has been elected a foreign associate of the National Academy of Sciences.

Dr. L. M. Brown of the University of Edinburgh, Dr. J. M. Hyslop of the University of Glasgow, Dr. J. M. Jackson of University College, Dundee, and Dr. A. J. Macintyre of the University of Aberdeen have been elected fellows of the Royal Society of Edinburgh.

Miss Mary L. Cartwright of the University of Cambridge and Professor T. G. Cowling of the University College of North Wales have been elected fellows of the Royal Society.

Dean R. D. Carmichael of the University of Illinois has retired with the title emeritus.

Professor O. E. Neugebauer of Brown University and Professor Hassler Whitney of Harvard University have been elected to membership in the American Philosophical Society.

Professor P. A. Smith of Columbia University has been elected to membership in the National Academy of Sciences.

Dr. Patricia Dolciani of Cornell University has been awarded the Sigma Delta Epsilon post doctoral fellowship for 1947–1948 and will study at the Institute for Advanced Study.

Messrs. Paul Olum of Harvard University and E. H. Spanier of the University of Michigan have been awarded Frank B. Jewett fellowships for 1947–1948 by the American Telephone and Telegraph Company.

Associate Professor J. B. D. Derksen of the Netherlands School of Economics at Rotterdam has joined the Statistical Office of the United Nations at Lake Success.

Dr. G. H. A. Grosheide and Dr. J. Haantjes of the Free University of Amsterdam have been promoted to professorships. This is a correction of an item in the March issue of this Bulletin.

Mr. K. J. Arrow of Columbia University has been appointed research associate with the Cowles Commission for Research in Economics.

Professor W. L. Ayres of Purdue University has been appointed Dean of the School of Science.

Mr. R. C. Buck of Harvard University has been appointed to an assistant professorship at Brown University.

Professor Samuel Eilenberg of Indiana University has been appointed to a professorship at Columbia University.

Dr. Abe Gelbart of Syracuse University will be on leave of absence at the Institute for Advanced Study.

Professor H. H. Germond of the University of Florida has accepted the position of director of research for F. W. Marshall Company, New York, New York.

Associate Professor H. S. Gould of Victoria College, University of Toronto, has been appointed to an assistant professorship at Purdue University.

Dr. E. J. Gumbel of Newark College of Engineering has been appointed to an associate professorship at Brooklyn College.

Dr. O. G. Harrold of Princeton University has been appointed to a professorship at the University of Tennessee.

Assistant Professor Edwin Hewitt of Bryn Mawr College has been appointed lecturer at the University of Chicago.

Dr. Carl Kossack of the Navy Department has been appointed to an associate professorship at Purdue University.

Professor C. G. Latimer of the University of Kentucky has been appointed to a professorship at Emory University.

Professor L. L. Lowenstein of Alfred University has been appointed to a professorship at Kent State University.

Assistant Professor Andrewa R. Noble of Montana State University has been appointed to an associate professorship at Pacific University, Forest Grove, Oregon.

Mr. J. I. Northam of the University of Wisconsin has been appointed to an assistant professorship at Kansas State College.

Assistant Professor A. M. Peiser of Rutgers University has been appointed mathematician with Hydrocarbon Research, Incorporated, New York, N. Y.

Professor H. B. Phillips of the Massachusetts Institute of Technology has retired.

Assistant Professor R. S. Phillips of New York University has been appointed to an associate professorship at the University of Southern California.

Associate Professor Arthur Rosenthal of the University of New Mexico has been appointed to a professorship at Purdue University.

Professor A. C. Schaeffer of Stanford University has been appointed to a professorship at Purdue University.

Assistant Professor Ruth G. Simond of Berea College has been appointed to an assistant professorship at Morningside College, Sioux City, Iowa.

Assistant Professor G. B. Van Schaack of Union College has been appointed to an assistant professorship at Washington University.

Assistant Professor G. L. Walker of Temple University has been appointed to an assistant professorship at Purdue University.

Assistant Professor A. D. Wallace of the University of Pennsylvania has been appointed to a professorship at Tulane University.

Assistant Professor Kenichi Watanabe of the University of Hawaii has been appointed to an assistant professorship at Wabash College, Crawfordsville, Indiana.

Associate Professor E. L. Welker of the University of Illinois has been appointed associate in mathematics in the Bureau of Medical Economic Research of the American Medical Association.

Dr. G. W. Whitehead of Princeton University has been appointed to an assistant professorship at Brown University.

The following promotions are announced:

Max Astrachan, Antioch College, to a professorship.

John Dyer-Bennet, Purdue University, to an assistant professorship.

Cleota G. Fry, Purdue University, to an assistant professorship.

W. H. Gottschalk, University of Pennsylvania, to an assistant professorship.

M. H. Heins, Brown University, to a professorship.

H. K. Hughes, Purdue University, to a professorship.

C. W. Jordan, Williams College, to an assistant professorship.

M. W. Keller, Purdue University, to a professorship.

R. R. Kuebler, Dickinson College, Carlisle, Pennsylvania, to an assistant professorship.

G. E. Markle, University of Detroit, to an assistant professorship.

R. D. Mindlin, Columbia University, to a professorship.

M. G. Moore, Bradley University, Peoria, Illinois, to an associate professorship.

W. D. Munro, University of Minnesota, to an assistant professorship.

Nilan Norris, Hunter College, to an assistant professorship.

Sam Perlis, Purdue University, to an assistant professorship.

E. K. Ritter, Postgraduate School, United States Naval Academy, to an associate professorship.

C. K. Robbins, Purdue University, to an associate professorship.

J. C. Smith, Lafayette College, Easton, Pennsylvania, to an assistant professorship.

D. W. Western, Brown University, to an assistant professorship.

The following appointments to instructorships are announced:

Brown University: Mrs. R. C. Buck, Dr. Kathleen E. Butcher; University of California: Dr. E. W. Barankin; United States Naval Academy: Mr. J. R. Gorman.

Professor P. J. Daniells of the University of Sheffield died May 25, 1946.

Professor L. G. Owen, formerly of the University of Rangoon, died February 23, 1947.

Mr. C. C. Andersen of Chicago, Illinois, died on April 21, 1947.

Professor H. J. Gay of Worcester Polytechnic Institute died May 1, 1947, at the age of forty-nine years. He had been a member of the Society since 1921.

Professor Emeritus C. J. Keyser of Columbia University died May 8, 1947 at the age of eighty-six years. He had been a member of the Society since 1897.

Professor W. T. Short of Oklahoma Baptist University died February 19, 1947.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

Announcement. Beginning with the report of the 1947 Summer Meeting, this BULLETIN will publish the abstracts of papers offered for presentation at a meeting of the Society as part of the report of the meeting. This arrangement will save considerable space in the BULLETIN due to the fact that it will no longer be necessary to print the title of a paper and the name of the author as part of the abstract and also as part of the report of a meeting. The present plan of publishing abstracts was inaugurated in 1930 in the expectation that abstracts would appear in the BULLETIN before the papers were read at meetings of the Society. Unfortunately a large proportion of the abstracts are not received in time to make such advance printing possible.

The Editors are pleased to announce in this connection that the Secretary of the Society plans to have available for distribution at as many meetings of the Society as possible mimeographed copies of the abstracts of papers to be presented. It is believed that such a distribution of the abstracts will be of considerable assistance to easier understanding of the papers presented.

ANALYSIS

288. M. Fekete: *On generalized transfinite diameter.*

The function $g(x)$ is called a generator function if (1) $g(x)$ is continuous for $0<x<\infty$; (2) $g(x)$ is strictly decreasing with x^{-1}; (3) $\lim_{x\to 0} g(x)=\infty$. Let C be an infinite limited-closed point set of a Euclidean space E_q of q dimensions and $g(x)$ a generator function. Put $C_{n,2}g(\delta_n)=$minimum of $\sum_{1\leq\mu<\nu\leq n} g([P_\mu P_\nu])$ when $n\geq 2$ and P_i, $1\leq i\leq n$, is a subset of C of $n\geq 2$ points and $[P_\mu P_\nu]$ denotes distance of P_μ, P_ν. Then $\delta_n=\delta_n(C, g)$, the "diameter of order n of C with respect to $g(x)$" cannot increase as n increases and is positive for $n\geq 2$. Call $\lim_{n\to\infty}\delta_n=\delta=\delta(C, g)$ the "transfinite diameter of C with respect to $g(x)$." In case $q=2$, $g(x)=\log x^{-1}$, δ coincides with the transfinite diameter introduced by the author (Math. Zeit. vol. 17 (1923)); for $q=3$, $g(x)=x^{-1}$, δ is the generalization of the former notion by Pólya-Szegö (J. Reine Angew. Math. vol. 165 (1931)). For arbitrary $q\geq 1$ and $g(x)$, $\delta_2(C, g)=s=s(C)=$the span of C. It is obvious that $\delta(C, g)\leq\delta(D, g)$ whenever C is contained in D (monotonicity) whence $\lim_{\rho\to 0}\delta(C(\rho); g)\geq\delta(C, g)$ when $C(\rho)$ is the ρ-neighborhood of C. It is easily shown that the former relation always holds with the equality sign (continuity). A deeper result is the "law of subadditivity": If $C = C' + C''$ and $\delta(C', g) = \delta'$, $\delta(C'', g) = \delta''$, $s(C) = s$ then $(g(\delta) - g(s))^{-1} \leqq (g(\delta') - g(s))^{-1} + (g(\delta'') - g(s))^{-1}$. This leads, in case $q = 3$, $g(x) = x^{-1}$, to an improved form of Kellog's rule of conductor capacities. (Received April 26, 1947.)

289. Philip Hartman and Aurel Wintner: *Töplerian* (L^2)*-bases.*

Let $\varphi(t)$ be an odd, periodic function of period 2π. The problem of conditions, sufficient or necessary, in order that the sequence $\phi(t), \phi(2t), \phi(3t), \cdots$ form an (L^2)-basis on $(0, \pi)$ is considered. The similar problem of the possibility of an (L^2)-expansion, $f(t) \sim \sum c_n \phi(nt)$, for arbitrary functions $f(t)$ of class (L^2) in $(0, \pi)$ is also treated. The problem is attacked by considering the infinite set of linear homogeneous equations $Dx = 0$, where D is the Toeplitz D-matrix associated with the Dirichlet series $\sum \phi_n n^{-s}$. (Received April 15, 1947.)

290. Mark Kac: *Distribution properties of certain gap sequences.*

Erdős and Fortet noticed that for $f(x) = \cos 2\pi x + \cos 4\pi x$ and $n_k = 2^k - 1$ the distribution function of (*) $m^{-1/2}\sum_1^m f(n_k x)$, $0 \leqq x \leqq 1$, cannot approach the normal distribution. In this paper it is shown that the measure of the set of x's for which (*) is less than ω approaches, as $m \to \infty$, the integral $\int_0^1 h(\omega, x)dx$, where $h(\omega, x) = \pi^{-1/2}\int_{-\infty}^{\omega/|\cos \pi x|} \exp(-v^2)dv$. Actually, a much more general theorem can be established but its statement is too complicated to be included here. (Received April 25, 1947.)

291. Otto Szász: *Quasi-monotone series.*

A sequence $\{a_n\}$ of positive numbers is called quasi-monotone if, for some $\alpha \geqq 0$, $a_{n+1} \leqq a_n(1 + \alpha/n)$, $n = 1, 2, \cdots$. A series is called quasi-monotone if its terms form a quasi-monotone sequence. Such series have many properties in common with monotone series (a_n decreasing). Thus, (1) if $\sum a_n < \infty$, then $na_n \to 0$, (2) the series $\sum a_n$ and $\sum 2^n a_{2^n}$ are either both convergent or both divergent. Analogous results hold for infinite integrals. (Received April 11, 1947.)

APPLIED MATHEMATICS

292. K. O. Friedrichs: *On the non-occurrence of a limiting line in transonic flow.*

In this paper the author treats gas flows around an air foil or in a duct which are subsonic at the beginning and the end and possess supersonic regions adjacent to the boundary. Consider a set of such flows depending on a parameter such as the free stream Mach number M. Then the problem arises whether for a certain value M^* of M a limit line can appear for the first time. It is proved that that cannot occur. To this end the linear hodograph differential equations for x and y as functions of u and v are considered and it is shown that if the Jacobian never vanishes for $M < M^*$, it does not vanish for $M = M^*$, neither at the boundary nor in the interior of the flow. (Received April 11, 1947.)

293. K. O. Friedrichs: *Water waves on a shallow sloping beach.*

The potential function of the motion of water waves on a beach whose slope is an integer fraction of a right angle was first derived by J. J. Stoker and H. Lewy. In the present paper a different representation of this potential is given from which it is possible, with the aid of the saddle point method, to derive an asymptotic representation valid for small slope angles. Simple formulas for the variation of wave length, amplitude and phase with the depth are obtained. The agreement with the results of exact calculations for a beach with a slope of 6° is perfect. (Received April 11, 1947.)

294. K. O. Friedrichs and Hans Lewy: *Dock problem.*

The problem of two-dimensional gravity waves in a half space filled with water, one-half of the surface being covered by a rigid plate called the "dock," requires the determination of an analytic function $\chi(s)$ defined in the lower half space such that $d\chi/ds$ is real on the positive real axis while $d\chi/ds+i\chi$ is negative on the negative real axis and such that χ behaves like ke^{-is} at infinity when approached in positive real direction while it vanishes at infinity when approached by any other direction. The solution is given explicitly in terms of two analytic functions represented explicitly as complex integrals. One of the functions is bounded at the origin, representing the edge between the dock and the free water surface, the other one having a logarithmic singularity there. The ratio of the amplitudes at the edge and at infinity for the former solution is $1/2^{1/2}$ in agreement with the general result for waves on sloping beaches and with a slope angle $p\pi/2q$ derived in a paper of the second named author. (Received April 11, 1947.)

295. George Pólya: *On virtual masses.*

A solid moves through an ideal incompressible fluid that fills the whole space outside the solid and is at rest at infinity. The motion of the solid is pure translation with velocity 1, the density of the fluid is 1 and its kinetic energy T. The so-called virtual mass of the solid, $2T=M$, depends on the direction of the velocity; the direction is assessed with respect to a coordinate system rigidly tied to the solid. Averaging M over all directions, we obtain $\overline{M}$. There are reasons to conjecture that *of all solids with given volume the sphere has the minimum average virtual mass* $\overline{M}$. This conjecture has been verified for a general ellipsoid and the analogous theorem in two dimensions has been completely proved. (Received April 11, 1947.)

296. J. L. Synge: *Apsidal angles for symmetrical dynamical systems with two degrees of freedom.*

The path of a particle on a surface of revolution corresponds to a geodesic on a certain surface of revolution S for which distance is defined by the Jacobi action. There is direct correspondence between apsides for the two systems, and the apsidal angles are the same. Upper and lower bounds for the apsidal angle are obtained. (Received April 10, 1947.)

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A TERNARY OPERATION IN DISTRIBUTIVE LATTICES

GARRETT BIRKHOFF AND S. A. KISS

It can be easily seen that the *graph* [1, p. 9],[1] of the Boolean algebra B^n of 2^n elements (consisting of the vertices and edges of an n-cube) has $2^n(n!)$ "link-automorphisms," whereas B^n has only $(n!)$ lattice-automorphisms. In an unpublished book,[2] one of us has developed new operations in B^n and other distributive lattices, which admit such a wider group of invariance. The purpose of this note is to show the role of the symmetric and self-dual ternary operation [1, p. 74]

$$(1)\qquad \begin{aligned}(a, b, c) &= (a \cap b) \cup (b \cap c) \cup (c \cap a)\\ &= (a \cup b) \cap (b \cup c) \cap (c \cup a)\end{aligned}$$

in a general distributive lattice L, with reference to the wider group of symmetries which it admits.

THEOREM 1. *In any metric distributive lattice* [1, p. 41], *the following conditions are equivalent*: (i) $a \cap b \leqq x \leqq a \cup b$, (ii) $|a-x| + |x-b| = |a-b|$, (iii) $(a, x, b) = x$.

PROOF. V. Glivenko [3, p. 819, Theorem V] has shown the equivalence of (i) and (ii); condition (i) says that x is metrically "between" a and b in the sense of Menger. But now if $a \cap b \leqq x \leqq a \cup b$, then $(a, x, b) = (a \cap b) \cup (b \cap x) \cup (x \cap a) = (a \cap b) \cup [x \cap (a \cup b)] = x$. Conversely, if $(a, x, b) = x$, then

$$x = (a \cap b) \cup (b \cap x) \cup (x \cap a) \geqq a \cap b,$$

and dually, $x \leqq a \cup b$. Hence (i) and (iii) are equivalent.

DEFINITION. The segment joining a and b is the set of x satisfying any (hence all) of the conditions of Theorem 1 (cf. Duthie [4]); we denote it $[a, b]$.

THEOREM 2. *The element* (a, b, c) *is the intersection of the sets* $[a, b]$ $[b, c]$ $[c, a]$.

PROOF. This is obvious from condition (i) and formula (1).

COROLLARY 1. *The element* (a, b, c) *minimizes*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] S. A. Kiss, *Transformations on lattices and structures of logic*, Bull. Amer. Math. Soc. Abstract 52-1-4.

$$|x-a|+|x-b|+|x-c|.$$

PROOF. In fact, $|x-a|+|x-b| \geqq |a-b|$ and symmetrically. Adding, we get for all x

$$2\{|x-a|+|x-b|+|x-c|\} \geqq |a-b|+|b-c|+|c-a| \tag{2}$$

By Theorems 1 and 2, equality holds if and only if $x=(a, b, c)$.

Since (a, b, c) can be defined in terms of distance and a, b, c, we get, further, the following corollary.

COROLLARY 2. *Any isometry of a metric distributive lattice preserves the operation* (a, b, c).

In neither nondistributive lattice of five elements, does every triple of elements a, b, c, determine a unique x minimizing $|x-a|+|x-b|+|x-c|$; it would be interesting to know what the lattices are in which this "midpoint" is uniquely determined.

By using a weighted dimension function [1, Theorem 3.9] on B^2, we can find easily lattice-automorphisms which are not isometries. However, in the finite-dimensional case, we can obtain a converse to the preceding corollary.

We define *distance* in the graph of a finite-dimensional lattice in the usual way, by making all segments have length one. Thus if $d[x]$ is the number of "links" in the longest chain joining O and x, $|x-y| = d[x\cup y]-d[x\cap y]$. This makes L a metric lattice [1, Theorem 3.9].

THEOREM 3. *The isometries of the graph of a finite-dimensional distributive lattice L are precisely the automorphisms with respect to the ternary operation* (a, b, c).

PROOF. Corollary 2 above proves one half. To prove the converse, since any isometry carries "linked" elements into "linked" elements, it suffices to note the following lemma.

LEMMA. *In L, a and b are "linked" (that is, a covers b or b covers a) if and only if $(a, x, b)=a$ or b for all x.*

PROOF. By Theorem 1, $(a, x, b)=a$ or b for all x if and only if $[a, b]$ consists only of a and b—that is, if and only if a and b are equal or linked.

DEFINITION. We shall call elements a, a' of a distributive lattice *complementary* if and only if $(a, x, a')=x$ for all x.

It is easily seen that a given a can have only one complement.

For if it has two, say a' and a'', then $a''=(a, a'', a')=(a, a', a'')=a'$, for any symmetric ternary operation. It is also easily seen that if a and a' are complementary, then we have a "double algebra" with an "extreme pair";[3] we denote $x\cap y=(x, a, y)$ and $x\cup y=(x, a', y)$, and get a distributive lattice.

More generally, the 2^n mutually distributive operations introduced by one of us[2] in B^n are the operations (x, a, y), one for each of the 2^n different elements a of B^n.

Added January 15, 1947. It may be shown that distributive lattices can be defined in terms of the ternary operation (a, b, c), by postulating

$$(0, a, I)=a \qquad \text{for a fixed pair } O, I, \tag{3.1}$$

$$(a, b, a)=a, \tag{3.2}$$

$$(a, b, c)=(b, a, c)=(b, c, a) \qquad \text{(symmetry)}, \tag{3.3}$$

$$((a, b, c), d, e)=((a, d, e), b, (c, d, e)). \tag{3.4}$$

The proof is straightforward, if we try to prove the right known set of postulates.[4] The greatest trouble comes in proving that (a, b, c) is actually defined by (1) in the distributive lattice defined by the binary operations $a\cap b=(a, 0, b)$ and $a\cup b=(a, I, b)$. This trouble is resolved by using Theorem 2 above. Law (3.4), which contains both ordinary distributive laws as special cases, may be proved by direct substitution.

The structure of the group of automorphisms of B^n with respect to the ternary operation (a, b, c) is of interest for its own sake, and because it is the structure of the group of symmetries of the n-cube.

For any fixed a, the correspondence

$$x \rightarrow (x\cap a')\cup(x'\cap a)=x+a \tag{4}$$

(here $+$ denotes addition in the additive group of the corresponding Boolean ring) is an automorphism for (a, b, c); this defines a *simply transitive, elementary Abelian* subgroup of order 2^n, of symmetries of the n-cube; the case $a=I$, $x\rightarrow x'$ gives the "antipodal reflection" generating the center of the group. We also have the subgroup "of stability" leaving the vertex O fixed, and consisting of the lattice-automorphisms of B^n. The entire group is the product of these two subgroups.

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[3] Cf. M. H. A. Newman, *A characterisation of Boolean lattices and rings*, J. London Math. Soc. vol. 16 (1941) p. 257.

[4] Namely, those of p. 7 of *Distributive postulates for systems like Boolean algebras*, by G. D. Birkhoff and G. Birkhoff, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 3–11. It might be possible to base postulates on the conditions of §§8–10 of E. Pitcher and M. F. Smiley, *Transitivities of betweenness*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 95–114.

There is an obvious analogy between the ternary operation (1) and the ternary operation $ab^{-1}c$ of a group G; in the latter case, the group of automorphisms for $ab^{-1}c$ is the holomorph[5] of G. This similarly is the product of the subgroup of group-automorphisms and a simply transitive subgroup of right-translations $x \to xa$. It would be interesting to extend our ternary operation to Newman's "double systems" (reference of footnote 2).

However, it should be noted that in doing this, we should not use the ternary operation $a+b+c$ of Baer-Certaine. For with respect to this, B^n has $2^n(2^n-1) \cdots (2^n-2^{n-1})$ automorphisms; hence we cannot even *define* (a, b, c) in terms of it.

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[5] J. Certaine, *The ternary operation* $(abc) = ab^{-1}c$ *of a group*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 869–877; this operation had previously been discussed by R. Baer. It satisfies $(aab) = b$, instead of $(aab) = a$.

ON RELATIONS EXISTING BETWEEN TWO KERNELS OF THE FORM $(a, b)+b$ AND $(b, a)+b$

P. HEBRONI

Let s and t be variables in the interval from 0 to 1, and let $a, b, c, \cdots,$ be functions of s and t. Putting, as is customary,

$$(a, b) = \int_0^1 a(s\lambda)b(\lambda t)d\lambda,$$

we have

$$(a \pm b, c) = (a, c) \pm (b, c),$$
$$(a, b \pm c) = (a, b) \pm (a, c),$$
$$((a, b), c) = (a, (b, c)) = (a, b, c).$$

From this follows readily the meaning of (a, b, c, d). Putting, again,

$$[a, b] = a + (a, b) + b,$$

we have

$$[0, a] = a, \qquad [a, 0] = a, \qquad [[a, b], c] = [a, [b, c]] = [a, b, c].$$

We put finally,

$$\{a, b, c\} = (a, b, c) + (a, b) + (b, c) + b.$$

The function a is said to be reciprocable if there exists a function $\bar{a}$ such that

$$(*) \qquad [a, \bar{a}] = 0 \quad \text{and} \quad [\bar{a}, a] = 0.$$

(Each of these equations, it is well known, implies the other.) We say then that $\bar{a}$ is the reciprocal of a. If a is reciprocable, then so is $\bar{a}$, and the reciprocal of $\bar{a}$ is a. In what follows we shall designate the Fredholm determinant of a function a by D_a, and the reciprocal of a by $\bar{a}$. Of the various relationships that exist among the symbols (a, b), (a, b, c), $[a, b]$, $[a, b, c]$ and $\{a, b, c\}$, we state here the following:

$$(1) \qquad [a, b, c] = \{a, b, c\} + [a, c],$$

$$(2) \qquad [a, b, \bar{a}] = \{a, b, \bar{a}\}.$$

The following relations also hold true:

(α) $\{a, b, 0\} = (a, b)+b \quad \{0, a, b\} = (a, b)+a \quad \{a, 0, b\} = 0,$

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(β) $\{a, (b, c)+c, d\} = \{[a, b], c, d\}$,
(γ) $((a, b)+a, \{\bar{b}, c, d\}) = (a, c, d)+(a, c)$,
(δ) $\{a, b \pm c, d\} = \{a, b, d\} \pm \{a, c, d\}$;
and more generally:
(φ) $\{a, \{b, c, d\}, e\} = \{[a, b], c, [d, e]\}$,
(ψ) $(\{a, b, c\}, \{d, e, f\}) = \{a, (b, [c, d], e)+(b, e), f\}$.

(β) can be derived from (φ). For we have by (α) and (φ),

$$\begin{aligned}\{a, (b, c) + c, d\} &= \{a, \{b, c, 0\}, d\} = \{[a, b], c, [0, d]\}\\ &= \{[a, b], c, d\}.\end{aligned}$$

(γ) could likewise be derived from (ψ). For we have, by (α), (ψ), and (δ),

$$\begin{aligned}((a, b) + a, \{\bar{b}, c, d\}) &= (\{0, a, b\}, \{\bar{b}, c, d\})\\ &= \{0, (a, [b, \bar{b}], c) + (a, c), d\}\\ &= \{0, (a, 0, c) + (a, c), d\}\\ &= \{0, (a, c), d\} = (a, c, d) + (a, c).\end{aligned}$$

(γ) and (δ) are thus seen to be special cases of (φ) and (ψ). For what follows, however, (γ) and (δ) will be amply sufficient.

Of the Fredholm determinant it is known that

$$D_{[a,b]} = D_a \cdot D_b \tag{3}$$

(v. G. Kowalewski, *Determinanten*, 1909, p. 467), from which relation follows easily:

$$D_{[a,b,c]} = D_a \cdot D_b \cdot D_c. \tag{4}$$

From (3) we derive the known fact:

$$D_a \cdot D_{\bar{a}} = D_{[a,\bar{a}]} = D_0 = 1. \tag{5}$$

Again, by (2), (4) and (5), we have

$$D_{\{a,b,\bar{a}\}} = D_{[a,b,\bar{a}]} = D_a \cdot D_b \cdot D_{\bar{a}} = D_b. \tag{6}$$

Let $D_a \neq 0$, so that $\bar{a}$ exists. We put $c = (a, b)+b$, $e = (b, a)+b$ and conclude that

$$D_c = D_e. \tag{7}$$

To prove (7), we put $w = \{a, e, \bar{a}\}$. We have, then, by (6)

$$D_e = D_w. \tag{8}$$

On the other hand we have:

$$\begin{aligned} w = \{a, e, \bar{a}\} &= (a, e, \bar{a}) + (a, e) + (e, \bar{a}) + e \\ &= (a, (b, a) + b, \bar{a}) + (a, (b, a) + b) + ((b, a) + b, \bar{a}) + (b, a) + b \\ &= (a, b) + b + ((a, b) + b, a + (a, \bar{a}) + \bar{a}) \\ &= (a, b) + b + ((a, b) + b, 0) = (a, b) + b = c; \end{aligned}$$

therefore $D_w = D_c$. From this and (8) follows (7).

Equation (7) holds true even when $D_a = 0$. For, putting

$$c' = (\lambda a, b) + b, \quad e' = (b, \lambda a) + b,$$

we have for all λ for which $D_{\lambda a} \neq 0$,

$$D_{c'} = D_{e'}. \tag{9}$$

$D_{c'}$ and $D_{e'}$, however, can easily be shown to be entire functions of λ, and, moreover, the zero points of $D_{\lambda a}$ accumulate nowhere. It follows, therefore, that (9) holds true for all λ, particularly for $\lambda = 1$, that is, (7) is true even in the case of $D_a = 0$.

Retaining the notation $c = (a, b) + b$, $e = (b, a) + b$, we state that if $D_a \neq 0$, and $D_c \neq 0$, so that both $\bar{a}$ and $\bar{c}$ exist, then there exists also $\bar{e}$, and we have

$$\bar{e} = \{\bar{a}, \bar{c}, a\}. \tag{10}$$

Proof. We have $c + (c, \bar{c}) + \bar{c} = 0$. From this follows (by (δ)),

$$\{\bar{a}, c, a\} + \{\bar{a}, (c, \bar{c}), a\} + \{\bar{a}, \bar{c}, a\} = 0. \tag{11}$$

But from (α) and (β), we obtain

$$\begin{aligned} \{\bar{a}, c, a\} &= \{\bar{a}, (a, b) + b, a\} = \{[\bar{a}, a], b, a\} \\ &= \{0, b, a\} = (b, a) + b = e. \end{aligned} \tag{12}$$

Again, by (β) we have,

$$\begin{aligned} \{\bar{a}, (c, \bar{c}), a\} &= \{\bar{a}, ((a, b) + b, \bar{c}), a\} = \{\bar{a}, (a, (b, \bar{c})) + (b, \bar{c}), a\} \\ &= \{[\bar{a}, a], (b, \bar{c}), a\} = \{0, (b, \bar{c}), a\} = (b, \bar{c}, a) + (b, \bar{c}). \end{aligned}$$

On the other hand we have by (γ)

$$(e, \{\bar{a}, \bar{c}, a\}) = ((b, a) + b, \{\bar{a}, \bar{c}, a\}) = (b, \bar{c}, a) + (b, \bar{c});$$

therefore $\{\bar{a}, (c, \bar{c}), a\} = (e, \{\bar{a}, \bar{c}, a\})$, from which, and (11) and (12), follows

$$e + (e, \{\bar{a}, \bar{c}, a\}) + \{\bar{a}, \bar{c}, a\} = 0,$$

and thus the statement above is proven.

In a similar way it can be shown that if $D_a \neq 0$ and $D_e \neq 0$, so that $\bar{a}$ and $\bar{e}$ exist, then $\bar{c}$ also exists and we have

$$\bar{c} = \{a, \bar{e}, \bar{a}\}.$$

The above results are summed up in the following:

THEOREM 1. *If a and b are any functions whatever of s and t, then the Fredholm determinants of $c = (a, b) + b$ and $e = (b, a) + b$ are equal.*

If $D_a \neq 0$ and $D_c \neq 0$, so that $\bar{a}$ and $\bar{c}$ exist, then $\bar{e}$ also exists, and we have $\bar{e} = (\bar{a}, \bar{c}, a) + (\bar{a}, \bar{c}) + (\bar{c}, a) + \bar{c}$; and similarly, if $D_a \neq 0$ and $D_e \neq 0$, so that $\bar{a}$ and $\bar{e}$ exist, then $\bar{c}$ also exists, and we have

$$\bar{c} = (a, \bar{e}, \bar{a}) + (a, \bar{e}) + (\bar{e}, \bar{a}) + \bar{e}.$$

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ON THE REPRESENTATION OF σ-COMPLETE BOOLEAN ALGEBRAS

L. H. LOOMIS

A σ-complete Boolean algebra is a Boolean algebra in which for every sequence of elements a_i, $i=1, \cdots$, there is an element $\bigcup_1^\infty a_n$, the countable union of the a_i, such that $a_i \subseteq \bigcup_1^\infty a_n$ for every i, and such that if $a_i \subseteq x$ for every i then $\bigcup_1^\infty a_n \subseteq x$. The dual operation, countable intersection, can be introduced through complementation, and the distributive law $a \cap \bigcup_1^\infty a_n = \bigcup_1^\infty (a \cap a_n)$ and its dual can be proved (see [3, p. 93]).[1] Certain types of Boolean algebras have representations as algebras of point sets, the representation preserving all the operations of the algebra. Among these are ordinary Boolean algebras with no further operations (Stone [1, p. 106]) and complete Boolean algebras for which very general operations and distributive laws are assumed (Tarski [2, pp. 197–198]). On the other hand it is well known that a σ-complete Boolean algebra has in general no such representation. For example, the quotient of the algebra of Lebesgue measurable subsets of $[0, 1]$ modulo the ideal of sets of measure zero is a σ-complete Boolean algebra which is not σ-isomorphic to any σ-complete Boolean algebra of point sets. The following theorem, which we shall prove in this note, shows that this example illustrates the general situation.

THEOREM. *Every σ-complete Boolean algebra is σ-isomorphic to a σ-complete Boolean algebra of point sets modulo a σ-ideal in that algebra.*

In particular, every abstract measure algebra can be considered as an algebra of point sets modulo sets of measure zero. Bischof [4] has recently obtained a proof of this special case of the theorem, but his proof leans heavily on the existence of a measure and does not generalize.

If we are given a representation of an abstract collection R of objects onto a family F of sets, then each point p of the representation space determines a subfamily of sets F_p and hence a subcollection R_p of R, namely, those objects in R whose image sets in F contain p. This suggests the way to define points in attempting to build up a representation. A point will be a certain kind of subset of R, the image $\mathfrak{a}$ of a will be the set of points containing a, and the representation $\mathfrak{R}$ of R

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

will be the collection of such sets $\mathfrak{a}$. If R is closed under a complementation operation and the representation is to preserve complements, then clearly a point must be a "selection subset" of R, which for each $a \in R$ contains a or a' but not both. If R is a Boolean algebra and the representation is to be an isomorphism with respect to Boolean operations, then a point P must be a selection subset and must have the further properties that (1) if $a \in R$ and $b \in P$ then $a \cup b \in P$, and (2) if $a_i \in P$, $i = 1, \cdots, n$, then $\bigcap_1^n a_i$ is in P and is not null. Such a subset of R is called a *dual prime ideal*. Stone's theorem asserts that the correspondence $a \rightarrow \mathfrak{a}$, where $\mathfrak{a}$ is the set of dual prime ideals containing a, is actually an isomorphism. If R is a σ-complete Boolean algebra and the representation is to be a σ-isomorphism, then the point P must be not only a dual prime ideal, but must also have the stronger multiplicative property that if $a_i \in P$, $i = 1, \cdots, \infty$, then $\bigcap_1^\infty a_i$ is in P and is not null. Such a dual ideal could be called a dual prime σ-ideal. It can easily be shown that *the correspondence $a \rightarrow \mathfrak{a}$ is a σ-isomorphism if and only if the set $\mathfrak{a}$ of dual prime σ-ideals containing a is nonvoid for every non-null a in R.* In general there are not enough dual prime σ-ideals. In fact, the above mentioned algebra of measurable subsets of $[0, 1]$ modulo sets of measure 0 has *no* dual prime σ-ideals. For let a_i be the set of numbers in $[0, 1]$ which in dyadic representation have 0 at the ith place. Then $m(a_i) = m(a_i') = 1/2$, and the measure of the product of any n such sets or their complements is at most 2^{-n}. Thus any dual prime σ-ideal would have to contain an infinite product of measure 0 and consequently, in the quotient space, a null product, which contradicts the definition of a dual prime σ-ideal. This proves the assertions we have made about this example, and it also shows that in defining points for our representation we cannot use the stringent requirements of dual prime σ-ideals.

We shall, in fact, go back to our weakest definition and take a point to be simply a selection subset with respect to complementation. We repeat that the set $\mathfrak{a}$ corresponding to the element a is the set of all points containing a, and that $\mathfrak{R}$ is the collection of such sets $\mathfrak{a}$. The correspondence $a \rightarrow \mathfrak{a}$ clearly preserves complementation. Let $\mathfrak{B}(\mathfrak{R})$ be the σ-complete Boolean algebra (Borel field) of point sets generated by $\mathfrak{R}$, that is, the smallest family of sets including $\mathfrak{R}$ and closed under the operations of complementation and countable union. Let $\mathfrak{N}$ be the family of sets of $\mathfrak{B}(\mathfrak{R})$ of the form $\bigcap_1^n \mathfrak{a}_i$ if $\bigcap_1^n a_i = 0$ or $\bigcap_1^\infty \mathfrak{a}_i$ if $\bigcap_1^\infty a_i = 0$. Let $\mathfrak{J}(\mathfrak{N})$ be the σ-ideal generated by $\mathfrak{N}$, that is, the smallest family of sets including $\mathfrak{N}$ and closed under the operations of countable union and of intersection with any set of $\mathfrak{B}(\mathfrak{R})$. Let $\overline{\mathfrak{R}}$ be the quotient σ-complete Boolean algebra $\mathfrak{B}(\mathfrak{R})/\mathfrak{J}(\mathfrak{N})$ and let $\tilde{\mathfrak{a}}$ be the element (coset) of $\bar{\mathfrak{R}}$ containing $\mathfrak{a}$. The theorem can be stated more exactly as follows.

THEOREM. *The σ-complete Boolean algebras R and $\bar{\mathfrak{R}}$ are σ-isomorphic under the correspondence $a \rightarrow \bar{\mathfrak{a}}$.*

First, the correspondence $a \rightarrow \bar{\mathfrak{a}}$ is a σ-homomorphism onto the whole of $\bar{\mathfrak{R}}$. For if $a = \bigcup_1^\infty a_n$ then $a \cap \bigcap_1^\infty a_n' = 0$ and $a' \cap a_i = 0$ for every i. Therefore, by the definition of $\mathfrak{N}$, $\mathfrak{a} \cap \bigcap_1^\infty \mathfrak{a}_n' \in \mathfrak{N}$ and $\mathfrak{a}' \cap \mathfrak{a}_i \in \mathfrak{N}$ for every i, so that $\bar{\mathfrak{a}} = \bigcup_1^\infty \bar{\mathfrak{a}}_n$, which proves the correspondence to be a σ-homomorphism. The images $\bar{\mathfrak{a}}$ of the elements of R thus form a σ-complete Boolean algebra and hence cover the whole of $\bar{\mathfrak{R}}$ (which is generated by these images). That this σ-homomorphism onto $\bar{\mathfrak{R}}$ is a σ-isomorphism is the obvious content of the following lemma, the proof of which depends ultimately upon an application of the diagonal process.

LEMMA. *If* $\bar{\mathfrak{a}} = 0$ (*that is,* $\mathfrak{a} \in \mathfrak{J}(\mathfrak{N})$) *then* $a = 0$.

Any element of $\mathfrak{J}(\mathfrak{N})$ is included in a countable sum of elements of $\mathfrak{N}$, for the elements of $\mathfrak{J}(\mathfrak{N})$ having this property clearly form a σ-ideal including $\mathfrak{N}$. Thus, to prove the lemma it is sufficient to prove that

$$\text{(1)} \qquad \text{if} \quad \mathfrak{a} \subseteq \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathfrak{a}_m^n \quad \text{then} \quad a \subseteq \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} a_m^n.$$

Because of the more intuitive distributive law involved we shall prove the complementary form

$$\text{(2)} \qquad \text{if} \quad \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathfrak{a}_m^n \subseteq \mathfrak{a} \quad \text{then} \quad \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} a_m^n \subseteq a.$$

The hypothesis of (2) implies that

$$\text{(3)} \qquad \bigcap_{n=1}^{\infty} \mathfrak{a}_{m(n)}^n \subseteq \mathfrak{a} \qquad \text{for every function } m(n).$$

This can occur only if (i) a occurs in the sequence $a_{m(n)}^n$, or if (ii) the sequence contains a complementary pair. For otherwise we can form a point P containing all the elements $a_{m(n)}^n$ and also containing a', invalidating the inclusion (3). But the conditions (i) or (ii) imply the conclusion of (2). For suppose on the contrary that

$$\text{(4)} \qquad a' \cap \left(\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} a_m^n \right) = b \neq 0.$$

Since $b\subseteq\bigcup_{m=1}^{\infty}a_m^n$ for every n, we can find an index $m(1)$ such that $b\cap a^1_{m(1)}\neq 0$, an index $m(2)$ such that $(b\cap a^1_{m(1)})\cap a^2_{m(2)}\neq 0$, and, inductively, indices $m(i)$ such that $b\cap\bigcap_1^n a^i_{m(i)}\neq 0$. The sequence $a^i_{m(i)}$ so chosen cannot have the property (ii), and since $b\subseteq a'$ it cannot have the property (i). The hypothesis (4) thus leads to a contradiction and (2) has been established.

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HARVARD UNIVERSITY

SOME REMARKS AND CORRECTIONS TO ONE OF MY PAPERS

PAUL ERDŐS

Professor Hartmann pointed out two inaccuracies in my paper *Some remarks about additive and multiplicative functions* (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 527–537) (see Mathematical Reviews vol. 7 (1946) p. 577).

His first objection is that my proof of Theorem 12 (see p. 535) assumes that $f(p^\alpha) \geqq 0$. The only place the error occurs is in the fifth formula line of p. 536. But the error is quite easy to correct, only a $O(1)$ term is missing. The correct version of the formula is

$$\sum_{m=1}^{n} g_k(m) \leqq n \sum_{d} \frac{h_k(d)}{d} + O(1) < n \prod_{p} \left(1 + \frac{h_k(p)}{p}\right) + O(1).$$

Otherwise the proof is unchanged.

His second objection is against Theorem 13 (pp. 536–537) and is more serious.

Theorem 13 was stated as follows: Let $g(n) \geqq 0$ be multiplicative. Then the necessary and sufficient condition for the existence of the distribution function is that

$$(1) \qquad \sum_{p} \frac{(g(p) - 1)'}{p} < \infty, \qquad \sum_{p} \frac{((g(p) - 1)')^2}{p} < \infty$$

where $(g(p) - 1)' = g(p) - 1$ if $|g(p) - 1| \leqq 1$ and 1 otherwise.

I try to prove this by putting $\log g(n) = f(n)$ and state that $g(n)$ has a distribution function if and only if $f(n)$ has a distribution function.

In his review Hartmann points out that first of all this implies $g(n) > 0$ (instead of $g(n) \geqq 0$), and in a letter he points out that my statement is incorrect if $g(n)$ has a distribution function but $\lim_{x \to +0} G(x) > 0$ ($G(x)$ being the distribution function of $g(x)$). (I seem to remember that in my mind I was somehow unwilling to admit these $G(x)$ as distribution functions, but neglected to state this.)

In fact it is easy to see that this case can occur. Put $g(p^\alpha) = 1/2$ for all p and α. Then $G(x) = 1$ for all $x \geqq 0$, but clearly $f(n)$ has no distribution function, and the series (1) do not converge. Thus Theorem 13 is incorrect as it stands. The correct version may be stated as follows:

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THEOREM 13′. *Let $g(n) \geqq 0$ be multiplicative. Assume that the series (1) converge. Then $g(n)$ has a distribution function. The converse is also true unless $G(x) = 1$ for all $x \geqq 0$.*

First of all we remark that if

$$\sum_{g(p)=0} \frac{1}{p} = \infty$$

we have $G(x) = 1$ for all $x \geqq 0$ (since almost all integers are divisible by a p with $g(p) = 0$). Thus this case can be neglected, and we can assume that the primes with $g(p) = 0$ can be neglected, since they do not influence the convergence of the series (1) or the existence of the distribution function.[1]

The first part of Theorem 13 follows as on p. 537 of my paper.

Next we investigate the converse. If we assume that $\lim_{x\to+0} G(x) = 0$ the convergence of (1) follows as on p. 537, since in this case it really is true that $g(n)$ has a distribution function if and only if $f(n)$ has a distribution function.

Assume now

$$\lim_{x\to+0} G(x) = c > 0. \tag{2}$$

We shall show $c = 1$. Suppose that $c < 1$, we shall show that this leads to a contradiction.

Denote by $F(x)$ the density of integers with $f(n) < x$ (where $f(n) = \log g(n)$). Clearly $F(x)$ exists and satisfies ($G(x)$ is a distribution function)

$$\lim_{x\to-\infty} F(x) = c > 0, \qquad \lim_{x\to+\infty} F(x) = 1 \qquad (c < 1). \tag{3}$$

From now on we make constant use of my joint paper with Wintner[1] (referred to as E.W.). It follows from (3) that there exist real numbers a and b such that

$$-\infty < a < b < \infty \quad \text{and} \quad F(b) - F(a) > 0. \tag{4}$$

From (4) and E.W. §9, p. 717 it follows that $|f(p)| < A$ (except for a sequence of primes q with $\sum 1/q < \infty$, which can be neglected).

Next we deduce (E.W. §3, pp. 714–715) that

$$\sum_{p} \frac{(f(p))^2}{p} < \infty. \tag{5}$$

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[1] Amer. J. Math. vol. 61 (1939) pp. 713–721.

Further it follows that (E.W. §4, p. 714)

$$\left| \sum_{p<x} \frac{f(p)}{p} \right| < B \qquad (B \text{ independent of } x). \tag{6}$$

In §6, p. 716 it is shown that from $|f(p)| < A$, (4) and (5) it follows that

$$\sum_{m=1}^{n} (f(m))^2 < Cn. \tag{7}$$

But clearly (7) contradicts (3) (since (3) implies that the density of integers with $f(m) > D$ is not less than c for every D), which completes the proof of Theorem 13′.

The following question can be raised: Let $f(n)$ be additive and assume that for some $a<b$ the density of the integers satisfying $a \leqq f(n) \leqq b$ exists and is different from 0. Does it then follow that $f(n)$ has a distribution function?

By the same methods as just used we can show that

$$|f(p)| < c, \qquad \sum_{p} \frac{(f(p)')^2}{p} < \infty, \qquad \sum_{p} \frac{f(p)'}{p} < \infty.$$

But at present I cannot decide whether the distribution function has to exist.

Professor Hartmann also pointed out the following misprints in my previous paper:

(1) The first sentence of Theorem 12 should read "Let $f(p^\alpha) \leqq C$."

(2) The inequality symbol in the two formula lines at the bottom of p. 535 should be "$\leqq$" instead of "$>$."

(3) On p. 537, in the line following the third formula line "$(\log g(p))^1 > \cdots$" should be "$(\log g(p))^2 > \cdots$."

(4) On p. 537, the fifth formula line should be "$\sum(1/p) \cdots$" instead of "$\sum(1/2) \cdots$."

(5) In the next to the last line of the paper, p. 537, "$\cdots f(n)$" should be "$\cdots g(n)$."

(6) The first formula on p. 529 should read "$\cdots \exp \exp (d\phi(n))$" instead of "$\cdots \exp \exp (\phi(n))$."

Syracuse University

RAMANUJAN'S FUNCTION $\tau(n)$—A CONGRUENCE PROPERTY

R. P. BAMBAH

In a recent paper[1] S. Chowla and I proved that

$$(1) \qquad \tau(n) \equiv n\sigma_3(n) \pmod 7$$

where

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\prod_{n=1}^{\infty}(1-x^n)^{24} \qquad (|x|<1)$$

and $\sigma_k(n)$ is the sum of the kth powers of the divisors of n.

The following is another short proof of (1).

Writing

$$P = 1 - 24\sum_1^{\infty}\sigma(n)x^n,$$

$$Q = 1 + 240\sum_1^{\infty}\sigma_3(n)x^n,$$

and

$$R = 1 - 504\sum_1^{\infty}\sigma_5(n)x^n,$$

Ramanujan proved that

$$1728\sum_1^{\infty}\tau(n)x^n = Q^3 - R^2.$$

From relation 10, Table III, p. 142, Ramanujan's *Collected Papers*, Cambridge, 1927, we derive

$$\begin{aligned} 3(1728)\sum_1^{\infty}\tau(n)x^n &= 3(Q^3 - R^2) \\ &= 41472\sum_1^{\infty} n^4\sigma_3(n)x^n - 7R^2 \\ &\quad - 7(P^4Q - 4P^3R + 6P^2Q^2 - 4PQR). \end{aligned}$$

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[1] Sent to the Quarterly Journal of Mathematics, Oxford.

Therefore comparing coefficients of x^n, we have

$$3(1728)\tau(n) \equiv 41472n^4\sigma_3(n) \pmod{7}$$

or

$$\begin{aligned}\tau(n) &\equiv n^4\sigma_3(n) \pmod{7}\\ &\equiv n\sigma_3(n) \pmod{7},\end{aligned}$$

for if[3]

$$(n/7) = -1, \qquad \sigma_3(n) \equiv 0 \pmod{7}$$

and if $(n/7) = +1$, or $n \equiv 0 \pmod{7}$

$$n^4 \equiv n \pmod{7}.$$

Note: The results for moduli 5 and 691 can be similarly obtained by using Relations 5, Table II, and 6, Table I, respectively, of Ramanujan's *Collected papers*, pages 141–142.

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[3] This is a particular case of the more general theorem: If p is a prime, and if $(k/p) = -1$, then $\sigma_{(p-1)/2}(pt+k) \equiv 0 \pmod{p}$.

A CONGRUENCE PROPERTY OF RAMANUJAN'S FUNCTION $\tau(n)$

R. P. BAMBAH, S. CHOWLA, AND H. GUPTA

We have recently proved[1] that

$$\tau(n) \equiv \sigma(n) \pmod 8 \tag{1}$$

if $(n, 2) = 1$, where

$$\sum_1^\infty \tau(n)x^n = x\prod_1^\infty (1 - x^n)^{24} \qquad (|x| < 1),$$

$$\sigma(n) = \sum_{d|n} d.$$

We give, in this note, a new and short proof of (1).

In the following we write J for any integral power series with integral coefficients.

We have $(1-x)^2 = (1-x^2)+2J$. Therefore

$$(1 - x)^8 = (1 - x^2)^4 + 8J.$$

Hence

$$\begin{aligned} \sum_1^\infty \tau(n)x^n &= x\{(1 - x^2)(1 - x^4)(1 - x^6) \cdots \}^{12} + 8J \\ &= x\left\{\sum_{u=0}^\infty (-1)^u(2u + 1)x^{u(u+1)}\right\}^4 + 8J \\ &= x\left\{\sum_{u=0}^\infty x^{u(u+1)} + 4J\right\}^4 + 8J \\ &= x\left\{\sum_0^\infty x^{u(u+1)}\right\}^4 + 8J. \end{aligned} \tag{2}$$

But[2]

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[1] Jointly with D. B. Lahiri (Calcutta) in a paper sent to the Quarterly Journal of Mathematics, Oxford.

[2] Whitaker and Watson, *Modern analysis*, 4th ed., p. 479, Example 3; and *Ramanujan* by Hardy, p. 132, for $(1+2x+2x^4+2x^9+\cdots)^4 = 1+\sum r_4(n)x^n$ and $r_4(n) = 8\sigma^*(n)$ where $\sigma^*(n) = \sigma(n)$ when n is odd and $\sigma^*(n) = 3$ (the sum of the odd divisors of n) when n is even, that is, $\sigma^*(n) = \sigma(n) - 4\sigma(n/4)$ for all n.

$$\begin{aligned}
16x\left\{\sum_{0}^{\infty} x^{n(n+1)}\right\}^4 &= 16x(1 + x^2 + x^6 + x^{12} + \cdots)^4\\
&= (1 + 2x + 2x^4 + 2x^9 + \cdots)^4\\
&\quad - (1 - 2x + 2x^4 - 2x^9 + \cdots)^4\\
&= 8\sum_{1}^{\infty}\left\{\sigma(n) - 4\sigma(n/4)\right\}x^n \qquad (3)\\
&\quad - 8\sum_{1}^{\infty}\left\{\sigma(n) - 4\sigma(n/4)\right\}(-1)^n x^n\\
&= 16\sum_{n=1}^{\infty}\sigma(n)x^n \qquad (n \text{ is odd}).
\end{aligned}$$

Therefore, from (2) and (3), we have

$$\sum_{n=1}^{\infty}\tau(n)x^n = \sum_{n=1}^{\infty}\sigma(n)x^n + 8J \qquad (n, 2) = 1.$$

From the above, comparing coefficients, we have if $(n, 2) = 1$

$$\tau(n) \equiv \sigma(n) \pmod 8$$

and if $2 | n$

$$\tau(n) \equiv 0 \pmod{8}.$$

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GOVERNMENT COLLEGE, LAHORE, AND
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A NEW CONGRUENCE PROPERTY OF RAMANUJAN'S FUNCTION $\tau(n)$

R. P. BAMBAH AND S. CHOWLA

In a joint sitting we guessed, with the help of Gupta's tables, that

$$\tau(n) \equiv n^2\sigma(n) \pmod 9, \tag{1}$$

where

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\prod_{n=1}^{\infty}(1-x^n)^{24} \qquad (|x| < 1)$$

and

$$\sigma_k(n) = \sum_{d|n} d^k, \qquad \sigma(n) = \sigma_1(n).$$

Each of us obtained a separate proof of (1). But we publish only one.

PROOF OF (1). Writing

$$P = 1 - 24\sum_1^{\infty}\sigma(n)x^n,$$

$$Q = 1 + 240\sum_1^{\infty}\sigma_3(n)x^n,$$

$$R = 1 - 504\sum_1^{\infty}\sigma_5(n)x^n,$$

Ramanujan proved that

$$1728\sum_1^{\infty}\tau(n)x^n = Q^3 - R^2.$$

Writing J_γ ($\gamma = 1, 2, 3, \cdots$) for an integral power series in x with integral coefficients, we derive from relations 8, 1, and 3, Table III, p. 142 of Ramanujan's *Collected papers*, Cambridge, 1927,

$$1728\sum_1^{\infty}\tau(n)x^n = Q^3 - R^2 = 5184\sum_1^{\infty}(n^3\sigma_5(n)x^n)$$
$$- 2R^2 - 6P^2Q^2 + 2P^3R + 6PQR$$

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$$= 5184\sum_{1}^{\infty}(n^3\sigma_5(n)x^n)$$

$$+ R(3PQ - 2R - P^3) - 3P(2PQ^2 - P^2R - QR)$$

$$= 5184\sum_{1}^{\infty} n^3\sigma_5(n)x^n$$

$$+ (1 - 504J_1)\left(1728\sum_{1}^{\infty} n^2\sigma(n)x^n\right)$$

$$- 3(1 - 24J_2)\left(1728\sum_{1}^{\infty} n^2\sigma_5(n)x^n\right)$$

$$= 5184\sum_{1}^{\infty} n^3\sigma_5(n)x^n + 1728\sum_{1}^{\infty} n^2\sigma(n)x^n$$

$$- 5184\sum_{1}^{\infty} n^2\sigma_5(n)x^n + 243J_3.$$

Comparing the coefficients of x^n, we have

$$27\tau(n) \equiv 81n^3\sigma_5(n) + 27n^2\sigma(n) - 81n^2\sigma_5(n) \pmod{243}$$

or

$$\begin{aligned}\tau(n) &\equiv n^2\sigma(n) + 3(n^3 - n^2)\sigma_5(n) \pmod{9}\\ &\equiv n^2\sigma(n) + 3(n^3 - n^2)\sigma(n) \pmod{9}\\ &\equiv n^2\sigma(n) \pmod{9},\end{aligned}$$

for if

$$n \equiv 0, 1 \pmod{3} \quad \text{then} \quad n^3 - n^2 \equiv 0 \pmod{3}$$

while if

$$n \equiv 2 \pmod{3} \quad \text{then} \quad \sigma(n) \equiv 0 \pmod{3}.$$

COROLLARY.

$$\tau(n) \equiv 0 \pmod{9} \quad \textit{for almost all } n.$$

THE UNIVERSITY OF THE PANJAB, LAHORE

TOPOLOGICAL ABELIAN GROUPS WITH ORDERED NORMS

R. C. JAMES, A. D. MICHAL, MAX WYMAN

The purpose of this paper is to study Abelian groups with a norm whose values are in another Abelian group having an order relation. Postulates on the ordering are given which are sufficient for the space to be a topological group under the neighborhood system of this norm. It is seen that the added assumption that the ordering is Archimedean implies the space is a subset of a normed linear space. A differential is defined in the general space. This is seen to be closely related to the Fréchet differential if the ordering is Archimedean

DEFINITION 1. An *S-space* is a set S which is an Abelian group with a relation $\alpha>\beta$ (or $\alpha<\beta$) defined for some pairs α, β of elements of S and satisfying the postulates:

1. If $\alpha>\beta$, and $\beta>\gamma$, then $\alpha>\gamma$.
2. If $\alpha_1>\alpha_2$ and $\alpha_3\geqq\alpha_4$, then $\alpha_1+\alpha_3>\alpha_2+\alpha_4$.
3. If $\alpha_1>0$, and $\alpha_2>0$, then there exists an $\alpha_3>0$ such that $\alpha_1>\alpha_3$ and $\alpha_2>\alpha_3$.
4. If $\alpha>0$, then $\alpha\neq 0$.

DEFINITION 2. A set T with operations $x+y$ and $\|x\|$ defined for all elements x, y of T shall be said to be a G_s-space if the following are true:

1. T is an Abelian group with respect to $x+y$.
2. To every x in T and every positive number n there exists a positive integer N and an x_N such that $x=Nx_N$, where $N>n$.
3. $\|x\|$ is a function from T to an S-space with the following properties:
   a. $\|x\|\geqq 0$, and $\|x\|=0$ if and only if $x=0$.
   b. $\|x+y\|\leqq\|x\|+\|y\|$.
   c. $\|nx\|=|n|\,\|x\|$ for all integers n.

By use of condition (2) of Definition 1, it is easily seen that an equivalent definition of S-spaces would result from replacing (4) by the condition that $\alpha>\beta$ implies the impossibility of $\alpha<\beta$. Also, the proof of Theorem 1 would be much simpler if it were assumed that $\alpha>0$ whenever $n\alpha>0$ for some positive integer n. Condition (2) of Definition 1 would then be unnecessary as far as this theorem is concerned. An S-space of a G_s-space is more restricted than the postulates of Definition 1 alone would indicate. This is shown by the following two theorems:

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THEOREM 1. *For any element* $\epsilon>0$ *of an S-space, define the neighborhood* $U_{x_0,\epsilon}$ *of an element* x_0 *of an associated* G_s*-space as the totality of* $x\in G_s$ *satisfying* $n\|x-x_0\|<n\epsilon$ *for some positive integer* n. *Then* G_s *is a topological Abelian group if the neighborhood system is taken as the totality of all such neighborhoods.*

PROOF. (a) Clearly x_0 is in every $U_{x_0,\epsilon}$.

(b) Suppose U_{x_0,ϵ_1} and U_{x_0,ϵ_2} are two neighborhoods of x_0. Then $\epsilon_1>0$, $\epsilon_2>0$, and there is an $\epsilon_3>0$ such that $\epsilon_1>\epsilon_3$ and $\epsilon_2>\epsilon_3$. If $n\|x-x_0\|<n\epsilon_3$, then $n\|x-x_0\|<n\epsilon_1$ and $n\|x-x_0\|<n\epsilon_2$. Thus $U_{x_0,\epsilon_3}\subset U_{x_0,\epsilon_1}\cap U_{x_0,\epsilon_2}$.

(c) If y_0 is in $U_{x_0,\epsilon}$, then $p\|y_0-x_0\|<p\epsilon$ for some positive integer p. If $\epsilon'=p[\epsilon-\|y_0-x_0\|]$, then $U_{y_0,\epsilon'}$ contains some element $y\neq y_0$ if $U_{y_0,\epsilon'}\not\subset U_{x_0,\epsilon}$. Choose y_1 and $N\geqq p$ so that $y-y_0=Ny_1$. Then there is a positive integer q such that $qN\|y_1\|<q\epsilon'=qp[\epsilon-\|y_0-x_0\|]$ and by (2) of Definition 1, $qp\|y_1\|<qp[\epsilon-\|y_0-x_0\|]$. Therefore $U_{y_0,\epsilon'}\subset U_{x_0,\epsilon}$.

(d) If $x_0\neq y_0$, then $\|x_0-y_0\|>0$ follows from (3a) of Definition 2. There is then by (2) and (3c) of Definition 2 an element $u\in G_s$ and an integer $n\geqq 2$ such that $x_0-y_0=nu$ and $n\|u\|=\|x_0-y_0\|$. If $\epsilon=\|u\|$, then $U_{x_0,\epsilon}\cap U_{y_0,\epsilon}=0$. For suppose z is in both of these neighborhoods. Then for some positive integer p, $pn\epsilon=p\|x_0-y_0\|\leqq p\|x_0-z\|+p\|z-y_0\|<2\epsilon p$, or $pn\|u\|<2p\|u\|$. Hence $p\|(n-2)u\|<0$, which contradicts (3a) of Definition 2.

This verifies that any G_s-space is a Hausdorff space [1, pp. 228–229, (A), (B), (C), (5)].[1] To show that $x+y$ is continuous, let $U_{x+y,\epsilon}$ be any neighborhood of $x+y$. If there is an element $z\neq x+y$ in $U_{x+y,\epsilon}$, then choose u with $z-(x+y)=nu$ and $n>2$. As in (d) above, it follows that if $\epsilon_1=\|u\|$, then $x'+y'\in U_{x+y,\epsilon}$ if $x'\in U_{x,\epsilon_1}$ and $y'\in U_{y,\epsilon_1}$. If the only element in $U_{x+y,\epsilon}$ is $x+y$, then $U_{x,\epsilon}$ contains only x and $U_{y,\epsilon}$ contains only y. For if $p\|z-x\|<\epsilon p$, then $p\|z+y-(x+y)\|<\epsilon p$ and $z+y\in U_{x+y,\epsilon}$; while if $q\|z-y\|<\epsilon q$, then $z+x\in U_{x+y,\epsilon}$. Thus $x+y$ is continuous. Since $y\in U_{x,\epsilon}$ implies $-y\in U_{-x,\epsilon}$, $-x$ is also continuous.

DEFINITION 3. An S-space is *Archimedian* if for every $\epsilon_1>0$ and $\epsilon_2>0$ of S there exists a positive integer n such that $\epsilon_2<n\epsilon_1$.

The following is an example of a G_s-space whose S-space is not Archimedian. Let S be the set of complex numbers, and G_s the same set. Let $\|x+yi\|=|x|+|y|i$, and $a+bi>c+di$ if and only if $b>d$ or $b=d$ and $a>c$. These spaces clearly satisfy the postulates of Definitions 1 and 2. But if $\alpha = 1 + 0\, i$ and $\beta = 1 + i$, then there is no number n such that $n\alpha > \beta$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Let S be the set of all real numbers, G_s the set of all complex numbers $x+yi$ with x and y rational, and $\|x+yi\| = |x+yi|$. This is a G_s-space whose S-space is Archimedian, but which is itself a subset of the normed linear space of all complex numbers. The meaning of assuming that an S-space is Archimedian is further shown by the following results.

Lemma 1. *A G_s-space whose S-space is Archimedian is a normable topological Abelian group.*

Proof. Let G_1 be a G_s-space whose S-space S_1 is Archimedian. Let U_ϵ be any neighborhood of zero. Then G_1 is normable if U_ϵ generates G_1 and U_ϵ is bounded and convex [2, Corollary 5.2].

(a) *U_ϵ generates G_1.* For let $x \in G_1$ and choose an integer n such that $n\epsilon > \|x\|$. There is then a number $N > n$ and an x_N such that $x = Nx_N$. But then $N\epsilon > \|x\| = N\|x_N\|$ and $N\|x_N\| < N\epsilon$. Thus $x_N \in U_\epsilon$ and $x \in U_\epsilon^N = U_\epsilon + \cdots U_\epsilon$.

(b) *U_ϵ is bounded* [2, Definition 2.3]. For let $U_{\epsilon'}$ be any other neighborhood of the identity and choose n so that $n\epsilon' > \epsilon$. Suppose that $mx \in U_\epsilon$ for an $m \geqq n$. Then $p\|mx\| < p\epsilon < pn\epsilon' \leqq pm\epsilon'$ for some positive integer p. Hence $pm\|x\| < pm\epsilon'$ and $x \in U_{\epsilon'}$.

(c) *U_ϵ is convex* [2, Definition 2.1]. For suppose that $nx \in U_\epsilon^n$ for a positive integer n. Then $nx = x_1 + x_2 + \cdots + x_n$, where for each x_i there is a positive integer p_i such that $p_i\|x_i\| < p_i\epsilon$. Thus $n(p_1p_2 \cdots p_n)\|x\| < m(p_1p_2 \cdots p_n)\epsilon$ and $x \in U_\epsilon$.

It follows from Lemma 1 that if S is Archimedian, then G_s is a normable topological Abelian group and therefore a subspace of a normed linear space [2, Theorem 4.1]. The following is a similar, but stronger, result. Hereafter a G_s-space whose S-space is Archimedian will be called an *Archimedian G_s-space.*

Theorem 2. *For a given Archimedian G_s-space G_1 there is a normed linear space T_1 which contains G_1 as a subset and which is contained in every normed linear space containing G_1. Also, every element of T_1 is a limit point of elements of G_1.*

Proof. As noted above, there is a normed linear space T with $G_1 \subset T$. Then $\sum_{i=1}^{n} a_i x_i \in T$ for any numbers a_i and elements x_i of G_1, such multiplication being defined in T. The set T_1 of all such elements of T is clearly a linear space and as a linear subset of T is a normed linear space. Because of (2) of Definition 2, for any number a and element $x_0 \in G_1$ there are numbers α_i such that $\lim_{i\to\infty}\alpha_i = a$ and $\alpha_i x_0$ is an element of G_1. Let $x = \sum_{i=1}^{n} a_i x_i \in T_1$, where $x_i \in G_1$, and let ϵ be any positive number. Choose numbers α_i such that $|a_i - \alpha_i| < \epsilon/n\|x_i\|$ and $\alpha_i x_i \in G_1$ for each i. Then $\|x - \sum_{i=1}^{n} \alpha_i x_i\| < \epsilon$ and $\sum_{i=1}^{n} \alpha_i x_i \in G_1$. Hence every element of T_1 is a limit point of elements of G_1. From this, it is clear that $T_1 \subset T'$ for any normed linear space T' containing G_1.

The usual definition of a linear function as being additive and continuous will be used in the following. If the G_s-space is Archimedian, then the theory of the differential as defined below becomes a consequence of the established theory of Fréchet differentiation of functions with arguments and values in normed linear spaces.

Definition 4. Let G_1 and G_2 be any two G_s-spaces. If $f(x)$ is a function on a neighborhood U_1 of $x_0 \in G_1$ to G_2, then $f(x)$ is differentiable at $x = x_0$ if there exists a function $f(x_0; \delta x)$ on G_1 to G_2, defined for all elements δx of G_1 and such that:

1. $f(x_0; \delta x)$ is linear in δx.
2. For any integer $n > 0$ there exists a $\rho > 0$ such that $n\|f(x_0 + \delta x) - f(x_0) - f(x_0; \delta x)\| < \|\delta x\|$ for all δx such that $0 < \|\delta x\| < \rho$.

In this case, $f(x_0; \delta x)$ is called the differential of $f(x)$ at x_0.

Theorem 3. *Let G_1 and G_2 be any two Archimedian G_s-spaces. If the function $f(x)$ on $U_1 \subset G_1$ to G_2 has the differential $f(x_0; \delta x)$ at $x_0 \in U_1$, and T_1 and T_2 are the smallest normed linear spaces containing G_1 and G_2, respectively, then there is a function F on T_1 to T_2 such that:*

1. *$F(x) = f(x)$ for all $x \in U_1$.*
2. *F has the differential $F(x_0; \delta x)$ at x_0, where $F(x_0; \delta x)$ is the unique linear function on T_1 to T_2 for which $F(x_0; \delta x) = f(x_0; \delta x)$ for all $x \in G_1$.*

Proof. Let x be any element of T_1. The existence of elements $x_i \in G_1$ with $x = \lim_{i\to\infty} x_i$ follows from Theorem 2. Define $F(x_0; x)$ as $\lim_{i\to\infty} f(x_0; x_i)$. Such sequences are clearly Cauchy sequences and converge to an element of any Banach space containing T_2. But $F(x_0; x)$ is then a linear function on T_1 to this Banach space, and is therefore homogeneous of degree one in x. Since $x = \sum_{i=1}^{n} a_i y_i$ for some elements $y_i \in G_1$ and numbers a_i, this implies that $F(x_0; \sum_{i=1}^{n} a_i y_i) = \sum_{i=1}^{n} a_i F(x_0; y_i)$ and is an element of T_2 [$F(x_0; y_i) = f(x_0; y_i) \in G_2$ if $y_i \in G_1$]. Thus $F(x_0; x)$ is a linear function on T_1 to T_2. Since G_1 is dense in T_1, it follows that this is the only linear function with $F(x_0; \delta x) = f(x_0; \delta x)$ for $\delta x \in G_1$. Now define F on T_1 by the relation $F(x) = f(x)$ if $x \in U_1$, and

$$F(x) = F(x_0) + F(x_0; x - x_0)$$

otherwise. If n is any positive integer, then there is a number $\rho>0$ such that $0<\|\delta x\|<\rho$ implies $x_0+\delta x\in U_1$ and $n\|f(x_0+\delta x)-f(x_0)-f(x_0;\delta x)\|<\|\delta x\|$ if $\delta x\in G_1$. But then $n\|F(x_0+\delta x)-F(x_0)-F(x_0;\delta x)\|<\|\delta x\|$ if $0<\|\delta x\|<\rho$ and $\delta x\in T_1$, since this is satisfied for δx if $\delta x\in G_1$ and $0<\|\delta x\|<\rho$, while $F(x_0+\delta x)=F(x_0)+F(x_0;\delta x)$ if δx is not in $U_1\subset G_1$. Hence F has the differential $F(x_0;\delta x)$ at x_0; while $F(x)=f(x)$ if $x\in U_1$ and $F(x_0;x)=f(x_0;x)$ if $x\in G_1$.

THEOREM 4. *Let T_1 and T_2 be any two normed linear spaces. If a function f on T_1 to T_2 has a differential $f(x_0;\delta x)$ at the point x_0 (in the sense of Definition 4), then $f(x_0;\delta x)$ is a Fréchet differential of f at x_0.*

PROOF. Let ϵ be any positive number and choose an integer n such that $0<1/n\leq\epsilon$. By assumption, there is a number ρ such that $n\|f(x_0+\delta x)-f(x_0)-f(x_0;\delta x)\|<\|\delta x\|$ for all δx with $0<\|\delta x\|<\rho$. But then $\|f(x_0+\delta x)-f(x_0)-f(x_0;\delta x)\|<\epsilon\|\delta x\|$ if $0<\|\delta x\|<\rho$. Hence $f(x_0;\delta x)$ is the Fréchet differential of $f(x)$ at x_0.

COROLLARY 1. *Let G_1 and G_2 be any two Archimedian G_s-spaces, and $f(x)$ be a function on a neighborhood of $x_0\in G_1$ to G_2. Then*: (1) *If a differential of $f(x)$ exists, it is unique.* (2) *If $f(x)$ is differentiable at $x=x_0$, then $f(x)$ is continuous at $x=x_0$.* (3) *If $f(x)$ is differentiable at $x=x_0$, then it is also differentiable in the sense of Michal* [3], *and the two differentials are equal.*

COROLLARY 2. *Let G_1, G_2, and G_3 be three Archimedian G_s-spaces. Suppose $\phi(x)$ is a function on a neighborhood of $x_0\in G_1$ to G_2 and is differentiable at x_0, while $f(\phi)$ is a function on a neighborhood of $\phi(x_0)\in G_2$ to G_3 which is differentiable at $\phi(x_0)$. Then $f[\phi(x)]$ is differentiable at x_0, and this differential is $f[\phi(x_0);\phi(x_0;\delta x)]$.*

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ALTERNATIVE REGULAR RINGS WITHOUT NILPOTENT ELEMENTS

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Introduction. Interest in N. Jacobson's result that (associative) rings R which satisfy

$$a^{n(a)} = a \qquad (n(a) \textit{ an integer greater than one}) \tag{1}$$

for every $a \in R$ are commutative[1] [4, p. 702] led Forsythe and McCoy to show that every (associative) regular ring without nilpotent[2] elements is a subdirect sum of division rings [3]. The principal tool they used was a very general theorem of G. Birkhoff on subdirect unions in universal algebra [1]. In this note we shall prove the one additional fact which is needed to extend their argument to the alternative case. Our interest is not, of course, directed to the commutativity of alternative rings which satisfy (1), since this is an immediate consequence of Jacobson's result and the Theorem of Artin. In fact, such rings are associative. This consequence is well known if the additive order of no element of the ring is divisible by three. Our main result enables us to settle this exceptional case also. A corollary of our principal theorem is that every alternative algebraic algebra which has no nilpotent elements is the subdirect sum of alternative division algebras (cf. [4]). Whether this fact will find its place in a general theory of the structure of alternative algebraic algebras remains to be seen.

1. Preliminaries. Max Zorn calls a ring R *alternative* in case, for every $a, b, c \in R$, the *associator* $(a, b, c) = a(bc) - (ab)c$ changes sign on interchange of two of its arguments [7, 8]. Interest in these rings currently stems from a fundamental result of R. Moufang in the foundations of projective geometry [5]. It is known to be sufficient to assume that $(a, a, b) = (b, a, a) = 0$ [7]. Emil Artin has proved the following fundamental theorem.

THEOREM OF ARTIN. *If R is an alternative ring, then the subring of R generated by two of its elements is an associative subring.*

In this note we shall need to use the following identities

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[1] Numbers in brackets denote the references at the end of the paper.

[2] By a *nilpotent* element of a ring R we mean a nonzero element $a \in R$ such that $a^n = 0$ for a positive integer n.

$$(x(yz))x = (xy)(zx), \tag{2.1}$$

$$x(y(zy)) = ((xy)z)y \tag{2.2}$$

which have been established for alternative rings by Zorn [8].

J. von Neumann [6] calls an (associative) ring R *regular* in case for each $a \in R$ there is an $x \in R$ for which $axa = a$. By the Theorem of Artin, this is a meaningful definition in alternative rings and we may speak henceforth of *alternative regular rings*.

2. The main result.

In this section we shall prove the following theorem.

THEOREM. *An alternative regular ring R is a subdirect sum of alternative division rings if and only if R has no nilpotent elements.*

PROOF. An examination of the argument of Forsythe and McCoy will show that associativity of three elements of R which are not necessarily in a subring of R generated by two elements of R is used only in the proof of their Lemma 2 and in this case one of the three elements is idempotent. Our theorem is, then, a direct consequence of the Theorem of Artin, the arguments of Forsythe and McCoy, and the following lemma.

LEMMA 1. *If e is an idempotent element of an alternative ring R such that $ex = xe$ for every $x \in R$, then $e(xy) = (ex)y = x(ye)$ for every $x, y \in R$.*

Before we prove this lemma we shall first prove the more general lemma.

LEMMA 2. *If x is an element of an alternative ring R for which $xy = yx$ for every $y \in R$, then $x^3y = yx^3$ and $x^3(yz) = (x^3y)z = y(zx^3)$ for every $y, z \in R$.*

PROOF. That $x^3y = yx^3$ for every $y \in R$ is an immediate consequence of the Theorem of Artin. Now, using the identities (2.1) and (2.2), we may compute

$$\begin{aligned}(yz)x^3 = x^3(yz) &= x[(x(yz))x] = x[(xy)(zx)] \\ &= [(yx)(xz)]x = y(zx^3).\end{aligned}$$

Thus $(y, z, x^3) = 0$, and consequently all associators involving x^3 vanish.

PROOF OF LEMMA 1. Take $x = e = x^3$ in Lemma 2.

Remark. Lemma 2 is related to a result of R. H. Bruck [2, p. 302] which states that the mapping $x \to x^3$ is an endomorphism of a commutative Moufang loop into its center. In our original version we noted that Bruck's result is valid for alternative rings in connection with our proof that alternative rings which satisfy (1) are associative. Our thanks are due to the referee who formulated Lemma 2, which, as we shall see, is useful in *both* situations.

3. The associativity of certain alternative rings. As we noted in the introduction, our main result is not needed to show that alternative rings which satisfy (1) are commutative. It is well known that a commutative alternative ring R is associative if $3a=0$ implies that $a=0$, for every $a \in R$. In this section we shall apply our main result to show that every alternative ring which satisfies (1) is *associative*. We first note that such a ring is regular and without nilpotent elements. By our theorem, such a ring is, then, a subdirect sum of commutative alternative division rings each of which satisfies (1). Now, an alternative division ring which satisfies (1) necessarily has finite prime characteristic (see [4, p. 702]). To prove the associativity desired it is sufficient to prove the following lemma.

LEMMA 3. *If R is a commutative alternative division ring whose characteristic is three and which satisfies* (1), *then R is associative.*

PROOF. Let $a \in R$ and consider the subfield generated by a. This subfield is certainly finite since it is contained in the field

$$K = [c_1 + c_2 a + \cdots + c_{n-1} a^{n-2};\ c_1, \cdots, c_{n-1} \in P_3],$$

where $n = n(a)$ and $P_3 = [0, 1, 2]$ is the prime field with three elements. Consequently, the field generated by a is finite and $a^q = a$, $q = 3^k$, with k a positive integer. But then $a = b^3$, $b = a^r$, with $r = 3^{k-1}$ and $b \in R$. The ring R is, therefore, associative by Lemma 2.

Note added in proof. (Received May 12, 1947.) R. H. Bruck has asked us to include the following theorem of his in our note because of its importance in geometric applications. *Every commutative alternative division ring R is associative and hence a field.* For, if a, b, $c \in R$, we may write $(ab)c = t(a(bc))$, from which $a^3b^3c^3 = t^3a^3b^3c^3$ follows by Lemma 2. It is well known that $3(a, b, c) = 3(t-1)(a(bc)) = 0$. If (a, b, c) is nonzero, then $t^3 = 1$ and $3(t-1) = 0$. But then $(t-1)^3 = t^3 - 3t(t-1) - 1 = 0$, $t = 1$, a contradiction. This result, together with our theorem, provides another proof of the conclusion reached in §3. Irving Kaplansky has given (in conversation) still other proofs. Bruck's result and our theorem also imply that *every regular commutative alternative ring is associative*, since a regular commutative alternative ring has no nilpotent elements [1, p. 767]. We can then also show that *every subdirectly irreducible commutative alternative ring* *without nilpotent elements is a field* (cf. [1, Lemma 2]), but we shall not do this here.

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A NOTE ON RELATIVELY PRIME SEQUENCES

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In volume 2 of Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, pp. 133 and 342, there occurs the following result (appearing also in Hardy-Wright, *Theory of numbers*, p. 14):

THEOREM 1. *No two numbers of the form* $2^{2^n}+1$, $n=1, 2, \cdots$, *have a common divisor greater than* 1.

The numbers $2^{2^n}+1$, $n=1, 2, \cdots$, are the well known Fermat numbers, which may be generated by iteration of the quadratic polynomial $\phi(x)=(x-1)^2+1$, choosing x equal to 3. This follows easily by induction, since, putting $\phi_1(x)=\phi(x)$, $\phi_{n+1}(x)=\phi(\phi_n(x))$, if $\phi_n(x)=2^{2^n}+1$, then $\phi_{n+1}(x)=2^{2^{n+1}}+1$.

The above observation leads to the following result of which Theorem 1 is a special case:

THEOREM 2. *Let* $\phi(x)$ *be a polynomial in* x *with integral coefficients*

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possessing the following properties:

$$(1) \qquad \phi_n(0) = \phi(0), \qquad n \geqq 1, \qquad \phi(0) \neq 0,$$

$$(2) \qquad (x, \phi(0)) = 1 \rightarrow (\phi(x), \phi(0)) = 1.$$

Then if x is an integer and $(x, \phi(0)) = 1$, no two of the numbers $x, \phi_1(x), \cdots, \phi_n(x), \cdots$, have a common divisor greater than 1.

PROOF. Let us assume that the theorem is false, so that for some $m>1$, $n>m$, we have $(\phi_n(x), \phi_m(x))>1$. Since $\phi_n(x)=\phi_{n-m}(\phi_m(x)) \equiv \phi_{n-m}(0) \bmod \phi_m(x) \equiv \phi(0) \bmod \phi_m(x)$, if $(\phi_m(x), \phi_n(x))>1$, $(\phi_n(x), \phi(0))>1$. However, since $(x, \phi(0))=1$, it follows that $(\phi(x), \phi(0))=1$, and thus that $(\phi_n(x), \phi(0))=1$, which is a contradiction.

The sequence $x, \phi_1(x), \cdots, \phi_n(x), \cdots$, will have an infinity of distinct prime divisors if there are an infinite number of terms of the sequence different from ±1. This is true if x is an integer such that for $y \geqq x$, $\phi(y)>y$,[1] or if x is such that $\phi(x)>x$ and x is greater than the roots of $\phi(x)=\pm 1$, or, finally, if $|\phi(\pm 1)|>1$.

It is easy to verify that $\phi(x)=(x-1)^2+1$ satisfies the conditions of Theorem 2. If we choose $x=3$, we obtain the Fermat numbers, as mentioned above. Another admissible polynomial is $(x-2)^4-12$, where we shall choose x satisfying the conditions $(x, 4)=1$, $x \geqq 5$.

Theorem 2 leads one to consider the following question:[2]

Consider an irreducible polynomial $f(x)$ with integral coefficients, and choose an integer x so that all the iterates $f_n(x)$ yield distinct numbers. Can all these numbers be primes?

That this question is probably very difficult to answer might be surmised from the fact that the primality of *all* the Fermat numbers was disproved by exhibiting a specific counter-example, and the behavior of the general term of the sequence $2^{2^n}+1$ is still undetermined.

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[1] These latter alternatives were suggested by the referee, who also pointed out some superfluous restrictions in the original statement of Theorem 2.

[2] The case where $f(x)$ is linear has been worked out by the author and H. N. Shapiro, and the answer is negative.

PARAMETRIC SOLUTION OF LINEAR HOMOGENEOUS DIOPHANTINE EQUATIONS

WALLACE GIVENS

1. **Introduction.** Certain parametric expressions were recently shown by L. W. Griffiths [1][1] to yield all the integral solutions of a system of linear homogeneous equations with integer coefficients in terms of integral values of the parameters. A simpler proof of this same result is given in the next section (cf. our equations (4) with Griffiths' [1] equations (2)). Following this we prove that the denominator in the formulas may be made independent of the values assigned the parameters, a possibility not discussed by Miss Griffiths. The proof of this stronger theorem is made to depend on one of its special cases, namely a result of Hermite's which states that the n minors of order $n-1$ of an $n-1$ by n matrix can have preassigned integral values for a suitable choice of integral values of the components of the matrix.

2. **Form of the solution.** Without essential loss of generality, we may assume that the system of equations is linearly independent and contains more unknowns than equations. Let the equations be

$$\sum_{i=1}^{n} a_{\alpha i}x_i = 0, \qquad \alpha = 1, 2, \cdots, r \geqq 0, \tag{1}$$

where the $a_{\alpha i}$ are rational integers, the matrix $A = \|a_{\alpha i}\|$ is of rank r and we set $n = r+s+1$ with $s \geqq 0$. We shall suppose for convenience that the x_i have been numbered so that the determinant of $\|a_{\alpha\beta}\|$ (α, $\beta = 1, \cdots, r$) is different from 0.

We now adjoin to (1) the system of equations

$$\sum_{i=1}^{n} p_{\rho i}x_i = 0, \qquad \rho = 1, 2, \cdots, s, \tag{2}$$

and let D_i equal $(-1)^{i+1}$ times the determinant obtained by omitting the ith column of the $n-1$ by n matrix

$$\left\| \begin{matrix} A \\ P \end{matrix} \right\|, \qquad \text{where } P = \|p_{\rho i}\|. \tag{3}$$

Then $x_i = D_i$ is an integral solution of (1) and (2) for any choice of

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[1] Numbers in brackets refer to the references cited at the end of the paper.

integral values of the p_{pi}. If at least one of the D_i is different from zero, a relatively prime solution of (1) and (2) is unique to within sign and hence is obtained by dividing the D_i by their greatest common divisor, d, which will depend on the matrix P as well as on A. That every relatively prime solution of (1) alone is obtained in this way by some choice of P is proved by letting ξ_i be a particular nonzero solution of (1) and taking

$$(2') \qquad P = \left\| \begin{matrix} 0 & 0 \cdots 0 & \xi_n & 0 & \cdots & 0 & -\xi_{r+1} \\ 0 & 0 \cdots 0 & 0 & \xi_n & \cdots & 0 & -\xi_{r+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 \cdots 0 & 0 & 0 & \cdots & \xi_n & -\xi_{n-1} \end{matrix} \right\|$$

where for convenience we have supposed the last $n-r$ variables renumbered, if necessary, so that $\xi_n \neq 0$. With this choice of P, $D_i = (-1)^{n+1}(\xi_n)^{s-1}$ (determinant of $\|a_{\alpha\beta}\|$)ξ_i. Hence an arbitrary solution of (1) is of the form

$$(4) \qquad x_1 = \frac{pD_1}{d}, \; x_2 = \frac{pD_2}{d}, \cdots, \; x_n = \frac{pD_n}{d},$$

where p is an arbitrary integer. This is Griffiths' result.

In case $r=0$, (2′) is an $n-1$ by n matrix with signed minors equal to $(-1)^{n+1}(\xi_n)^{n-2}\xi_i$. Modifying (2′) in this case we can obtain a matrix having the ξ_i for signed minors. For, making even permutations on the rows and adding suitable integral multiples of one row to another, we can arrange to have all zeros in the last column except perhaps for the last element in it. Moreover, the values of the minors are unchanged and the elements of the first $n-1$ columns remain divisible by ξ_n. Dividing each of the first $n-2$ rows of the reduced matrix by ξ_n and multiplying one row by $(-1)^{n+1}$, we obtain a matrix having its signed minors equal to the (arbitrary) integers ξ_i. This proves the result due to Hermite [2] and stated in section one. (Hermite's original proof is also elementary in character.)

3. **The stronger theorem.** By Laplace's expansion, the D_i are polynomials in the elements of P with coefficients which are the signed r-rowed minors of A. The greatest common divisor of these minors, say a, is therefore a divisor of each D_i for every choice of the parameters. We now show that the denominator d in (4) can be replaced by a, which is independent of the parameters, thus proving the following theorem.

If $r<n-1$, the polynomials D_i/a in the $n(n-r-1)$ parameters p_{pi}

have the two properties: (1) $x_i = D_i/a$ *is an integral solution of* (1) *for arbitrary integral values of the parameters and* (2) *every integral solution of* (1) *is obtained in this way for some integral values of the parameters. If* $r = n-1$, *the expressions* pD_i/a *in the single parameter* p *have these two properties.*

The case $r = n-1$ was proved above so we shall suppose $s = n-$ $-1 > 0$ in what follows. The proof then consists essentially in redu ing the given system of equations to an equivalent system in cano ical form (r of the variables $=0$, the others arbitrary) and obtainin the parameter matrix of the original from that of the reduced systen the existence of the latter being guaranteed by Hermite's result.

PROOF OF THE THEOREM. Kronecker [3] proved that there exist square matrices M and N with integral elements, having determinants $+1$ or -1, of orders r and n respectively, and such that

$$B = MAN = \left\| \begin{matrix} e_1 & 0 & 0\cdots 0 & 0\cdots 0 \\ 0 & e_2 & 0\cdots 0 & 0\cdots 0 \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ 0 & 0 & 0\cdots e_r & 0\cdots 0 \end{matrix} \right\|, \tag{5}$$

where $e_\beta = (d_\beta/d_{\beta-1})$, d_β is the greatest common divisor of the β-rowed minors of A and $d_0 = 1$. It is clearly possible to choose the determinant of M to be $+1$ and we do this.

If, now, ξ_i is a particular integral solution of (1) and we set

$$\eta_i = \sum_{j=1}^{n} n_{ij}\xi_j, \tag{6}$$

where $\|n_{ij}\| = N^{-1}$ has integral elements, η_i will be an integral solution of the equations

$$\sum_{i=1}^{n} b_{\alpha i}\eta_i = 0, \qquad \alpha = 1, 2, \cdots, r. \tag{7}$$

Hence $\eta_1 = \eta_2 = \cdots = \eta_r = 0$. Adjoining to (7) the equations

$$\sum_{i=1}^{n} q_{\rho i}\eta_i = 0, \qquad \rho = 1, 2, \cdots s, \tag{8}$$

and designating the signed $(n-1)$-rowed minors of the matrix

$$\left\| \begin{matrix} B \\ Q \end{matrix} \right\|, \qquad \text{where } Q = \|q_{\rho i}\|, \tag{9}$$

by Δ_i, it turns out to be possible to find values for the elements of Q so that

$$\eta_i = \Delta_i/a, \tag{10}$$

where $a = d_r = e_1e_2 \cdots e_r =$ the greatest common divisor of the r-rowed minors of A. Indeed (10) reduces for $i \leqq r$ to $\eta_1 = \eta_2 = \cdots = \eta_r = 0$ and for $i > r$ to the equality of η_i and $(-1)^{i+1}$ times the determinant obtained by omitting the ith column of the matrix $\|q_{\rho\tau}\|$ ($\rho = 1, 2, \cdots, s$; $\tau = r+1, \cdots, n$). The existence of such integral values of the $q_{\rho\tau}$ is guaranteed by the theorem of Hermite proved in §2. The values of the parameters $q_{\rho\alpha}$ ($\rho = 1, \cdots, s$; $\alpha = 1, \cdots, r$) may be assigned arbitrarily.

We now define a parameter matrix $P = QN^{-1}$, and consider the signed $(n-1)$-rowed minors of the three matrices:

$$\left\|\begin{matrix} A \\ P \end{matrix}\right\| = \left\|\begin{array}{r} M^{-1}\ BN^{-1} \\ QN^{-1} \end{array}\right\|, \quad \left\|\begin{matrix} B\ N^{-1} \\ Q\ N^{-1} \end{matrix}\right\| = \left\|\begin{matrix} B \\ Q \end{matrix}\right\| N^{-1}, \text{ and } \left\|\begin{matrix} B \\ Q \end{matrix}\right\|. \tag{11}$$

A particular minor of the first of these matrices is equal to the corresponding minor of the second, multiplied by the determinant of M^{-1}, which we chose to be $+1$. Denoting as before the common value of these signed minors by D_i we then find that they are related to the Δ_i by the formula

$$\Delta_i = \pm \sum_{j=1}^{n} n_{ij} D_j, \tag{12}$$

the plus sign being used if the determinant of N is $+1$ and the minus sign if it is -1. Using (10) and (6) we find that

$$\xi_i = \pm D_i/a, \tag{13}$$

and the sign may be absorbed into one row of the parameter matrix.

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MERTENS' THEOREM AND SEQUENCE TRANSFORMATIONS

HUGH J. HAMILTON

The purposes of this note are to prove by sequence transformation theory a known [1, Theorem 6][1] form of Mertens' theorem which admits of a valid converse, and by extending this method to supplement some recent results of Sheffer [2] on extensions of Mertens' theorem to higher dimensions.

Let $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ be two convergent series, with sums A and B, respectively. We shall refer to these series as "a-series" and "b-series," respectively. By definition, their "Cauchy-product series" is $\sum_{k=0}^{\infty}\sum_{i=0}^{k} a_i b_{k-i}$. We shall refer to this series as the "c-series" for the a- and b-series. Now Mertens' theorem states that *a sufficient condition that the c-series converge to AB is that either the a- or the b-series converge absolutely.* Examples show that the condition is not necessary. On the other hand, J. D. Hill has rephrased the theorem so that its converse is true.[2]

THEOREM 1 *A necessary and sufficient condition that the c-series for an a-series and each b-series converge to AB is that the a-series converge absolutely.*

In two dimensions our a-, b-, and c-series become, respectively, $\sum_{i,j=0}^{\infty} a_{i,j}$, $\sum_{i,j=0}^{\infty} b_{i,j}$, and $\sum_{k,l=0}^{\infty}\sum_{i,j=0}^{k,l} a_{i,j} b_{k-i,l-j}$. We shall again suppose that the first two converge,[3] and shall denote their sums by A and B, respectively. Sheffer has recently shown [2, Theorem 1] that *a sufficient condition that the c-series converge to AB is that either the a- or the b-series converge absolutely and the other series converge boundedly.*[4] In our Theorem 3 (below) we show that *a sufficient condition that the c-series converge to AB is that either the a- or the b-series have only a finite number of nonzero terms.* Sheffer's results and ours may be regarded as generalizations of Mertens' theorem, his being the more interesting and ours at first sight trivial.

More interesting than either generalization is the problem of rephrasing it so as to obtain a valid converse. Sheffer in fact does this

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[1] Numbers in brackets refer to the papers cited at the end of this article.

[2] We defer proofs to the end.

[3] Convergence of multiple series shall be in the sense of Pringsheim.

[4] That is, with bounded partial sums.

[2, Theorems 2 and 3] with his generalization to secure the following result.

THEOREM (Sheffer). *A necessary and sufficient condition that the c-series for a b-series and each absolutely convergent a-series converge to AB is that the b-series converge boundedly.*

Unfortunately, this phrasing extends itself neither to the classical Mertens' theorem nor to our generalization. However, the phrasing of Theorem 1 (above) extends itself readily to both Sheffer's and our generalizations, as follows:

THEOREM 2. *A necessary and sufficient condition that the c-series for an a-series and each boundedly convergent b-series converge to AB is that the a-series converge absolutely.*

The duality between Theorem 2 and Sheffer's theorem is interesting. Our simple proof (below) of Theorem 2 of course establishes Sheffer's generalization of Mertens' theorem [2, Theorem 1].

THEOREM 3. *A necessary and sufficient condition that the c-series for an a-series and each convergent b-series converge to AB is that the a-series have only a finite number of nonzero terms.*

We now give proofs, remarking by the way that both Sheffer's and our theorems are equally valid for n-tuple series with $n>2$ in general [2, proofs of Theorems 1 and 2]. Our proofs are applications of linear sequence transformation theory. First, letting $C_m = \sum_{k=0}^{m}\sum_{i=0}^{k} a_i b_{k-i}$ and $B_k = \sum_{i=0}^{k} b_i$, we have

$$C_m = \sum_{i=0}^{m} a_i \sum_{k=i}^{m} b_{k-i} = \sum_{i=0}^{m} a_i B_{m-i} = \sum_{i=0}^{m} a_{m-i} B_i.$$

Similarly,[5] letting $C_{mn} = \sum_{k,l=0}^{m,n}\sum_{i,j=0}^{k,l} a_{ij} b_{k-i,l-j}$ and $B_{kl} = \sum_{i,j=0}^{k,l} b_{ij}$, we have

$$C_{mn} = \sum_{i,j=0}^{m,n} a_{ij} \sum_{k,l=i,j}^{m,n} b_{k-i,l-j} = \sum_{i,j=0}^{m,n} a_{ij} B_{m-i,n-j} = \sum_{i,j=0}^{m,n} a_{m-i,n-j} B_{ij}.$$

Thus the partial sums of the c-series are linear transformations of the partial sums of the b-series, both in one and in two dimensions.

Now the well known Silverman-Toeplitz conditions that a summability method be regular provide the following lemma.

LEMMA 1. *A necessary and sufficient condition that* $\lim_{m\to\infty}\sum_{i=0}^{m} \alpha_{m-i} B_i$

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[5] See also [2, p. 1038, (10)].

$=\lim_{i\to\infty}B_i$ *whenever the latter exists is that* $\sum_{i=0}^{\infty}\alpha_i=1$ *with* $\sum_{i=0}^{\infty}|\alpha_i|<\infty$.

Similarly, we have [3, pp. 59–60, $BC\to C$ with preservation of the limit] the following lemma.

LEMMA 2. *A necessary and sufficient condition that*

$$\lim_{m,n\to\infty}\sum_{i,j=0}^{m,n}\alpha_{m-i,n-j}B_{ij}=\lim_{i,j\to\infty}B_{ij}$$

whenever the latter exists boundedly is that $\sum_{i,j=0}^{\infty}\alpha_{ij}=1$ *with* $\sum_{i,j=0}^{\infty}|\alpha_{ij}|<\infty$.

Again, we have [3, pp. 59–60, $C\to C$ with preservation of the limit] the following lemma.

LEMMA 3. *A necessary and sufficient condition that*

$$\lim_{m,n\to\infty}\sum_{i,j=0}^{m,n}\alpha_{m-i,n-j}B_{ij}=\lim_{i,j\to\infty}B_{ij}$$

whenever the latter exists is that $\sum_{i,j=0}^{\infty}\alpha_{ij}\neq 1$ *with all but a finite number of the* α_{ij} *equal to zero.*

We assume for the moment that $A=0$.

PROOF[6] OF THEOREM 1. Note that $C_m/A=\sum_{i=0}^{m}(a_{m-i}/A)B_i$ and apply Lemma 1.

PROOF OF THEOREM 2. Note that $C_{mn}/A=\sum_{i,j=0}^{m,n}(a_{m-i,n-j}/A)B_{ij}$ and apply Lemma 2.

PROOF OF THEOREM 3. Note that $C_{mn}/A=\sum_{i,j=0}^{m,n}(a_{m-i,n-j}/A)B_{ij}$ and apply Lemma 3.

If $A=0$ we replace the lemmas by similar ones referring to [3, pp. 49, 29, and 32] and reach our conclusions directly.

In conclusion we remark that the boundedness condition in Lemma 2 and the condition in Lemma 3 that nearly all of the α_{ij} vanish are necessary that the transforms in the respective lemmas be bounded even for all sufficiently large m and n, to say nothing of convergence [3, p. 41, 3 and 4].[7]

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[6] See also proof of [1, Theorem 6].

[7] Compare [1], last phrase in statement of Theorem 2.

A NOTE ON TRANSFORMS OF UNBOUNDED SEQUENCES

PAUL ERDÖS AND GEORGE PIRANIAN

During an evening session at the recent Ithaca meeting it was conjectured that it is possible to construct a regular Toeplitz matrix $A\equiv\|a_{nk}\|$ with the property that for every sequence s_n the transformed sequence

$$(1)\qquad t_n=\sum_{k=0}^{\infty}a_{nk}s_k$$

possesses at least one limit point in the finite plane; and it was counter-conjectured that for every regular Toeplitz matrix A there exists a sequence s_n such that the sequence t_n of equation (1) tends to infinity monotonically. It is the purpose of the present note to report that both conjectures are false and to prove a consolation theorem regarding the first conjecture. The notation of equation (1) will be used throughout the paper.

THEOREM 1. *If A is a row-finite regular Toeplitz matrix, there exist sequences s_n such that the corresponding sequences $|t_n|$ tend to infinity with arbitrary rapidity.*

Let A be a row-finite regular Toeplitz matrix. If n_0 is sufficiently large, each row whose index exceeds n_0 contains a nonzero element, and therefore a last nonzero element (a *row-terminal* element). Let $k_1, k_2, \cdots$ be the indices of the columns that contain row-terminal elements ($k_1<k_2<\cdots$), and let the terms s_k ($k\neq k_1, k_2, \cdots$) be chosen arbitrarily. Regularity of the matrix A implies that each column contains at most a finite number of row-terminal elements, that is, that for each column the row-terminal elements are bounded away from zero. It is now clear that if $f(n)$ is any arbitrary real function, the terms $s_{k_1}, s_{k_2}, \cdots$ can be chosen large enough so that $|t_n|>f(n)$ ($n>n_0$), and Theorem 1 is proved.

THEOREM 2. *If A is a regular Toeplitz matrix, there exists a sequence s_n such that the sequence t_n has no limit point in the finite plane.*

Here the matrix A is not required to be row-finite, and if the sequence s_n is chosen (as in the proof of Theorem 1) so as to tend to infinity with reckless rapidity, there is danger that the transformation A does not apply to the sequence (*the transformation A applies to the*

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sequence s_n if the sum $\sum_{k=0}^{\infty} a_{nk}s_k$ exists in the ordinary sense when n is sufficiently large). Let c be a constant such that $\sum_{k=0}^{\infty}|a_{nk}| < c/5$ for all n. (Note: this implies that $c \geqq 5$.) Integers $m_1, n_1, m_2, n_2, \cdots, m_r, n_r, \cdots$ can be chosen successively so that

$$\sum_{k=m_1+1}^{\infty} |a_{0k}| < 1/c,$$

$$\sum_{k=0}^{m_1} |a_{nk}| < 1/5 \qquad \text{when } n > n_1,$$

$$\sum_{k=m_2+1}^{\infty} |a_{nk}| < 1/c^2 \qquad \text{when } n \leqq n_1,$$

$$\sum_{k=0}^{m_2} |a_{nk}| < 1/5 \qquad \text{when } n > n_2,$$

and so that generally

$$\sum_{k=m_r+1}^{\infty} |a_{nk}| < 1/c^r \qquad \text{when } n \leqq n_{r-1} \ (r = 1, 2, \cdots; n_0 = 0),$$

$$\sum_{k=0}^{m_r} |a_{nk}| < 1/5 \qquad \text{when } n > n_r \ (r = 1, 2, \cdots).$$

The sequence s_n is now chosen according to the rule

$$s_k = (1 + 1/c)^r \quad \text{when } m_{r-1} < k \leqq m_r \ (r = 1, 2, \cdots; m_0 = -1).$$

To see that the transform of this sequence tends to infinity, observe that when $n_{r-1} < n \leqq n_r$

$$\text{(i)} \quad \sum_{0}^{m_{r-1}} |a_{nk}| s_k < \frac{1}{5}(1 + 1/c)^{r-1} \qquad (r = 2, 3, \cdots);$$

$$\text{(ii)} \quad \begin{aligned} \sum_{m_{r+1}+1}^{\infty} |a_{nk}| s_k &< \frac{1}{c^{r+1}}(1 + 1/c)^{r+2} + \frac{1}{c^{r+2}}(1 + 1/c)^{r+3} + \cdots \\ &= \frac{1}{c^{r+1}}(1 + 1/c)^{r+2} \frac{1}{1 - (1/c)(1 + 1/c)} \\ &< \frac{1}{c^{r+1}}(1 + 1/c)^{r+2} \frac{1}{1 - 2/5} \\ &\leqq \frac{1}{3c^r}(1 + 1/c)^{r+2} \qquad (r = 0, 1, \cdots); \end{aligned}$$

$$\text{(iii)} \quad \sum_{m_{r-1}+1}^{m_{r+1}} a_{nk}s_k = (1 + 1/c)^r[\alpha_n + \beta_n(1 + 1/c)]$$
$$= (1 + 1/c)^r[\alpha_n + \beta_n + \beta_n/c] \quad (r = 1, 2, \cdots)$$

where

$$\alpha_n = \sum_{m_{r-1}+1}^{m_r} a_{nk}, \qquad \beta_n = \sum_{m_r+1}^{m_{r+1}} a_{nk}, \qquad |\beta_n|/c < 1/5,$$

and, when n is sufficiently large,

$$|\alpha_n + \beta_n| > 3/5.$$

It follows that for all sufficiently large values of n

$$\left|\sum_0^\infty a_{nk}s_k\right| > (1 + 1/c)^r\left[\frac{2}{5} - \frac{1}{5(1 + 1/c)} - \frac{1 + 1/c}{3c^r}\right]$$
$$> \frac{1}{10}(1 + 1/c)^r,$$

and the proof is complete.

Only slight modifications in the last construction are necessary for the following result:

THEOREM 2a. *If A is a regular Toeplitz matrix, there exists a sequence s_n such that $t_n = \rho_n e^{i\theta_n}$, with* $\lim_{n\to\infty}\rho_n = \infty$ *and* $\lim_{n\to\infty}\theta_n = 0$. *If the matrix A is also real, the sequence s_n can be chosen so that the sequence t_n is real and positive.*

In the inequality governing the choice of n_r, the constant $1/5$ must be replaced by a quantity that tends to zero as r becomes large. Also, the growth of s_n must be so slow that the ratio between the coefficients of α_n and β_n respectively tends to unity as r becomes large (this can be achieved by choosing $s_k = (1+1/c)^{r^{1/2}}$ when $m_{r-1} < k \leqq m_r$). Since $\lim_{n\to\infty}(\alpha_n+\beta_n) = 1$ and β_n is bounded, the theorem then follows immediately.

In every regular Toeplitz matrix, the early and the late elements in each row are small. Roughly speaking, the rate of growth of the sequence constructed above (and therefore the rate of growth of the transformed sequence) is retarded by the slowness with which the early and the late elements tend to zero. The proof of the following theorem indicates that this retarding effect is not a mere accident of the construction; the theorem itself salvages a portion of the conjecture that gave occasion to this note.

THEOREM 3. *If $f(n)$ is a real function such that $\lim_{n\to\infty} f(n) = \infty$, there exists a regular Toeplitz matrix A such that, for every sequence s_n to which the transformation A is applicable, the inequality*

$$|t_n| < f(n) \tag{2}$$

is satisfied for infinitely many values of n.

The theorem asserts that there exist regular transformations that transform every sequence to which they are applicable either into a sequence with at least one finite limit point, or else into a sequence whose terms tend to infinity (in absolute value) with arbitrary slowness independent of the sequence. The following matrix satisfies the promise of the theorem: If n is even, $a_{nk} = 1$ when $k = n/2$, and otherwise $a_{nk} = 0$. If n is odd, $a_{nk} = 1$ when $k = (n-1)/2$; $a_{nk} = 0$ when $k < (n-1)/2$; when $k > (n-1)/2$, a_{nk} takes the value zero except when k has one of the values $k_1, k_2, \cdots$ about to be described. The number k_r $(r = 1, 2, \cdots)$ is chosen so that $f(k_r) > 2^r$; and if n is odd and $k_r > (n-1)/2$, $a_{nk_r} = 2^{-r}$.

Suppose now that s_n is a sequence such that the inequality (2) does not hold infinitely often. Because of those rows of A that contain only one nonzero element, it is necessary that $|s_k| \geq f(k)$ for all sufficiently large k. This implies that for odd values of n the formal series $\sum_{k=0}^{\infty} a_{nk}s_k$ contains infinitely many terms whose absolute value exceeds unity, and the transformation A cannot be applicable to the sequence s_n.

THEOREM 4. *There exists a regular Toeplitz matrix A such that the sequence $|t_n|$ does not increase monotonically for any sequence s_n.*

Example: If n is even, choose $a_{nk} = 1$ when $k = n/2$, otherwise $a_{nk} = 0$. If n is odd, choose $a_{nk} = 1 - 1/n$ when $k = (n-1)/2$, otherwise $a_{nk} = 0$.

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NOTE ON POWER SERIES

MAX A. ZORN

1. **The problem.** The following question was raised by Bochner. Let $\sum \alpha_{ik}\xi^i\eta^k$ be a power series with complex coefficients, such that substitution of convergent power series $\sum_1^\infty \alpha_i\zeta^i$ and $\sum_1^\infty \beta_i\zeta^i$ for ξ and η produces always a convergent power series in ζ. Is the double series $\sum \alpha_{ik}\xi^i\eta^k$ convergent?

The answer is yes; we present a proof which presupposes from function theory only the Cauchy estimate for the coefficients of polynomials in a complex variable:

$$\text{(C)} \qquad |\gamma_k\zeta_0^k| \leqq (|\zeta| = |\zeta_0|) \sup \left|\sum \gamma_k\zeta^k\right|.$$

We note that this estimate is also valid in certain types of fields with non-Archimedian valuations, namely, those for which the values are dense and the index is infinite; this was shown by Schoebe in [1].[1]

2. **Homogeneous polynomials.** We denote a vector (ξ, η) by x and introduce as the norm $\|x\|$ of x the maximum of $|\xi|$ and $|\eta|$. A complex Banach space results which, as a complete metric space, is of the second category with respect to itself. We then consider homogeneous polynomials $P(x) = \sum_{i+k=n} \alpha_{ik}\xi^i\eta^k$; it is clear that $P(\zeta x) = \zeta^n P(x)$, that $P(x+\zeta x_0)$ is a polynomial in ζ, and that P is a continuous function of x.

The following three lemmata are immediate consequences of the estimate (C).

(2.1) LEMMA. *If* $|P(x)| \leqq M$ *for* $\|x\| \leqq \zeta$, *then* $|\alpha_{ik}\xi^i\eta^k| \leqq M$ *for* $|\xi|, |\eta| \leqq \zeta$.

(2.2) LEMMA. $|P(x)| \leqq (|\zeta| = 1) \sup |P(x+\zeta x_0)|$.

This special case of the principle of the maximum is a special case of (C), applied to the constant term of $P(x+\zeta x_0)$, considered as a polynomial in ζ. It is used in the proof of (2.3).

(2.3) LEMMA. *If* $|P(x)| \leqq M$ *for* $\|x-x_0\| \leqq \zeta$, *then* $|P(x)| \leqq M$ *for* $\|x\| \leqq \zeta$.

PROOF (compare [2, p. 590]): $|P(x)| \leqq (|\zeta| = 1) \sup |P(\zeta x_0 + x)| = (|\zeta| = 1) \sup |P(x_0 + \zeta^{-1}x)| \leqq (\|x_1 - x_0\| \leqq \|x\|) \sup |P(x_1)|$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

3. Proof of the theorem.

We may now dispose of the problem by proving a slightly stronger result.

(3.1) THEOREM. *If the substitution* $\xi = \alpha\zeta$, $\eta = \beta\zeta$ *produces from* $\sum \alpha_{ik}\xi^i\eta^k$ *always a power series with a nonvanishing radius of convergence then the series* $\sum |\alpha_{ik}\xi^i\eta^k|$ *converges for sufficiently small* $|\xi|$ *and* $|\eta|$.

PROOF. The result of the substitution is, with $(\alpha, \beta) = a$,

$$\sum_{n=0}^{\infty}\Bigl(\sum_{i+k=n} \alpha_{ik}\alpha^i\beta^k\Bigr)\zeta^n = \sum_{0}^{\infty} P_n(a)\zeta^n.$$

Now let δ be a complex number[2] for which $0 < |\delta| < 1$; there will exist, for every vector a, an integer m such that $\sum P_n(a)(\delta^m)^n = \sum P_n(a\delta^m)$ converges. We say that the set D of the vectors x for which $\sum P_n(x)$ converges is of the second category. For every vector is in one of the sets $\delta^{-m}D$; if D were of the first category, the sets $\delta^{-m}D$ and therefore the whole space would be of the same character.

By virtue of the continuity of the functions P_n there will exist (compare [3, p. 19]) a sphere $\|x - x_0\| \leqq \rho$, $\rho > 0$, and an M such that $|P_n(x)| \leqq M$ holds in it for all n. By Lemma (2.3), the same inequality is valid for $\|x\| \leqq \rho$; therefore, $|P_n(x)| \leqq M/2^n$ will be true[3] for $\|x\| \leqq \rho/2$. By Lemma (2.1) we have, for $|\xi|$, $|\eta| \leqq \rho/2$, the inequalities

$$|\alpha_{ik}\xi^i\eta^k| \leqq M/2^n = M/2^{i+k},$$

which establish the absolute convergence of the double series.

4. Comments.

The main point of our arrangement was the weakening of the premise; this procedure was possible essentially because we worked with the complex numbers. It would be interesting to know whether the original conjecture holds in the real case.

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[2] The introduction of the powers of a number in the place of the integers is a concession to the non-Archimedean case and otherwise not relevant.

[3] This inference can also be made in the non-Archimedean case, provided the valuation is dense.

ON THE CONVERGENCE OF DOUBLE SERIES

ALBERT WILANSKY

1. **Introduction.** We consider three convergence definitions for double series, denoted by (p), (σ), and (reg), which are respectively Pringsheim, Sheffer, and regular convergence. Definitions will be given in §2.

Convergence (σ) has been defined by I. M. Sheffer in a paper[1] which will be referred to as [S]. Convergence (p) and (reg) are well known, the latter having been discussed by G. H. Hardy[2] and others.

[S] established the relation $(\sigma)\subseteq(\text{reg})$; that is, every series which is convergent (σ) is also convergent (reg) and to the same sum. The question arises as to whether the relation between these two types of convergence is actually equivalence. It is part of the purpose of this paper to answer this question in the negative. An infinite set of convergence definitions will be presented, denoted by (σ_n), $n=1, 2, \cdots$, with the property:

$$(\sigma) \subseteq (\sigma_{n+1}) \subseteq (\sigma_n) \subseteq (\sigma_1) \subseteq (\text{reg}), \qquad n = 1, 2, \cdots,$$

and it will then be shown that every inclusion sign but the first may be replaced by equivalence. Of these the most difficult to prove is $(\sigma_2)\equiv(\sigma_1)$. The others just escape being trivial.

The words "to the same sum" will always be understood in the symbols $\subset$ and $\equiv$.

2. **Definitions.** Definitions 3 to 3.2 are adapted from [S].

DEFINITION 1. *A region is a* finite *set of values of the indices.* Examples of regions are the following:

(i) Triangular region: the set of indices (p, q) with $p+q\leqq N$ for a given N.

(ii) Rectangular region: the set of (p, q) with $p\leqq M$, $q\leqq N$, for given M, N.

By varying N in (i) (M, N, in (ii)) we obtain a set of triangular (rectangular) regions. These two examples have in common the following important property:

PROPERTY (1). *Given any square region containing* (0, 0) *there is a region of the set under consideration which includes the square region.*

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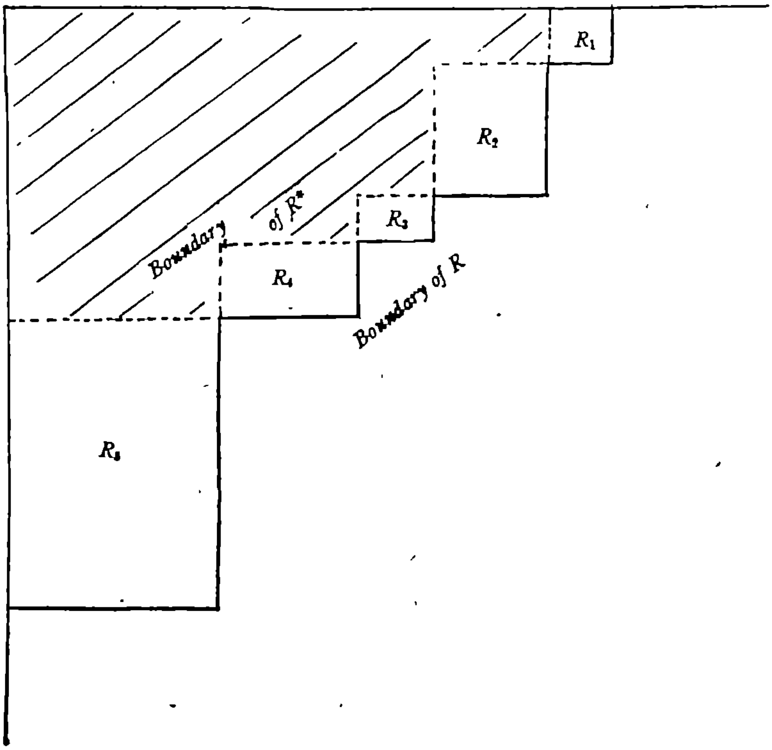
[1] Amer. Math. Monthly vol. 52 (1945) pp. 365–376.

[2] Proc. Cambridge Philos. Soc. vol. 19 (1916–1919) pp. 86–95; Lemmas γ and δ are wrong but Theorem 10 is correct.

The statement "R is a region containing (p, q)" will be written "R is a region (p, q)."

DEFINITION 2. *If R is a region, $\sum(R)$ will denote $\sum_{(i,j)\in R} a_{ij}$, $|R|$ will denote $|\sum(R)|$.*



DEFINITION 3. *Let $\{P\}$ be a set of regions with Property (1). Convergence of $\sum a_{ij}$ to the sum A by means of $\{P\}$ is defined as: to every $\epsilon > 0$ corrrespond indices p and q such that $|A - \sum(R)| < \epsilon$ for every region $R \in \{P\}$ which is a region (p, q).*

DEFINITION 3.1. *Convergence* (p): *The set $\{P\}$ (of Definition 3) is the set of rectangular regions* $(0, 0)$ *of the example in Definition* 1.

DEFINITION 3.2. *A region with the following property will be called a σ-region, or Sheffer region: "if it contains (p, q), then it contains every (i, j), $i \leq p$, $j \leq q$."* An equivalent definition of a σ-region is that it is the logical sum of a finite number of rectangular regions $(0, 0)$. The lower right boundary of a σ-region may be thought of as a "staircase" rising to the right. It should be noted that the set of σ-regions has Property (1), and that each region is finite by definition.

DEFINITION 3.21. *Convergence* (σ): *The set* $\{P\}$ *is the set of all* σ*-regions.*

DEFINITION 3.3. *A* σ*-region whose lower right boundary rises to the right in exactly* n *steps is called a* σ_n*-region.* The figure illustrates $n=5$. It is convenient to count the corners sticking out into the excluded portions of the array. *These are called outer corners. The other* $n-1$ *corners are called inner corners.* It is obvious that $n \geqq 1$.

DEFINITION 3.31. *Convergence* (σ_n), $n \geqq 2$: *The set* $\{P\}$ *is the set of all* σ_k*-regions,* $k \leqq n$.

LEMMA 1. *For* $m \geqq n \geqq 2$, *we have* $(\sigma_m) \subseteq (\sigma_n)$.

For every σ_k-region, $k \leqq n$, is also a σ_k-region, $k \leqq m$.

DEFINITION 3.32. *Convergence* (σ_1): *This is defined as convergence* (p) *together with summability by rows and columns to the same sum.* In other words $\sum a_{ij}$ converges (p) to the sum

$$A = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{ij}.$$

The extra condition makes possible a more unified theory.

DEFINITION 3.4. *Convergence* (reg): *A double series is defined to be convergent* (reg) *if it is convergent* (p), *and if every row and column has a finite sum.*

LEMMA 2. $(\sigma_1) \equiv$ (reg).

This well known fact was proved by Pringsheim.[3] There is a proof in Bromwich, *Infinite series*, 1926, chap. 5, p. 81.

2.1. Consequences. The following lemmas are given in [S] for convergence (σ). The proofs apply formally unchanged to convergence (σ_n). We suppose throughout that $n \geqq 1$ is a fixed integer.

LEMMA 3. *A convergent* (σ_n) *series has a unique sum.*

LEMMA 4. *If* $\sum a_{ij}$ *converges* (σ_n) *to the sum* A, *then* $\sum m \cdot a_{ij}$ *converges* (σ_n) *to* mA.

LEMMA 5 (The general principle of convergence). *For* $n \geqq 2$, *a necessary and sufficient condition that a series converge* (σ_n) *is that to every* $\epsilon > 0$ *correspond indices* p *and* q *such that*

$$\left| \sum (R_1) - \sum (R_2) \right| < \epsilon$$

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[3] *Elementare theorie der unendliche Doppelreihen*, Sitzungsberichte der Akademie der Wissenschaften der München pp. 101–152, especially Satz II, p. 117.

for all R_1, R_2, which are σ_k-regions (p, q), $k \leqq n$.

LEMMA 6. *If $\sum a_{ij}$ converges (σ_n), then to every $\epsilon > 0$ correspond indices p and q such that $|a_{ij}| < \epsilon$ if $i > p$ or $j > q$.*

LEMMA 7. *A series which converges (σ_n) is summable by rows and columns to the same sum.*

For $n = 1$ this is a restatement of part of the definition. For $n \geqq 2$, Sheffer's proof of Theorem 7 in [S] holds.

Since a series which is convergent (σ_n) is obviously convergent (p) we have:

COROLLARY. $(\sigma_n) \subseteq (\sigma_1)$, $n \geqq 1$.

At this stage the following relations are clear:

$$(1) \qquad (\sigma) \subseteq (\sigma_{n+1}) \subseteq (\sigma_n) \subseteq (\sigma_1) = (\text{reg}) \subset (\text{p}), \qquad n = 1, 2, \cdots .$$

That the last relation is strict inclusion is shown by the simple example: $a_{0j} = 1$, $a_{1j} = -1$, $a_{ij} = 0$ if $i > 1$.

3. **Equivalence of all (σ_n).** If R is a σ_{n+1}-region, let R^* denote that σ_n-region whose outer corners are the inner corners of R. (In the figure R^* is indicated by shading.) Let $R - R^* = R_1 + R_2 + \cdots + R_{n+1}$ as shown in the figure. R and R^* will be thought of as including their boundaries; the R_j must be construed accordingly.

THEOREM I. $(\sigma_m) = (\sigma_n)$ *for all* m, $n \geqq 1$.

For definiteness we suppose $m > n$. In view of (1) we have only to prove $(\sigma_n) \subseteq (\sigma_m)$. Suppose that $\sum a_{ij}$ is convergent (σ_n), $n \geqq 1$; we now show that it is convergent (σ_{n+1}) to the same sum. Let us suppose that the sum in question is zero; this can be brought about by the addition of a suitable constant to a_{00}.

Given $\epsilon > 0$, we may by hypothesis choose p, q depending on ϵ, such that $|S| < \epsilon/3$ for any region S that is a σ_k-region (p, q), $k \leqq n$. Now let R be an arbitrary σ_{n+1}-region (p, q). There are two possibilities:

Case I: $(p, q) \in R^*$.

Consider the identity $R = (R - R_1) + [(R - R_2) - (R - R_1 - R_2)]$. It is seen that $|R| \leqq |R - R_1| + |R - R_2| + |R - R_1 - R_2| < \epsilon$, the last inequality holding since the three regions are σ_k-regions (p, q), $k \leqq n$.

Case II: $(p, q) \in R_r$, $1 \leqq r \leqq n+1$.

Suppose at first that $n \geqq 2$. In this case it is possible to chose s, t distinct from r and each other such that $1 \leqq s$, $t \leqq n+1$. Then we can write $R = (R - R_s) + [(R - R_t) - (R - R_s - R_t)]$, and noting that the regions on the right-hand side are again σ_k-regions (p, q), $k \leqq n$, we have $|R| < \epsilon$.

Case II *when* $n = 1$. We require two lemmas.

LEMMA 8. *If* $\sum a_{ij}$ *is summable by rows and columns to the sum* A, *then a necessary and sufficient condition that it be convergent* (p) *to the same sum is the following: to every* $\epsilon > 0$ *correspond indices* p, q *such that* $\left|\sum_{i=r+1}^{\infty}\sum_{j=0}^{s} a_{ij}\right| < \epsilon$ *if* $r > p$ *and* $s > q$.

This follows immediately from consideration of the identity

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} - \sum_{i=0}^{r}\sum_{j=0}^{s} a_{ij} = \sum_{i=r+1}^{\infty}\ \sum_{j=0}^{s} a_{ij} + \sum_{i=0}^{\infty}\ \sum_{j=s+1}^{\infty} a_{ij}.$$

The inversion is justified by the hypothesized summability by rows and columns to the same sum. The series is convergent (p) if and only if the left-hand side can be made smaller than ϵ in absolute value for sufficiently large r, s. The last term on the right-hand side is the remainder of a convergent simple series whose general term is $a_j = \sum_{i=0}^{\infty} a_{ij}$.

Note that in the statement of Lemma 8, the symbol (p) can be replaced by (σ_1) without any alteration in the meaning of the lemma.

LEMMA 9. *If* $\sum a_{ij}$ *is convergent* (σ_1) *the following condition is satisfied: to every* $\epsilon > 0$ *corresponds an index* p *such that if* $r > p$, *then* $\left|\sum_{i=r+1}^{\infty}\sum_{j=0}^{s} a_{ij}\right| < \epsilon$, *for all* s.

First choose p' and q' as in Lemma 8, then choose p'' depending on q' and ϵ such that for $r > p''$ we have $\left|\sum_{i=r+1}^{\infty} a_{ij}\right| < \epsilon/(q'+1)$ for $j = 0, 1, 2, \cdots, q'$. Finally let $p = \max(p', p'')$. Then if $r > p$, we have

$$\left|\sum_{i=r+1}^{\infty}\ \sum_{j=0}^{s} a_{ij}\right| \leqq \sum_{j=0}^{q'}\left|\sum_{i=r+1}^{\infty} a_{ij}\right| < \epsilon, \qquad s \leqq q';$$

$$\left|\sum_{i=r+1}^{\infty}\ \sum_{j=0}^{s} a_{ij}\right| < \epsilon, \qquad s > q'.$$

This proves Lemma 9

To complete the discussion of Case II with $n = 1$, we assume $(p, q) \in R_1$. In view of the symmetry in all formulae, i and j can always be interchanged; thus the following argument will cover the case $(p, q) \in R_2$.

From $R = (R - R_2) + R_2$ follows $|R| < \epsilon/3 + |R_2|$ since the first region is a σ_1-region (p, q). Next

$$|R_s| = \left|\sum_{i=r}^{\infty}\sum_{j=0}^{s} a_{ij} - \sum_{i=t}^{\infty}\sum_{j=0}^{s} a_{ij}\right|$$

$$\leqq \left|\sum_{i=r}^{\infty}\sum_{j=0}^{s} a_{ij}\right| + \left|\sum_{i=t}^{\infty}\sum_{j=0}^{s} a_{ij}\right|,$$

where r, t, s are determined by R_s and $t>r>p$. Lemma 9 shows the possibility of making p so large (depending only on ϵ) that each of the quantities on the right is less than $\epsilon/3$. Thus $|R|<\epsilon$. This completes Case II. The relation $(\sigma_n)\subseteq(\sigma_{n+1})$ having been proved it is clear that Theorem I is established.

4. **On convergence** (σ). Let us introduce the following notations:

Let S_k be a σ_k-region defined as follows: Its inner corners are at points (p, q) with $p+q=k$; its outer corners are at points (p, q) with $p+q=k+1$, with the exception of the two points $(k+1, 0)$ and $(0, k+1)$. S_k is bounded by a staircase which rises to the right in k equal steps from $(k, 0)$ to $(0, k)$.

Let T_k be the "diagonal" region of points (p, q) with $p+q=k$. Thus

$$(2)\qquad \sum (S_k - S_{k-1}) = \sum (T_{k+1}) - a_{k+1,0} - a_{0,k+1} + a_{k,0} + a_{0,k}.$$

LEMMA 10. *A necessary condition for convergence* (σ) *is that the diagonal sums* $|T_k|$ *approach zero as* $k\to\infty$.

This follows from (2) and Lemma 6, taking account of (1).

THEOREM II. $(\sigma)\subset(\sigma_n)$, *strictly, for every* $n\geqq 1$.

The relation of inclusion is a trivial consequence of the definitions since every σ_k-region, $k\leqq n$, is a σ-region. We now exhibit a class of double series each of which converges (σ_1) and thus (σ_n) for every n. This class will be shown to contain a member which does not converge (σ).

Let $a_{ij}=v_{i+j}-v_{i+j+1}$ where $\{v_n\}$ is a sequence such that $\sum v_n$ converges to the sum v. Such double series are considered by Pringsheim.[4] Verification of (σ_1) convergence is trivial, the sum being v. We have

$$\sum (T_k) = \sum_{r=0}^{k} a_{r,k-r} = \sum_{r=0}^{k} (v_k - v_{k+1}) = (k+1)(v_k - v_{k+1})$$

and if we choose $v_n=(-1)^n/(n+1)$ we have $\lim |T_k| = 2$ and, by Lemma 10, $\sum a_{ij}$ is not convergent (σ).

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[4] Loc. cit. pp. 127–128.

4.1 Remarks. Suppose that the hypothesis of a theorem involves convergence (σ). It may happen (and does indeed in the case of all theorems in [S]) that convergence (σ_1) may be substituted in the hypothesis as follows: each σ-region which occurs in the proof is a σ_n-region for some n; if there is a largest such n call it N. Then convergence (σ_N) may be substituted in the hypothesis and may, by Theorem I, be replaced by convergence (σ_1). Lemma 10 is an example of an exception to this statement; there is no largest n and convergence (σ) is essential.

If a series is convergent (σ_1) then, for a given n, the series is convergent (σ_n). Hence, given $\epsilon > 0$ there are indices (p, q) such that $|A - \sum(R)| < \epsilon$ for every σ_n-region R which is a region (p, q). In general (p, q) will depend on n; if the choice does not depend on n then the series is convergent (σ).

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THE DIFFERENTIABILITY AND UNIQUENESS OF CONTINUOUS SOLUTIONS OF ADDITION FORMULAS

NELSON DUNFORD AND EINAR HILLE

The problem of representing a one-parameter group of operators (that is, a family T_ξ, $-\infty < \xi < \infty$, of bounded linear operators on a Banach space which satisfies $T_{\xi+\eta} = T_\xi T_\eta$) reduces according to several well known methods of attack to establishing differentiability of the function T_ξ at $\xi = 0$. The derivative $Ax = \lim_{\xi\to 0} \xi^{-1}(T_\xi - I)x$ exists as a closed operator with domain $D(A)$ dense, providing T_ξ is continuous in the *strong* operator topology (that is, $\lim_{\xi\to\xi_0} T_\xi x = T_{\xi_0} x$, $x \in \mathfrak{X}$). It is then possible to assign a meaning to exp (ξA) in a natural way and so that $T_\xi = \exp(\xi A)$, $-\infty < \xi < \infty$. The operator A is bounded if and only if T_ξ is continuous in ξ in the *uniform* operator topology (that is, $\lim_{\xi\to\xi_0} |T_\xi - T_{\xi_0}| = 0$) in which case $A = \lim_{\xi\to 0} \xi^{-1}(T_\xi - I)$ exists in the uniform topology. This implies that T_ξ is an entire function of ξ; conversely, if T_ξ is analytic anywhere, then A is bounded. These considerations extend to the semi-group case in which $T_{\xi+\eta} = T_\xi T_\eta$ is known to hold only for positive values of the parameters, although

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the number of distinct cases is much larger and, in particular, analyticity does not imply that A is bounded.

It is a matter of natural curiosity to ask whether or not similar results hold if the semi-group law $f(\xi+\zeta)=f(\xi)f(\zeta)$ is replaced by an arbitrary addition formula. The results presented in the following correspond to the case of an analytical group (that is, continuity in the uniform topology). The strong operator topology leads to particular difficulties which have been overcome only in part, but we hope to return to this case on a future occasion.

In this note we consider the differentiability and uniqueness of continuous solutions $f(\xi)$ of the equation

$$f(\xi+\zeta)=G[f(\xi),f(\zeta)],\qquad 0\leqq\xi,\zeta,\xi+\zeta\leqq\omega, \tag{1}$$

where $G[\alpha,\beta]$ is a symmetric complex function analytic for α, β in the closure of a domain Δ bounded by a rectifiable Jordan curve. The solutions considered are functions $f(\xi)$ on $0\leqq\xi\leqq\omega$ to a commutative complex Banach algebra B with unit e. We define $G[u,v]$ only for those u, $v\in B(\Delta)$, the subset of B consisting of elements x whose spectrum $\sigma x\subset\Delta$. For such u, v we define $G[u,v]$ by the double resolvent integral

$$G[u,v]=\frac{1}{(2\pi i)^2}\int_{\Gamma_u}\int_{\Gamma_v}G[\alpha,\beta]R(\alpha,u)R(\beta,v)\,d\alpha\,d\beta, \tag{2}$$

where $R(\alpha,u)=(\alpha e-u)^{-1}$ and Γ_u, Γ_v are oriented envelopes in Δ of σu, σv respectively. Thus by a solution of (1) is meant a function $f(\xi)$ on $0\leqq\xi\leqq\omega$ to $B(\Delta)$ which satisfies (1).

THEOREM. *If $f(\xi)$ is a continuous B-valued solution of* (1) *and if $G_1[f(0),f(0)]$ has an inverse ($G_1[\alpha,\beta]=(\partial/\partial\alpha)G[\alpha,\beta]$), then $f(\xi)$ has derivatives of all orders and $f'(\xi)=G_1[f(0),f(\xi)]f'(0)$. If $g(\xi)$ is any other continuous solution of* (1) *with $g(0)=f(0)$ and $g'(0)=f'(0)$, then $g(\xi)=f(\xi)$. If $\phi(\xi)$ is a nonconstant scalar analytic solution of* (1) *then the only continuous B-valued solution of* (1) *with $f(0)=\phi(0)e$ is given by $f(\xi)=\phi(f'(0)\xi)$ in a neighborhood $0\leqq\xi\leqq\rho$.*

LEMMA. *Let $Q[\alpha,\beta,\gamma]=(G[\alpha,\gamma]-G[\beta,\gamma])(\alpha-\beta)^{-1}$, $\alpha,\beta,\gamma\in\Delta$. If $f(\xi)$ is continuous on $0\leqq\xi\leqq\omega$ to $B(\Delta)$ and $0\leqq\eta$, $\zeta\leqq\omega$ then uniformly with respect to ζ we have*

$$\lim_{\xi\to\eta}Q[f(\xi),f(\eta),f(\zeta)]=G_1[f(\eta),f(\zeta)], \tag{3}$$

$$Q(u,v,w)(u-v)=G(u,w)-G(v,w),\qquad u,v,w\in B(\Delta). \tag{4}$$

First note that since $f(\xi)$ is continuous in $[0, \omega]$ the range of $f(\xi)$ is a closed connected compact set $R\subset B(\Delta)$. If $\Phi = \bigcup \sigma x$, $x\in R$, then Φ is a closed subset of Δ and if Γ is an oriented envelope of Φ in Δ, then $R[\alpha, f(\xi)]$ is a continuous function of (α, ξ) for $\alpha\in\Gamma$, $0\leqq\xi\leqq\omega$. There is consequently a finite positive constant $M=M(\Gamma)$ such that

(5) $$|R[\alpha, f(\xi)]| \leqq M(\Gamma), \qquad \alpha\in\Gamma, 0\leqq\xi\leqq\omega.$$

Further, $\lim_{\xi\to\eta} R[\alpha, f(\xi)] = R[\alpha, f(\eta)]$ uniformly with respect to α on Γ. It follows that uniformly in ζ, $0\leqq\zeta\leqq\omega$,

$$\begin{aligned}
&\lim_{\xi\to\eta} Q[f(\xi), f(\eta), f(\zeta)]\\
&= -\frac{1}{8\pi^3 i}\int_\Gamma\int_\Gamma\int_\Gamma Q[\alpha, \beta, \gamma]R[\alpha, f(\eta)]R[\beta, f(\eta)]R[\gamma, f(\zeta)]d\alpha d\beta d\gamma\\
&= \frac{1}{8\pi^3 i}\int_\Gamma\int_\Gamma\int_\Gamma Q(\alpha, \beta, \gamma)\frac{R[\alpha, f(\eta)] - R[\beta, f(\eta)]}{\alpha-\beta}R[\gamma, f(\zeta)]d\alpha d\beta d\gamma\\
&= \frac{1}{8\pi^3 i}\int_{\Gamma_1}\int_{\Gamma_1} d\beta d\gamma R[\gamma, f(\zeta)]\left\{\int_\Gamma \frac{Q(\alpha, \beta, \gamma)R[\alpha, f(\eta)]}{\alpha-\beta}d\alpha\right.\\
&\qquad\qquad\left. - R[\beta, f(\eta)]\int_\Gamma \frac{Q(\alpha, \beta, \gamma)}{\alpha-\beta}d\alpha\right\}.
\end{aligned}$$

Here Γ, Γ_1, are oriented envelopes of Φ in Δ and Γ_1 is interior to Γ. Now

$$\int_\Gamma \frac{Q(\alpha, \beta, \gamma)}{\alpha-\beta}d\alpha = 2\pi i Q(\beta, \beta, \gamma) = 2\pi i G_1(\beta, \gamma)$$

so

$$\begin{aligned}
&\lim_{\xi\to\eta} Q[f(\xi), f(\eta), f(\zeta)]\\
&\qquad = \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_1} G_1(\beta, \gamma)R[\beta, f(\eta)]R[\gamma, f(\zeta)]d\beta d\gamma + U\\
&\qquad = G_1[f(\eta), f(\zeta)] + U.
\end{aligned}$$

Here

$$\begin{aligned}
U &= \frac{1}{8\pi^3 i}\int_{\Gamma_1}\int_{\Gamma_1} d\beta d\gamma R[\gamma, f(\zeta)]\int_\Gamma \frac{Q(\alpha, \beta, \gamma)}{\alpha-\beta}R[\alpha, f(\eta)]d\alpha\\
&= \frac{1}{8\pi^3 i}\int_{\Gamma_1} d\gamma R[\gamma, f(\zeta)]\int_\Gamma d\alpha R[\alpha, f(\eta)]\int_{\Gamma_1}\frac{Q(\alpha, \beta, \gamma)}{\alpha-\beta}d\beta.
\end{aligned}$$

But for $\alpha \in \Gamma$ the last integral is zero so $U=0$. This completes the proof of (3). Equation (4) may be proved in a fashion analogous to the method used for establishing the multiplicative law in the operational calculus involving functions of one variable.

We now proceed to the proof of the theorem. From the contour integral definition of $G_1[f(0), f(\zeta)]$ we see that it is continuous in ζ and hence

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_0^\alpha G_1[f(0), f(\zeta)]\, d\zeta = G_1[f(0), f(0)].$$

Thus we may fix $\alpha < \omega$ so that the integral on the left has an inverse in B. From the lemma we have

$$\lim_{\xi \to 0} \frac{1}{\alpha} \int_0^\alpha Q[f(\xi), f(0), f(\zeta)]\, d\zeta = \frac{1}{\alpha} \int_0^\alpha G_1[f(0), f(\zeta)]\, d\zeta.$$

Hence

$$\begin{aligned} &\lim_{\xi \to 0} \left\{ \frac{1}{\alpha} \int_0^\alpha Q[f(\xi), f(0), f(\zeta)]\, d\zeta \right\}^{-1} \\ &\qquad = \left\{ \frac{1}{\alpha} \int_0^\alpha G_1[f(0), f(\zeta)]\, d\zeta \right\}^{-1}. \end{aligned} \tag{6}$$

From (4) we have

$$[f(\xi) - f(\eta)]Q[f(\xi), f(\eta), f(\zeta)] = G[f(\xi), f(\zeta)] - G[f(\eta), f(\zeta)] \tag{7}$$

whence

$$\begin{aligned} \frac{1}{\xi}[f(\xi) - f(0)] &\frac{1}{\alpha} \int_0^\alpha Q[f(\xi), f(0), f(\zeta)]\, d\zeta \\ &= \frac{1}{\alpha} \int_0^\alpha \frac{1}{\xi} \{G[f(\xi), f(\zeta)] - G[f(0), f(\zeta)]\}\, d\zeta \\ &= \frac{1}{\alpha\xi} \int_0^\alpha [f(\xi + \zeta) - f(\zeta)]\, d\zeta \\ &= \frac{1}{\alpha\xi} \left\{ \int_\alpha^{\alpha+\xi} f(\tau)\, d\tau - \int_0^\xi f(\tau)\, d\tau \right\} \\ &\to \frac{1}{\alpha}[f(\alpha) - f(0)] \qquad \text{as } \xi \to 0. \end{aligned} \tag{8}$$

Thus (6) and (8) give the existence of

$$\lim_{\xi\to 0}\frac{1}{\xi}\,[f(\xi)-f(0)] = \left\{\frac{1}{\alpha}\int_0^{\alpha} G_1[f(0), f(\zeta)]d\zeta\right\}^{-1}\frac{1}{\alpha}\,[f(\alpha)-f(0)].$$

Thus $f(\xi)$ is differentiable at $\xi=0$. Applying the lemma once more we have

$$\begin{aligned}\frac{1}{\eta}\,[f(\xi+\eta)-f(\xi)] &= \frac{1}{\eta}\,\{G[f(\eta), f(\xi)] - G[f(0), f(\xi)]\}\\ &= \frac{1}{\eta}\,[f(\eta)-f(0)]Q[f(\eta), f(0), f(\xi)]\\ &\to f'(0)G_1[f(0), f(\xi)]\end{aligned}$$

uniformly for $0\leqq\xi\leqq\omega$. The existence of the higher derivatives is readily established. We shall indicate the argument for the case of the second derivative. We have

$$f'(\xi) = f'(0)\,\frac{1}{(2\pi i)^2}\int_\Gamma\int_\Gamma G_1[\alpha,\beta]R[\alpha, f(0)]R[\beta, f(\xi)]d\alpha d\beta.$$

It is readily shown that uniformly for β on Γ we have as $\eta\to 0$

$$\frac{1}{\eta}\,\{R[\beta, f(\xi+\eta)] - R[\beta, f(\xi)]\} \to -f'(\xi)\{R[\beta, f(\xi)]\}^2.$$

Hence

$$f''(\xi) = -f'(0)f'(\xi)\,\frac{1}{(2\pi i)^2}\int_\Gamma\int_\Gamma G_1(\alpha,\beta)R[\alpha, f(0)]R[\beta, f(\xi)]^2 d\alpha d\beta$$

and a contour integral argument of familiar type shows that this expression equals $f'(0)f'(\xi)G_{11}[f(0), f(\xi)]$ where $G_{11}[\alpha,\beta] = (\partial^2/\partial\alpha\partial\beta)G[\alpha,\beta]$ and $G_{11}(u, v)$ is defined by the usual contour integral for $u, v\in B(\Delta)$. We see in particular that $f''(0) = [f'(0)]^2G_{11}[f(0), f(0)]$ and hence is uniquely determined by $f(0)$ and $f'(0)$. Similarly it may be shown that all higher derivatives exist and are uniquely determined by $f(0)$ and $f'(0)$.

Now suppose that $g(\xi)$ is another continuous solution of equation (1) with $g(0)=f(0)$ and $g'(0)=f'(0)$. From the preceding remarks we see that $g^{(n)}(0)=f^{(n)}(0)$, $n=1, 2, \cdots$. From the contour integral representation of the function $Q[f(\xi), g(\xi), f(\eta)]$ combined with (5) we see that there is a finite $K=K(f, g)$ such that for $0\leqq\xi, \eta\leqq\omega$

$$|Q[f(\xi), g(\xi), f(\eta)]|\leqq K,\qquad |Q[g(\eta), f(\eta), g(\xi)]|\leqq K.$$

Since

$$G[f(\xi), f(\eta)] - G[g(\xi), f(\eta)] = Q[f(\xi), g(\xi), f(\eta)][f(\xi) - g(\xi)]$$

we have

$$|G[f(\xi), f(\eta)] - G[g(\xi), f(\eta)]| \leq K|[f(\xi) - g(\xi)]| \tag{9}$$

and a similar inequality in which f, g and ξ, η are interchanged. Placing $h(\xi) = f(\xi) - g(\xi)$ we have

$$\begin{aligned} h(\xi + \eta) &= f(\xi + \eta) - g(\xi + \eta) \\ &= G[f(\xi), f(\eta)] - G[g(\xi), g(\eta)] \\ &= G[f(\xi), f(\eta)] - G[g(\xi), f(\eta)] \\ &\quad + G[g(\xi), f(\eta)] - G[g(\xi), g(\eta)]. \end{aligned}$$

Hence from (9) we have

$$|h(\xi + \eta)| \leq K\{|f(\xi) - g(\xi)| + |f(\eta) - g(\eta)| = K\{|h(\xi)| + |h(\eta)|\}$$

whence

$$|h(2\xi)| \leq 2K|h(\xi)|.$$

By repeated use of this inequality we get

$$h(\xi) \leq (2K)^m |h(\xi 2^{-m})|, \qquad m = 1, 2, 3, \cdots. \tag{10}$$

Now consider

$$g_n(\xi) = g(\xi) - \sum_{\nu=0}^{n} g^{(\nu)}(0) \frac{\xi^\nu}{\nu!}$$

and let

$$M_n[g] = \max |g_n^{(n)}(\xi)| = \max |g^{(n)}(\xi) - g^{(n)}(0)|$$

in the interval $[0, \omega]$. Since

$$g_n(\xi) = \int_0^\xi \cdots \int_0^{\xi_{n-2}} \int_0^{\xi_{n-1}} g_n^{(n)}(\xi_n) d\xi_n d\xi_{n-1} \cdots d\xi_1,$$

we have

$$|g_n(\xi)| \leq \frac{\xi^n}{n!} M_n[g], \qquad 0 \leq \xi \leq \omega.$$

Since

$$|h(\xi)| = |f(\xi) - g(\xi)| = |f_n(\xi) - g_n(\xi)| \leq |f_n(\xi)| + |g_n(\xi)|,$$

we see that $|h(\xi)| \leq 2\xi^n M_n/n!$, where M_n is the larger of $M_n[f]$ and

$M_n[g]$. A combination of this inequality with (10) yields

$$|k(\xi)| \leq \frac{2}{n!} M_n\omega^n(2^{1-n}K)^m, \qquad 0 \leq \xi \leq \omega,\ m, n = 1, 2, \cdots .$$

Here we fix n so that $2^{1-n}K < 1$. Since m is independent of n and may be taken arbitrarily large we see that $k(\xi) = 0$ and thus that $g(\xi) = f(\xi)$, $0 \leq \xi \leq \omega$.

Suppose now that $\phi(\xi)$ is a nonconstant analytic scalar solution of (1). Since ϕ is analytic at $\zeta = 0$ we have $\phi(\zeta) = \sum \alpha_n \zeta^n |\zeta| < \rho$ and

$$\phi(\zeta_1 + \zeta_2) = G[\phi(\zeta_1), \phi(\zeta_2)], \qquad |\zeta_1|, |\zeta_2|, |\zeta_1 + \zeta_2| < \rho. \tag{11}$$

Differentiating $\phi(\zeta) = G[\phi(\zeta), \phi(0)]$ we get $\phi'(\zeta) = \phi'(\zeta)G_1[\phi(\zeta), \phi(0)]$ and since $\phi'(\zeta) \neq 0$ we have $G_1[\phi(\zeta), \phi(0)] = 1$ and, in particular,

$$G_1[\phi(0), \phi(0)] = 1.$$

Differentiating (11) by parts with respect to ζ_1 and putting $\zeta_1 = 0$ in the result one gets

$$\phi'(\xi) = \phi'(0)G_1[\phi(0), \phi(\xi)]$$

whence $\phi'(0) \neq 0$. But if $\phi(\zeta)$ satisfies the conditions stated so does $\phi(\alpha\zeta)$ for any $\alpha \neq 0$. We can consequently normalize $\phi(\zeta)$ by assuming that $\phi'(0) = 1$. Suppose now that $f(\xi)$ is a continuous, and hence differentiable, solution of (1) such that $f(0) = \phi(0)e$, $f'(0) = a$. Then $G_1[f(0), f(0)] = G_1[\phi(0), \phi(0)]e = e$ and hence it has an inverse. On the other hand the function $\phi(a\zeta)$ is given by the series $\sum \alpha_n a^n \zeta^n$ at least for $|\zeta| < \rho/|a|$ and for such values of ζ we have also $\sigma\phi(a\zeta) = \phi(\zeta\sigma a) \subset \Delta$ since $|\sigma a| \leq |a|$. From the construction of the series and the properties mentioned it follows that it satisfies (1) for $|\zeta_1|$, $|\zeta_2|$, $|\zeta_1 + \zeta_2| < \rho/|a|$. Further, $\phi(0) = \phi(0)e = f(0)$, $\phi'(0) = a = f'(0)$ and thus $f(\xi) = \phi(a\xi)$.

THE INSTITUTE FOR ADVANCED STUDY AND
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SOME GENERALIZED HYPERGEOMETRIC POLYNOMIALS

SISTER MARY CELINE FASENMYER

1. Introduction. We shall obtain some basic formal properties of the hypergeometric polynomials

$$
\begin{aligned}
f_n(a_i;\, b_j;\, x) &= f_n(a_1, a_2, \cdots, a_p;\, b_1, b_2, \cdots, b_q;\, x) \\
&= {}_{p+2}F_{q+2}\left[\begin{matrix} -n, & n+1, & a_1, \cdots, a_p; \\ 1/2, & 1, & b_1, \cdots, b_q; \end{matrix}\; x\right]
\end{aligned} \tag{1}
$$

(n a non-negative integer) in an attempt to unify and to extend the study of certain sets of polynomials which have attracted considerable attention. Some special cases of the $f_n(a_i;\, b_j;\, x)$ are:[1]

(a) $f_n(1/2;\, —;\, x) = P_n(1-2x)$ (Legendre).
(b) $f_n(1;\, —;\, x) = [n!/(1/2)_n]P_n^{(-1/2,1/2)}(1-2x)$ (Jacobi).
(c) $f_n(1,\, 1/2;\, b;\, x) = [n!/(b)_n]P_n^{(b-1,1-b)}(1-2x)$ (Jacobi).
(d) $f_n(1/2,\, \zeta;\, p;\, v) = H_n(\zeta,\, p,\, v)$ [12].
(e) $f_n[1/2,\, (1+z)/2;\, 1;\, 1] = F_n(z)$ [3].
(f) $f_n(1/2;\, 1;\, t) = Z_n(t)$ [4].
(g) $f_n[1/2,\, (z+m+1)/2;\, m+1;\, 1] = F_n^m(z)$ [8].

2. A generating function. Let $G(y)$ be analytic at $y=0$,

$$
G(y) = \sum_{n=0}^{\infty} c_n y^n,
$$

and define $f_n(x)$ by the relation

$$
\frac{1}{1-w}\, G\left[\frac{-4xw}{(1-w)^2}\right] = \sum_{n=0}^{\infty} f_n(x) w^n. \tag{2}
$$

If w is sufficiently small, the left member of (2) may be expanded in an absolutely convergent double series and rearranged so as to give a convergent power series in w. Let that be done. Then it is easily shown that

$$
f_n(x) = \sum_{r=0}^{n} \frac{(-n)_r (n+1)_r c_r x^r}{(1/2)_r r!} \tag{3}
$$

in which $(a)_r = a(a+1) \cdots (a+r-1)$; $(a)_0 = 1$.

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[1] A dash indicates the absence of parameters. A number in brackets relates to the references.

Thus, if $G(y)$ is a generalized hypergeometric function,[2]

$$G(y) = {}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; y), \tag{4}$$

then the $f_n(x)$ defined by (2) are precisely the polynomials $f_n(a_i; b_j; x)$ given by (1). This is a special case of a known summation formula given by T. W. Chaundy [5, p. 62]. It is necessary to note that in using the generating function we implicitly demand that the parameters a_i and b_j be independent of n.

Starting with (1) in its expanded form,

$$f_n(a_i; b_j; x) = \sum_{r=0}^{n} \frac{(-n)_r(n+1)_r(a_1)_r \cdots (a_p)_r x^r}{(1/2)_r(b_1)_r \cdots (b_q)_r r!}, \tag{7}$$

we may, by replacing r by $n-r$ and reversing the order of summation, write

$$\begin{aligned} f_n(a_i; b_j; x) = {} & \frac{(-4x)^n(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!} \\ & {}_{q+3}F_{p+1}\left[\begin{matrix} -n, -n, 1/2-n, 1-b_1-n, \cdots, 1-b_q-n; \\ -2n, \qquad 1-a_1-n, \cdots, 1-a_p-n; \end{matrix} (-1)^{q-p+1}/x\right]. \end{aligned} \tag{6}$$

3. **A differential recurrence formula.** Let us return to (2) without the restriction that $G(y)$ be of hypergeometric type. Two equations each involving $G'(y)$ may be found by separate differentiations of (2) with respect to x and to w. Then $G'(y)$ may be eliminated and

$$\theta[f_n(x) + f_{n-1}(x)] = n[f_n(x) - f_{n-1}(x)], \tag{7}$$

in which θ is the operator xd/dx.

Direct application of (7) to (d) and (f) of §1 yields the possibly new relations $v[H_n'(\zeta, p, v) + H_{n-1}'(\zeta, p, v)] = n[H_n(\zeta, p, v) - H_{n-1}(\zeta, p, v)]$ (in which primes denote differentiation with respect to v) and

$$t[Z_n'(t) + Z_{n-1}'(t)] = n[Z_n(t) - Z_{n-1}(t)].$$

4. **Pure recurrence relations.**[3] Let $p \leqq q+1$, the parameters a_i and b_j be independent of n, no a_i be equal to any b_j, one, or one-half and let no b_j be equal to a nonpositive integer. Then, for $n \geqq q+4$, $f_n(a_i; b_j; x)$ satisfies a pure linear recurrence relation of exactly $q+4$ terms of the following type:

---

[2] Sufficient conditions for the convergence of the series for $G(y)$ in this form for $p \leqq q+1$ are given in [2] and will not be restated here.

[3] The proofs and details of the material in §§4 and 5 can be found in my dissertation at the University of Michigan, 1945.

$$
\begin{aligned}
&f_n + (A_1 + A_2x)f_{n-1} + (A_3 + A_4x)f_{n-2} + \cdots \\
&\quad + (A_{2k-1} + A_{2k}x)f_{n-k} + \cdots \\
&\quad + (A_{2q+3} + A_{2q+4}x)f_{n-q-2} + A_{2q+5}f_{n-q-3} = 0,
\end{aligned} \tag{8}
$$

in which the numerators of the rational functions A_{2k} are functions of n and a_i only. If we change the hypothesis slightly so as to permit a_1 to equal either one or one-half, we obtain a linear recurrence relation of $q+3$ terms. If we allow a_1 to equal one and a_2 to equal one-half, there exists a linear recurrence relation of $q+2$ terms.

As an example, Bateman's $Z_n(t)$ [4] has the following recurrence relation:

$$
\begin{aligned}
n^2(2n-3)Z_n(t) &- (2n-1)[3n^2 - 6n + 2 - 2(2n-3)t]Z_{n-1}(t) \\
&+ (2n-3)[3n^2 - 6n + 2 + 2(2n-1)t]Z_{n-2}(t) \\
&- (2n-1)(n-2)^2Z_{n-3}(t) = 0,
\end{aligned} \tag{9}
$$

which we have not been able to find elsewhere.

Now let $p > q+1$ but retain the remaining conditions in the opening sentence of this section. Then, for $n \geqq p+3$, $f_n(a_i; b_j; x)$ satisfies a pure linear recurrence relation of exactly $p+3$ terms and of the same type as (8) above.

5. **Contiguous polynomial relations.** For the sake of simplicity, when discussing contiguous polynomials we use the shortened notation $f_n(a_1+)$ to signify $f_n(a_1+1, \cdots, a_p; b_j; x)$ while $f_n(a_m-)$, $f_n(b_m+)$, and $f_n(b_m-)$ have similar meanings. Using Rainville's results [11] it can be shown that the $f_n(a_i; b_j; x)$ have the following canonical set of $2p+q-2$ contiguous polynomial relations:

For $p \leqq q+1$,

$$(a_1 - a_m)f_n = a_1f_n(a_1+) - a_mf_n(a_m+); \qquad m = 2, 3, \cdots, p, \tag{10}$$

$$(a_1 - b_m + 1)f_n = a_1f_n(a_1+) - (b_m - 1)f_n(b_m-); m = 1, 2, \cdots, q. \tag{11}$$

For $p < q$, (10) and (11) with

$$
\begin{aligned}
&4(a_1 - a_m)(a_1 + a_m - 1)f_n \\
&\quad = a_1(2a_1 - 1)f_n(a_1+) - a_m(2a_m - 1)f_n(a_m+) \\
&\quad + (a_1 - 1)(2a_1 - 1)f_n(a_1-) - (a_m - 1)(2a_m - 1)f_n(a_m-) \\
&\quad - 2(a_1 - a_m)x\sum_{j=1}^{q} T_{j;1,m}f_n(b_j+); \qquad m = 2, 3, \cdots, p,
\end{aligned} \tag{12}
$$

in which

$$T_{j;k,m} = \frac{-(n+b_j)(n+1-b_j)\prod_{s=1,(k,m)}^{p}(a_s-b_j)}{b_j\prod_{s=1,(j)}^{q}(b_s-b_j)},$$

in which $\prod_{s=1,(k,m)}^{p} c_s$ indicates the product of p factors, from which c_k and c_m have been deleted.

For $p=q$, (10) and (11) with

$$\begin{aligned} &2(a_1-a_m)[2(a_1+a_m-1)+x]f_n \\ &\quad = a_1(2a_1-1)f_n(a_1+) - a_m(2a_m-1)f_n(a_m+) \\ &\quad + (a_1-1)(2a_1-1)f_n(a_1-) - (a_m-1)(2a_m-1)f_n(a_m-) \\ &\quad - 2(a_1-a_m)x\sum_{j=1}^{q} T_{j;1,m}f_n(b_j+); \qquad m = 2, 3, \cdots, p. \end{aligned} \tag{13}$$

For $p=q+1$, (10) and (11) with

$$\begin{aligned} &(a_1-a_m)[4(a_1+a_m-1)-(4a_1+4a_m-2C+2D-3)x]f_n \\ &\quad = a_1(2a_1-1)(1-x)f_n(a_1+) - a_m(2a_m-1)(1-x)f_n(a_m+) \\ &\quad + (a_1-1)(2a_1-1)f_n(a_1-) - (a_m-1)(2a_m-1)f_n(a_m-) \\ &\quad - 2(a_1-a_m)x\sum_{j=1}^{q} T_{j;1,m}f_n(b_j+); \qquad m = 2, 3, \cdots, p, \end{aligned} \tag{14}$$

in which

$$C = \sum_{s=1}^{p} a_s \quad \text{and} \quad D = \sum_{s=1}^{q} b_s.$$

If we admit $f_{n-1}(a_i; b_j; x)$ and polynomials contiguous to it, we have another contiguous polynomial relation, namely,

$$\begin{aligned} &(n+a_1)f_n - (n-b_1+1)f_{n-1} - a_1f_n(a_1+) \\ &\qquad - (b_1-1)f_{n-1}(b_1-) = 0. \end{aligned} \tag{15}$$

By an extension of the method used by Rainville for the derivation of his formula (30) in [11], it is comparatively easy to obtain a similar set of contiguous function relations applicable to any terminating generalized hypergeometric function in which the number of numerator parameters exceeds the number of those in the denominator by two or more.

6. Integral relations. Using standard methods[4] we obtain the integral relations

$$(16)\qquad f_n(a_i;\, b_j;\, x) = \frac{i}{2\pi^{1/2}} \int_{\infty}^{(0+)} (-y)^{-1/2} e^{-y} f_n(1/2,\, a_i;\, b_j;\, -x/y)\, dy,$$

$$(17)\qquad f_n(a_i;\, b_j;\, x) = \frac{1}{\pi^{1/2}} \int_{0}^{\infty} y^{-1/2} e^{-y} f_n(a_i;\, 1/2,\, b_j;\, xy)\, dy,$$

and

$$(18)\qquad \begin{aligned} f_n(a_i;\, b_j;\, x) &= \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1 - a_1)} \\ &\quad \cdot \int_0^1 y^{a_1-1}(1-y)^{b_1-a_1-1} f_n(a_2, \cdots, a_p;\, b_2, \cdots, b_q;\, xy)\, dy, \end{aligned}$$

in which $R(b_1) > R(a_1) > 0$.

Some special cases of interest are:

$$(19)\qquad f_n(-;\, -;\, x) = \frac{i}{2\pi^{1/2}} \int_{\infty}^{(0+)} (-y)^{-1/2} e^{-y} P_n[(2x + y)/y]\, dy.$$

$$(20)\qquad P_n(1 - 2x) = \frac{1}{\pi^{1/2}} \int_{0}^{\infty} y^{-1/2} e^{-y} f_n(-;\, -;\, xy)\, dy.$$

$$(21)\qquad H_n(\zeta,\, p,\, v) = \frac{1}{\pi^{1/2}} \int_{0}^{\infty} y^{-1/2} e^{-y} f_n(\zeta;\, p;\, vy)\, dy.$$

$$(22)\qquad f_n(\zeta;\, p;\, v) = \frac{i}{2\pi^{1/2}} \int_{\infty}^{(0+)} (-y)^{-1/2} e^{-y} H_n(\zeta,\, p,\, -v/y)\, dy.$$

7. An algebraic relation. Let us return to the generating function (2). After substituting $1/s$ for w, multiplying both sides of the resulting equation by $1/s$, and taking the inverse Laplace transform of both sides, we have

$$(23)\qquad L^{-1}\left[\frac{1}{s-1}\, G\left(\frac{-4xs}{(s-1)^2}\right)\right] = \sum_{n=0}^{\infty} f_n(x)\, \frac{t^n}{n!}.$$

Next consider the polynomials $h_n(x)$ defined by

$$(24)\qquad G(2xz - z^2) = \sum_{n=0}^{\infty} h_n(x) z^n/n!,$$

---

[4] The methods of this section were suggested by those used by Rice in [12] for obtaining relations for $H_n(\zeta, p, v)$.

where $G(y)$ is the same analytic function which appears in (2). Replacing s by $(-2x)/(s-1)$, we obtain

$$(25)\qquad L^{-1}\left[\frac{1}{s-1}G\left(\frac{-4x^2s}{(s-1)^2}\right)\right] = e^t\sum_{n=0}^{\infty} h_n(x)\frac{(-2xt)^n}{(n!)^2}.$$

From (23) and (25) it follows that

$$(26)\qquad f_n(x^2) = \sum_{k=0}^{\infty} C_{n,k}(-2x)^k h_k(x)/k!.$$

In a compact symbolic notation[5] (26) appears as

$$(26)'\qquad f_n(x^2) \doteq L_n(2xh(x)),$$

in which $L_n(x)$ is the Laguerre polynomial.

If the function $G(y)$ in (24) is taken to be ${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; y)$, then the resulting $h_n(x)$ are the generalized hypergeometric polynomials

$$(27)\qquad h_n(a_i; b_j; x) = n!\sum_{k=0}^{[n/2]}\frac{(a_1)_{n-k}\cdots(a_p)_{n-k}(2x)^{n-2k}(-1)^k}{(b_1)_{n-k}\cdots(b_q)_{n-k}(n-2k)!k!}.$$

These polynomials are generalizations of the Hermite and the ultraspherical polynomials,

$$h_n(-;-;x) = H_n(x), \qquad h_n(a;-;x) = n!P_n^{(a)}(x).$$

Hence our f_n without parameters is related by (26) to the ordinary Laguerre and Hermite polynomials by

$$f_n(-;-;x^2) \doteq L_n(2xH(x)).$$

8. **Bateman's $Z_n(t)$ and the Laguerre polynomials.** From (17) above it follows at once that

$$(28)\qquad f_n(1/2;1;t) = \frac{1}{\pi^{1/2}}\int_0^\infty y^{-1/2}e^{-y}f_n(-;1;ty)dy.$$

Ramanujan's formula[6] [6] which gives the product of two ${}_1F_1$'s as an ${}_2F_2$ and which was proved by C. T. Preece [9] is useful here. It yields at once

$$f_n(-;1;x^2/4) = L_n(x)L_n(-x).$$

But Bateman's $Z_n(t)$ is given by

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[5] Rainville [10, p. 244].

[6] Bailey [2, p. 97].

$$Z_n(t) = f_n(1/2; 1; t).$$

With these results it is easy to write (28) in the interesting form

$$Z_n(t^2) = \frac{1}{\pi^{1/2}}\int_0^\infty e^{-\alpha^2/4}L_n(\alpha t)L_n(-\alpha t)d\alpha.$$

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MERCYHURST COLLEGE AND
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SOME INEQUALITIES RELATING TO CONFORMAL MAPPING UPON CANONICAL SLIT-DOMAINS

BERNARD EPSTEIN

Let D be a domain of the extended z-plane ($z=x+iy$) of finite connectivity n, which contains the point $z=\infty$ and is bounded by n proper[1] continua. According to a fundamental theorem in the theory of conformal mapping of multiply-connected domains [4, 7][2] there exists one and only one function $\zeta=s_\theta(z)$ which in the neighborhood of $z=\infty$ has a Laurent expansion of the form

$$\zeta = s_\theta(z) = z + \frac{a_\theta}{z} + \cdots \tag{1}$$

and which maps D conformally and bi-uniformly upon a domain D_θ of the ζ-plane bounded by n rectilinear slits each of which makes the angle θ with the positive direction of the real axis. The domain D_θ is itself also uniquely determined for each value of θ.

In the present paper we shall derive two inequalities involving the coefficient a_θ appearing in (1) and the outer measure A of the complement (with respect to the entire plane) of the domain D—that is, the greatest lower bound of the total area enclosed by a set of analytic curves surrounding the boundary continua. The first of these inequalities is the following:

$$\operatorname{Re}\,(a_\theta e^{-2i\theta}) \geq \frac{A}{2\pi}\cdot \tag{2}$$

The second inequality, which will be derived by using the theory of orthonormal systems of analytic functions [1, 2, 9, 10], constitutes a strengthening of (2), namely:

$$\operatorname{Re}\,(a_\theta e^{-2i\theta}) - \frac{|a_\theta|^2}{a_0 - a_{\pi/2}} \geq \frac{A}{2\pi}\cdot \tag{3}$$

It suffices to prove the inequalities (2) and (3) for the case when the boundary continua of D are closed analytic curves $C_1, C_2, \cdots, C_n$, for it is known that D can be approximated by an increasing sequence of domains having such boundaries for which the mapping functions corresponding to (1) will converge to $s_\theta(z)$, so that (2) and

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[1] A proper continuum is one which does not consist of a single point.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

(3) will continue to hold in the limit, when A is interpreted in the manner explained above.

Since the boundary of D is for the present assumed to consist of analytic curves, it follows that $s_\theta(z)$ remains analytic there as well as in the interior of D. It then follows, taking account of the form of $s_\theta(z)$ in the neighborhood of $z=\infty$, that $d(s_\theta - z)/dz$ is quadratically integrable over D, that is,

$$I_\theta = \iint_D \left| \frac{d}{dz}(s_\theta - z) \right|^2 dxdy \tag{4}$$

exists. I_θ is real and non-negative, vanishing if and only if $s_\theta = z$, that is, if and only if D is identical with the domain into which it is to be mapped. Now the double integral appearing in (4) can be transformed into an integral taken along the boundary curves of D, as follows:

$$I_\theta = -\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \overline{(s_\theta - z)}\left(\frac{ds_\theta}{dz} - 1\right)dz, \tag{5}$$

the sense of integration being positive with respect to the *interior* of each boundary curve. By multiplying out the integrand, I_θ can be expressed as the sum of four terms, namely:

$$-\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \bar{s}_\theta \frac{ds_\theta}{dz}\,dz, \tag{6a}$$

$$\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \bar{s}_\theta dz, \tag{6b}$$

$$\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \bar{z}\frac{ds_\theta}{dz}\,dz, \tag{6c}$$

$$-\frac{1}{2i}\sum_{k=1}^{k=n}\int_{C_k} \bar{z}dz. \tag{6d}$$

The integrals (6a), (6b), (6c) will now be evaluated by employing an artifice which has been used with great success by Grunsky [4]. One observes that on each curve C_k the expression $e^{-i\theta}(s_\theta - \zeta_k)$ is real, where ζ_k is any point on or collinear with the image of C_k, so that

$$\bar{s}_\theta = e^{-2i\theta}s_\theta + \text{constant}.^3 \tag{7}$$

Replacing $\bar{s}_\theta$ in (6a) by the right-hand side of (7) and recalling that $s_\theta(z)$ is single-valued in D, one sees that (6a) vanishes. By performing the same substitution in (6b) and then integrating around a

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[3] In general, of course, this constant will be different for each C_k.

large circle lying entirely within the domain of convergence of the expansion (1) instead of around the curves C_k (which is justified by the Cauchy integral theorem), one finds that (6b) is equal to $\pi a_\theta e^{-2i\theta}$.

(6c) may be evaluated most easily by integration by parts; the integrated part vanishes, leaving an integral which is the conjugate of (6b). Thus, (6c) is equal to $\pi \bar{a}_\theta e^{2i\theta}$.

Finally, (6d) is easily seen to be equal to $-A$. Thus, equation (4) may be rewritten in the form

$$I_\theta = -A + 2\pi \operatorname{Re} (a_\theta e^{-2i\theta}). \tag{8}$$

Since I_θ is of positive definite character, vanishing, as mentioned above, if and only if $s_\theta = z$, the inequality (2) follows directly from (8).[4] Since the real part of a complex number is at most equal to its absolute value, there is immediately obtained from (2) the following lower bound on $|a_\theta|$, valid for all values of θ:

$$|a_\theta| \geq \frac{A}{2\pi}. \tag{9}$$

That the factor $1/2\pi$ appearing in (2) cannot be replaced by any larger number is easily seen by considering the mapping of the exterior of an ellipse upon the exterior of a slit parallel to the major axis and letting the eccentricity of the ellipse approach unity. For example, for $\theta = 0$, the exterior of the ellipse $x^2R^4/(R^2+1)^2 + y^2R^4/(R^2-1)^2 = 1$ $(R>1)$ is mapped upon the exterior of the slit $[-2, 2]$ of the ζ-plane by eliminating w between the pair of equations:

(10a) $z = w + 1/R^2 w$, (10b) $\zeta = w + 1/w$.

It is easily found that the Laurent expansion for ζ at $z = \infty$ is of the form

$$\zeta = z + \frac{(1 - 1/R^2)}{z} + \cdots \tag{11}$$

so that, in this case:

$$\frac{\operatorname{Re} (a_\theta e^{-2i\theta})}{A} = \frac{1 - 1/R^2}{\pi(1 - 1/R^4)} = \frac{R^2}{\pi(1 + R^2)}. \tag{12}$$

Letting R (and hence the eccentricity) approach unity, the right-hand side of (12) is seen to approach $1/2\pi$.

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[4] The evaluation of positive-definite integrals has been used frequently as a means of obtaining inequalities in the theory of conformal mapping. To mention only one example, one may cite the "Flächensatz" of Bieberbach [3].

A more refined estimate than (9) is easily obtained in the following manner. First, by setting $\theta=0$, $\theta=\pi/2$, $\theta=\pi/4$, and $\theta=3\pi/4$, respectively, there are obtained from (2) the inequalities:

$$\text{(13a)} \qquad \operatorname{Re} a_0 \geqq \frac{A}{2\pi}, \qquad \text{(13c)} \qquad \operatorname{Im} a_{\pi/4} \geqq \frac{A}{2\pi},$$

$$\text{(13b)} \qquad \operatorname{Re} a_{\pi/2} \leqq -\frac{A}{2\pi}, \qquad \text{(13d)} \qquad \operatorname{Im} a_{3\pi/4} \leqq -\frac{A}{2\pi}.$$

(Grunsky, in [5], and Schiffer in [8] and [10], showed not only that $a_0-a_{\pi/2}$ (the "span" of D) is real, but that

$$\text{(14)} \qquad a_0 - a_{\pi/2} \geqq \frac{2A}{\pi}.$$

While neither of these results can be obtained from the considerations presented here, it is of interest that previously no estimates like (13a) and (13b) for these coefficients separately had been obtained.)

Now it is known [4] that for each value of θ, $s_\theta(z)$ can be expressed as a simple combination of $s_0(z)$ and $s_{\pi/2}(z)$, namely

$$\text{(15)} \qquad s_\theta(z) = e^{i\theta}\{s_0 \cos\theta - i s_{\pi/2} \sin\theta\}.$$

Combining (15) with (1), the following equation is obtained for a_θ:

$$\text{(16)} \qquad a_\theta = e^{i\theta}\{a_0 \cos\theta - i a_{\pi/2} \sin\theta\}.$$

Now, if we set $a_0=\alpha+i\beta$ and $a_{\pi/2}=\gamma+i\beta$ (here we use the fact that $a_0-a_{\pi/2}$ is real), it is found that (1) may be written in the form

$$\text{(17)} \qquad \frac{\alpha-\gamma}{2}+\frac{\alpha+\gamma}{2}\cos 2\theta+\beta\sin 2\theta \geqq \frac{A}{2\pi}.$$

Since this inequality holds for all values of θ, it holds when θ is so chosen as to minimize the left-hand side of (17). The result obtained is

$$\text{(18)} \qquad \frac{\alpha-\gamma}{2}-\left(\left(\frac{\alpha+\gamma}{2}\right)^2+\beta^2\right)^{1/2} \geqq \frac{A}{2\pi},$$

which is completely equivalent to (17).

The inequality (18) admits an interesting geometrical interpretation, which may be seen in the following manner. From (16) and the real character of $a_0-a_{\pi/2}$, it is easily shown that the point a_θ, when plotted in the usual manner, describes the circumference of a circle with radius $R=(\alpha-\gamma)/2$ and with center at the point $(\alpha+\gamma)/2+i\beta$, whose distance r from the origin is precisely the second term of the left-hand side of (18). Thus, it is seen that the left-hand side of (18) is, both in magnitude and sign, equal to the distance between the origin and the nearest point of the circle described by the coefficients a_t. The inequality (9) may be seen geometrically from (18), while it is clear from either (13a–d) or (18) that the origin lies inside (or perhaps on) the aforementioned circle. This latter fact has previously been obtained in [4] by Grötzsch, who showed that for the class of all biuniform conformal mappings of a fixed domain D (of the type described at the beginning of this paper) having at infinity a Laurent expansion of the form

$$\zeta = z + \frac{c}{z} + \cdots, \tag{19}$$

the coefficients c cover a circular area of the complex plane whose circumference is described by the points a_t; since $\zeta = z$ is a mapping belonging to the aforementioned class, it follows that the origin cannot be exterior to the circle. By combining this observation with the fact that $a_0 - a_{\pi/2}$ is real and equal to the diameter of the circle described by the coefficients a_t, the inequalities

(20a) $\operatorname{Re} a_0 \geqq 0$, (20b) $\operatorname{Re} a_{\pi/2} \leqq 0$

are obtained. These results are contained in (13a) and (13b), which were obtained by much simpler considerations than those used by Grötzsch.

As stated previously, we can strengthen the inequality (2) by the use of the theory of complete systems of orthonormal functions. As shown in [1], there can be chosen for every domain B whose boundary does not consist entirely of isolated points—that is, contains at least one proper continuum—in infinitely many ways an orthonormal[5] sequence of functions $\{f_k(z)\}$, each of which is analytic and uniform, and possesses a uniform integral in B, such that every function $f(z)$ defined in B which satisfies the above conditions and is quadratically integrable over the entire domain can be expanded in a series of the form

$$f(z) = \sum_{k=1}^{k=\infty} c_k f_k(z) \tag{21}$$

which converges uniformly to $f(z)$ in every closed subdomain of B.

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[5] Orthonormality is defined by the usual condition: $\iint_B f_m(z)\overline{f_n(z)}\,dx\,dy = \delta_{mn}$.

Such a sequence of functions is called a complete system for the domain B.

It is known that the kernel function of the domain, defined as follows:

$$K(z, \bar{z}_1) = \sum_{k=1}^{k=\infty} f_k(z)\,\overline{f_k(z_1)}, \tag{22}$$

is independent of the particular choice of the complete orthonormal system $\{f_k(z)\}$. Identifying B with the domain D considered in §2, let two different complete orthonormal systems be constructed, beginning with the functions:[6]

$$g(z) = \frac{ds_\phi/dz - 1}{(2\pi \operatorname{Re}(a_\phi e^{-2i\phi}) - A)^{1/2}} \tag{23a}$$

and

$$h(z) = \frac{d(s_0 - s_{\pi/2})/dz}{(2\pi(a_0 - a_{\pi/2}))^{1/2}} \tag{23b}$$

respectively. (It follows from (8) that $g(z)$ is normalized, while Schiffer [10] has shown this for $h(z)$.)

Letting $z_1 = z$, it is obvious from (22) that

$$K(z, \bar{z}) \geq |g(z)|^2. \tag{24}$$

Multiplying both sides of (24) by $|z|^4$ and letting $|z| \to \infty$, one obtains, taking account of (1) and (2),

$$\lim_{|z|\to\infty} |z|^4 K(z, \bar{z}) \geq \frac{|a_\phi|^2}{2\pi \operatorname{Re}(a_\phi e^{-2i\phi}) - A} \geq 0. \tag{25}$$

On the other hand, it is proven in [10] that every function which belongs to the class described at the beginning of this section and which is orthogonal to $h(z)$ vanishes at $z = \infty$ to at least the third order, so that

$$\lim_{|z|\to\infty} |z|^4 K(z, \bar{z}) = \lim_{|z|\to\infty} |z|^4 |h(z)|^2 = \frac{1}{2\pi}(a_0 - a_{\pi/2}). \tag{26}$$

Combining (25) with (26) one obtains the inequality (3), which is seen to constitute a strengthening of (2). Replacing a_ϕ in (3) by the right-hand side of (16), one finds that the left-hand side of (3) is independent of θ, so that (3) may be rewritten in the simpler form

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[6] Here we disregard the special case $s_\phi = s$, for which (3) is directly seen to hold.

$$-\frac{\alpha\gamma+\beta^2}{\alpha-\gamma} \geqq \frac{A}{2\pi}. \tag{27}$$

(Here, as before, $a_0=\alpha+i\beta$, $a_{\pi/2}=\gamma+i\beta$.) If we note that $(\alpha\gamma+\beta^2)$ is simply the scalar product of the vectors representing the numbers a_0 and $a_{\pi/2}$, (27) may be written in the form

$$|a_0|\,|a_{\pi/2}|\cos\tau \leqq -\frac{A}{2\pi}(\alpha-\gamma), \tag{28}$$

where τ is the angle between the two vectors.

It is of interest to note that, while (2) reduces to an equality only in the degenerate case when the domain D coincides with D_0, (3) and its equivalents (27) and (28) becomes equalities for a nondegenerate class of domains. In particular, one can easily show, by actually computing s_0 and $s_{\pi/2}$, that the equality sign may be taken in (3), (27), and (28) if D is taken to be the exterior of any ellipse.

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NOTE ON AFFINELY CONNECTED MANIFOLDS

SHIING-SHEN CHERN

The aim of this note is to prove some statements concerning the differential geometry in the large of an affinely connected manifold.

Let M be an orientable differentiable manifold of dimension n and class two. We say that an affine connection is defined in M, if a set of quantities[1] Γ^k_{ij} is defined in each allowable coordinate system x^i such that under change of the allowable coordinate system they are transformed according to the following law:

$$\bar{\Gamma}^r_{pq} = \frac{\partial^2 x^k}{\partial \bar{x}^p \partial \bar{x}^q} \frac{\partial \bar{x}^r}{\partial x^k} + \Gamma^k_{ij} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial \bar{x}^r}{\partial x^k}. \tag{1}$$

The connection may be symmetric or asymmetric.

It is well known that from Γ^k_{ij} the covariant derivative of a contravariant vector X^i can be defined as follows:

$$X^i_{,k} = \frac{\partial X^i}{\partial x^k} + X^l \Gamma^i_{lk}. \tag{2}$$

We also recall that the affine curvature tensor is given by

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^i_{ml}\Gamma^m_{jk} + \Gamma^i_{mk}\Gamma^m_{jl}. \tag{3}$$

We put

$$R_{kl} = R^i_{ikl} \tag{4}$$

and introduce the exterior differential form

$$P = R_{kl}\, dx^k dx^l. \tag{5}$$

Then the main theorem of this note can be stated as follows:

THEOREM. *The integral of P over any two-dimensional cycle is equal to zero.*

To prove this theorem we consider n linearly independent contravariant vectors $X^i_{(1)}, \cdots, X^i_{(n)}$ and their determinant

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[1] All indices in this paper run from 1 to n and we agree as usual that repeated indices mean summation.

$$\Delta = \epsilon_{i_1 \cdots i_n} X^{i_1}_{(1)} \cdots X^{i_n}_{(n)}, \tag{6}$$

where $\epsilon_{i_1 \cdots i_n}$ is the Kronecker index. When the vectors are parallelly transported, we have

$$-\frac{d\Delta}{\Delta} = \Gamma^{i}_{ik} dx^{k}. \tag{7}$$

The differential form $-d\Delta/\Delta$ we shall denote for simplicity by ϕ. It now follows immediately from (3) that its exterior derivative is

$$d\phi = 2P. \tag{8}$$

It is sufficient for the proof of this theorem to assume that the domain of integration is a simplicial two-dimensional cycle in a sufficiently fine simplicial decomposition of M such that each simplex lies in one coordinate neighborhood of M, as every cycle over which the integral of P is defined can be approximated by a simplicial cycle with the above property. On such a simplicial cycle we shall define a continuous field of n independent contravariant vectors at each point. According to a standard procedure in the theory of fibre bundles,[2] the sets of vectors are at first defined at the vertices of the cycle. They can be extended over the one-dimensional simplexes, because M is orientable. To extend the field over the two-dimensional simplexes and hence over the whole cycle, we notice that the sets of n independent contravariant vectors at a point whose determinant is positive (or negative) form a topological space which is the group space of the group of linear transformations in n variables with positive (or negative) determinant and is hence simply-connected.[3] As the sets of vectors defined on the boundary of a simplex give rise to a mapping of the boundary into the group space in question, the simply-connectedness of this group space implies the possibility of extension over the two-dimensional simplexes. The integral of P over the cycle can now be interpreted as the integral of P over this field of independent contravariant vectors in the space of all sets of n independent contravariant vectors of the manifold M. From (8) it follows that the integral is zero and the theorem is proved.

With the terminology of the cohomology theory the theorem can be stated as follows: *The cohomology class to which P belongs is the zero class.*

We shall state the following corollaries of our theorem:

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[2] Cf., for instance, N. Steenrod, *Topological methods for the construction of tensor functions*, Ann. of Math. vol. 43 (1942) pp. 116–131.

[3] A proof is given later in this paper.

COROLLARY 1. *Let M be a compact orientable affinely-connected manifold of class two and even dimension n. Then*

$$\int_M P^{n/2} = 0, \tag{9}$$

or

$$\int_M \epsilon^{i_1\cdots i_n} R_{i_1 i_2} R_{i_3 i_4} \cdots R_{i_{n-1} i_n} dx^1 \cdots dx^n = 0. \tag{9a}$$

The integrand of the integral (9a):

$$I = \epsilon^{i_1\cdots i_n} R_{i_1 i_2} R_{i_3 i_4} \cdots R_{i_{n-1} i_n}$$

is a tensor density of the affine connection.

COROLLARY 2. *Let M have the same meaning as in Corollary 1. Then I can not be always positive nor always negative in M.*

Remark. In terms of the notation of Elie Cartan, the affine connection can be written as the infinitesimal displacement of affine frames $pe_1 \cdots e_n$:

$$dp = \omega^i e_i, \qquad de_i = \omega_i^j e_j, \tag{10}$$

with the equations of structure:

$$d\omega^i = \omega^j \omega_j^i + \Omega^i, \qquad d\omega_j^i = \omega_j^k \omega_k^i + \Omega_j^i. \tag{11}$$

Then we have

$$P = \Omega_i^i. \tag{12}$$

I shall conjecture that all the differential forms

$$P_\alpha = \Omega_{i_1}^{i_2} \Omega_{i_2}^{i_3} \cdots \Omega_{i_\alpha}^{i_1}, \qquad 0 \leqq \alpha \leqq n/4, \tag{13}$$

define cohomology classes which are topological invariants of M. I hope to return to this in a later paper.

For completeness I indicate here a proof of the well known fact that the space of all matrices (with a natural topology) (a_{ij}), $|a_{ij}| > 0$, is simply connected, as I am not able to give the exact reference. This space is a group manifold and the theorem in question follows from the following theorem which I quote from Chevalley in a slightly different version:[4]

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[4] C. Chevalley, *Theory of Lie groups*, p. 59.

Let G be a connected Lie group and H a subgroup of G. If the homogeneous space G/H is simply connected, the fundamental group of G is isomorphic to a factor group of the fundamental group of H.

Using this theorem we shall give the proof by induction on the order n of the matrices. Let G be the group of all these matrices, H the subgroup of all matrices

$$\begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

and K the subgroup of all matrices

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

so that

$$K \subset H \subset G.$$

The induction hypothesis states that K is simply connected. It is easy to verify that both homogeneous spaces H/K (the space of right cosets) and G/H (the space of left cosets) are Euclidean spaces and hence are simply connected. It therefore follows from the above theorem that H and hence also G are simply connected.

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RATIONAL HARMONIC CURVES

EDWARD KASNER AND JOHN DECICCO

1. Introduction. We shall study curves related to rational fractional functions of a complex variable. This will generalize known results of curves related to rational integral functions.

Curves defined by setting the real part of a polynomial (rational integral function) in the complex variable $u=x+iy$ equal to zero are well known. These had been studied initially by Briot and Bouquet, and Bôcher; and finally characteristic properties have been given by Kasner. These curves have been called *algebraic potential curves* by Kasner, and this term is employed in later papers by Loria and in the German *Encyclopedia*. But we shall find it more convenient to use the term: *polynomial harmonic or polynomial potential curves.*

We define a *rational harmonic or rational potential curve* to be the locus obtained by setting the real part of a rational fractional function of a complex variable $u=x+iy$ equal to zero. The class of rational harmonic curves of course includes the class of polynomial harmonic curves.

We shall obtain various geometric properties of rational harmonic curves. These generalize corresponding results of Briot and Bouquet, and Kasner concerning the polynomial potential curves. We shall prove that *the real asymptotes of a rational potential curve are concurrent and make equal angles with one another*; *the remaining asymptotes are minimal.* This condition is only necessary but not sufficient. We do find a characteristic property of rational potential curves by studying the related focal properties. In the final part of our paper, we study the Schwarzian reflection with respect to a rational harmonic curve. *The satellite of a rational harmonic curve is itself.* This result gives the largest known class of self-satellite algebraic curves.

2. Theorems of Briot and Bouquet, and Kasner concerning polynomial potential curves. For purposes of contrast, certain theorems concerning polynomial harmonic curves will be stated.

The theorem of Briot and Bouquet concerning the asymptotes of a polynomial harmonic curve is as follows.[1]

The n asymptotes of a polynomial potential curve of degree n are all

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[1] Briot and Bouquet, *Théorie des fonctions elliptiques*, vol. 4, Paris, Gauthier-Villar, 1875, chap. 2, p. 226. See also Bôcher, Göttingen prize memoir.

real, concurrent, and disposed symmetrically about their common point, the angle between consecutive asymptotes being π/n.

In all cases, the point of concurrence O of the asymptotes is a *center* of the curve. That is, if any line is drawn through O, the sum of the distances measured from O on one side of O is the same as the corresponding sum for the points on the other side. This follows from the fact that when the origin of coordinates is taken at O, all the terms of degree $(n-1)$ disappear.

Kasner noted that this result is not characteristic for polynomial harmonic curves of degree $n \geqq 4$. Some of Kasner's characterizations of polynomial harmonic curves are here stated.[2]

An algebraic curve is polynomial potential if and only if it is apolar to the fundamental conic of euclidean geometry which consists of the circular points I and J at infinity.

The polar curves of a circular point I (or J) with respect to a polynomial potential curve degenerate into sets of straight lines passing through the other circular point J (or I), and conversely.

An algebraic curve is polynomial harmonic when and only when the polar conics with respect to the curve are rectangular hyperbolas.

All the polars of polynomial potential curves are also polynomial potential curves.

Any curve of degree n is polynomial potential if and only if it passes through the n^2 foci of a curve of class n.

Each polynomial harmonic curve of degree n passes through the foci of an infinite number of systems of confocal curves of class n.

Two algebraic curves of degree n are conjugate polynomial potential curves if and only if they intersect orthogonally in the foci of a curve of class n.

Kasner has given also various geometric characterizations of polynomial harmonic surfaces in space.[3]

3. Our theorem concerning the asymptotes of a rational potential curve. Consider a rational conformal transformation of the form

$$(1)\qquad \begin{aligned} U &= \frac{f(u)}{g(u)} = \frac{a_0u^r + a_1u^{r-1} + \cdots + a_r}{b_0u^s + b_1u^{s-1} + \cdots + b_s}, \\ V &= \frac{F(v)}{G(v)} = \frac{A_0v^r + A_1v^{r-1} + \cdots + A_r}{B_0v^s + B_1v^{s-1} + \cdots + B_s} \end{aligned}$$

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[2] Kasner, *On the algebraic potential curves*, Bull. Amer. Math. Soc. vol. 7 (1901) pp. 392–399.

[3] Kasner, *Some properties of potential surfaces*, Bull. Amer. Math. Soc. vol. 8 (1902) pp. 243–248.

in the conjugate complex variables $u=x+iy$ and $v=x-iy$. Of course, A_j is the conjugate of a_j for $j=0, 1, 2, \cdots, r$, and B_k is the conjugate of b_k for $k=0, 1, 2, \cdots, s$. The leading coefficients (a_0, b_0, A_0, B_0) are assumed to be all different from zero. The polynomials $f(u)$ and $g(u)$ (and also $F(v)$ and $G(v)$) have no common factors.

We define a *rational harmonic curve* as the locus obtained by setting the real part of the rational fractional function $U=f(u)/g(u)$ equal to zero. Thus a rational harmonic curve is the image in the (x, y)-plane of the Y-axis defined in the (X, Y)-plane by $X=0$ or $U+V=0$ under the rational conformal transformation (1). Therefore *a rational harmonic or potential curve of total degree not exceeding $n=r+s$ is defined by the equation*

$$\begin{aligned}(2)\qquad &(a_0u^r + a_1u^{r-1}+\cdots+a_r)(B_0v^s+B_1v^{s-1}+\cdots+B_s)\\ &+(A_0v^r+A_1v^{r-1}+\cdots+A_r)(b_0u^s+b_1u^{s-1}+\cdots+b_s)=0,\end{aligned}$$

where we may assume that the integers r and s are such that $r\geqq s\geqq 0$ so that the degree of $U=f(u)/g(u)$ (or $V=F(v)/G(v)$) is $r\geqq 1$. For otherwise, (1) followed by an inversion and reflection will define the same rational harmonic curve (2). (2) may be written in the form

$$(3)\qquad f(u)G(v)+F(v)g(u)=0.$$

Of course, the left-hand member of equation (2) or (3) is a polynomial in (u, v) or (x, y), of total degree which is not greater than $n=r+s$. *This polynomial does not satisfy the Laplace equation; but we have proved elsewhere that it does obey a certain partial differential equation of fourth order.*[4]

In particular, we can show by (2) that *the only real rational harmonic conic sections are the circles and the rectangular hyperbolas.* Of course, the latter are the only real polynomial harmonic curves of second degree.

Now we shall state the following result. The proof is given in §4.

FUNDAMENTAL THEOREM. *Let a rational potential curve be defined by setting equal to zero the real part of a rational fractional function of a complex variable $u=x+iy$ of degree $r\geqq 1$. This is given by an equation of the form $P(x, y)=0$, where $P(x, y)$, in general not harmonic, is a polynomial of degree $n=2r-k$ where $0\leqq k\leqq r$. There are k real asymptotes all of which pass through a common point and make equal angles*

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[4] Kasner and DeCicco, *A partial differential equation of fourth order connected with rational functions of a complex variable*, Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 326–328. See also a forthcoming paper, *Partial differential equations related to rational functions of a complex variable*. Duke Math. J. (1947).

with one another. The angle between consecutive asymptotes is π/k. *The remaining* $2(r-k)$ *asymptotes are minimal.*

Of course, if $k=0$ there are no real asymptotes, and all the asymptotes are minimal. On the other hand, if $k=1$ there is only one real asymptote.

When $k=r$, this result reduces to the theorem of Briot and Bouquet concerning the asymptotes of polynomial potential curves. Thus our fundamental theorem is an extension to rational fractional functions of the theorem of Briot and Bouquet.

4. **Proof of the fundamental theorem.** It is noted that if $r>s$, the degree of the rational potential curve (2) is exactly $n=r+s$. In this case, let $k=r-s$, so that the degree is $n=2r-k$. On the other hand, if $r=s$ the degree of the curve is equal to or less than $r+s=2r$ according as the expression $(a_0B_0+A_0b_0)$ is not or is zero.

If $r>s>0$, so that $n=r+s=2r-k$, the rational potential curve (2) passes through each of the circular points I and J at infinity. In general, there are $s=r-k$ branches of the curve which pass through each of these circular points. The curve intersects the line at infinity $s=r-k$ times in the circular point I, $s=r-k$ times in the circular point J, and in $n-2s=r-s=k$ other real points.

If $r=s>0$, and $a_0B_0+A_0b_0\neq 0$, the degree of the curve is $n=2r$ so that $k=0$. The curve intersects the line at infinity r times in the circular point I and r times in the circular point J.

Let $r=s>0$ and let $a_0B_0+A_0b_0=0$. There must exist a least integer k such that $0<k\leqq r$ for which the expression $a_0B_k+A_kb_0$, which is the coefficient of u^rv^{r-k} in (2), is different from zero. For otherwise, this would mean that (1) is degenerate. Since the conditions

$$
\begin{gathered}
a_0B_j+A_jb_0=0,\ A_0b_j+a_jB_0=0,\ \text{for } j=0,1,2,\cdots,k-1;\\
a_0B_k+A_kb_0\neq 0,\qquad A_0b_k+a_kB_0\neq 0
\end{gathered}
\tag{4}
$$

are satisfied, we find that the equation (2) may be written in the form

$$
\begin{aligned}
&(a_0u^r+a_1u^{r-1}+\cdots+a_r)(B_kv^{r-k}+B_{k+1}v^{r-k-1}+\cdots+B_r)\\
&\quad+(A_0v^r+A_1v^{r-1}+\cdots+A_r)\\
&\qquad\cdot(b_ku^{r-k}+b_{k+1}u^{r-k-1}+\cdots+b_r)\\
&\quad+(B_0v^r+B_1v^{r-1}+\cdots+B_{k-1}v^{r-k+1})\\
&\qquad\cdot(a_ku^{r-k}+a_{k+1}u^{r-k-1}+\cdots+a_r)\\
&\quad+(b_0u^r+b_1u^{r-1}+\cdots+b_{k-1}u^{r-k+1})\\
&\qquad\cdot(A_kv^{r-k}+A_{k+1}v^{r-k-1}+\cdots+A_r)=0.
\end{aligned}
\tag{5}
$$

From (4) and (5), we find that the degree of the curve (5) is exactly $n=2r-k$. The curve intersects the line at infinity $(r-k)$ times in the circular point I, $(r-k)$ times in the circular point J, and in k other real points.

Thus unless our rational harmonic curve reduces to a polynomial harmonic curve, it is found that the rational harmonic curves always pass through the circular points I and J.

It can be proved by the equation (2) of a rational potential curve that if $r>s\geqq 1$ so that $n=r+s=2r-k$, the curve has $s=r-k$ asymptotes through the circular point I and $s=r-k$ asymptotes through the circular point J. Similarly the curve (5) where $r=s\geqq 1$ and the degree is $n=2r-k$, $0\leqq k\leqq r$, has $r-k$ asymptotes passing through I and $r-k$ asymptotes passing through J.

Next we consider the real asymptotes of the rational potential curve (2) where $r>s\geqq 0$ so that its degree is $n=r+s=2r-k$. The equation of any non-minimal line may be written in the form $v=mu+p$, where $m\neq 0$ and the inclination to the x-axis is $\theta=-(1/2i)\log m$. If this line is to be an asymptote of the curve (2), we find, upon substituting this into equation (2) and setting the coefficients of u^{r+s} and u^{r+s-1} equal to zero, the following conditions on m and p.

$$
\begin{aligned}
&a_0B_0m^s + A_0b_0m^r = 0,\\
(6)\qquad &a_1B_0m^s + a_0m^{s-1}(sB_0p + B_1)\\
&\qquad\qquad + A_0b_1m^r + b_0m^{r-1}(rA_0p + A_1) = 0.
\end{aligned}
$$

These equations are found to be equivalent to the system

$$
\begin{aligned}
&a_0B_0m^s + A_0b_0m^r = 0,\\
(7)\quad &A_0B_0(a_0b_1 - a_1b_0)m + a_0b_0A_0B_0(r - s)p\\
&\qquad\qquad + a_0b_0(A_1B_0 - A_0B_1) = 0.
\end{aligned}
$$

Thus if $r>s\geqq 0$ so that the degree of the curve is $n=r+s=2r-k$, there are $r-s=k$ real asymptotes, all of which pass through a common point and make equal angles with one another. The angle between consecutive asymptotes is π/k.

It is noted that by a translation, any rational harmonic curve is changed into a rational harmonic curve. Upon translating (2) so that the new origin O is the common point of intersection of the real asymptotes, it is found that the rational potential curve (2) assumes a similar form with the conditions

$$
(8)\qquad a_0b_1 - a_1b_0 = 0, \qquad A_1B_0 - A_0B_1 = 0.
$$

It follows by these conditions and (2) that *the point O is not in general a center of the curve.* The origin O is a center if and only if

$$\text{(9)} \qquad \text{either } a_0B_0 + A_0b_0 = 0 \quad \text{or} \quad a_1 = A_1 = b_1 = B_1 = 0.$$

For a polynomial potential curve, the latter conditions of (9) are valid, and thus the point O is a center.

If $r = s$, there are no non-minimal asymptotes when $a_0B_0 + A_0b_0 \neq 0$. Let k be the integer where $0 < k \leq r$, for which the conditions (4) hold. The equation of the curve is given by (5). Its degree is $n = 2r - k$. Upon eliminating v between the equation of the line: $v = mu + p$, where $m \neq 0$, and (5), it is found that if this line is to be an asymptote of (5) the quantities m and p must satisfy the two conditions

$$\text{(10)} \quad \begin{aligned} &(a_0B_k + A_kb_0)m^{r-k} + (A_0b_k + a_kB_0)m^r = 0, \\ &m^{r-k-1}[(r-k)(a_0B_k + A_kb_0)p + (a_0B_{k+1} + A_{k+1}b_0)] \\ &\qquad + m^{r-k}[a_1B_k + A_kb_1] + m^r[A_0b_{k+1} + a_{k+1}B_0] \\ &\qquad + m^{r-1}[r(A_0b_k + a_kB_0)p + (A_1b_k + a_kB_1)] = 0. \end{aligned}$$

These equations are equivalent to the system

$$\text{(11)} \quad \begin{aligned} &(a_0B_k + A_kb_0) + (A_0b_k + a_kB_0)m^k = 0, \\ &k(A_0b_k + a_kB_0)(a_0B_k + A_kb_0)p \\ &\quad + [(a_0B_k + A_kb_0)(A_0b_{k+1} + a_{k+1}B_0) \\ &\qquad\qquad - (A_0b_k + a_kB_0)(a_1B_k + A_kb_1)]m \\ &\quad + [(a_0B_k + A_kb_0)(A_1b_k + a_kB_1) \\ &\qquad\qquad - (A_0b_k + a_kB_0)(a_0B_{k+1} + A_{k+1}b_0)] = 0. \end{aligned}$$

Thus for the rational potential curve (5) *of degree $n = 2r - k$, there are k real asymptotes, all of which pass through a common point and make equal angles with one another. The angle between consecutive asymptotes is π/k.*

It is found by (11) that the common center of the real asymptotes is not in general a center of the curve (5).

This completes the proof of our fundamental theorem.

5. Other properties of rational potential curves. An examination of equation (3) which defines a rational harmonic curve yields:

An algebraic curve of degree n not exceeding $(r+s)$, where $r > 1$ and $s > 1$, is rational harmonic if and only if it passes through the r^2 foci of a curve C_r of class r and the s^2 foci of a curve C_s of class s such that no minimal line contains a focus of C_r and a focus of C_s.

Of course, each rational harmonic curve of degree $n \leq r + s$ passes through the r^2 foci of an infinite number of confocal curves C_r of class r and the s^2 foci of an infinite number of confocal curves C_s of class s.

If the real and imaginary parts of an analytic function of a complex variable are set equal to zero, the resulting curves may be called conjugate harmonic. Thus a pair of algebraic curves are *conjugate rational harmonic* if their equations are of the form

$$(12)\qquad f(u)G(v)+F(v)g(u)=0,\qquad f(u)G(v)-F(v)g(u)=0,$$

where the various functions appearing in these equations are the polynomials defined by equations (1).

A pair of algebraic curves of degree $n\leqq r+s$ are conjugate rational harmonic if and only if they intersect orthogonally in the r^2 foci of a system of confocal curves C_r of class r and in the s^2 foci of another system of confocal curves C_s of class s, such that no minimal line contains a focus of C_r and a focus of C_s.

It is remarked that the polar curves of a rational harmonic curve are not in general rational harmonic.[5]

6. **Satellite theory.** A Schwarzian reflection or conformal symmetry may be defined as a reverse conformal transformation of period two. It results that a Schwarzian reflection leaves fixed the points of a unique analytic curve. If the curve is given in minimal coordinates by the equation

$$(13)\qquad \phi(u,v)=0,$$

the Schwarzian reflection T with respect to this curve is obtained by solving for (U, V) the equations

$$(14)\qquad \phi(U,v)=0,\qquad \phi(u,V)=0.$$

Kasner developed the geometry in the large of the Schwarzian reflection T with respect to a general algebraic curve; that is, when $\phi(u, v)$ is a general polynomial of degree n. *In general, the degree of the Schwarzian reflection T is n^2.* In general, the image of the algebraic curve (13) under the Schwarzian reflection T as given by the equation (14) is not only the curve itself but also a new algebraic curve which Kasner termed the *satellite* of the given curve (13). In particular, the satellite of a conic is a confocal conic.[6]

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[5] Kasner and DeCicco, *Rational functions of a complex variable and related potential curves*, Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 280–282.

[6] Kasner, *La satelite conforme de una curva algebraica general*, Revista de la Unión Matemática Argentina vol. 2 (1946) pp. 77–83. Also *Algebraic curves, symmetries, and satellities*, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 250–252.

The Schwarzian reflection T with respect to the rational harmonic curve (3) is

$$(15)\qquad f(U)G(v) + F(v)g(U) = 0,\quad f(u)G(V) + F(V)g(u) = 0.$$

The Schwarzian reflection T with respect to a rational harmonic curve of degree $n = 2r - k$, where $0 \leq k \leq r$, is of degree r^2.

In order to find the satellite of the rational potential curve (3), we have to eliminate (u, v) from the equations (3) and (15). It is found that the result yields the equation (3) where u is replaced by U and v is replaced by V.

The satellite of a rational potential curve is the original curve itself.

These new results are exact extensions of the corresponding theorems developed by Kasner concerning polynomial potential curves, to our rational potential curves.

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HARMONIC TRANSFORMATION THEORY OF ISOTHERMAL FAMILIES

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1. Harmonic transformations. Let a transformation T

$$X + \phi(x, y), \qquad Y = \psi(x, y), \tag{1}$$

with the jacobian

$$J = \phi_x\psi_y - \phi_y\psi_x \neq 0, \tag{2}$$

be such that the components ϕ and ψ satisfy the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0, \qquad \psi_{xx} + \psi_{yy} = 0, \tag{3}$$

in a certain region of the real (or complex) cartesian plane. We shall term any such correspondence T a *harmonic transformation.*

The harmonic transformations form an infinite set (H) of $\infty^{4f(1)}$ correspondences since they are defined essentially by four independent functions of a single variable. The totality of harmonic correspondences of course do not constitute a group.[1]

A subset of the class (H) of harmonic transformations is the conformal group. The components ϕ and ψ of a conformal map are of course conjugate-harmonic, that is, they satisfy the direct or reverse Cauchy-Riemann equations

$$\phi_x = \pm \psi_y, \qquad \phi_y = \mp \psi_x. \tag{4}$$

Thus a harmonic transformation is conformal if and only if its components are conjugate-harmonic. In general, the components of a harmonic transformation are not interrelated in any way whatsoever.

We have proved that in the real domain the only groups contained in the infinite set (H) of $\infty^{4f(1)}$ harmonic transformations are the group of $\infty^{2f(1)}$ conformal maps, the group of ∞^6 affinities, and the subgroups of these two. In the imaginary domain, we have found in addition two extra infinite groups each consisting of $\infty^{1+2f(1)}$ transformations.

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[1] Harmonic functions and harmonic transformations appear in the theory of minimal surfaces and the Plateau problem. In particular see the fundamental papers of Schwarz and Douglas. Schwarz discusses the case where the jacobian of a harmonic transformation vanishes and studies possible singularities. See *Abhandlungen*, vol. 1, p. 293. Douglas studies the inverse of harmonic transformations. See abstracts in Bull. Amer. Math. Soc. vol. 49 (1943) and vol. 50 (1944).

In the present paper, we shall study the isothermal properties of harmonic transformations. We obtain characterizations not only of our set (H) of harmonic correspondences but also new characterizations of the conformal group. See our Theorems 1, 2, 3, 4 and 5.

2. **Minimal coordinates.** In order to derive our results simply, it is found advantageous to introduce the minimal coordinates $u=x+iy$, $v=x-iy$. The transformation T as defined by equations (1) may be written in the form

$$U = U(u, v), \qquad V = V(u, v). \tag{5}$$

Of course each of the components $U=\phi+i\psi$ and $V=\phi-i\psi$ represents a *polygenic function* of the complex variable $u=x+iy$.

If the components ϕ and ψ of (1) can be expressed as Taylor series of (x, y), we shall say that each of the expressions U and V of (5) is an *analytic polygenic function*. The set of holomorphic functions defining the conformal group is a proper subset of the class of analytic polygenic functions.

It can be shown that for any transformation T defined by an analytic polygenic function, the right-hand sides of (5) can be expressed as Taylor series of (u, v). Only for analytic transformations of this nature can we think of u and v as being independent complex variables, thus giving rise to the study of the four-dimensional complex plane. For any other transformation T, these minimal coordinates (u, v) must be considered as conjugate complex variables.

3. **The mean and phase derivatives.** For any polygenic function $w=w(u, v)$, we consider the two linear differential operators

$$\begin{aligned} w_u &= \frac{\partial w}{\partial u} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)w = \frac{1}{2}(w_x - iw_y), \\ w_v &= \frac{\partial w}{\partial v} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)w = \frac{1}{2}(w_x + iw_y). \end{aligned} \tag{6}$$

The operator $\partial/\partial u$ indicates performing the linear differential operation $(1/2)(\partial/\partial x - i\partial/\partial y)$, and is called the *mean derivative*. Similarly $\partial/\partial v$ means the application of the linear differential operation $(1/2)(\partial/\partial x + i\partial/\partial y)$, and is termed the *phase derivative*. In the case of analytic polygenic functions, these signify the formal partial derivatives with respect to u and v.

The mean and phase derivatives are commutative. The Laplacian operator can be expressed as the product of these two operators except for a constant factor. That is

$$(7)\qquad \begin{aligned} w_{uv} = w_{vu} = \frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial v \partial u} &= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w \\ &= \frac{1}{4}(w_{xx} + w_{yy}). \end{aligned}$$

By (6), it is found that the jacobian J as given by (2) of any transformation T (analytic or not) can be written in the form

$$(8)\qquad J = \phi_x\psi_y - \phi_y\psi_x = U_uV_v - U_vV_u \neq 0.$$

A harmonic transformation T can be written in the form (5) where $U_{uv} = V_{uv} = 0$; or in the more explicit form

$$(9)\qquad U = f(u) + g(v), \qquad V = F(v) + G(u),$$

where the four functions appearing on the right-hand sides are all analytic functions of one variable. In the real domain, the coefficients of F and G are the conjugates of those of f and g respectively.

By (8), it is found that the jacobian J of a harmonic transformation (9) is

$$(10)\qquad J = f_uF_v - g_vG_u \neq 0.$$

By this, it can be shown that any pair of harmonic functions, one of which is not a linear integral function of the other with constant coefficients, defines a nondegenerate transformation of our harmonic set (H).

4. Discussion of the harmonic transformation theory of certain isothermal families of curves. We shall study the problem of characterizing harmonic transformations by means of converting some isothermal families in the (x, y)-plane into isothermal families in the transformed (X, Y)-plane. Emphasis must be placed on the word *some* as we have proved elsewhere that the only point transformations carrying every isothermal family of curves into an isothermal family are the conformal ones.

It is evident by (9) that under any harmonic transformation T, a parallel pencil of straight lines in the (X, Y)-plane corresponds to an isothermal family of curves in the (x, y)-plane. *We shall prove that this is characteristic of the infinite set (H) of harmonic transformations, and of the products of conformal maps by circle-to-line transformations.* The following precise result will be demonstrated.

THEOREM 1. *If, by a transformation T from the (x, y)-plane to the (X, Y)-plane, more than four distinct parallel pencils of straight lines* *in the (X, Y)-plane correspond to isothermal families in the (x, y)-plane, then in the real domain T is a harmonic transformation or else T is the product of a conformal map by a circle-to-line transformation; then necessarily every parallel pencil of straight lines in the (X, Y)-plane corresponds to an isothermal family in the (x, y)-plane.*

It is remarked that a *circle-to-line transformation* is a point correspondence whereby every straight line in the transformed (X, Y)-plane corresponds to a circle or straight line in the (x, y)-plane. The circle-to-line transformations form an eleven-parameter set which includes the collineation group and the Moebius inversive group.

To discuss this result, we shall adopt minimal coordinates (u, v). The transformation T, which is not necessarily an analytic polygenic function, can be written in the form (5) with nonvanishing jacobian J as given by (8). Whenever u (or v) appears as a subscript, the meaning is the application of the mean (or phase) derivative (6). Only where T is defined by an analytic polygenic function will the subscripts denote formal partial differentiation with respect to u (or v).

In minimal coordinates, a parallel pencil of straight lines in the (X, Y)-plane can be written in the form $V-mU=\text{const.}$, where m is a fixed constant, essentially defining the direction of the parallel pencil. By the transformation T, this parallel pencil corresponds in the (x, y)-plane to the family of curves

$$V(u, v) - mU(u, v) = \text{const.} \tag{11}$$

For this family to be isothermal, the left-hand side must be a function of a harmonic function. If Δ denotes the expression

$$\Delta = \frac{V_{uv} - mU_{uv}}{(V_u - mU_u)(V_v - mU_v)}, \tag{12}$$

the condition that (11) be isothermal is

$$\Delta_u/(V_u - mU_u) = \Delta_v/(V_v - mU_v). \tag{13}$$

Substituting (12) into (13), this condition becomes

$$\begin{aligned} (V_v - mU_v)^2[(V_u - mU_u)(V_{uuv} - mU_{uuv}) \\ - (V_{uv} - mU_{uv})(V_{uu} - mU_{uu})] \\ = (V_u - mU_u)^2[(V_v - mU_v)(V_{uvv} - mU_{uvv}) \\ - (V_{uv} - mU_{uv})(V_{vv} - mU_{vv})]. \end{aligned} \tag{14}$$

Since this is of the fourth degree in m, it is obvious that, under a given transformation T, there can be at most four parallel pencils of straight lines in the (X, Y)-plane which correspond to isothermal families in the (x, y)-plane, unless (14) is an identity.

Now let the transformation T be such that there are more than four such distinct parallel pencils of straight lines in the (X, Y)-plane which correspond to isothermal families in the (x, y)-plane. Therefore the preceding equation is an identity in m.

Since the jacobian $J \neq 0$, it follows that at least one of the expressions U_u or U_v is not zero. Therefore the quantities $(V_u - mU_u)$ and $(V_v - mU_v)$ are relatively prime in the indeterminate m. Thus since each of these quantities are repeated factors of the identity (14), they must each be factors of the bracketed expressions in (14). The full discussion of the resulting identities is too complicated and will not be given here.

In conclusion, we find that in the imaginary domain the set of transformations

$$
\begin{aligned}
U &= a(u)f(v) + h, \qquad & V &= b(u)f(v) + k; \\
U &= a(v)f(u) + h; \qquad & V &= b(v)f(u) + k;
\end{aligned}
\tag{15}
$$

where (h, k) are constants, possess the property of Theorem 1 besides those there stated.

5. **Discussion of harmonic maps of isothermal systems of curves.** We shall consider those systems of curves which are converted into isothermal systems under harmonic transformations.

By appropriately changing the arbitrary constant to an isothermal parameter, it is known that any isothermal system of curves in the (X, Y)-plane can be written in the form

$$
F(U) + G(V) = \text{const.} \tag{16}
$$

We shall obtain a condition on F and G that this system shall correspond to an isothermal family in the (x, y)-plane under a harmonic transformation T.

Substituting the harmonic transformation T: $U = U(u, v)$, $V = V(u, v)$, where $U_{uv} = 0$ and $V_{uv} = 0$, into (16), we see that the corresponding family of curves in the (x, y)-plane is defined by (16) also, in which the left-hand side is considered to be a function of (u, v).

The condition that this system of curves in the (x, y)-plane be isothermal is

$$
\begin{aligned}
&(F_U U_v + G_V V_v)^2 \big[(F_U U_u + G_V V_u)(F_{UUU} U_u^2 U_v + G_{VVV} V_u^2 V_v) \\
&\qquad - (F_{UU} U_u U_v + G_{VV} V_u V_v)(F_{UU} U_u^2 + G_{VV} V_u^2) \\
&\qquad + (U_{uu} V_u - U_u V_{uu})(F_{UU} G_V U_v - F_U G_{VV} V_v)\big] \\
&= (F_U U_u + G_V V_u)^2 \big[(F_U U_v + G_V V_v)(F_{UUU} U_u U_v^2 + G_{VVV} V_u V_v^2) \\
&\qquad - (F_{UU} U_u U_v + G_{VV} V_u V_v)(F_{UU} U_v^2 + G_{VV} V_v^2) \\
&\qquad + (U_{vv} V_v - U_v V_{vv})(F_{UU} G_V U_u - F_U G_{VV} V_u)\big].
\end{aligned}
\tag{17}
$$

THEOREM 2. *The only isothermal systems of curves in the (X, Y)-plane which correspond to an isothermal family in the (x, y)-plane by every transformation of our harmonic set (H) are the parallel pencils of straight lines.*

This is deduced from (17) upon observing that it must be an identity in the partial derivatives of $U(u, v)$ and $V(u, v)$. Hence upon setting the coefficients of U_{uu}, V_{uu}, U_{vv}, V_{vv} equal to zero, we find $F_{UU}=0$ and $G_{VV}=0$. Hence both F and G are linear integral functions of U and V respectively. Therefore the family in the (X, Y)-plane is a parallel pencil of straight lines.

6. **Transformations whereby every pencil of straight lines corresponds to isothermal families.** We wish to discuss in the real or imaginary domain the transformations whereby every pencil (parallel or not) of straight lines in the (X, Y)-plane corresponds to isothermal families of curves in the (x, y)-plane. Simultaneously we shall consider the corresponding problem with respect to every parallel pencil of straight lines and every set of concentric circles.

For this purpose, let $F=\log h$, $G=\epsilon \log k$ where $h=U-U_0$, $k=V-V_0$ and $\epsilon^2=1$. Thus (16) represents a pencil with finite vertex of straight lines or a concentric set of circles according as $\epsilon=-1$ or 1. Substituting these into (17) and simplifying the result, we obtain

$$
\begin{aligned}
&(kU_v+\epsilon hV_v)^2[2(kU_u+\epsilon hV_u)(k^3U_u^2U_v+\epsilon h^3V_u^2V_v)\\
&\qquad -(k^2U_uU_v+\epsilon h^2V_uV_v)(k^2U_u^2+\epsilon h^2V_u^2)\\
&\qquad -\epsilon(U_{uu}V_u-U_uV_{uu})(kU_v-hV_v)]\\
&\quad =(kU_u+\epsilon hV_u)^2[2(kU_v+\epsilon hV_v)(k^3U_uU_v^2+\epsilon h^3V_uV_v^2)\\
&\qquad -(k^2U_uU_v+\epsilon h^2V_uV_v)(k^2U_v^2+\epsilon h^2V_v^2)\\
&\qquad -\epsilon(U_{vv}V_v-U_vV_{vv})(kU_u-hV_u)].
\end{aligned}
\tag{18}
$$

This is a polynomial of the sixth degree in (h, k). We shall find all harmonic transformations for which this is an identity in (h, k).

Upon setting the sixth degree terms equal to zero and simplifying, we find

$$
\begin{aligned}
(1+\epsilon)\ [&hU_uU_v\{k(U_uV_v+U_vV_u)+2kV_uV_v\}\\
&-hV_uV_v\{2kU_uU_v+h(U_uV_v+U_vV_v)\}]=0.
\end{aligned}
\tag{19}
$$

Upon setting the remaining terms of third degree in (18) equal to zero, we find

$$(kU_v + \epsilon hV_v)^2(kU_v - kV_v)(U_{uu}V_u - U_uV_{uu}) \tag{20}$$
$$= (kU_u + \epsilon hV_u)^2(kU_u - kV_u)(U_{vv}V_v - U_vV_{vv}).$$

THEOREM 3. *The only transformations in the real or imaginary domain whereby every pencil of straight lines (parallel or not) in the (X, Y)-plane corresponds to an isothermal family of curves in the (x, y)-plane are the conformal maps, circle-to-line transformations, and products of conformal correspondences by circle-to-line transformations.*

The set of harmonic transformations of Theorem 3 is

$$U = a_1f(u) + b_1g(v), \qquad V = a_2f(u) + b_2g(v), \tag{21}$$

where f and g are analytic functions of u and v respectively with $f_u \neq 0$ and $g_v \neq 0$, and (a_1, b_1, a_2, b_2) are complex constants such that $a_1b_2 - a_2b_1 \neq 0$. In the real domain, the coefficients of g are the conjugates of those of f, and (b_2, a_2) are the conjugates of (a_1, b_1) respectively.

To prove that (21) is the total set of harmonic transformations of Theorem 3, it is observed that $\epsilon = -1$ in the identity (18). It follows that (19) is satisfied identically. From (20), we deduce

$$U_{uu}V_u - U_uV_{uu} = 0, \qquad U_{vv}V_v - U_vV_{vv} = 0, \tag{22}$$

since the jacobian J of the transformation T is not zero.

From these equations, we find

$$\frac{U_u}{a_1} = \frac{V_u}{a_2} = f_u(u) \neq 0, \qquad \frac{U_v}{b_1} = \frac{V_v}{b_2} = g_v(v) \neq 0, \tag{23}$$

where f and g are analytic functions of u and v respectively, and (a_1, b_1, a_2, b_2) are constants such that $a_1b_2 - a_2b_1 \neq 0$. Thus it follows that any harmonic transformation T of Theorem 3 is a conformal map, or an affinity, or the product of a conformal correspondence by an affine transformation. That the harmonic transformation T can be written in the form (21) follows from the fact that any affine transformation can be factored into the product of a translation by an affinity which leaves the origin fixed.

The validity of Theorem 3 is established as a consequence of Theorem 1 and the set (21).

7. **Characterization of the conformal group.** In this final section, we shall give a characterization of the conformal group in the real domain and also in the imaginary domain.

THEOREM 4. *The only transformations in the real domain whereby every parallel pencil of straight lines and also every concentric set of* *circles in the* (X, Y)*-plane corresponds to an isothermal family of curves in the* (x, y)*-plane are those of the conformal group.*

In the imaginary domain, the complete set of transformations possessing the property of Theorem 4 consists not only of conformalities but also of correspondences of the imaginary subset of harmonic transformation

$$U = a_1[f(u) + g(v)], \qquad V = a_2[f(u) - g(v)], \tag{24}$$

where f and g are analytic functions of u and v separately with $f_u \neq 0$ and $g_v \neq 0$ and (a_1, a_2) are complex constants such that $a_1 \neq 0$ and $a_2 \neq 0$. Any transformation of this set is the product of a conformal transformation followed by an affinity of the special form: $U = a_1(u+v)$, $V = a_2(u-v)$.

The proof of the above result is as follows. By Theorem 1, the required transformation T is necessarily harmonic or else T is the product of a conformal map by a circle-to-line transformation, or of the form (15). However, it can be shown that a circle-to-line transformation or one of the form (15) will possess the property of Theorem 4 if and only if it is conformal. Hence only harmonic transformations need be considered for which (18) must be an identity and $\epsilon = 1$. From (19) and (20), we deduce the conditions

$$\begin{aligned} &U_u U_v (U_u V_v + U_v V_u) = 0, && V_u V_v (U_u V_v + U_v V_u) = 0, \\ &U_{uu} V_u - U_u V_{uu} = 0, && U_{vv} V_v - U_v V_{vv} = 0. \end{aligned} \tag{25}$$

If $U_u V_v + U_v V_u \neq 0$, we find from these that our transformation T is conformal. Otherwise the transformation T must be necessarily of the form (21) where $U_u V_v + U_v V_u = 0$. Hence $b_1 = r a_1$, $b_2 = -r a_2$, where $r \neq 0$. Upon replacing $g(v)$ by the new analytic function $g(v)/r$ we obtain our set (24). Of course, this set is imaginary, since for any real transformation T, the expression $(U_u V_v + U_v V_u)$ is always positive and can never be zero. This completes our discussion of Theorem 4.

THEOREM 5. *The only transformations in the real or imaginary domain which convert every pencil of circles into an isothermal family of curves are the conformal maps.*

Let the pencils of circles be in the (X, Y)-plane. By Theorem 4, the transformation T must be either a conformality or a correspondence of the set (24). To prove our new result, it remains to show that, by any transformation of this set (24), not every pencil of circles in the (X, Y)-plane corresponds to an isothermal family of curves in the (x, y)-plane.

Let us suppose that, by a transformation T of the set (24), a parabolic pencil of circles P_0 in the (X, Y)-plane corresponds to an isothermal family in the (x, y)-plane. Now T can be factored into the product of a conformal map Γ by an affinity of the special form: $U = a_1(u+v)$, $V = a_2(u-v)$. Since the inverse of Γ carries the given isothermal family of curves into an isothermal family, it follows that, by this special affinity, the pencil P_0 must correspond to an isothermal family.

However, if the parabolic pencil of circles P_0 is taken to consist of all the circles tangent to the Y-axis at the origin, it is an easy matter to show that P_0 corresponds to a linear family of conic sections which is not isothermal, under this special affinity. Therefore, by any transformation T of (24), the parabolic pencil of circles P_0 can not correspond to an isothermal family.

The preceding arguments show that the only transformations T with the desired property are the conformalities. Theorem 5 is proved.

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A GENERALIZATION OF STEINER'S FORMULAE

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Let C be an arbitrary convex curve in the plane of length L and area F; and let C_ρ be a curve parallel to C at a distance ρ from it, of length L_ρ and area F_ρ. Then according to Steiner's classical result:

$$L_\rho = L + 2\pi\rho, \qquad F_\rho = F + \rho L + \pi\rho^2.$$

In this paper we develop a generalization of these formulae for curves lying on a curved surface whose curvature $K(v^1, v^2)$ (referred to geodesic parallel coordinates) is a function of v^1 alone. Explicit formulae are derived in the case of surfaces of constant curvature. In this treatment it is necessary to put certain restrictions on the curve C and the distance ρ to replace Steiner's assumption of convexity. These restrictions (which are discussed below) are stated in their most obvious form, and a discussion of methods of relaxing them is deferred to a later paper. Our chief results are contained in the formulae (12) and (15) below.

Let the curve C be a simple, closed, bounding, and differentiable curve on the surface S. Choose a coordinate system in which $v^1 = 0$ is the curve C, and in which $v^2 =$ constant are the geodesics orthogonal to C. Further let v^2 be the arc length of C measured positively for motion on the curve which keeps the bounded area to the left, and let v^1 be the arc length of geodesics normal to C measured positively outward from C. Choose the unit normals to C so that they point toward the interior of C. Then we have:

$$ds^2 = (dv^1)^2 + g_{22}(v^1, v^2)(dv^2)^2; \qquad g_{22}(0, v^2) = 1. \tag{1}$$

For the moment we ignore the question of determining the region of S within which such a coordinate system is valid, and proceed to compute $(g_{22})^{1/2}$. In this coordinate system we have the following relations (see L. P. Eisenhart *An introduction to differential geometry*, pp. 181 and 188)

$$\frac{\partial^2 (g_{22})^{1/2}}{\partial v^1 \partial v^1} + K(g_{22})^{1/2} = 0, \tag{2}$$

$$\kappa_g(v^2) = \left[\frac{\partial (g_{22})^{1/2}}{\partial v'}\right]_{v^1=0}, \tag{3}$$

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where κ_g is the geodesic curvature of C.

We assume that K is a differentiable function of v^2, that it is independent of v^1, and that it is never zero. Then the integration of (2) gives:

$$(g_{22})^{1/2} = f(v^2) \sin \left[v^1(K(v^2))^{1/2}\right] + h(v^2) \cos \left[v^1(K(v^2))^{1/2}\right]. \tag{4}$$

If $K(v^2)$ is negative, complex numbers are introduced, and f and h must be so chosen that the resulting value of $(g_{22})^{1/2}$ is real. From (3) we find that

$$\begin{aligned} \kappa_g(v^2) = \Big\{ & f(v^2)(K(v^2))^{1/2} \cos \left[v^1(K(v^2))^{1/2}\right] \\ & - h(v^2)(K(v^2))^{1/2} \sin \left[v^1(K(v^2))^{1/2}\right]\Big\}_{v^1=0} \end{aligned} \tag{5}$$

or $\kappa_g(v^2) = f(v^2)(K(v^2))^{1/2}$. Hence

$$f(v^2) = \frac{\kappa_g(v^2)}{(K(v^2))^{1/2}}. \tag{6}$$

Furthermore equation (4) must be valid along C, on which $v^1=0$ and $(g_{22})^{1/2}=1$. Therefore from (4), $h(v^2)=1$. Hence:

$$(g_{22})^{1/2} = \frac{\kappa_g(v^2)}{(K(v^2))^{1/2}} \sin \left[v^1(K(v^2))^{1/2}\right] + \cos \left[v^1(K(v^2))^{1/2}\right]. \tag{7}$$

The chosen coordinate system will fail to be valid whenever (1) $(g_{22})^{1/2} \leqq 0$; or (2) when v^1 is so large that the region described overlaps itself. The second difficulty may be overcome by considering overlapping portions to be on separate covering sheets of S (as in a Riemann surface), but we must assume that $(g_{22})^{1/2}>0$. We let C_ρ be the closed curve $v^1=\rho$ (const.) and restrict ourselves to the interior of C_ρ. Hence we require that:

$$\frac{\kappa_g(v^2)}{(K(v^2))^{1/2}} \sin \left[v^1(K(v^2))^{1/2}\right] + \cos \left[v^1(K(v^2))^{1/2}\right] > 0 \tag{8}$$

for all v^2 and for $0 \leqq v^1 \leqq \rho$. Without further assumptions on K, κ_g, and ρ no simplification of (8) is possible. However, for constant K the validity of (8) may be inferred from other simple assumptions as follows:

Case 1. $K=\text{constant}>0$. Then if $\kappa_g(v^2)>0$ and $0 \leqq \rho \leqq \pi 2K^{1/2}$ each term of (8) is positive, so (8) holds.

Case 2. $K=\text{constant}<0$. Then (8) can more properly be written:

$$\frac{\kappa_g(v^2)}{(-K)^{1/2}} \sinh \left[v^1(-K)^{1/2}\right] + \cosh \left[v^1(-K)^{1/2}\right] > 0. \tag{8'}$$

And here if $\kappa_g(v^2)>0$ and $\rho \geqq 0$, the inequality is valid.

These are the assumptions which correspond to Steiner's requirement that C be convex, and henceforth we consider only curves C and values of ρ for which they are verified.

Then from (7) the length of C_ρ is given by:

$$L_\rho = \int_C (g_{22}(\rho, v^2))^{1/2} dv^2 \tag{9}$$

or

$$L_\rho = \int_C \frac{\kappa_g(v^2)}{(K(v^2))^{1/2}} \sin\left[\rho(K(v^2))^{1/2}\right] dv^2 + \int_C \cos\left[\rho(K(v^2))^{1/2}\right] dv^2. \tag{10}$$

(We note that (10) holds even if C does not bound. However, the assumption that C bounds is essential for further developments.)

When K is constant, (10) may be simplified by the use of the Gauss-Bonnet formula:

$$\int_C \kappa_g(v^2) dv^2 = 2\pi - K \iint_{\text{Interior of } C} (g_{22})^{1/2} dv^1 dv^2 = 2\pi - KF. \tag{11}$$

Hence

$$L_\rho = 2\pi \frac{\sin\left[\rho K^{1/2}\right]}{K^{1/2}} - FK^{1/2} \sin\left[\rho K^{1/2}\right] + L \cos\left[\rho K^{1/2}\right]. \tag{12}$$

When K is negative (12) may more appropriately be written:

$$\begin{aligned} L_\rho = 2\pi \frac{\sinh\left[\rho(-K)^{1/2}\right]}{(-K)^{1/2}} &+ F(-K)^{1/2} \sinh\left[\rho(-K)^{1/2}\right] \\ &+ L \cosh\left[\rho(-K)^{1/2}\right]. \end{aligned} \tag{12'}$$

We note that as $K \to 0$, (12) and (12′) approach Steiner's formula.

Finally to find F_ρ, the area of C_ρ, we consider

$$F_\rho = F + \int_C \left\{ \int_0^\rho (g_{22}(v^1, v^2))^{1/2} dv^1 \right\} dv^2 \tag{13}$$

or

$$\begin{aligned} F_\rho = F &+ \int_C \left\{ \int_0^\rho \kappa_g(v^2) \frac{\sin\left[v^1(K(v^2))^{1/2}\right]}{(K(v^2))^{1/2}} dv^1 \right\} dv^2 \\ &+ \int_C \left\{ \int_0^\rho \cos\left[v^1(K(v^2))^{1/2}\right] dv^1 \right\} dv^2. \end{aligned} \tag{14}$$

When K is constant, (14) simplifies as follows owing to (11):

$$(15)\quad F_\rho = L\frac{\sin[\rho K^{1/2}]}{K^{1/2}} - 2\pi\left(\frac{\cos[\rho K^{1/2}]-1}{K}\right) + F\cos[\rho K^{1/2}].$$

When K is negative (15) may more appropriately be written:

$$(15')\quad \begin{aligned} F_\rho = {} & L\frac{\sinh[\rho(-K)^{1/2}]}{(-K)^{1/2}} - 2\pi\left(\frac{\cosh[\rho(-K)^{1/2}]-1}{K}\right) \\ & + F\cosh[\rho(-K)^{1/2}]. \end{aligned}$$

We again note that if $K \to 0$, formulae (15) and (15′) approach Steiner's result.

OBSERVATORIE DE SANTIAGO DE COMPOSTELA

THE ESSENTIAL PART OF A SURFACE

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This paper attempts to place the concept of the essential part of the projection of a continuous surface upon a plane in its historical and mathematical setting, and to outline the role this concept is playing in the solution of the area problem. To that purpose those events most closely related to this concept will be sketched, and less relevant facts will be suppressed.[1]

1. **The length of a curve.** Consider a continuous path curve C given by a representation

$$C: \quad x = x(u), \ y = y(u), \ z = z(u), \qquad 0 \leqq u \leqq 1, \tag{1.1}$$

where the functions $x(u)$, $y(u)$, $z(u)$ are defined, single-valued, real-valued, and continuous on the closed unit interval $0 \leqq u \leqq 1$. It is well known that the length $L(C)$ of C may be defined as the limit of the lengths of inscribed polygons which converge to C—if $0 = u_0 < u_1 < \cdots < u_i < \cdots < u_n = 1$ be a subdivision of $0 \leqq u \leqq 1$, then

$$L(C) = \lim_{n\to\infty} \sum_{i=1}^{n} \{[x(u_i) - x(u_{i-1})]^2 + [y(u_i) - y(u_{i-1})]^2 + [z(u_i) - z(u_{i-1})]^2\}^{1/2}, \tag{1.2}$$

the limit[2] being taken with respect to subdivisions of $0 \leqq u \leqq 1$ for which the maximum value of $|u_i - u_{i-1}|$ for i between 1 and n converges to zero with $1/n$. Observe that (1.2) gives an expression for the length of the curve C in terms of its representation (1.1), standard algebraic operations, and one limit process.

2. **The area of a surface.** Consider a continuous path surface S

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[1] A list of papers closely related to the facts to be presented in the sequel is included at the end of this paper. Numbers in brackets refer to this bibliography. For an exhaustive treatment of the concepts of length and area and an extensive bibliography upon the subject, the reader should consult Radó [4]. This volume of the Colloquium publications is now in the process of being published. The writer had the privilege of reading the manuscript.

[2] Of course, it is necessary to discuss the existence of this limit and its independence of the representation chosen for C. See Radó [4, III. 3].

given by a representation[3]

$$(2.1)\qquad S:\ x = x(u, v),\ y = y(u, v),\ z = z(u, v), \qquad 0 \leqq u, v \leqq 1,$$

where the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are defined, single-valued, real-valued, and continuous on the closed unit square $0 \leqq u, v \leqq 1$. Before the twentieth century mathematicians generally defined the area of S in a way analogous to that for the length of a curve—namely, as the limit of the areas of inscribed polyhedra which converge to S. In 1880 H. A. Schwarz inscribed sequences of polyhedra in a right circular cylinder to show these interesting facts (see Schwarz [1]): there are sequences of polyhedra inscribed in the cylinder and converging to it whose areas have no limit; if r be any real number not less than the accepted value for the area of the surface of the cylinder, there are sequences of polyhedra inscribed in the cylinder and converging to it whose areas converge to r; in particular there are sequences of polyhedra inscribed in the cylinder and converging to it whose areas converge to plus infinity.

3. **The Lebesgue area.** Stimulated by Schwarz's discovery, many mathematicians, including Peano, Lebesgue, and Geöcze (see Radó [2, 4]), set forth definitions for the area of a continuous surface. Lebesgue's definition has gained wide acceptance, and is used in the sequel (see Lebesgue [1]). Given a continuous path surface S as in (2.1), consider a sequence of polyhedra P_n—not necessarily inscribed in S—which converge to S. If $E(P_n)$ denotes the elementary area of the polyhedron P_n—that is, the sum of the areas of its triangular faces—then lim inf $E(P_n)$ is an upper bound for the Lebesgue area $L(S)$ of S, and $L(S)$ is defined to be the greatest lower bound of all the upper bounds obtained in this way.[4]

4. **The area problem.** The reader's attention is invited to the following facts: (i) Given a continuous path curve C as in (1.1), consider a sequence of polygons p_n—not necessarily inscribed in C—which converge to C. If $e(p_n)$ denotes the elementary length of the polygon p_n—that is, the sum of the lengths of its linear segments—then lim inf $e(p_n)$ is an upper bound for the length $L(C)$ of C, and $L(C)$ is the greatest lower bound of all the upper bounds obtained in this way (see Radó [4, III. 3]). Thus Lebesgue's definition for the area of a surface is analogous to a possible definition for the length of a curve.

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[3] For a thorough discussion of the representations for continuous curves and surfaces, see Radó [4, II. 3].

[4] Again, the independence of $L(S)$ upon the representation chosen for S must be discussed. See Radó [4, V. 2].

(ii) The polyhedron P_n is not required to be inscribed in the surface S; this permits one to prove that the Lebesgue area $L(S)$ is a lower semi-continuous functional of S—that is, if S_n be any sequence of continuous path surfaces converging[5] to S, then $\liminf L(S_n)$ is an upper bound for $L(S)$. (iii) In all cases where an area for a given surface has been assigned and widely accepted, the Lebesgue area agrees with the accepted value—in particular, the Lebesgue area $L(P)$ of a polyhedron P is equal to its elementary area $E(P)$. (iv) While the definition of the Lebesgue area $L(S)$ of a continuous surface S guarantees the existence of a sequence of polyhedra P_n such that the P_n converge to S and the elementary areas $E(P_n)$ converge to $L(S)$, it gives no clue as to a way of constructing such a sequence. (v) Let there be assigned in any manner whatsoever a real number $A(S)$ to every continuous path surface S so that the following three conditions are fulfilled: $A(S)$ is a lower semi-continuous functional of S; for every polyhedron, P, $A(P)$ has the same value as the elementary area $E(P)$; for every continuous surface S there exists a sequence of polyhedra P_n such that the P_n converge to S and the $A(P_n)$ converge to $A(S)$. Then for every continuous surface S, $A(S)$ must have the same value as the Lebesgue area $L(S)$. (vi) The Lebesgue area of a surface S is not directly expressible in terms of its representation (2.1) using standard operations (cf. §1). Hence one of the outstanding problems is the following:

AREA PROBLEM: *Given a continuous path surface S as in* (2.1), *express its area in terms of its representation* (2.1), *using standard operations.*

5. A solution to the area problem for a non-parametric surface. In one important special case—the so-called nonparametric case—the area problem is solved. Suppose the continuous path surface S has a representation of the form (see Radó [4, V. 3])

$$S:\quad x = u,\ y = v,\ z = z(u, v), \qquad 0 \leqq u, v \leqq 1. \tag{5.1}$$

Under no additional hypotheses, Radó has shown (see Radó [1]) that $L(S)$ may be given in terms of the representation (5.1) and standard operations as follows. Consider any sequence of subdivisions D_n of the unit square $0 \leqq u, v \leqq 1$ into oriented rectangles $r: u' \leqq u \leqq u''$, $v' \leqq v \leqq v''$ for which the maximum of the diameters of the rectangles in D_n converges to zero with $1/n$. Then[6]

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[5] A sequence of continuous surfaces S_n converges to a continuous surface S given as in (2.1) if the S_n admit of representations $S_n: x = x_n(u, v)$, $y = y_n(u, v)$, $z = z_n(u, v)$, $0 \leqq u, v \leqq 1$, $n = 1, 2, \cdots$, such that x_n, y_n, z_n converge uniformly on $0 \leqq u, v \leqq 1$ to x, y, z respectively. For a treatment of the concept of convergence for continuous curves and surfaces, see Radó [4, II. 3].

$$L(S) = \lim_{n\to\infty} \sum_{r\in D_n} \left\{ \left[\int_{v'}^{v''} \left| z(u'', v) - z(u', v) \right| dv \right]^2 + \left[\int_{u'}^{u''} \left| z(u, v'') - z(u, v') \right| du \right]^2 + \left[(u'' - u')(v'' - v') \right]^2 \right\}^{1/2}.$$

6. **An expression for the length of a curve.** One approach to the area problem in case S is any continuous path surface is to investigate various possible expressions for the length of a curve, expecting that these may afford a clue to a method for attacking the area problem. An expression for the length of a curve which has been especially fruitful will now be considered (see Banach [1]). Let C be a continuous path curve given by the representation (1.1). The projection of C on the x-axis is then represented by $x = x(u)$, $0 \leqq u \leqq 1$. For each interval $i: u' \leqq u \leqq u''$, where $0 \leqq u' < u'' \leqq 1$, and for each point x_0, let $N(x_0, i)$ denote the number—possibly zero, possibly plus infinity—of points u_0 in i such that $x(u_0) = x_0$. Then for each interval i, $N(x, i)$ is a non-negative measurable function of x which is zero outside of a certain interval, and[7] $\int N(x, i)$ may be regarded as the length of the one-dimensional curve represented by $x = x(u)$, $u' \leqq u \leqq u''$—in fact, $\int N(x, i)$ is equal to the total variation of x over the interval i (see Radó [4, III. 2]). Functions $N(y, i)$, $N(z, i)$ are defined similarly and have analogous properties. Now let D_n be any sequence of subdivisions of the interval $0 \leqq u \leqq 1$ into intervals $i: u' \leqq u \leqq u''$ for which the maximum of the lengths of the intervals in D_n converges to zero with $1/n$. Then[8]

$$L(C) = \lim_{n\to\infty} \sum_{i\in D_n} \left\{ \left[\int N(x, i) \right]^2 + \left[\int N(y, i) \right]^2 + \left[\int N(z, i) \right]^2 \right\}^{1/2}. \tag{6.1}$$

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[6] For other possible solutions of the area problem in the nonparametric case, see Radó [4, V. 3].

[7] Let I be any interval on the x-axis such that $N(x, i)$ is zero for x not in I. If $N(x, i)$ is summable on I, then $\int N(x, i)$ denotes the Lebesgue integral of $N(x, i)$ on I; if $N(x, i)$ is not summable on I, then $\int N(x, i)$ is defined to be plus infinity. Clearly $\int N(x, i)$ is independent of the choice of I, so long as $N(x, i)$ is zero for x not in I.

[8] For other possible expressions for $L(C)$ see Radó [4, III. 3].

For purposes of comparison, another formula for $L(C)$ is presented. If D be the generic notation for a subdivision of the interval $0 \leqq u \leqq 1$, then

$$L(C) = \text{l.u.b.} \sum_{i \in D} \left\{ \left[\int N(x, i) \right]^2 + \left[\int N(y, i) \right]^2 + \left[\int N(z, i) \right]^2 \right\}^{1/2}, \tag{6.2}$$

the least upper bound being taken with respect to all subdivisions D. Formula (6.1) is preferable to (6.2) because it states that $L(C)$ may be computed as the limit of sums over any sequence of subdivisions D_n of $0 \leqq u \leqq 1$ for which the maximum of the lengths of the intervals in D_n converges to zero with $1/n$.

7. An expression for the area of a surface.

Now let S be a continuous path surface given by the representation (2.1). The projection of S on the xy-plane is then represented by

$$x = x(u, v), \qquad y = y(u, v), \qquad 0 \leqq u, v \leqq 1.$$

For each set E in $0 \leqq u, v \leqq 1$, and for each point (x_0, y_0), let $N(x_0, y_0, E)$ denote the number—possibly zero, possibly plus infinity—of points (u_0, v_0) in E such that $x(u_0, v_0) = x_0$, $y(u_0, v_0) = y_0$. If one tries to parallel the reasoning used to obtain (6.1), one encounters trouble—the functions $N(x, y, E)$ are generally too large. The following example illustrates the difficulty. Let $\mathfrak{C}$ be a Peano curve given by

$$\mathfrak{C}: \quad x = x(u), \ y = y(u), \ z = 0, \qquad 0 \leqq u, v \leqq 1,$$

which fills the unit square $0 \leqq x, y \leqq 1$. If $\mathfrak{C}$ is regarded as a path surface, and Q denotes the unit square $0 \leqq u, v \leqq 1$, it is clear that $N(x, y, Q)$ is everywhere plus infinity for $0 \leqq x, y \leqq 1$. On the other hand, it is easily seen that the Lebesgue area $L(\mathfrak{C})$ is zero (see §3). So there is no hope for a formula for the Lebesgue area analogous to (6.1) for the length of a curve.

8. The essential multiplicity with respect to a region.

A study of the works of Geöcze (see Geöcze [1, 2, 3, 4]) led Radó (see Radó [2, 3]) to replace the function $N(x, y, E)$ defined in §7 by a generally smaller function which may be described as follows. Let $\mathfrak{R}$ be a bounded finitely connected Jordan region,[8] and let $x(u, v)$, $y(u, v)$ be two functions which are defined, single-valued, real-valued, and continuous on $\mathfrak{R}$. These determine a continuous transformation T from $\mathfrak{R}$ in the uv-plane into the xy-plane,

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[8] A region is a connected open set plus its boundary. A bounded finitely connected Jordan region is a bounded region whose boundary consists of a finite number of simple closed curves.

$$(8.1) \qquad T: x = x(u, v), \ y = y(u, v), \qquad (u, v) \in \mathfrak{R}.$$

Observe that T may be regarded as a representation for a flat continuous surface—that is, a continuous surface lying in the xy-plane (see §2). For any set E in $\mathfrak{R}$, the function $N(x, y, E)$ defined in §7 will be more completely denoted by $N(x, y, T, E)$—it is known as the crude multiplicity of the point (x, y) under the transformation T with respect to the set E (see Radó and Reichelderfer [1], Radó [4, IV. 1]). If T_* be any other continuous transformation defined on $\mathfrak{R}$,

$$T_*: \ x = x_*(u, v), \ y = y_*(u, v), \qquad (u, v) \in \mathfrak{R},$$

then the distance between T and T_* on $\mathfrak{R}$ is the least upper bound of the distances between the points $[x(u, v), y(u, v)]$ and $[x_*(u, v), y_*(u, v)]$ for (u, v) in $\mathfrak{R}$. If (x_0, y_0) be any point in the xy-plane, the essential multiplicity $\kappa(x_0, y_0, T, \mathfrak{R})$ of (x_0, y_0) under T with respect to $\mathfrak{R}$ is defined as follows. $\kappa(x_0, y_0, T, \mathfrak{R})$ has the non-negative integral value k if (i) all transformations T_* sufficiently close to T have a crude multiplicity $N(x_0, y_0, T_*, \mathfrak{R})$ not less than k, but (ii) there are continuous transformations $T_\sharp$ arbitrarily close to T for which the crude multiplicity $N(x_0, y_0, T_\sharp, \mathfrak{R})$ is exactly k. $\kappa(x_0, y_0, T, \mathfrak{R})$ is set equal to plus infinity if for every positive integer k all transformations T_* sufficiently close to T have a crude multiplicity $N(x_0, y_0, T_*, \mathfrak{R})$ not less than k (see Radó and Reichelderfer [1], Reichelderfer [2]). Clearly the essential multiplicity $\kappa(x, y, T, \mathfrak{R})$ never exceeds the crude multiplicity $N(x, y, T, \mathfrak{R})$—in fact,[10] if $\mathfrak{R}^0$ denotes the set of interior points of $\mathfrak{R}$ then $\kappa(x, y, T, \mathfrak{R})$ cannot exceed $N(x, y, T, \mathfrak{R}^0)$. The essential multiplicity $\kappa(x, y, T, \mathfrak{R})$ is a lower semi-continuous function of (x, y)—given any sequence of points (x_n, y_n) converging to a point (x_0, y_0), it follows that $\liminf \kappa(x_n, y_n, T, \mathfrak{R})$ is not less than $\kappa(x_0, y_0, T, \mathfrak{R})$. The essential multiplicity $\kappa(x, y, T, \mathfrak{R})$ is a lower semi-continuous functional of T—given any sequence of continuous transformations T_n defined on $\mathfrak{R}$ and converging to a continuous transformation T_0 on $\mathfrak{R}$, it follows that $\liminf \kappa(x, y, T_n, \mathfrak{R})$ is not less than $\kappa(x, y, T_0, \mathfrak{R})$.

9. **An intrinsic characterization for the essential multiplicity.** If the essential multiplicity $\kappa(x, y, T, \mathfrak{R})$ defined in §8 is to be used to solve

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[10] For a discussion and proof of the properties of the essential multiplicity, see Radó and Reichelderfer [1], or Radó [4, IV. 1].

the area problem (see §4), it is clear that one must find a characterization—that is, an equivalent definition—for $\kappa(x, y, T, \mathfrak{R})$ in terms of the continuous transformation T alone (see (8.1)). Such an intrinsic characterization has been found (see Radó and Reichelderfer [1], Radó [4, IV. 1]), and will be described presently. If C be any simple closed curve in $\mathfrak{R}$ its image $\bar{C}$ under T is a continuous closed curve in the xy-plane. Now if (x_0, y_0) be any point not on $\bar{C}$, the angle which the vector with end points (x_0, y_0) and $[x(u, v), y(u, v)]$, $(u, v) \in C$, makes with a fixed direction has an algebraic change of the form $2\pi n$ as (u, v) traverses C once, where n is an integer depending only upon (x_0, y_0), T, C, and the direction in which C is traversed. Let $\mathfrak{r}$ be any bounded finitely connected Jordan region in $\mathfrak{R}$; it is bounded by a finite number of simple closed curves $C_0, C_1, \cdots, C_m$ (see §8); denote by $\bar{C}_0, \bar{C}_1, \cdots, \bar{C}_m$ the respective images of these curves under T. Let (x_0, y_0) be any point of the xy-plane not on any one of the continuous curves $\bar{C}_0, \bar{C}_1, \cdots, \bar{C}_m$, and let n_i denote the integer associated with (x_0, y_0), T, C_i as (u, v) traverses C_i in the positive sense relative to $\mathfrak{r}$, in the manner just described. The index $\mu(x_0, y_0, T, \mathfrak{r})$ is defined as the sum $n_0+n_1+ \cdots +n_m$ (see Radó [4, II. 4]). Given any point (x_0, y_0) in the xy-plane, let $T^{-1}(x_0, y_0)$ denote the set of all points (u_0, v_0) in $\mathfrak{r}$ for which $x(u_0, y_0)=x_0$, $y(u_0, v_0)=y_0$. The continuity of the transformation T insures that $T^{-1}(x_0, y_0)$ is a closed set, which may be empty. If $T^{-1}(x_0, y_0)$ is not empty, let σ denote any component—a maximal connected subset—of $T^{-1}(x_0, y_0)$. Then σ is termed essential if in every neighborhood of σ in $\mathfrak{R}$ there exists a finitely connected Jordan region $\mathfrak{r}$ containing σ in its interior and such that (x_0, y_0) does not lie on the image under T of any one of the simple closed curves bounding $\mathfrak{r}$ and the index $\mu(x_0, y_0, T, \mathfrak{r})$ is not zero; otherwise σ is not essential. Observe that the property of being an essential component of $T^{-1}(x_0, y_0)$ is a local property—if T_* is any continuous transformation defined on $\mathfrak{R}$ which is identical with T in a neighborhood of σ, then σ is also a component of $T_*^{-1}(x_0, y_0)$, and σ is an essential component of $T_*^{-1}(x_0, y_0)$ if and only if σ is an essential component of $T^{-1}(x_0, y_0)$. It is proven that for any point (x_0, y_0) in the xy-plane, the essential multiplicity $\kappa(x_0, y_0, T, \mathfrak{R})$ is equal to the number of components of the set $T^{-1}(x_0, y_0)$ which are essential (see Radó and Reichelderfer [1]; Radó [4, IV. 1]). Thus $\kappa(x, y, T, \mathfrak{R})$ may be defined in terms of the transformation T alone.

10. **The essential multiplicity with respect to a domain.** In order to use the essential multiplicity to effect a solution of the area problem, it seems necessary to have the concept defined when the continuous transformation is defined over a domain—a connected open

set. Let $\mathfrak{D}$ be any bounded domain in the uv-plane, and let $x(u, v)$, $y(u, v)$ be two functions which are defined, single-valued, real-valued, bounded, and continuous on $\mathfrak{D}$—nothing is assumed about the behavior of these functions, or even their definition, on the boundary of $\mathfrak{D}$. These determine a bounded continuous transformation T from $\mathfrak{D}$ in the uv-plane into the xy-plane,

$$(10.1) \qquad T: \quad x = x(u, v), \; y = y(u, v), \qquad (u, v) \in \mathfrak{D}.$$

Now there are sequences of finitely connected Jordan regions $\mathfrak{R}_n$ in $\mathfrak{D}$ such that any given closed set in $\mathfrak{D}$ is contained in $\mathfrak{R}_n$ for all n sufficiently large—such sequences are said to fill up $\mathfrak{D}$ from the interior (see Radó and Reichelderfer [1]). For each value of n, consider the continuous transformation

$$T_n: \quad x = x(u, v), \; y = y(u, v), \qquad (u, v) \in \mathfrak{R}_n.$$

For each point (x_0, y_0) in the xy-plane the essential multiplicity $\kappa(x_0, y_0, T_n, \mathfrak{R}_n)$ has been defined in §8. As n tends to infinity, it is shown that the numbers $\kappa(x_0, y_0, T_n, \mathfrak{R}_n)$ have a limit k—this limit being either a non-negative integer or plus infinity. Clearly k is independent of the choice of the sequence of Jordan regions $\mathfrak{R}_n$ which fill up $\mathfrak{D}$ from the interior. The essential multiplicity $\kappa(x_0, y_0, T, \mathfrak{D})$ of (x_0, y_0) under T with respect to $\mathfrak{D}$ is defined to have this value k. From the intrinsic characterization of $\kappa(x_0, y_0, T_n, \mathfrak{R}_n)$ described in §9, it follows easily that $\kappa(x_0, y_0, T, \mathfrak{D})$ is equal to the number of components of the set[11] $T^{-1}(x_0, y_0)$ which are essential (see Radó and Reichelderfer [1]; Radó [4, IV. 1]). Thus $\kappa(x, y, T, \mathfrak{D})$ may be defined in terms of the transformation T alone. Further, $\kappa(x, y, T, \mathfrak{D})$ is lower semicontinuous with respect to both (x, y) and T (cf. §8).

11. A comparison of the definitions for the essential multiplicity. If one compares the definition of the essential multiplicity $\kappa(x, y, T, \mathfrak{D})$ when the range of T is a domain (see §10) with the definition of the essential multiplicity $\kappa(x, y, T, \mathfrak{R})$ when the range of T is a Jordan region (see §8), a question naturally arises. It is clear that the definition for $\kappa(x, y, T, \mathfrak{D})$ might have been made, without using the definition for the essential multiplicity with respect to Jordan regions, in a way completely analogous to that for $\kappa(x, y, T, \mathfrak{R})$ in §8. Are these two possible definitions equivalent, or are two basically different ideas involved here? Let T be a bounded continuous transformation as in (10.1), and let (x_0, y_0) be any point in the xy-plane. If k be any non-negative integer not greater than $\kappa(x_0, y_0, T, \mathfrak{D})$, it is clear from the definitions of the essential multiplicities in §§8 and 10 that all continuous transformations T_* sufficiently close to T on $\mathfrak{D}$ have a crude multiplicity $N(x_0, y_0, T_*, \mathfrak{D})$ not less than k. But if $\kappa(x_0, y_0, T, \mathfrak{D})$ has the value k, do there exist continuous transformations $T_\sharp$ defined on $\mathfrak{D}$ and arbitrarily close to T on $\mathfrak{D}$ for which the crude multiplicity $N(x_0, y_0, T_\sharp, \mathfrak{D})$ is exactly k? In view of what one is trying to do—namely, to replace the crude multiplicity of a point (x_0, y_0) under a continuous transformation T by a generally smaller multiplicity which has significance (cf. §§7, 8), this question gains importance. It has recently been answered in the affirmative (see Reichelderfer [2])—given an arbitrary positive number ϵ, the continuous transformation T has been modified to yield a continuous transformation $T_\sharp$ defined on $\mathfrak{D}$ whose distance from T on $\mathfrak{D}$ is less than ϵ, and for which the crude multiplicity $N(x_0, y_0, T_\sharp, \mathfrak{D})$ is equal to the essential multiplicity $\kappa(x_0, y_0, T, \mathfrak{D})$. Thus the definition for the essential multiplicities as given in §§8 and 10, though differently phrased, are basically the same.

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[11] As in §9, $T^{-1}(x_0, y_0)$ denotes the set of all points (u_0, v_0) in $\mathfrak{D}$ such that $x(u_0, v_0) = x_0$, $y(u_0, v_0) = y_0$.

12. **A measure for the essential part of a surface.** Again, let T be a continuous transformation defined over a bounded finitely connected Jordan region $\mathfrak{R}$ (see §8). Let $\mathfrak{D}$ be any domain in $\mathfrak{R}$, and consider the bounded continuous transformation (cf. §10)

$$T(\mathfrak{D}):\quad x = x(u, v),\ y = y(u, v), \qquad (u, v) \in \mathfrak{D}. \tag{12.1}$$

For each point (x_0, y_0) in the xy-plane, $\kappa(x_0, y_0, T, (\mathfrak{D}), \mathfrak{D})$—denote it simply by $\kappa(x_0, y_0, T, \mathfrak{D})$—is equal to the number of components of $T^{-1}(x_0, y_0)$ which lie in $\mathfrak{D}$ and are essential (see §10). Now $\kappa(x, y, T, \mathfrak{D})$ is a non-negative measurable function of (x, y) which is zero outside of a certain rectangle, and[13] $\iint\kappa(x, y, T, \mathfrak{D})$ may be regarded as a measure for the essential part of the flat continuous surface represented by (12.1) (see §§6, 8).

13. **A partial solution to the area problem.** Now consider a continuous path surface S given by the representation (2.1), and suppose the Lebesgue measure $L(S)$ of S is finite (see §3). Under these hypotheses, an expression for $L(S)$ analogous to that for the length of a curve given in (6.2) is now described. Let

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[13] Let R be any rectangle in the xy-plane such that $\kappa(x, y, T, \mathfrak{D})$ is zero for (x, y) not in R. If $\kappa(x, y, T, \mathfrak{D})$ is summable on R, then $\iint\kappa(x, y, T, \mathfrak{D})$ denotes the Lebesgue integral of $\kappa(x, y, T, \mathfrak{D})$ on R; if $\kappa(x, y, T, \mathfrak{D})$ is not summable on R, then $\iint\kappa(x, y, T, \mathfrak{D})$ is defined to be plus infinity. Clearly $\iint\kappa(x, y, T, \mathfrak{D})$ is independent of the choice of R, so long as $\kappa(x, y, T, \mathfrak{D})$ is zero for (x, y) not in R.

$$
\begin{aligned}
T_x&: \quad y = y(u, v), \quad z = z(u, v), && 0 \leqq u, v \leqq 1;\\
T_y&: \quad z = z(u, v), \quad x = x(u, v), && 0 \leqq u, v \leqq 1;\\
T_z&: \quad x = x(u, v), \quad y = y(u, v), && 0 \leqq u, v \leqq 1,
\end{aligned}
$$

be representations for the flat continuous surfaces obtained by projecting the representation for S upon the coordinate planes yz, zx, xy respectively. If K be the generic notation for a finite system of disjoint domains $\mathfrak{D}$ in $0 \leqq u, v \leqq 1$, then[13]

$$
\begin{aligned}
L(S) = \text{l.u.b.}_{K} \sum_{\mathfrak{D}\in K} \Bigg\{ &\left[\iint \kappa(y, z, T_x, \mathfrak{D})\right]^2 \\
&+ \left[\iint \kappa(z, x, T_y, \mathfrak{D})\right]^2 \\
&+ \left[\iint \kappa(x, y, T_z, \mathfrak{D})\right]^2 \Bigg\}^{1/2},
\end{aligned}
\tag{13.1}
$$

the least upper bound being taken with respect to all finite systems K of disjoint domains in $0 \leqq u, v \leqq 1$. Observe that the essential multiplicity functions described in §§8–10 play the role of the crude multiplicity functions in §6. It is not yet possible to describe a sequence of finite systems K_n of disjoint domains such that $L(S)$ is the limit of sums of the above radical over the K_n—that is, so that (13.1) may be replaced by a formula analogous to (6.1)—although such a sequence of K_n must exist. Nor is it possible to assert that the expression in the right member of (13.1) must be plus infinity if the Lebesgue area $L(S)$ is plus infinity, although it has been shown that the expression in the right member of (13.1) cannot then be zero.[14]

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[13] See Radó [4, V. 2] for an extensive discussion of this partial solution to the area problem.

[14] Since this address was delivered, Professor Radó has informed the writer that by using certain results of Cesari—see L. Cesari, *Caratterizzazione analitica delle superficie continue di area finita secondo Lebesgue*, Annali R. Scuola Normale Superiore Pisa (2) vol. 10 (1941) pp. 253–294 and vol. 11 (1942) pp. 1–42—it is now possible to establish (13.1) even if the Lebesgue area $L(S)$ is plus infinity.

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SUBDIRECT SUMS OF RINGS

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1. **Introduction.** In much of the rather extensive theory of rings, the rings considered are restricted by certain finiteness assumptions. Under these assumptions the notion of *direct sum* plays an important role, but for general rings the usual theorems about direct sums are no longer valid. However, in certain important cases it turns out that the concept of *subdirect sum* is an appropriate generalization of that of direct sum. The purpose of this address is to summarize some of the more important known results concerning subdirect sums of rings. Following the introduction of the necessary definitions, we shall give an indication of the nature of these results.

2. **Notation and preliminary concepts.** Let S_i $(i=1, 2, \cdots, n)$ be given rings, and denote by S the set of all symbols

$$(s_1, s_2, \cdots, s_n) \quad (s_i \in S_i,\ i = 1, 2, \cdots, n).$$

If we define addition and multiplication of these symbols by

$$(s_1, s_2, \cdots, s_n) + (s_1', s_2', \cdots, s_n') = (s_1 + s_1', s_2 + s_2', \cdots, s_n + s_n'),$$

and

$$(s_1, s_2, \cdots, s_n)(s_1', s_2', \cdots, s_n') = (s_1s_1', s_2s_2', \cdots, s_ns_n'),$$

it is easy to verify that S is a ring which we call the *direct sum* of the rings S_i $(i=1, 2, \cdots, n)$. The set of all elements of S of the form

$$(s_1, 0, \cdots, 0) \qquad (s_1 \in S_1)$$

is clearly a subring S_1' of S which is isomorphic to S_1 under the correspondence

$$s_1 \leftrightarrow (s_1, 0, \cdots, 0).$$

Furthermore, S_1' is closed under multiplication on either side by elements of S and hence is not only a subring of S but actually a *two-sided ideal* in S. If, in like manner, we define S_i' $(i=2, 3, \cdots, n)$, it is clear that every element of S is expressible uniquely in the form

$$s_1' + s_2' + \cdots + s_n' \qquad (s_i' \in S_i',\ i = 1, 2, \cdots, n),$$

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the addition being ring addition in S. In the literature, S is usually said to be the direct sum of its ideals S_i' and this is sometimes indicated by writing

$$S = S_1' + S_2' + \cdots + S_n'.$$

However, for our purposes, it is usually more convenient to use the definition given above, that is, to consider S as the direct sum of the rings S_i which, although not subrings of S, are respectively isomorphic to the ideals S_i' in S. Obviously, the two concepts coincide up to an isomorphism.

The direct sum of the rings S_i $(i=1, 2, \cdots, n)$ may be reformulated as the ring of all functions defined on a set of n "points," say $\{1, 2, \cdots, n\}$, such that at the point i the functional values are in S_i. This immediately suggests the following generalization. Let $\mathfrak{A}$ be an arbitrary set of "points" and suppose that with each point α of $\mathfrak{A}$ there is associated a unique ring S_α. The ring of all functions f defined on $\mathfrak{A}$ such that $f(\alpha) \in S_\alpha$ $(\alpha \in \mathfrak{A})$ is the *direct sum* of the rings S_α $(\alpha \in \mathfrak{A})$. In considering the direct sum of such a set of rings we shall find it convenient to use a notation and terminology which may suggest that the set $\mathfrak{A}$ is denumerable. However, we do not mean to imply any restriction on the cardinal number of the set $\mathfrak{A}$, or even that the set is necessarily well-ordered.

With this understanding, let S_i $(i=1, 2, \cdots)$ be an arbitrary set of rings, distinct or identical, and let us denote an element s of the direct sum S of this set of rings by

$$s = (s_1, s_2, \cdots) \qquad (s_i \in S_i).$$

The zero of S is clearly $(0, 0, \cdots)$, the 0 in the ith place being the zero of S_i. Now let T be a *subring* of S. If

$$t = (t_1, t_2, \cdots) \qquad (t_i \in S_i)$$

is an arbitrary element of T, the correspondence

$$t \rightarrow t_i$$

defines a homomorphism of T *into* S_i, in other words, a homomorphism of T *onto* a subring S_i^* of S_i. The case of principal interest is that in which $S_i^* = S_i$ $(i=1, 2, \cdots)$, that is, under each homomorphism $t \rightarrow t_i$ *every* element of S_i is the image of some element of T. If this is true we say that T is a *subdirect sum* of the rings S_i $(i=1, 2, \cdots)$, and that each S_i is a *component* of T.

If the ring R is isomorphic to a subdirect sum T of rings S_i $(i=1, 2, \cdots)$, T may be called a *representation* of R as a subdirect sum of the rings S_i ($i=1, 2, \cdots$). If the isomorphism of R and T is given by

$$r \leftrightarrow t,$$

it is clear that the correspondence

$$r \leftrightarrow t \rightarrow t_i$$

defines a "natural" homomorphism of R onto S_i.

Various problems now suggest themselves as follows. Given a ring R, find representations of R as a subdirect sum. Do such representations always exist? Or, find necessary and sufficient conditions that a ring R have a representation as a subdirect sum of rings of some specified kind. The main part of this address will be devoted to giving partial answers to certain specific questions of this general type.

The earliest references to infinite direct sums of rings or to subdirect sums of any type are Prüfer [24],[1] Krull [14] and Köthe [11, 12]. Later on, we shall mention some of the essential contributions of these authors. However, much of the credit for the recent interest in this subject goes to Stone [26] for his fundamental work on the representations of Boolean rings. At this point it may be in order to quote one of Stone's results since it will illustrate the type of theorem to be expected.

A ring R of more than one element is a *Boolean ring* if $a^2=a$ for every element a of R. It is not difficult to show, either directly as in Stone [26] or as a special case of a well known general theorem, that a Boolean ring with a finite number of elements is isomorphic to a direct sum of a finite number of two-element fields, each of which we may consider to be the field $I/(2)$ of integers modulo 2. For unrestricted Boolean rings this result does not carry over but instead we have the following result of Stone [26]:

THEOREM 1. *A ring R is a Boolean ring if and only if it is isomorphic to a subdirect sum of fields $I/(2)$.*

To show that the word "subdirect" in the statement of the theorem can not be replaced by "direct," it is only necessary to observe that the direct sum of any number of fields has a unit element but that there exist Boolean rings without unit element.

As a matter of fact, most of the known theorems about subdirect sum representations are generalizations of Theorem 1 in the sense that they reduce to this theorem in case the rings considered are specialized to be Boolean rings. An equivalent, and perhaps more familiar, formulation of Theorem 1 is that a ring is a Boolean ring if and only if it is isomorphic to a ring of subsets of some set. However, we shall not discuss this aspect of the theorem since it is the subdirect sum formulation which lends itself to algebraic generalization.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

3. A fundamental theorem. A problem of central interest is that of determining conditions under which a ring is isomorphic to a subdirect sum of given rings S_i. A solution of this problem is given by the next theorem which, although implicit in earlier work of Prüfer [24], Krull [14], Köthe [11], and Stone [26], was apparently first formulated explicitly in something like its present form by McCoy and Montgomery [17].

THEOREM 2. *A ring R is isomorphic to a subdirect sum of rings S_i if and only if for each i there exists a homomorphism h_i of R onto S_i such that if r is any nonzero element of R, $h_i(r)\neq 0$ for at least one i.*

PROOF. Let R be isomorphic to a subdirect sum of rings S_i, and h_i the natural homomorphism of R onto S_i. Since the zero elements must correspond to each other under an isomorphism, not all $h_i(r)$ can be zero unless $r=0$. Conversely, if h_i is a homomorphism of R onto S_i $(i=1, 2, \cdots)$ and not all $h_i(r)$ are zero for nonzero r, it is easy to verify that

$$r \leftrightarrow (h_1(r), h_2(r), \cdots)$$

defines an isomorphism of R with a subdirect sum of the rings S_i.

From this theorem it follows easily that if R is homomorphic to each of a set of rings S_i^* with the added condition that R is *isomorphic* to at least one of the S_i^*, then R has a representation as a subdirect sum of the rings S_i^*. For this reason, if in a representation of R as a subdirect sum of rings S_i the natural homomorphism of R onto S_i is an isomorphism for at least one i, we say that this is a *trivial representation*; otherwise it is a *nontrivial representation*.

In view of the close relation between homomorphisms and two-sided ideals, the preceding theorem may be reformulated as follows:

THEOREM 3. *A ring R is isomorphic to a subdirect sum of rings S_i if and only if there exist in R two-sided ideals $\mathfrak{a}_i$ such that $R/\mathfrak{a}_i \cong S_i$, and* $\Pi\mathfrak{a}_i=0$.

Although Theorems 2 and 3 are fundamental in the sense that most results on subdirect sums are obtained as applications of one or the other of these theorems, it will be apparent that their principal contribution to the theory is that of stating the underlying problem in convenient form. In any case, the main difficulty is that of establishing the *existence* of the required homomorphisms or ideals.

As an application of Theorem 3, let R be a commutative ring without nonzero nilpotent elements, that is, no positive integral power of a nonzero element is zero. A theorem of Krull [14] states that the set of all *prime* ideals in R has zero intersection. Furthermore, if $\mathfrak{p}$ is a prime ideal, $R/\mathfrak{p}$ has no proper divisors of zero and is therefore an *integral domain*. The following result of the writer [19] then is an immediate consequence of Theorem 3:

THEOREM 4. *A necessary and sufficient condition that a commutative ring R be isomorphic to a subdirect sum of integral domains is that R contain no nonzero nilpotent element.*

4. Subdirectly irreducible rings.

A ring which has no nontrivial representation as a subdirect sum of any rings is said to be *subdirectly irreducible*. Since a homomorphic image of a *field* is either a field isomorphic to the given field or a one-element ring consisting of the zero alone, it is clear that every field is subdirectly irreducible. More generally, Theorem 3 shows that *a ring R is subdirectly irreducible if and only if the intersection of all nonzero two-sided ideals in R is different from zero.* Thus a subdirectly irreducible ring R has a unique minimal two-sided ideal $J\neq 0$ which is contained in every nonzero two-sided ideal.

Birkhoff [2] introduced the concept of subdirect irreducibility and proved the following fundamental result:[3]

THEOREM 5. *Any ring R is isomorphic to a subdirect sum of subdirectly irreducible rings.*

PROOF. Let a be any nonzero element of R, and consider the set of all two-sided ideals in R which do not contain a. Zorn's Lemma asserts the existence of a maximal ideal $\mathfrak{m}_a$ in this set. If now a varies over the nonzero elements of R the intersection of all $\mathfrak{m}_a$ is zero and hence, by Theorem 3, R is isomorphic to a subdirect sum of the rings $R/\mathfrak{m}_a$. However, every nonzero two-sided ideal in $R/\mathfrak{m}_a$ contains $\bar{a}\neq 0$, where $\bar{a}$ is the residue class to which a belongs modulo $\mathfrak{m}_a$, and thus $R/\mathfrak{m}_a$ is subdirectly irreducible.

This theorem suggests the desirability of characterizing the subdirectly irreducible rings. Any *simple* ring, that is, a ring containing no nonzero two-sided ideal except the entire ring, is clearly subdirectly irreducible. Beyond this, not much is known about the noncommutative subdirectly irreducible rings. However, in the commutative case, the subdirectly irreducible rings have been completely characterized in [21]. These results may be stated in the next two theorems, in both of which the ideal J is the unique minimal ideal.

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[3] In McCoy [19] it was shown that any ring with unit element is isomorphic to a subdirect sum of *irreducible* rings, an irreducible ring being one which can not be expressed as the direct sum of two proper two-sided ideals. The irreducible components constructed in proving this are actually subdirectly irreducible, but the significance of this concept was first observed by Birkhoff [2].

THEOREM 6. *Let R be a commutative ring with at least one element which is not a divisor of zero, and let D be the set of all divisors of zero in R. Then R is subdirectly irreducible if and only if it has the following properties:*

(i) *The set of all elements x of R such that $Dx=0$ is a principal ideal $J=(j)\neq 0$.*

(ii) *D is an ideal and R/D is a field.*

(iii) *If d_1 is any element of D which is not in J, there exists an element d_2 of D not in J such that $d_1d_2\in J$.*

A typical example of a ring satisfying these conditions is the ring $I/(p^m)$ of integers modulo a power of a prime. In this case, $J=(p^{m-1})$, $D=(p)$.

THEOREM 7. *Let R be a commutative ring of more than one element in which all elements are divisors of zero. Then R is subdirectly irreducible if and only if it has the following properties:*

(i) *There exists a fixed prime p such that if $aR=0$, then $p^ka=0$ for some positive integer k, depending on a.*

(ii) *The set of all elements x of R such that $xR=0$, $px=0$, is a principal ideal $J=(j)\neq 0$.*

(iii) *If $bR\neq 0$, there exists an element c such that $bc\in J$.*

The ring of integral multiples of a prime p taken modulo p^m is an example of a ring satisfying these conditions.

In view of Theorem 5, *any* commutative ring of more than one element is isomorphic to a subdirect sum of rings, each of which satisfies the conditions of Theorem 6 or those of Theorem 7.

We now observe that the representation of a ring as a subdirect sum of subdirectly irreducible rings is in no sense unique. For example, the set of all ideals (p_i) in the ring I of integers, where each p_i is a prime of the form $4n-1$, has zero intersection and hence I is isomorphic to a subdirect sum of the fields $I/(p_i)$. On the other hand, it is clear that also I is isomorphic to a subdirect sum of the rings $I/(q_i^2)$, where the q_i are the primes of the form $4n+1$. We thus have two representations of I as a subdirect sum of subdirectly irreducible rings, all the components being fields in one representation and none being a field in the other. Actually, no component in one representation has even a subring isomorphic to a component in the other.

The condition (ii) of Theorem 7 shows that $j^2=0$, while condition (i) of Theorem 6 implies that $j^2=0$ unless $D=0$. We therefore have the following important special case due to Birkhoff [**2**]:

COROLLARY. *A subdirectly irreducible commutative ring of more than one element and without nonzero nilpotent elements is a field.*

By this result, a subdirectly irreducible Boolean ring is a field, which is clearly isomorphic to $I/(2)$. Theorem 1 thus follows immediately from Theorem 5.

Now let R be a commutative ring of more than one element which is a *p-ring* as defined in McCoy and Montgomery [**17**]; that is, p is a fixed prime and

$$x^p = x, \qquad px = 0,$$

for every element x of R. It will be pointed out presently that the commutativity of R need not be assumed since it is a consequence of $x^p=x$ for every x. Clearly, a p-ring can have no nonzero nilpotent elements and the following result of McCoy and Montgomery [**17**] follows from the corollary and Theorem 5:

THEOREM 8. *A ring R is a p-ring if and only if it is isomorphic to a subdirect sum of fields $I/(p)$.*

In a Boolean ring, $2x=0$ for every element x and thus a Boolean ring is just a 2-ring. Theorem 8 is therefore a generalization of Theorem 1.

It is easy to extend Theorem 8 to the case in which

$$x^{p^n} = x, \qquad px = 0,$$

for every element x of R. This extension, as well as some related results, will be found in [**18**].

Another interesting application of Theorem 5 is to the regular rings of von Neumann [**27, 28**]. A ring R is a *regular ring* if for each element a of R there exists an element x of R such that

$$axa = a.$$

We do not require the existence of a unit element. Clearly, regular rings are generalizations of division rings since the above equation is certainly satisfied if a has an inverse and we put $x=a^{-1}$.

We now consider regular rings without nonzero nilpotent elements.

Although, in general, a homomorphic image of a ring without nonzero nilpotents may have such nilpotents, it can be shown that this does not happen in the case of regular rings. Furthermore, a subdirectly irreducible regular ring of more than one element and without nonzero nilpotent elements is a division ring. Hence we have the following result of Forsythe and McCoy [4]:

THEOREM 9. *A regular ring of more than one element and without nonzero nilpotent elements is isomorphic to a subdirect sum of division rings.*

As a special case of this theorem, we mention the following result of Köthe [13]:

COROLLARY 1. *Any commutative regular ring of more than one element is isomorphic to a subdirect sum of fields.*

Since an *algebraic algebra* without nonzero nilpotent elements, as defined by Jacobson, is regular, we also have at once the result of Jacobson [9]:

COROLLARY 2. *An algebraic algebra of more than one element and without nonzero nilpotent elements is isomorphic to a subdirect sum of division rings.*

One further special case may be of interest. If the ring R has the property that for each element a of R there exists an integer $n>1$, depending on a, such that

$$a^n = a,$$

then clearly R is a regular ring without nonzero nilpotents and is therefore isomorphic to a subdirect sum of division rings, each of which has the same property as R. Jacobson [9] has proved that such a division ring is necessarily a field, that is, it is commutative. Accordingly, we have the following result (Jacobson [9], Forsythe and McCoy [4]) which is an extension of a classical theorem of Wedderburn to the effect that a division ring with a finite number of elements is necessarily commutative:

THEOREM 10. *A ring R with the property that for each element a there exists an integer $n>1$, depending on a, such that $a^n=a$, is necessarily commutative.*

In particular, this result shows that the commutativity of a p-ring is a consequence of $x^p=x$ for every element x.

In addition to the references already given, other results related to the material of this section will be found in Brown and McCoy [3], McCoy [20], Moisil [22] and Wade [29].

5. **The radical of a ring with minimum condition.** The important concept of *radical* of a ring has usually been defined only under assumption of certain finiteness conditions, in particular, under assumption of the *minimum condition*. A ring R satisfies the minimum condition (for right ideals) if and only if each set of right ideals in R contains a minimal one, that is, an ideal containing no other one of the set. An equivalent restriction is the *descending chain condition* for right ideals which requires that in each sequence of right ideals I_i such that

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots,$$

the equality must hold from some point on.

An ideal is a *nil ideal* if each of its elements is nilpotent. Now if R is a ring which satisfies the minimum condition, the *radical* $N = N(R)$ of R is the join (sum) of all nil right ideals in R. This means that N consists of all elements of R which are expressible in the form of a finite sum $\sum a_i$, where each a_i is an element of a nil right ideal.

The following facts about the radical are to be found in the standard texts, and we state them briefly in preparation for some generalizations to be discussed presently. The radical of R is a two-sided ideal in R which contains every nil ideal (right, left, or two-sided). If $N(R) = 0$, R is said to be *semi-simple*, and the structure of semi-simple rings is given by the following classical results of Wedderburn and Artin. *A ring R satisfying the minimum condition is semi-simple if and only if it is the direct sum of (a finite number of) simple rings. Furthermore, each of these simple rings is isomorphic to a complete matrix ring over a division ring* since a simple ring with minimum condition is necessarily of this type. It is also known that the residue class ring R/N is always semi-simple and thus has the structure just described.

If we relax the requirement that R must satisfy the minimum condition, the results as stated are no longer valid. Various attempts have been made to define a useful radical for an arbitrary ring, most of them quite naturally in terms of nilpotence.[3] However, it now appears that other concepts are probably more fundamental in the definition of radical. We shall presently define two different radicals of a general ring, both of which coincide with the radical as defined above in case the minimum condition is assumed, and both of which lead to

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[3] See Jacobson [8] for references.

subdirect sum decompositions which generalize the *direct* sum decomposition mentioned above.

Before proceeding, we may remark that a regular ring which satisfies the minimum condition is necessarily semi-simple (von Neumann [28]). Thus the regular rings, although not involving the concept of radical, do furnish a generalization of the notion of semi-simple ring.

6. **The Jacobson radical.** We now turn to a brief discussion of the radical of a general ring as defined by Jacobson [8], based partly on previous work of Perlis [23] and Baer [1].

If a is an element of the arbitrary ring R, the set of all elements of the form $x+ax$, $x\in R$, is a right ideal $\{x+ax\}$ in R. If $a\in\{x+ax\}$, a is said to be *quasi-regular*, and an ideal is *quasi-regular* if and only if each of its elements is quasi-regular. The *Jacobson radical* $N_1=N_1(R)$ is defined to be the join of all quasi-regular right ideals in R.

If $z^n=0$, it is easy to verify that $z=x+zx$, where

$$x = -\sum_{i=1}^{n-1}(-z)^i$$

and thus every nilpotent element is quasi-regular. Hence every nil ideal is a quasi-regular ideal and N_1 contains every nil right ideal of R. However, N_1 may well contain elements which are not nilpotent—in contrast to the radical N as defined in the preceding section for rings with minimum condition. For example, if S is the ring of rational numbers of the form r/s, where s is odd, it is easy to verify that $N_1(S)=(2)\neq 0$, and S has no nonzero nilpotent elements.

Jacobson [8] has proved the following facts. The radical $N_1=N_1(R)$ is a two-sided ideal in R, and R/N_1 is *semi-simple* in the sense that $N_1(R/N_1(R))=0$. If R satisfies the minimum condition, then $N_1(R)=N(R)$, and thus the Jacobson radical is actually a generalization of the ordinary radical N. If R_n denotes the complete matrix ring of order n with elements in R, then $N_1(R_n)=(N_1(R))_n$. Furthermore, if R is a regular ring, then $N_1(R)=0$, that is, a regular ring is semi-simple.

Before stating the result which shows the fundamental connection of the radical with the notion of subdirect sum, we need some preliminary definitions of Jacobson [7]. Let $\mathfrak{M}$ be a vector space, of any finite or infinite dimension, over a division ring D, and let S be a ring of linear transformations of $\mathfrak{M}$. If $a\in S$, the effect of applying the linear transformation a to the vector x of $\mathfrak{M}$ may be denoted by xa. Suppose now that in $\mathfrak{M}$ there exist at least k vectors which are linearly independent relative to D. The ring S is said to be *k-fold* *transitive* if for every set $x_1, x_2, \cdots, x_k$ of k linearly independent vectors and arbitrary vectors $y_1, y_2, \cdots, y_k$, there exists an element b of S such that $x_ib = y_i$ $(i = 1, 2, \cdots, k)$. The ring S is *dense in* $\mathfrak{M}$ *over* D if it is k-fold transitive for all $k = 1, 2, \cdots$ (up to the dimension of $\mathfrak{M}$ if $\mathfrak{M}$ has finite dimension). The abbreviated statement that S is a *dense ring* is to be interpreted as meaning that S is a dense ring of linear transformations of some vector space over a division ring.

We can now state the following fundamental theorem of Jacobson [8]:

THEOREM 11. *A necessary and sufficient condition that a ring R be isomorphic to a subdirect sum of dense rings is that $N_1(R) = 0$.*[4]

Jacobson [7] has pointed out that a dense ring S satisfies the minimum condition if and only if its vector space $\mathfrak{M}$ has finite dimension. Furthermore, a dense ring of linear transformations of an n-dimensional vector space is the ring of *all* linear transformations of the space and hence is isomorphic to the ring of all matrices of order n with elements in a division ring which is anti-isomorphic to the underlying division ring. This shows that the dense rings generalize the complete matrix rings over division rings. Theorem 11 is thus an appropriate analogue of one of the Wedderburn-Artin theorems for rings with minimum condition.

It is easy to see that a commutative dense ring is necessarily a field and Theorem 8, for example, may be obtained as a corollary to Theorem 11 since a p-ring is necessarily semi-simple.

Before proceeding to still another definition of radical, we may make a few additional remarks. In the first place, we point out that a dense ring need not be simple since the (dense) ring of *all* linear transformations of an infinite dimensional vector space contains as a proper two-sided ideal the set of all *finite valued* transformations a, that is, transformations a such that $\mathfrak{M}a$ has finite dimension. However, an important result of Jacobson [7] shows that a simple ring with a maximal right ideal—in particular, a simple ring with unit element—is isomorphic to a dense ring. In this connection, we may also observe that a dense ring need not have a unit element since the ring of all finite valued linear transformations of an infinite dimensional vector space is itself a dense ring without unit element.

In a recent paper, Goldman [5] has discussed the radical of a general ring, using a definition which he attributes to Chevalley. According to this definition, the radical of R is the intersection of the annihilators of all simple R-modules. It can be shown, however, that this coincides with $N_1(R)$ and hence we have an interesting alternative definition of $N_1(R)$.

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[4] Jacobson has formulated this result in terms of *primitive* rings rather than *dense* rings, but has shown in [8] that the concepts are equivalent. If I is a right ideal in R, we may denote by $I:R$ the set of all elements a of R such that $Ra \subseteq I$. The ring R is said to be primitive if there exists in R a maximal right ideal I such that $I:R = 0$.

7. Another definition of radical. In preparation for the definition to be stated presently, let us give alternate characterizations of the previously defined radicals as follows.

If R satisfies the minimum condition, the radical $N(R)$ consists of the elements a of R such that every element of the principal two-sided ideal (a) is nilpotent.

For arbitrary R, the Jacobson radical $N_1(R)$ consists of the elements a of R such that every element of the principal two-sided ideal (a) is quasi-regular. That is, for every element b of (a), we have $b \in \{x+bx\}$.

We now denote by $G(b)$ the two-sided ideal in R generated by the elements of the right ideal $\{x+bx\}$, that is,

$$G(b) = \{t + bt + \sum y_i z_i + \sum y_i b z_i\},$$

where t, y_i, z_i vary over R, the sums naturally being finite. We define the *radical* $N_2(R)$ to consist of the elements a of R such that $b \in G(b)$ for every element b of the principal two-sided ideal (a). This definition and the results to be presently stated concerning this radical are to be found in Brown and McCoy [3].

It is clear that always $N_1(R) \subseteq N_2(R)$, and if R happens to be commutative, $N_1(R) = N_2(R)$. Furthermore, it can be shown that the equality also holds if R satisfies the minimum condition. However, we shall presently give an example which will show that in general $N_1(R)$ and $N_2(R)$ do not coincide; hence they are different generalizations of the radical $N(R)$.

It can be shown that $N_2(R)$ is a two-sided ideal in R, and that $R/N_2(R)$ is semi-simple in the sense that $N_2(R/N_2(R)) = 0$. Also, if R_n is the complete matrix ring of order n over R, then $N_2(R_n) = (N_2(R))_n$. Actually, $N(R)$, $N_1(R)$ and $N_2(R)$ have these properties in common.

If R has more than one element and a unit element 1, $G(-1) = 0$; and hence $N_2(R) \neq R$. If S is a *simple* ring with unit element it therefore follows that $N_2(S) = 0$. The following result shows that simple rings with unit element are of fundamental interest in a study of this radical:

THEOREM 12. *A necessary and sufficient condition that a ring R be* *isomorphic to a subdirect sum of simple rings with unit element is that* $N_2(R)=0$.

The main step in the proof of this theorem is that of showing that $N_2(R)$ may be characterized as the intersection of all two-sided ideals $\mathfrak{a}$ in R such that $R/\mathfrak{a}$ is a simple ring with unit element.[5] One part of the theorem then follows at once from Theorem 3, and the other part follows from the observation that if R is homomorphic to S, then $N_2(R)$ goes *into* $N_2(S)$ under the given homomorphism.

It can be shown that a subdirect sum of a *finite* number of simple rings is isomorphic to the direct sum of some or all of these rings. From this, and the preceding theorem, it is easy to obtain the following corollary.

COROLLARY. *Let R be a ring in which the descending chain condition is satisfied for two-sided ideals. A necessary and sufficient condition that R be isomorphic to a direct sum of a finite number of simple rings with unit element is that* $N_2(R)=0$.

An example may clarify the relation between $N_1(R)$ and $N_2(R)$. Let $\mathfrak{L}$ be the ring of *all* linear transformations of a vector space $\mathfrak{M}$ with a *denumerable* basis over a division ring, and $\mathfrak{F}$ the set of all finite-valued transformations of $\mathfrak{L}$. Jacobson [7] has shown that $\mathfrak{F}$ is a two-sided ideal of $\mathfrak{L}$ which is contained in every nonzero two-sided ideal of $\mathfrak{L}$, and hence $\mathfrak{L}$ is subdirectly irreducible. Furthermore, Johnson and Kiokemeister [10] have pointed out that $\mathfrak{F}$ is the *only* proper two-sided ideal in $\mathfrak{L}$. Now $N_2(\mathfrak{L})\neq\mathfrak{L}$ since $\mathfrak{L}$ has a unit element, and hence $N_2(\mathfrak{L})=\mathfrak{F}$ or $N_2(\mathfrak{L})=0$. If $N_2(\mathfrak{L})$ were zero, the preceding theorem would show that $\mathfrak{L}$ is isomorphic to a subdirect sum of simple rings with unit element. However, since $\mathfrak{L}$ is subdirectly irreducible, this would imply that $\mathfrak{L}$ is itself a simple ring with unit element, which is obviously false. Thus $N_2(\mathfrak{L})=\mathfrak{F}$, and it is also easy to verify that $N_2(\mathfrak{F})=\mathfrak{F}$. Since $\mathfrak{L}$ and $\mathfrak{F}$ are dense rings, we have $N_1(\mathfrak{L})=N_1(\mathfrak{F})=0$, and therefore N_1 is properly contained in N_2 in the case of both the rings $\mathfrak{L}$ and $\mathfrak{F}$.

Incidentally, Johnson and Kiokemeister [10] have shown that the ring of *all* linear transformations of any vector space is a regular ring, and since $N_2(\mathfrak{L})\neq 0$ we see that a regular ring need not be semi-simple with reference to the last defined radical. We may also remark that $\mathfrak{L}$ is a regular ring which can not be isomorphic to a subdirect sum of simple regular rings.[6]

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[5] Cf. Segal [25] who essentially takes this as the *definition* of the radical of a group algebra. We may point out that a one-element ring is here considered as a simple ring with unit element, so that $N_2(R)=R$ in case there exists no maximal two-sided ideal $\mathfrak{a}$ such that $R/\mathfrak{a}$ has a unit element. If we complete Segal's definition by agreeing that his radical is to be the entire ring R in the case just described, then Segal's radical and $N_2(R)$ actually coincide for every ring R.

As another example, it may be of interest to state that in the ring A of all bounded operators in Hilbert space, $N_1(A)=0$ while $N_2(A)$ is the ideal of all completely continuous operators.

In the definition of $N_2(R)$ one may just as well use in place of $G(b)$ the ideal

$$G'(b) = \left\{t + tb + \sum y_iz_i + \sum y_ibz_i\right\},$$

or the ideal

$$G''(b) = \left\{s + bs + t + tb + \sum y_iz_i + \sum y_ibz_i\right\}.$$

Each of these lead to precisely the same radical $N_2(R)$. In case R happens to have a unit element 1, clearly

$$G(b) = G'(b) = G''(b) = \left\{\sum y_i(1 + b)z_i\right\}.$$

8. **The F-radical of a ring.** Most of the results of the preceding section have been generalized as follows by Brown and McCoy [**3**]. Let $b \rightarrow F(b)$ be a mapping F of R into the set of two-sided ideals in R which is defined for all rings and has the property that if $b \rightarrow \bar{b}$ is a homomorphism of R onto a ring S, then

$$F(\bar{b}) = \overline{F(b)}.$$

The mapping $b \rightarrow G(b)$, where $G(b)$ is the two-sided ideal defined above, clearly has this property. If in the definition of $N_2(R)$ we replace $G(b)$ by $F(b)$ we get what we shall call the F-radical N_F of R.

It can be shown that N_F is a two-sided ideal in R which may be characterized as the intersection of all two-sided ideals $\mathfrak{a}$ such that $R/\mathfrak{a}$ is subdirectly irreducible and has zero F-radical. Also, R/N_F has zero F-radical. Concerning subdirect sums we have the following theorem.

THEOREM 13. *A ring has zero F-radical if and only if it is isomorphic to a subdirect sum of subdirectly irreducible rings of zero F-radical.*

These results show that the F-radical has a number of "radical-like" properties. However, in general, it need not reduce to the radical N in case the minimum condition is satisfied in R, and hence is not an actual generalization of that radical.

Let us now consider a few illustrations. If $F(b)$ is defined by

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[6] This apparently contradicts a statement of Köthe [**13**, p. 141].

$F(b) = 0$ for all elements b of a ring, then every ring has zero F-radical and Theorem 13 reduces to Birkhoff's Theorem 5. If we define $F(b) = G(b)$, it is not difficult to show that a subdirectly irreducible ring of zero F-radical is a simple ring with unit element, and Theorem 13 yields Theorem 12.

As another illustration, suppose that

$$F_1(b) = G(b^2) = \left\{t + b^2t + \sum y_i z_i + \sum y_i b^2 z_i\right\}.$$

In this case it turns out that a subdirectly irreducible ring of zero F_1-radical is a simple ring with unit element 1 and an element whose square is -1. Hence Theorem 13 gives us the following rather curious result:

THEOREM 14. *A ring has zero F_1-radical if and only if it is isomorphic to a subdirect sum of simple rings, each with a unit element* 1 *and an element whose square is* -1.

A corresponding result will be obtained if in the definition of $F_1(b)$ we replace b^2 by any fixed polynomial in b with integral coefficients and zero constant term. For other illustrations of the general theory, see Brown and McCoy [**3**].

9. **Special subdirect sums.** The earliest work on subdirect sums was not concerned with subdirect sums as we have so far discussed them but rather with subdirect sums in which additional restrictions are imposed. In the remainder of this address we shall give a brief exposition of some results concerning these restricted subdirect sums. The material of the present section is primarily due to Köthe [**11**].

Let S be the direct sum of rings S_i, the elements of which we may denote as usual by

$$(s_1, s_2, \cdots) \qquad (s_i \in S_i,\ i = 1, 2, \cdots).$$

Now let S_1' be the subring of S consisting of all elements of S of the form

$$(s_1, 0, 0, \cdots) \qquad (s_1 \in S_1),$$

and similarly S_i' is the subring of S consisting of all elements with an element of S_i in the ith "place" and zeros elsewhere. We shall say that a subdirect sum T of the rings S_i is a *special subdirect sum*[7] if T contains all S_i' $(i = 1, 2, \cdots)$. In this case, the rings S_i' are actually two-sided ideals in T. Furthermore, the set of all elements of T with

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[7] Köthe calls this a "transcendental direct sum."

0 in the ith "place" is also a two-sided ideal T_i' and, for each i,

$$T = S_i' + T_i',$$

that is, T is the direct sum of its ideals S_i' and T_i'.

Before stating the conditions under which a ring R is isomorphic to a special subdirect sum of given two-sided ideals in R, we need a result of a very special nature. If x is an element of R such that $xR = Rx = 0$, we may say that x is an *annihilator* of R.

Lemma 1. *Let R be a ring with no nonzero annihilator. If* $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{b}_1$, $\mathfrak{b}_2$ *are two-sided ideals in R such that* $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$, *and* $R = \mathfrak{a}_1 + \mathfrak{b}_1 = \mathfrak{a}_2 + \mathfrak{b}_2$, *then* $\mathfrak{a}_1 \subseteq \mathfrak{b}_2$.

Proof. Let a_1 be any element of $\mathfrak{a}_1$. Since $R = \mathfrak{a}_2 + \mathfrak{b}_2$, we may write

$$a_1 = a_2 + b_2,$$

where $a_2 \in \mathfrak{a}_2$, $b_2 \in \mathfrak{b}_2$. If x is any element of $\mathfrak{a}_2$, then $xa_1 \in \mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$, and similarly $xb_2 = 0$ since $\mathfrak{a}_2 \cap \mathfrak{b}_2 = 0$. It follows that $xa_2 = 0$, that is, $\mathfrak{a}_2 a_2 = 0$. From $R = \mathfrak{a}_2 + \mathfrak{b}_2$, it then follows that $Ra_2 = 0$. In like manner, it can be shown that $a_2 R = 0$, and hence we must have $a_2 = 0$ and $a_1 = b_2 \in \mathfrak{b}_2$.

By a suitable change of notation, the lemma also shows that under the given hypothesis on R, $R = \mathfrak{a}_1 + \mathfrak{b}_1 = \mathfrak{a}_1 + \mathfrak{b}_2$ implies that $\mathfrak{b}_1 = \mathfrak{b}_2$.

The following theorem is due to Köthe [11] except for the restriction on R which seems to be necessary at one point in the proof.

Theorem 15. *Let R be a ring with no nonzero annihilator, and* $\mathfrak{a}_i$ $(i = 1, 2, \cdots)$ *a set of two-sided ideals in R. Necessary and sufficient conditions that R be isomorphic to a special subdirect sum of the* $\mathfrak{a}_i$ *are*:

(i) $\mathfrak{a}_i \cap \mathfrak{a}_j = 0$ *for* $i \neq j$.

(ii) *For each i,* $\mathfrak{a}_i$ *is a direct component of R, that is, there exists a (unique) two-sided ideal* $\mathfrak{b}_i$ *in R such that* $R = \mathfrak{a}_i + \mathfrak{b}_i$.

(iii) $\prod \mathfrak{b}_i = 0$.

The necessity of these conditions follows readily from the observations made above—without any restriction on R. Clearly, these conditions are satisfied for the special subdirect sum T by identifying S_i' with $\mathfrak{a}_i$, T_i' with $\mathfrak{b}_i$. The same conditions obviously hold for the corresponding ideals of any ring R which is isomorphic to T.

To prove the converse, suppose that ideals $\mathfrak{a}_i$ in R satisfy the three conditions of the theorem. If a is any element of R, then by (ii) we can express a uniquely in the form

$$a = a_i + b_i \qquad (a_i \in \mathfrak{a}_i,\ b_i \in \mathfrak{b}_i).$$

Let us denote the homomorphism $a \rightarrow a_i$ of R onto $\mathfrak{a}_i$ by h_i, and write $h_i(a) = a_i$. If a is different from zero, a is not in some $\mathfrak{b}_i$ by (iii) and hence, for some i, $a_i = h_i(a) \neq 0$. As in the proof of Theorem 2 it follows that

$$a \leftrightarrow (h_1(a), h_2(a), \cdots)$$

is an isomorphism of R with a subdirect sum of the $\mathfrak{a}_i$. Moreover, Lemma 1 shows that $\mathfrak{a}_i \subseteq \mathfrak{b}_j$ for $j \neq i$, and hence if $c \in \mathfrak{a}_i$ then $h_i(c) = c$, $h_j(c) = 0$ for $j \neq i$. Thus we actually have a *special* subdirect sum, and the proof is completed.

It may be of interest to point out that the theorem is not true without some restriction on R. As an example, let R be the ring with the four distinct elements 0, a, b, $a+b$; it being given that $xy = 0$ for all elements x, y of R and that $2x = 0$ for every element x. Then the ideals $\mathfrak{a}_1 = \{0, a\}$, $\mathfrak{a}_2 = \{0, b\}$, $\mathfrak{a}_3 = \{0, a+b\}$ satisfy the three conditions of the theorem with $\mathfrak{b}_1 = \{0, b\}$, $\mathfrak{b}_2 = \{0, a+b\}$, $\mathfrak{b}_3 = \{0, a\}$. In this case, however, the $\mathfrak{b}_i$ are not unique and furthermore R can not be isomorphic to a special subdirect sum of the ideals $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ since a special subdirect sum of a *finite* number of rings is actually the direct sum, and the direct sum of these three rings has eight distinct elements.

We shall presently apply Theorem 15 to prove another result, but first we need a few preliminary remarks. If, considered as a ring, the two-sided ideal $\mathfrak{a}$ in R has a unit element e, we may say that e is the unit element of the ideal $\mathfrak{a}$.

LEMMA 2. *If the two-sided ideal* $\mathfrak{a}$ *has a unit element,* $\mathfrak{a}$ *is a direct component of* R.

PROOF. If e is the unit element of $\mathfrak{a}$, then $ey = eye = ye$ for every y in R, and hence e commutes with all elements of R. If $\mathfrak{b} = \{x - ex\}$, it is easy to show that $R = \mathfrak{a} + \mathfrak{b}$. Furthermore, $\mathfrak{b}$ may be characterized as the set of all elements b of R such that $eb = be = 0$.

A nonzero two-sided ideal in R which does not properly contain a nonzero two-sided ideal of R may be called a *simple ideal*. Thus a simple ideal is just a minimal nonzero two-sided ideal. The next result is an immediate consequence of Lemma 2:

LEMMA 3. *A simple ideal of R which has a unit element is a simple ring.*

We may now prove the following theorem.

THEOREM 16. *A necessary and sufficient condition that a ring R be* *isomorphic to a special subdirect sum of simple rings with unit element is that every nonzero two-sided ideal in R contain a simple ideal with unit element.*

We first establish the sufficiency of this condition. If a is an annihilator of R, the principal two-sided ideal (a) can contain no simple ideal with unit element; hence R has no nonzero annihilator. Let $\mathfrak{a}_i$ $(i=1, 2, \cdots)$ be the set of all simple ideals in R with unit element. By Lemma 2, $R=\mathfrak{a}_i+\mathfrak{b}_i$, where $\mathfrak{b}_i$ consists of all elements x of R such that $e_ix=xe_i=0$, e_i being the unit element of $\mathfrak{a}_i$. Furthermore, $\mathfrak{a}_i\cap\mathfrak{a}_j=0$ for $i\neq j$, since $\mathfrak{a}_i\cap\mathfrak{a}_j$ is an ideal in the simple ideal $\mathfrak{a}_i$ (or $\mathfrak{a}_j$). Thus the conditions (i) and (ii) of Theorem 15 are satisfied, and there remains to verify that $\prod\mathfrak{b}_i=0$. If $\prod\mathfrak{b}_i\neq 0$, it is a nonzero two-sided ideal of R and hence, by hypothesis, it contains some $\mathfrak{a}_i$, say $\mathfrak{a}_1$. Hence $\prod\mathfrak{b}_i$ contains the unit element e_1 of $\mathfrak{a}_1$. But $\prod\mathfrak{b}_i\subseteq\mathfrak{b}_1$ and thus $e_1\prod\mathfrak{b}_i=0$, from which it follows that $e_1^2=e_1=0$, and $\mathfrak{a}_1=0$. This is impossible since, by definition, a simple ideal is different from zero. Hence $\prod\mathfrak{b}_i=0$, and Theorem 15 shows that R is isomorphic to a special subdirect sum of the simple rings $\mathfrak{a}_i$ with unit element.

Conversely, suppose that R is isomorphic to a special subdirect sum T of simple rings S_i with unit element. We shall show that every nonzero ideal in T contains a simple ideal with unit element, and the same will therefore be true for R. Let $\mathfrak{b}$ be a nonzero two-sided ideal in T, and b a nonzero element of $\mathfrak{b}$, say

$$b=(b_1, b_2, b_3, \cdots),$$

where $b_i\in S_i$ $(i=1, 2, \cdots)$. Since $b\neq 0$, some $b_i\neq 0$ and suppose, for convenience of notation, that $b_1\neq 0$. Then $\mathfrak{b}$ contains

$$(b_1, b_2, b_3, \cdots)(1, 0, 0, \cdots)=(b_1, 0, 0, \cdots),$$

and therefore contains the two-sided ideal generated by $(b_1, 0, 0, \cdots)$. However, since S_1 is a simple ring this is just the set of all elements $(x, 0, 0, \cdots)$, $x\in S_1$. This is therefore a simple ring with unit element, and the proof is completed.

We conclude our remarks on special subdirect sums with a result of Köthe [11]. A ring with minimum condition which contains no nonzero nil ideal is sometimes said to be *completely reducible*. More generally, a *two-sided ideal* $\mathfrak{a}$ in the arbitrary ring R may be said to be *completely reducible* if it contains no nonzero nil ideal of R and each set of right ideals of R which are contained in $\mathfrak{a}$ has a minimal one. It can be shown that a completely reducible two-sided ideal $\mathfrak{a}$ of R has a unit element and is therefore a direct component of R. Thus $\mathfrak{a}$ is also as a ring completely reducible and hence, by the results stated in §5, is a direct sum of simple rings each of which is isomorphic to a complete matrix ring over a division ring.

THEOREM 17. *A necessary and sufficient condition that a ring R be isomorphic to a special subdirect sum of simple rings, each of which is a complete matrix ring over a division ring, is that every two-sided ideal in R contain a completely reducible two-sided ideal of R.*

The proof follows by arguments similar to those used in the proof of Theorem 16 except that the set of all simple ideals satisfying the minimum condition takes the place of the set of all simple ideals with unit element which was used in the proof of that theorem.

10. **Discrete direct sums.** As usual, let S be the direct sum of rings S_i, with elements

$$s = (s_1, s_2, \cdots) \qquad (s_i \in S_i).$$

The subring T of S which consists of all elements s such that s_i is different from zero for only a finite number of values of i may be called the *discrete direct sum* of the rings S_i. It will be observed that a discrete direct sum is a restricted case of a special subdirect sum, and that a discrete direct sum of a *finite* number of rings is actually the direct sum. Furthermore, if R is isomorphic to a special subdirect sum of its ideals $\mathfrak{a}_i$ $(i = 1, 2, \cdots)$, R will clearly be isomorphic to the discrete direct sum of the $\mathfrak{a}_i$ if and only if R is the join of the ideals $\mathfrak{a}_i$. Therefore, a ring R without nonzero annihilator is isomorphic to a discrete direct sum of ideals $\mathfrak{a}_i$ if and only if R is the join of these ideals and the three conditions of Theorem 15 are satisfied. As an application of these observations, we can readily prove the following theorem.

THEOREM 18. *A necessary and sufficient condition that a ring R be isomorphic to a discrete direct sum of simple rings with unit element is that R be the join of its simple two-sided ideals with unit element.*

The necessity of the condition is almost obvious and we therefore consider only the sufficiency. Let R be the join of its simple two-sided ideals $\mathfrak{a}_i$ with unit element e_i. Conditions (i) and (ii) of Theorem 15 are satisfied as shown in the proof of Theorem 16, and $\mathfrak{b}_i$ is the set of all elements b of R such that $e_ib = be_i = 0$. Now each element a of R is expressible as a finite sum

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_k} \qquad (a_{i_s} \in \mathfrak{a}_{i_s},\ s = 1, 2, \cdots, k).$$

Since $\mathfrak{a}_i e_j \subseteq \mathfrak{a}_i \cap \mathfrak{a}_j$, we see that $\mathfrak{a}_i e_j = 0$ for $i \neq j$. Thus

$$a(e_{i_1} + e_{i_2} + \cdots + e_{i_k}) = a,$$

and R can have no nonzero annihilator. Suppose now that $a \in \prod \mathfrak{b}_i$. Since $ae_{i_1} = a_{i_1}$, it follows that $a_{i_1} \in \prod \mathfrak{b}_i \subseteq \mathfrak{b}_{i_1}$. But $\mathfrak{a}_{i_1} \cap \mathfrak{b}_{i_1} = 0$, and hence $a_{i_1} = 0$. Similarly, each $a_{i_s} = 0$, and therefore $a = 0$, that is, $\prod \mathfrak{b}_i = 0$. Thus all the conditions of Theorem 15 are satisfied, and this completes the proof.

If R happens to be a Boolean ring, a simple ideal is a field of two elements, and R is the join of its simple ideals with unit element if and only if R has an *atomic basis* as defined by Stone. The case of Theorem 18 in which R is restricted to be a Boolean ring was proved by Stone [26].

For another result which is related to the material of this section, see Jacobson [8, p. 319].

11. **Ring imbedding.** If a ring R is isomorphic to a subring of a direct sum S of rings S_i, we may identify R with this isomorphic subring and consider that S contains R. Furthermore, there may be important subrings of S which also contain R.

In the first place, we may point out that one may sometimes get an interesting result by simply considering R as imbedded in the direct sum S itself. For example, since a ring which is dense in a vector space $\mathfrak{M}$ over a division ring is a subring of the regular ring of *all* linear transformations of $\mathfrak{M}$, and a direct sum of regular rings is a regular ring, Theorem 11 yields the following result:

THEOREM 19. *If $N_1(R) = 0$, R is a subring of a regular ring.*

Since the radical as defined by Goldman [5] coincides with $N_1(R)$, the results of Goldman [6] furnish necessary and sufficient conditions that a ring be a subring of a regular ring. It may be of interest to remark that from these conditions it is easy to show that there exist commutative rings which can be imbedded in regular rings but not in *commutative* regular rings.

We now describe an imbedding procedure of a different nature. Let R be a ring which is isomorphic to a subdirect sum of rings S_i, and let h_i be the natural homomorphism of R onto S_i. We consider the set T_1 of all elements

$$t = (t_1, t_2, \cdots)$$

of the direct sum of the rings S_i with the property that if $i_1, i_2, \cdots, i_n$

is *any* finite set of distinct indices, there exists an element r of R (depending on the set of indices) such that

$$h_{i_k}(r) = t_{i_k} \qquad (k = 1, 2, \cdots, n).$$

That is, at any finite number of "places," t coincides with some element of R. It is easily verified that T_1 is a subring of S which contains R.

In principle, the method by which R has been imbedded in T_1 is due to Prüfer [24]. Although his point of view was somewhat different, he was apparently the first to use any kind of subdirect sum representation. Prüfer was concerned with the case in which R is the ring of algebraic integers of a finite algebraic number field and, in our terminology, pointed out that R is isomorphic to a subdirect sum of rings $R/(a)$, where a varies over the nonzero elements of R. He then showed that the ring T_1, constructed as described above, is a ring containing R and that in T_1 each element of R is uniquely expressible as a product of prime factors. In other words, the elements of T_1 are "ideal numbers" whose adjunction to R restores unique factorization of elements of R.

By an imbedding process somewhat similar to that just described, it is possible to imbed the ring of rational integers in the ring of p-adic integers. This is carried out in detail in Krull [16] where reference is made to Prüfer [24]. More recently, the same general idea has been used in various connections—in particular, in certain parts of the class field theory.

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SMITH COLLEGE

FIRST SYMPOSIUM IN APPLIED MATHEMATICS

The first annual Symposium in Applied Mathematics of the American Mathematical Society was held at Brown University, Providence, Rhode Island, Saturday to Monday, August 2–4, 1947. The subject of the Symposium was *Non-linear problems in mechanics of continua.* The following 265 persons, including 143 members of the Society, registered:

C. R. Adams, R. B. Adams, E. J. Akutowicz, H. J. Ansoff, R. C. Archibald, D. L. Arenberg, G. M. Arsove, M. G. Arsove, J. L. Barnes, I. L. Battin, Louis Bauer, Leonard Becker, Reinier Beeuwkes, J. C. Bell, A. I. Bellin, Richard Bellman, A. A. Bennett, Stefan Bergman, Lipman Bers, Garrett Birkhoff, Gertrude Blanch, B. C. Blanco, W. E. Bleick, A. D. Booth, L. R. Boyle, F. R. Brand, P. J. Bray, R. C. Briant, S. R. Brinkley, K. H. Britten, J. H. Bronzo, Felix Buckens, A. S. Campbell, R. M. Canning, E. W. Cannon, M. E. Carlen, G. F. Carrier, K. V. Casey, T. E. Caywood, A. I. Chalfant, H. A. Cheilek, P. P. Chen, Herman Chernoff, Peter Chiarulli, Charles Chin, W. S. Clement, R. F. Clippinger, Nathaniel Coburn, H. G. Cohen, Charles Concordia, D. P. Congdon, C. H. Coogan, E. A. Cook, John Crawford, J. E. Crawford, L. L. Cronvich, C. W. Crook, W. J. R. Crosby, J. M. Crowley, C. R. DePrima, J. B. Díaz, G. H. Dixon, C. L. Dolph, M. F. Dowell, W. G. Driscoll, D. C. Drucker, R. P. Eddy, Franz Edelman, Nat Edmonson, F. E. Ehlers, H. W. Emmons, O. C. Enikeleff, H. J. Ettlinger, Herbert Federer, Morton Finston, F. G. Fisher, J. R. Foote, R. M. Foster, F. H. Fowler, A. H. Fox, J. C. Freeman, Bernard Friedman, K. O. Friedrichs, D. L. Fuller, C. A. Garabedian, P. R. Garabedian, B. E. Gatewood, Abe Gelbart, J. H. Giese, Joseph Gillis, R. E. Gilman, Sydney Goldstein, Rosa Goldstein, H. H. Goode, H. J. Greenberg, H. R. J. Grosch, A. R. Gruber, Harold Gumbel, Shirley Gumbel, N. S. Hall, G. H. Handelman, Duncan Harkin, M. G. Harrison, G. K. Hartman, G. E. Hay, L. B. Hedge, G. S. Heller, Joseph Hilsenrath, D. L. Holl, T. R. Hollcroft, D. F. Hornig, P. T. Hsu, E. M. Hull, L. C. Hutchinson, J. S. Isenberg, Fritz John, B. G. Johnston, Robert Kahal, E. L. Kaplan, Wilfred Kaplan, S. N. Karp, Stanley Katz, Sara Kaufman, C. E. Kerr, W. M. Kincaid, Bertram Klein, Knud-Endre Knudsen, B. O. Koopman, G. L. Kreezer, J. A. Krumhansl, M. Z. Krzywoblocki, C. F. Ksanda, Gustav Kuerti, Y. H. Kuo, S. H. Lachenbruch, O. E. Lancaster, H. G. Landau, Philip LeCorbeiller, Patrick Leehey, Lester Lees, Norman Levinson, Samuel Levy, Henry Lew, J. A. Lewis, P. A. Libby, C. C. Lin, R. B. Lindsay, D. P. Ling, J. N. B. Livingood, Charles Loewner, A. N. Lowan, R. D. Luce, S. W. McCuskey, S. S. McNeary, J. K. L. MacDonald, D. P. MacDougall, C. E. Mack, Paul Maeder, L. E. Malvern, J. H. Marchant, W. T. Martin, R. C. Meacham, George Melrose, Irving Michelson, C. C. Miesse, R. B. Montgomery, Melvin Mooney, Morris Morduchow, H. H. Mostafa, Wolfe Mostow, M. M. Munk, F. D. Murnaghan, P. F. Nemenyi, E. N. Nilson, Lawrence Nilson, Yves Nubar, Lars Onsager, Isaac Opatowski, Elias Orshanson, C. J. Osborn, W. H. Pell, P. M. Pepper, A. D. Perry, F. V. Pohle, Harry Polachek, Hillel Poritsky, William Prager, R. C. Prim, M. H. Protter, Gordon Raisbeck, Mina Rees, Eric Reissner, H. J. Reissner, J. A. Rice, J. M. Richardson, R. G. D. Richardson, J. K. Riess, R. F. Rinehart, Jacques Rivaud, R. J. Rivlin, R. E. Roberson, R. C. Roberts, H. P. Robertson, S. I. Rubinow, B. T. Ruley, J. P. Russell, Edward Saibel, Charles Saltzer,

J. A. Saur, S. A. Schaaf, A. B. Schacknow, M. M. Schiffer, G. V. Schliestett, R. W. Schmied, R. N. Schwartz, W. R. Sears, C. H. W. Sedgewick, R. J. Seeger, Hyman Serbin, D. M. Seward, F. S. Shaw, S. F. Shen, W. L. Sherman, Marlow Sholander, S. S. Shu, D. N. Silver, David Singer, D. G. Skillman, C. V. L. Smith, Harold E. Smith, D. E. Spencer, V. E. Spencer, F. D. Stockton, J. J. Stoker, Henry Stommel, E. G. Straus, E. W. Suppiger, P. S. Symonds, J. L. Synge, A. H. Taub, Edward Temple, Feodor Theilheimer, G. B. Thomas, A. N. Tifford, C. B. Tompkins, Deonisie Trifan, Rohn Truell, C. A. Truesdell, J. W. Tukey, M. P. Tulin, Judith Ullman, Ernst Weber, J. V. Wehausen, Herschel Weil, Alexander Weinstein, E. T. Welmers, D. W. Western, F. J. Weyl, Alice Winzer, H. A. Wood, Dorothy Wrinch, A. W. Wundheiler, C. S. Yih, A. T. Yu, P. W. Zettler-Seidel.

The Symposium was cosponsored by the American Institute of Physics, the American Society of Mechanical Engineers, and the Institute of Aeronautical Sciences.

The dormitories of Brown University and the Faunce House Cafeteria and Lounges were available to those attending the Symposium.

An address of welcome was given at 10:00 A.M. Saturday at the opening session of the Symposium by Dean S. T. Arnold of Brown University.

Professor J. L. Synge, Chairman of the Committee on Applied Mathematics, presided at the opening session. Following the address of welcome, Professor Synge gave a brief review of the plans for Symposia in Applied Mathematics of which this was the first.

The ladies attending were conducted through Pendleton House on Saturday afternoon. Tea was served by the ladies of the Departments of Mathematics of Brown University in Pendleton House on Saturday afternoon and in John Carter Brown Library on Sunday afternoon. At 4:15 P.M. Monday, there was an excursion to Newport with dinner at the Viking Hotel.

A photograph of those attending the symposium was taken Saturday on the steps of the John Carter Brown Library.

The four principal addresses given by invitation of the Committee on Applied Mathematics were as follows: at 10:00 A.M. Saturday, *The foundations of the theory of elasticity* by Professor F. D. Murnaghan of Johns Hopkins University; at 10:00 A.M. Sunday, *Surface waves in shallow water* by Professor J. J. Stoker of New York University; at 10:00 A.M. Monday, *The edge effect in bending and buckling with large deflections* by Professor K. O. Friedrichs of New York University; at 2:20 P.M. Monday, *Non-linear problems in the theory of fluid motion with free boundaries* by Professor Alexander Weinstein of Carnegie Institute of Technology.

The Symposium was divided into three morning and three afternoon sessions. At four of these sessions, the principal addresses were given, each followed by a discussion period. In addition, twenty-one twenty-minute papers were given. A group (usually three) of these papers in a certain field was followed by a half-hour discussion period.

The twenty-one papers were presented in the following order:

1. Samuel Levy: *Large deflection theory for rectangular plates.*

2. Eric Reissner: *Remarks on finite deflections of elastic plates.*

3. G. F. Carrier: *On dynamic structural stability.*

4. Hillel Poritsky: *An approximate method of integrating the hodograph equations of a compressible fluid.*

5. Lipman Bers: *Some problems in two dimensional gas dynamics.*

6. Stefan Bergman: *Operator methods in the theory of compressible fluids.*

7. C. L. Dolph and Nathaniel Coburn: *The method of characteristics in three-dimensional steady supersonic flow of compressible fluids.*

8. Isaac Opatowski: *Stream lines of compressible and rotational flows.*

9. S. R. Brinkley: *Theory of the propagation of shock waves from cylindrical charges of explosive.*

10. Harry Polachek and R. J. Seeger: *On shock wave phenomena.*

11. A. H. Taub: *Some aspects of the interaction of shock waves in gases.*

12. Sydney Goldstein: *Singularities in the solution of the boundary layer equations at separation.*

13. Lester Lees: *Stability of the laminar boundary layer in a compressible fluid.*

14. W. R. Sears: *Some three-dimensional boundary layer flows.*

15. C. C. Lin: *Remarks on the spectrum of turbulence.*

16. H. W. Emmons: *A numerical treatment of the turbulence problem.*

17. Y. H. Kuo: *On the stability of transsonic flow.*

18. Wilfred Kaplan: *Numerical methods in the solution of problems of non-linear elasticity.*

19. D. C. Drucker: *Stress-strain relations for strain-hardening materials.*

20. William Prager: *Discontinuous solutions in the theory of plasticity.*

21. Garrett Birkhoff: *Recent applications of free boundary theory.*

The two following papers were on the program but were not presented due to the absence of the authors:

Richard Courant: *Methods of solving non-linear hyperbolic partial differential equations.*

Arpad Nadai: *Some remarks concerning the theory of the formation of surfaces of slip in plastic bodies.*

The presiding officers at the sessions were, respectively: Professors

J. L. Synge, Ernst Weber, Sydney Goldstein, H. P. Robertson, W. T. Martin, S. W. McCuskey, D. L. Holl and J. W. Tukey.

At the close of the Monday afternoon session, a resolution of thanks and appreciation to Brown University for its hospitality was read by Professor H. J. Ettlinger and unanimously approved.

The addresses and papers presented at the Symposium are to be published by the Society in a volume of the Proceedings of the First Symposium in Applied Mathematics.

WILLIAM PRAGER,
Secretary of the Committee on Applied Mathematics

T. R. HOLLCROFT,
Associate Secretary of the Society

HANS FREDERIK BLICHFELDT

1873–1945

Beginning with his doctor's thesis in 1898 under Sophus Lie at Leipzig, *On a certain class of groups of transformation in space of three dimensions* (Amer. J. Math. vol. 22 (1900) pp. 113–120), Blichfeldt was for forty years one of America's leading research men in groups and theory of numbers.

Born in the village Illar, Denmark, January 9, 1873, he was a tall strong youth at 15 when he came to America. For six years he did manual labor, once as a section hand on a railroad, "I worked with my hands doing everything, East and West the country across." But he saved enough money to carry him through two years at Stanford University, where he received his B.A. in 1896. He was appointed as instructor in the department of mathematics at that university in 1897 and subsequently he was Assistant Professor from 1901 to 1906, Associate Professor from 1906 to 1913, Professor from 1913 to 1938, and Professor Emeritus from 1938 to 1945.

He lectured as visiting professor at the University of Chicago, summer of 1911, and at Columbia University, summers of 1924, 1925. He received numerous calls to high permanent positions in leading universities, but remained at Stanford where he was head of his department from 1927 until his retirement at 65 in 1938. He was elected to membership in the National Academy of Sciences in 1920.

He made several trips to Europe in order to maintain close scientific relations with the mathematical life of that continent. He was a faithful friend to a number of outstanding mathematicians, to mention here only such names as Frobenius, Landau, Schur, and Harald Bohr. He represented the National Academy of Sciences at the International Mathematical Congress at Zurich in 1932 and the United States Government and the American Mathematical Society at the International Mathematical Congress at Oslo in 1936. All his life he kept a very intensive interest in his native land, visited that country several times, and in 1938 the King of Denmark made him a Knight of the Order of Dannebrog. He played a leading role in Danish affairs in United States. He died on November 16, 1945, at the age of 72 from a heart attack following an operation.

Professor Harold M. Bacon rendered me effective help in compiling the following list of publications and also in obtaining some of the biographical data.

PUBLICATIONS OF H. F. BLICHFELDT

Triangles with rational sides having rational areas, Ann. of Math. vol. 7 (1896) pp. 57–60.

On a certain class of groups of transformation in space of three dimensions, Amer. J. Math. vol. 22 (1900) pp. 113–120.

A new determination of the primitive continuous groups in two variables, Trans. Amer. Math. Soc. vol. 2 (1901) pp. 249–258.

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L. E. DICKSON

BOOK REVIEWS

The theory of Lie groups, I. By Claude Chevalley. (Princeton Mathematical Series, no. 8.) Princeton University Press, 1946. 9+217 pp. $3.00.

In this masterpiece of concise exposition, the concept of Lie group is put together with all the craftsmanship of an expert. The finished product is a fascinating thing to contemplate, equipped as it is with three inter-related structures—algebraic, topological and analytic. It has no loose ends, no doubtful regions: it can be explored freely without the usual necessity of having to stay within a safe distance of the identity.

The reviewer was particularly struck that so much has been accomplished in a volume of such modest size. One of the things which makes this possible is the effectiveness of the definitions—invariably they emphasize the property which can be most quickly put to work and which is most suitable to the logic of the situation. This insistence on calling things by their right names can be disconcerting at times since it tends to ignore intuitive meanings. A tangent vector at the point p on an analytic manifold, for example, is defined as a certain type of mapping of the family of functions which are analytic at p. But in the end, this book should be easier for most readers, and far more satisfying, than any exposition of the same material proceeding on a less exact level; it should be a rugged base for volumes II, III, $\cdots$ which are to follow.

The book begins with a brief account of the orthogonal, unitary and symplectic groups of matrices and the generation of these groups by their infinitesimal elements. Then comes the important chapter on topological groups. This includes a remarkably sophisticated treatment of the covering concept. A covering of the space $\mathfrak{V}$ is a pair $(\tilde{\mathfrak{V}}, f)$ where f is a mapping $\tilde{\mathfrak{V}} \rightarrow \mathfrak{V}$ satisfying certain conditions of evenness. The coverings of topological groups are themselves converted into topological groups in a natural manner. A connected, locally connected space is called simply connected if it has no coverings other than itself (with f the identity mapping). If $\mathfrak{V}$ admits a covering $(\tilde{\mathfrak{V}}, f)$ in which $\tilde{\mathfrak{V}}$ is simply connected, it admits only one and therefore the group of automorphisms of $\tilde{\mathfrak{V}}$ which preserve f is determined only by $\mathfrak{V}$. This, by definition, is the Poincaré group of $\mathfrak{V}$. An existence theorem shows that a connected, locally connected space in which every point has a simply connected neighborhood has a simply connected covering. The author formulates and proves for simply connected spaces a "principle of monodromy" which he regards as the basic theorem for such spaces. A neat application yields at once the theorem that every local isomorphism of a simply connected group can be extended to a full isomorphism. As one might expect in this lofty treatment, it requires a certain amount of argument to show that a linear interval, for example, is simply connected. On the other hand, it is interesting to follow the development of these ideas without encountering anything as banal as a path or a homotopy. (It must be admitted, however, that paths are still very useful in finding one's way about in an unfamiliar space.)

Analyticity now enters the picture. It is perfectly possible to define an analytic group as a topological group such that some neighborhood of the identity can be covered with an n-dimensional coordinate system in terms of which the functions of group composition are analytic. This coordinate system, and those which are analytically related to it, can be transferred to all parts of the group by group translations and in this way the group becomes an analytic manifold in the sense of Whitney. The author prefers, for reasons of completeness and artistry, to develop the concept of analytic manifold *a priori* without the fortuitous aid of group properties. The definition is based on local families of analytic functions rather than local coordinates. An analytic group is now defined as a topological group whose underlying space is identified with an analytic manifold in such a way that the mapping $\mathfrak{B}\times\mathfrak{B}\rightarrow\mathfrak{B}$ given by $(\sigma, \eta)\rightarrow\sigma\eta^{-1}$ is everywhere analytic. A Lie group is a locally connected topological group $\mathfrak{G}$ in which the connected component which contains the identity is the underlying topological group of an analytic group. This analytic group is uniquely determined by the algebraic-topological structure of $\mathfrak{G}$, a remarkable fact which the author asserts does not hold when complex coordinate systems are allowed.

The chapter on analytic manifolds contains a careful treatment of the integral varieties of involutorial families of vector fields—that is, vector fields which are closed under the "bracket operation." In the case of an analytic group $\mathfrak{G}$, the vector fields which are important are those which are left-invariant under $\mathfrak{G}$; they form an involutorial system which is called the Lie algebra of $\mathfrak{G}$. The fundamental theorems of Lie show how a given group is related to its Lie algebra. This material is here developed and expanded in a manner wholly unhampered by tradition.

There follows a chapter on the Cartan calculus of differential forms. In an analytic group, the forms which are of especial interest are those which are left-invariant. In particular, the left-invariant

Pfaffian forms—called the forms of Maurer-Cartan—form a system which is dual to the left-invariant vector fields. The equations of Maurer-Cartan involve these forms and are a sort of dual to the equations of the so-called first fundamental theorem of Lie. This chapter on forms closes with the theory of invariant integration.

The last chapter deals with the representation theory for compact Lie groups and seems to the reviewer to be the most interesting from the scientific point of view. Let $\mathfrak{G}$ be a compact Lie group and let $\mathfrak{R} = \{P\mathfrak{B}\}$ be the totality of representations P of $\mathfrak{G}$ over the field C of complex numbers. $\mathfrak{R}$ has an algebraic structure of its own since it admits the operations $P_1 \dotplus P_2$ (summation), $P_1 \times P_2$ (Kronecker multiplication), $\gamma P \gamma^{-1}$ (equivalence), $\overline{P}$ (passage to the complex conjugate). The coefficients of any representation are continuous complex-valued functions over $\mathfrak{G}$ and the totality of finite linear combinations of all coefficients of all representations is a ring $\mathfrak{o}$, called the representative ring of $\mathfrak{G}$. The two main objectives are (1) to show how $\mathfrak{G}$ can be reconstructed from $\mathfrak{R}$ and (2) to construct with the aid of $\mathfrak{o}$ an algebraic group in which $\mathfrak{G}$ can be isomorphically imbedded. We describe briefly the steps by which the author arrives at these goals simultaneously. A representation of $\mathfrak{R}$ is a function which associates to each P a matrix $\zeta(P)$ of the same size as P and this in such a way that sums, Kronecker products and equivalences are preserved. Matrix multiplication converts the totality of representations of $\mathfrak{R}$ into a group. Now every ring-homomorphism ω of $\mathfrak{o}$ into C defines a representation $\zeta(P)$ of $\mathfrak{R}$ (by assigning a complex number to each coefficient of each P, hence a matrix to each P) and conversely, every representation of $\mathfrak{R}$ comes from a homomorphism ω. Thus the totality $\{\omega\}$ is in one-one correspondence with the representations of $\mathfrak{R}$. This totality $\{\omega\}$ of homomorphisms is called the algebraic manifold $M(\mathfrak{G})$ associated with $\mathfrak{G}$. This may puzzle those who are not familiar with current notions in algebraic geometry. But the reason is simple. There exists at least one representation P of $\mathfrak{G}$ such that its d^2 coefficients $f_{ij}(\sigma)$ generate $\mathfrak{o}$. The f's will in general satisfy a number of algebraic equations and the complex numbers to which they correspond under a homomorphism ω must of course satisfy the same equations. Hence these numbers define a point on an algebraic variety in d^2-dimensional space and this variety is a "model" for $M(\mathfrak{G})$.—The correspondence which exists between $M(\mathfrak{G})$ and the representations of $\mathfrak{R}$ turns $M(\mathfrak{G})$ into a group, "the associated algebraic group" of $\mathfrak{G}$. Consider now the special homomorphisms ω in $M(\mathfrak{G})$ which preserve the passage to conjugate imaginaries. It is shown that there exists a natural isomorphism between this subgroup of $M(\mathfrak{G})$ and the original group $\mathfrak{G}$.

Hence $\mathfrak{G}$ can be thought of as imbedded isomorphically in its associated algebraic group $M(\mathfrak{G})$. The representations of $\mathfrak{R}$ to which these special ω's correspond are precisely those which preserve the passage to conjugate imaginaries (that is $\zeta(\overline{P})=\overline{\zeta(P)}$). This subgroup of the representations of $\mathfrak{R}$ is therefore in one-one correspondence with $\mathfrak{G}$. This is the duality theorem of Tannaka by which we regain $\mathfrak{G}$ from $\mathfrak{R}$. The usual theorems about reducibility, orthogonality and the approximation of continuous functions on $\mathfrak{G}$ are needed in the foregoing development; they are established with characteristic efficiency.

The book is dedicated, appropriately, to Elie Cartan and Hermann Weyl.

P. A. SMITH

Lectures on differential equations. By Solomon Lefschetz. (Annals of Mathematics Studies, no. 14.) Princeton University Press; London, Humphrey Milford, Oxford University Press, 1946. 8+210 pp. $3.00.

This book is a welcome addition to the literature of differential equations in the real domain, for in it one finds certain basic parts of the theory treated in a refreshingly modern manner. The principal topics considered include the fundamental existence and continuity theorems, critical points, periodic solutions, and the stability of solutions. Poincaré's geometrical theory of the qualitative properties of the general solution of an equation of the first order is developed at considerable length. One chapter is devoted particularly to systems of linear differential equations. It is the author's intention to furnish the necessary background for the modern work on the theory of nonlinear dynamical systems, and I believe that the judicious selection of the material, together with the thoroughness of the treatment, will cause the book to serve this purpose admirably. Some of the simpler physical applications are discussed briefly in the final chapter.

The consistent use of the terminology and properties of vector spaces, matrices, and matrix differential equations enables the author to handle quite general situations without any unduly complicated symbolism. The relevant parts of matrix theory are carefully explained in the first chapter; and the subsequent use of these notions results, for the most part, in a very perspicuous treatment of the subject matter. In a few places, mostly in the fourth and fifth chapters, the highly condensed notation, combined perhaps with some typographical errors, makes for rather difficult reading. (The book is lithoprinted from a typewritten manuscript and, as is usual in such cases, there are a goodly number of typographical errors. Most of these, however, are quite trivial, and will cause no difficulty.)

In order to secure a high degree of precision in the statments of the theorems and in the proofs, the author makes much use of concepts and theorems belonging to general topology. Although some brief explanations of these concepts and theorems are included in the first chapter, I believe that most readers will find this explanatory material insufficient, and will be forced to make frequent reference to other books. This task would have been lightened considerably if a greater number of precise references had been given to places where the necessary topological theory is to be found.

Aside from these matters of exposition, my chief criticism concerns the discussion of linear differential equations with periodic coefficients. The treatment, which contains no illustrative material, seems to be too brief, condensed, and general to give an adequate idea of the difficult problems which are presented by these important equations. Thus, although the entire discussion centers around the characteristic exponents, nothing whatever is said about the problem of calculating these numbers effectively. At the very least, one would have expected to see the general theory illustrated by some discussion of the familiar Mathieu equation.

Although some parts of the contents might have been dealt with advantageously in a more ample and leisurely fashion, the book remains an interesting and valuable exposition of a part of differential equation theory which has been too much neglected in American and British works. It will be exceedingly useful to people working on the theory of nonlinear dynamical systems; and it should do much toward attracting mathematicians to a fascinating field, where many further advances are urgently needed.

L. A. MacColl

Sur les bases du group symétrique et les couples de substitutions qui engendrent un groupe régulier. By Sophie Piccard. (Mémoires de l'Université de Neuchâtel, vol. 19.) Paris, Vuibert, 1946. 223 pp.

This book is divided into two main parts. The first is a collection of the author's previously published results on pairs of substitutions generating the symmetric and alternating groups. It has appeared in the Polish, French, and German journals during the years 1938–1942 (see Mathematical Reviews vol. 1 (1940) p. 161, vol. 4 (1943) p. 1, vol. 7 (1946) p. 410 and also Zentralblatt für Mathematik und ihre Grenzgebiete vols. 19, 21, 22). Besides the paper of Hoyer (Math. Ann. (1895)) referred to in her bibliography, the only other work having any close bearing on that of the author, which the reviewer was able to find, is a paper by Hadwiger (Tôhoku Math. J. vol. 49

(1942) pp. 87–89). The second part of the book is entirely new work.

Two permutations S, T of the set N of symbols $(1, 2, \cdots n)$ are said to be *connected* if for every separation of N into two parts, there is either a cycle of S or of T containing symbols from both parts. S, T are *primitive* if the complex S, T is primitive, which is equivalent to saying that the group (S, T) (group generated by S and T) is primitive in the usual sense. If (S, T) is the symmetric group $\mathfrak{S}_n$ on n symbols, or the alternating group $\mathfrak{A}_n$ on n symbols, then it is shown that S, T are connected and primitive. The first 39 propositions give explicitly pairs of elements generating $\mathfrak{S}_n$ and $\mathfrak{A}_n$. Designate the substitution $(12 \cdots n)$ by S_1, $(12 \cdots m)\ (m+1 \cdots n)$ by S_2, $(12 \cdots m_1)(m_1+1 \cdots m_2)(m_{k-1}+1 \cdots m_k)$ by S_k and cycles of order i by C_i. Then, given S_1, conditions are found on C_2, C_3, C_4, C_5, C_r,

$$C_2 \cdot C_3,\ C_2 \cdot C_5,\ C_2 \cdot C_{m_1} \cdots C_{m_r},\ C_3 \cdot C_4,\ C_3 \cdot C_5,\ C_3 \cdot C_4 \cdot C_5,\ C_2 \cdot C_3 \cdot C_5$$

so that S_1 and one of these latter substitutions generate $\mathfrak{S}_n$ or $\mathfrak{A}_n$. The parity of n frequently determines whether it is $\mathfrak{S}_n$ or $\mathfrak{A}_n$ that is generated. Given S_2, similar conditions are found on C_3, C_4, C_5 so that S_2 and one of them generate $\mathfrak{S}_n$ or $\mathfrak{A}_n$. This is also done with S_3 and C_3, C_4, C_5; with S_4 and C_4; and with S_5 and C_5. Using five of these theorems, the author proceeds to show that for each S there is a T such that S, T generate $\mathfrak{S}_n$ (similarly for $\mathfrak{A}_n$), except in the case of $\mathfrak{S}_4$, where the double transpositions are contained in the ϕ group and so cannot form a part of any basis of two elements. It is also shown for any even (odd) n that for any circular permutation on n letters there exists another circular permutation such that the two generate $\mathfrak{S}_n$ $(\mathfrak{A}_n)$.

In §V of Part I the possible bases (S, T) of $\mathfrak{S}_n$ and $\mathfrak{A}_n$ are enumerated and found to be multiples of $n!/2$ and $n!/4$ in number, respectively. In §VI a complete listing of all pairs of substitutions generating $\mathfrak{S}_n$ and $\mathfrak{A}_n$ is given for $n = 3, 4, 5, 6$.

The second part of the book is an original study of the author. The problem proposed is to find conditions on two substitutions which will make the permutation group generated by them regular. A more concise definition of *connectedness* is given in this part. It is shown that if S, T generate a regular group, then they must both be regular of the same degree on the same symbols, and also connected.

The whole work rests on the important concept of "property p_r."

Definition. Let

$$T = \begin{pmatrix} a_1 & a_2 \cdots & a_n \\ b_1 & b_2 \cdots & b_n \end{pmatrix}$$

be regular of degree n, and let S, regular and of the same degree, be written as a product of cycles. Then TST^{-1} can be obtained by replacing a_i by b_i in S. (Multiplication is on the right.) We shall say that T transforms a_i into b_i in S. If T transforms the elements of each cycle of S into elements of r other cycles of S, $1 \leqq r \leqq n$, then we shall say that T has property p_r with respect to S (in symbols Tp_rS).

If Tp_rS, then T transforms S^r into a power of itself, and r is a divisor of the order of S. The proof of this is involved and could be shortened (as in some other places in the book) by more direct appeal to abstract principles. If (S, T) is regular, then Tp_aS and Sp_bT for some numbers a, b. After the manner in which T transforms the elements of S has been particularized further, this section is closed with the exhibition of a pair S, T which generate a regular group of order $2mr$ such that Tp_rS, r a divisor of m.

Let Tp_1S, S regular. Necessary and sufficient conditions are found for T so that (S, T) is regular. Similar conditions are found so that (S, T) is regular and also abelian.

Let Tp_2S, S regular. Necessary and sufficient conditions are found for T, provided T is already of a certain form, so that (S, T) is regular. When S is of order m, T of order k, both S and T of degree mk, then sufficient conditions of a complicated number congruence nature are found for (S, T) to be regular.

Let Tp_mS, S regular of order mt, T of order 2, and both S and T of degree mkt. The cases for $m=2, 3, 4, 5$ are analyzed when either $TST=S^{m-1}TS^{m-1}$ or $TST=S^{-1}TS^{m-1}$. It is shown that (S, T) need not be regular under these circumstances when $m=6$.

Let Tp_mS, S regular of order m and degree m^2+m. If $T^2=1$, (S, T) is regular if and only if for every l there exist a pair of numbers (a, b), distinct for every l, and such that

$$\text{(A)} \qquad TS^lT = S^aTS^b.$$

Illustrations are given when $m=3, 4, 5, 6, 7, 8, 12$, and 16. The relations (A) are usually dependent, but unfortunately the nature of this dependence is not analyzed. If T is of order greater than 2, (S, T) is regular if and only if $T^2=S^iTS^j$, s^i, $s^j \neq 1$, and either $TS^lT=S^a$ or $TS^lT=S^aTS^b$ for every l, a and b functions of l.

Let Tp_mS, S regular of order m and of degree $mk \geqq m^2+m$. Let $T^2=1$. (S, T) is regular if and only if, when T transforms the elements of S in a certain way, identities of the form $\prod_i^k(TS^{\alpha_i}) = \prod_i^k(S^{\beta_i}T)$ are satisfied.

The following misprints were noted: p. 86, line 33, "proposition 44" should read "proposition 43"; p. 116, line 6, $(\mu/\ )$ should read $(\mu/2)$; p. 123, line 12, "$(jm+km'+i(m'+j-1)m+Km'+j+1)$" should read "$(jm+km'+i\ (m'+j-1)m+km'+j+1)$." There are too many substitutions given with the spacing not clear to enumerate them all. We content ourselves with a general warning on this point.

A. W. JONES

An index of mathematical tables. By A. Fletcher, J. C. P. Miller, and L. Rosenhead. New York, McGraw-Hill; London, Scientific Computing Service, 1946. 8+451 pp. $16.00.

The compilation of this index of tables has been achieved by a tremendous outlay of painstaking work over a period of many years. Many tables of value have appeared in scientific and engineering articles and bringing these to light constitutes by itself a most valuable endeavor. The material is listed according to functions, there being twenty-four sections of which two, for example, are logarithms of trigonometrical functions and Gamma function psi function, Polygamma function, Beta function, Incomplete gamma and Beta functions. There are also sections devoted to primes, binomial coefficients, Bernouilli numbers, mathematical constants, and so on.

The authors have indicated with each table the number of decimals, the range of the argument and the intervals of tabulation, the facilities for interpolation (that is, whether first or second differences are given), and the authorship with date. For example, under §21.31—Incomplete elliptic integrals of the first and second kinds—we find that $F(\theta, \phi)$ and $E(\theta, \phi)$ were tabulated as follows:

12dec. $\theta = 0(1^\circ)90^\circ$ $\phi = 45^\circ$ Δ^5 or Δ^6 Legendre 1816, 1826.

Similarly, under Bernouilli numbers the entries for exact values for B_n begin with: $n=1(1)5$ Bernouilli (1713) and continue with some 30 other tables.

There is a 70 page bibliography and an introduction in which the authors describe in detail the interpretation to be given to their notations and remarks.

E. R. LORCH

Tables of the Bessel functions of the first kind of orders zero, one, two, and three. (The Annals of the Computation Laboratory of Harvard University, vols. 3–4.) Cambridge, Harvard University Press; London, Oxford University Press, 1947. $20.00.

With the help of the automatic sequence controlled calculator, the

Computation Laboratory of Harvard University has prepared these tables of certain Bessel functions. The functions here given are $J_n(x)$, $n=0$, 1, 2, and 3. It is hoped to publish, during the coming years, tables for integral values of n ranging up to 100. In the present volumes the functions are given to 18 figures. For the range $0<x<25$, tabulation was carried out with the interval of the argument equal to 0.001. For $25<x<100$ the argument interval is 0.01. The computer originally gave the results to 23 places which were then rounded off to 18 for publication. Various checks such as differencing devices and comparison with older tables suggest that the probability of correctness of these values to 3 units in the 23rd place is very high.

E. R. LORCH

Table des solutions de la congruence $x^4+1\equiv 0$ (mod. p) *pour* $350.000<p<500.000$. By A. Gloden. Luxembourg, 1946. 40 pp. 50 fr.

The congruence $x^4+1\equiv 0(p)$ has solutions only for primes of the form $p=8k+1$. For such primes one solves the two congruences $l^2+1\equiv 0(p)$ and $2m^2+1\equiv 0(p)$ and then the four roots of the original congruence are $m(l\pm 1)$, $-m(l\pm 1)$. The author tabulates here the two smaller of these roots for primes ranging from 350,000 to 500,000. For $p<350{,}000$ the results have been compiled by other computers. One of the applications of these tables is to the factorization of numbers of the form x^4+1.

E. R. LORCH

Tables of spherical Bessel functions. Vol. 1. Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1947. 28+379 pp. $7.50.

The Mathematical Tables Project presents here tables of the Bessel functions of order $n+1/2$ where n is an integer. More precisely, it is the functions $(\pi/2x)^{1/2}J_{n+1/2}(x)$ which are tabulated. These functions arise in the separation of the wave equation for certain types of coordinate systems. Strangely enough, no adequate tables had been available previous to this publication. We have here 28 tables corresponding to $n+1/2=\pm 1/2, \pm 3/2, \cdots, \pm 27/2$. The entries are given to 8, 9, or 10 significant figures for $x\leqq 10$ and to 7 figures for $x>10$. The argument range is $0<x<10$ in intervals of 0.01 and $10<x<25$ in intervals of 0.1. Second central differences are also tabulated except near $x=0$.

E. R. LORCH

The theory of mathematical machines. By F. J. Murray. New York, Columbia University Press, 1947. 8+116 pp. $3.00.

There is a fairly extensive but widely scattered literature on computing instruments of various types, ranging from simple devices such as slide rules to such highly complex ones as the ENIAC. The present interest in all these types of computers makes Professor Murray's book especially apropos. He has written on a level which should make the book of equal interest to the applied mathematician and to the engineer. Due, unfortunately, to the secrecy restrictions of wartime the author was not, at the time he wrote this book, able to mention all the new developments in the field of digital computers, particularly the electronic ones. Subsequent editions will undoubtedly contain additional chapters describing the new instruments and indicating the present trends in the field.

The first part deals with the field of digital machines—subject to the hiatus mentioned above. It gives a careful discussion of how coincidence noting devices, adders and multipliers can be realized by means of mechanical elements such as gears as well as by electromechanical devices such as relays. Thus the reader is given a picture of how the familiar types of desk multipliers and related devices function in principle, without having to follow intricate engineering details. The chapter concludes with a brief discussion of the principles underlying the operation of the familiar punch card machines.

After this brief but reasonably complete discussion of the field of digital computers prior to the war the author turns his attention in Part II to so-called measurement machines. These are devices in which numerical quantities are represented by the magnitudes of physical quantities such as rotations, displacements, voltages, and so on. They are, in contradistinction to digital instruments, usually special purpose in character. Perhaps the most widely known and general machine in this class is the differential analyzer. They are almost invariably remarkably ingenious and represent quite interesting attempts to use known physical or chemical principles to solve mathematical problems. One particularly interesting machine in this category is the one which solves Laplace's equation in a given region subject to boundary conditions by shaping a container with non-conducting walls to be the given region. The container is then filled with an electrolytic solution and the boundary conditions are satisfied approximately by placing conductors on the boundaries. The electric potential function then satisfies the equation subject to the boundary conditions and is measured at any point in the solution by means of a probe.

Several mechanical adders are discussed including the familiar differential using bevel or spur gears, in which both the addend and augend are represented by rotations of shafts connected to gears and the sum is again such a motion. The use of Kirchhoff's laws also permits one to construct ingenious electrical adders, as is shown. The second chapter of Part II is devoted to a detailed discussion of various methods for achieving the product of two quantities by means of measurement devices. Many of these methods are, of course, now well known from their use in fire control computers and directors.

Having described adders and multipliers, the author next proceeds to a discussion of various means, both mechanical and electrical, for integrating and differentiating. This naturally leads to an inquiry into the nature of amplifiers since the power output levels at which the arithmetic or analytic components of a measurement computer function are usually below those needed for their inputs, for example, the familiar types of integrators, such as the wheel-disk types used in differential analyzers, depend for their output on a small, light wheel rotating as a result of frictional contact with a glass disk. The rotations of this wheel must be used to drive very long and massive shafts. It is clear that some amplification device must intervene between the integrator output and the driven shaft. The chapter on amplifiers gives a very readable account of electronic amplifiers as well as of the mechanical torque amplifiers which played so significant a role in making the original differential analyzer possible. (In this connection it might be remarked that a somewhat more elegant mathematical discussion of this device is given in W. D. MacMillan, *Statics and dynamics of a particle*, New York, 1927, p. 86.) The account of electronic amplifiers should be of particular interest to those not well conversant with circuit theory since the author gives a very lucid and complete introduction to the subject. As he suggests, this chapter should be read in conjunction with the Radio Amateur's Handbook. The second part closes with a discussion on how a function can be expressed as a Fourier series by measurement mechanisms. This includes an account of so-called selsyn systems.

Part III is devoted to showing how the components previously described can be assembled into machines for handling mathematical problems. The author gives extremely interesting accounts of linear equations solvers (including a preliminary account of a machine designed by him of which a small model has been built and a larger one is now in the process of being constructed), of machines for solving linear elliptic equations and of differential analyzers. He incorporates the essentials of a paper by Shannon which gives a complete analysis of the problems to which an analyzer is applicable but from a formal point of view. However this formal treatment presupposes an idealized analyzer, that is, one of unlimited precision, containing as many elements as desired. The difficulties with a differential analyzer however lie precisely in those aspects of the mechanism removed by the idealization.

The book closes with an account in Part IV of various instruments such as planimeters, integrometers and harmonic analyzers. This discussion includes a description of the important cinema integraph of Wiener, Hazen and Brown which may be used to evaluate integrals of the form $\int_0^a f(x+y)g(x)dx$. These, of course, play an important role in the Wiener-Kolmogoroff theory of time series—when $g(x)=f(x)$ the integral above is the familiar auto-correlation function.

Professor Murray and the King's Crown Press are to be highly commended for this book which makes a unique contribution to our literature. There now exists in one thoroughly readable volume a careful discussion by a professional mathematician of the many beautiful instruments that have been designed as tools for the applied mathematician. This book can be recommended to all those who desire to become acquainted with this fascinating subject as well as to the specialists in the field.

H. H. GOLDSTINE

NOTES

The following have received Guggenheim fellowship appointments: Professor Warren Ambrose of the Massachusetts Institute of Technology; Professor Garrett Birkhoff of Harvard University; Professor P. R. Halmos of the University of Chicago; Professor Saunders MacLane of the University of Chicago.

The following have received National Research Council fellowship appointments in mathematics: Mr. Hing Tong of Columbia University; Mr. Herman Rubin of the University of Chicago; Dr. Daniel Zelinsky of the University of Chicago.

Dr. Samuel Karlen of Princeton University has been awarded the first Harry Bateman Research Fellowship by California Institute of Technology.

Professor E. J. Cartan of the University of Paris has been elected to membership in the Royal Society.

Professor Emeritus G. H. Hardy of Cambridge University has been elected to membership in the French Academy of Science.

Professor H. A. Kramers of the University of Leyden has been elected correspondant of the French Academy of Science.

Professor E. J. McShane of the University of Virginia has been awarded an honorary doctorate of science by Tulane University of Louisiana.

Professor C. C. MacDuffee of the University of Wisconsin has been awarded an honorary doctorate of science by Colgate University.

Professor J. H. Van Vleck of Harvard University has received an honorary doctorate of science from the University of Wisconsin.

Professor Hermann Weyl of the Institute for Advanced Study and Professor George Pólya of Stanford University have been elected correspondants of the French Academy of Science.

Professor Myrtie Collier of Immaculata Heart College, Los Angeles, California, has retired with the title emeritus.

Professor Bird M. Turner of West Virginia University has retired with the title emeritus.

Professor P. A. M. Dirac of Cambridge University has been appointed visiting professor at the Institute for Advanced Study for the current academic year.

Dr. Y. C. Wong of the University of Pennsylvania has been appointed to a professorship at the National Sun Yat-Sen University, Canton, China.

In the March issue of this Bulletin it was reported in error that

Mr. J. B. Greeley had been appointed to an instructorship at Utica College, Syracuse University. It should have been reported that Mr. Greeley had been appointed to an assistant professorship.

Assistant Professor Florence E. Allen of the University of Wisconsin has retired.

Dr. Harriet W. Allen of Air Reduction Company, Stamford, Connecticut, has been appointed to an associate professorship in physics at Connecticut College.

Assistant Professor Warren Ambrose of the University of Michigan has been appointed to an assistant professorship at Massachusetts Institute of Technology.

Assistant Professor B. H. Arnold of Montana State College has been appointed to an assistant professorship at Oregon State College.

Associate Professor T. A. Bancroft of the University of Georgia has accepted a position as director of the Statistical Laboratory, Alabama Polytechnic Institute.

Assistant Professor M. T. Bird of Allegheny College has been appointed to an assistant professorship at San Jose State College, San Jose, California.

Associate Professor Z. W. Birnbaum of the University of Washington has been appointed director of the Institute of Mathematical Statistics which has been established at the University of Washington.

Professor C. C. Bramble of the Postgraduate School, United States Naval Academy, has accepted a position as director of computation and ballistics at the Naval Proving Ground, Dahlgren, Virginia.

Dr. E. L. Buell of Northwestern University has been appointed mathematician at the Aerial Measurements Laboratory, Northwestern Technical Institute, Evanston, Illinois.

Associate Professor G. P. Burns of Marshall College, Huntington, West Virginia, has accepted a position as research physicist at the Naval Research Laboratory, Washington, D. C.

Mr. Edmund Churchill of Rutgers University has been appointed to an assistant professorship at Antioch College.

Professor A. B. Coble of the University of Illinois has been appointed to a professorship at Haverford College.

Mr. G. R. Costello has accepted a position as mathematician with the National Advisory Committee on Aeronautics, Cleveland, Ohio.

Dr. Norman Davids of Johns Hopkins University has been appointed to an associate professorship at Pennsylvania State College.

Dr. W. W. Denton of Minden City, Michigan, has been appointed to an assistant professorship at the University of Arizona.

Assistant Professor J. B. Díaz of Carnegie Institute of Technology has been appointed to an assistant professorship at Brown University.

Dr. Bernard Dimsdale has accepted a position as mathematician at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Associate Professor T. L. Downs of the United States Naval Academy has been appointed to an associate professorship at Washington University.

Dr. Jacques Dutka of Princeton University has been appointed to an assistant professorship at Rutgers University.

Mr. M. W. Eudey of the University of California has accepted a position as operations analyst with the Army Air Force, Washington, D. C.

Assistant Professor C. J. Everett of the University of Wisconsin is on leave of absence and will be at the Los Alamos Scientific Laboratory.

Mr. Albert Furman of the University of Chicago has been appointed to an assistant professorship at Kansas State College of Agriculture and Mechanical Arts.

Associate Professor M. E. Gillis of the University of Florida has been appointed to a professorship at Blue Mountain College, Blue Mountain, Mississippi.

Dr. M. A. Girshick of the Bureau of the Census has accepted a position as research mathematician with Douglas Aircraft Corporation, Santa Monica, California.

Associate Professor Wallace Givens of the Illinois Institute of Technology has been appointed to a professorship at the University of Tennessee.

Dr. C. H. Graves, formerly of the Office of Price Administration, has accepted a position as operations analyst with the Army Air Forces, Air Defense Command, Mitchell Field, New York.

Associate Professor P. C. Hammer of Oregon State College has accepted a position as mathematician at the Los Alamos Scientific Laboratory.

Mr. I. H. Harris of the University of Illinois has been appointed to a professorship at Oklahoma Baptist University.

Associate Professor M. R. Hestenes of the University of Chicago has been appointed to a professorship at the University of California at Los Angeles.

Mr. W. C. Hoffman of the University of California at Los Angeles has accepted a position as mathematician at the Naval Electronics Laboratory, San Diego, California.

Professor Ralph Hull of the University of Nebraska has accepted a position as mathematician at Boeing Aircraft Company, Seattle, Washington.

Associate Professor E. D. Jenkins of Eastern Kentucky State Teachers College has been appointed to an associate professorship at Kent State University.

Assistant Professor Walter Jennings of Virginia Polytechnic Institute has been appointed to an assistant professorship at the Postgraduate School, United States Naval Academy.

Assistant Professor R. E. Johnson of Mount Holyoke College has been appointed to an associate professorship at Smith College.

Assistant Professor J. L. Kelley of the University of Chicago has been appointed to an associate professorship at the University of California.

Mr. E. H. Larguier of the University of Michigan has been appointed to a professorship at Spring Hill College, Spring Hill, Alabama.

Mr. M. M. Lemme of Purdue University has been appointed to an associate professorship at Illinois Wesleyan University. He will also be assistant dean of admissions.

Associate Professor C. C. Lin of Brown University has been appointed to an associate professorship at Massachusetts Institute of Technology.

Associate Professor H. W. Linscheid of the College of Emporia, Emporia, Kansas, has been appointed to an associate professorship at the Southwestern Institute of Technology, Weatherford, Oklahoma.

Mr. R. T. Luginbuhl of the City Bank Farmers Trust Company, New York City, has been promoted to the position of actuarial assistant.

Professor J. C. C. McKinsey of Oklahoma Agricultural and Mechanical College has accepted a position with Douglas Aircraft Company, Santa Monica, California.

Professor Saunders MacLane of Harvard University has been appointed to a professorship at the University of Chicago. He will be on leave of absence during the current academic year.

Dr. N. S. Mendelsohn of Queens University has been appointed to an assistant professorship at the University of Manitoba.

Dr. W. A. Mersman of Taylor Instrument Company, Rochester, New York, has accepted a position as mathematician with the National Advisory Committee on Aeronautics, Moffett Field, California.

Dr. D. S. Miller of Yale University has been appointed to an assistant professorship at the University of Rochester.

Mr. Leo Moser of the University of Toronto has been appointed lecturer.

Associate Professor Ivan Niven of Purdue University has been appointed to an associate professorship at the University of Oregon.

Mr. Edward Paulson has been appointed lecturer at the University of Washington.

Dr. B. J. Pettis of Yale University has been appointed to an associate professorship at Tulane University of Louisiana.

Professor H. P. Robertson of Princeton University has been appointed to a professorship at California Institute of Technology.

Dr. H. M. Schaerf of Montana State College has been appointed to an assistant professorship at Washington University.

Associate Professor Edith R. Schneckenburger of Michigan State Normal College has been appointed to an assistant professorship at the University of Buffalo.

Dr. H. M. Schwartz of the Franklin Institute has accepted a position as scientist with the Brookhaven National Laboratory, Patchogue, New York.

Assistant Professor G. E. Schweigert of Purdue University has been appointed to an associate professorship at the University of Pennsylvania.

Dr. I. E. Segal of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor Domina E. Spencer of Tufts College has been appointed to an assistant professorship of physics at Brown University.

Associate Professor N. E. Steenrod of the University of Michigan has been appointed to an associate professorship at Princeton University.

Professor M. H. Stone of the University of Chicago has been lecturing at the University of Brazil.

Mr. W. M. Stone of Iowa State College of Agriculture and Mechanical Arts has been appointed to an assistant professorship at Oregon State College.

Mr. F. W. Thalgott of the University of Illinois has accepted a position as mechanical engineer at the Clinton Laboratories, Oak Ridge, Tennessee.

Dr. E. B. Tolsted of Brown University has been appointed to an assistant professorship at Pomona College, Claremont, California.

Assistant Professor H. C. Trimble of Iowa State Teachers College has been appointed to an associate professorship at Florida State College.

Professor G. R. Trott of Blue Mountain College, Blue Mountain, Mississippi, has been appointed to an associate professorship at the University of Mississippi.

Professor André Weil of the University of São Paulo has been appointed to a visiting professorship at the University of Chicago.

Associate Professor F. P. Welch of Mississippi State College has been appointed to a professorship at Washington and Lee University.

Mr. R. L. Wine of the University of Oklahoma has been appointed to an assistant professorship at Washington and Lee University.

Assistant Professor G. S. Young of Purdue University has been appointed to an assistant professorship at the University of Michigan.

Professor Oscar Zariski of the University of Illinois has been appointed to a professorship at Harvard University.

Professor Antoni Zygmund of the University of Pennsylvania has been appointed to a professorship at the University of Chicago.

The following promotions are announced:

D. H. Ballou, Middlebury College, Middlebury, Vermont, to an associate professorship.

J. D. Bankier, McMaster University, to an assistant professorship.

Joseph Barnett, Oklahoma Agricultural and Mechanical College, to an associate professorship.

Grace E. Bates, Mount Holyoke College, to an assistant professorship.

Lipman Bers, Syracuse University, to an associate professorship.

E. E. Betz, United States Naval Academy, to an associate professorship.

F. C. Biesele, University of Utah, to an associate professorship.

J. C. Brixey, University of Oklahoma, to a professorship.

G. W. Brown, Iowa State College of Agriculture and Mechanical Arts, to a professorship.

R. H. Bruck, University of Wisconsin, to an associate professorship.

G. F. Carrier, Brown University, to an associate professorship.

I. S. Carroll, Syracuse University, to a professorship.

J. A. Clarkson, University of Pennsylvania, to an associate professorship.

I. S. Cohen, University of Pennsylvania, to an assistant professorship.

S. H. Crandall, Massachusetts Institute of Technology, to an assistant professorship.

Georgia K. DelFranco, University of Miami, to an assistant professorship.

W. W. Dolan, University of Oklahoma, to an assistant professorship.

H. L. Dorwart, Washington and Jefferson College, to a professorship.

William H. Durfee, Dartmouth College, to an assistant professorship.

Abe Gelbart, Syracuse University, to an associate professorship.

Gladys Gibbens, University of Minnesota, to an associate professorship.

H. J. Greenberg, Brown University, to an assistant professorship.

Bernard Greenspan, Drew University, to an assistant professorship.

Philip Hartman, Johns Hopkins University, to an associate professorship.

May N. Harwood, Syracuse University, to an associate professorship.

G. E. Hay, University of Michigan, to an associate professorship.

J. J. Hayes, University of Utah, to an assistant professorship.

Katherine E. Hazard, New Jersey College for Women, Rutgers University, to an assistant professorship.

Anna S. Henriques, University of Utah, to an associate professorship.

Mark Kac, Cornell University, to a professorship.

Irving Kaplansky, University of Chicago, to an assistant professorship.

William Karush, University of Chicago, to an assistant professorship.

R. J. Levit, University of Georgia, to an associate professorship.

A. J. Lorenz, St. Louis University, to an assistant professorship.

R. E. Luce, Rutgers University, to an assistant professorship.

F. A. McMahon, Manhattan College, to an assistant professorship.

Harriet Montague, University of Buffalo, to a professorship.

A. M. Mood, Iowa State College of Agriculture and Mechanical Arts, to a professorship.

Anne F. O'Neill, Smith College, to an assistant professorship.

Everett Pitcher, Lehigh University, to an associate professorship.

E. J. Purcell, University of Arizona, to a professorship.

Haim Reingold, Illinois Institute of Technology, to an associate professorship.

Helene Reschovsky, Russell Sage College, Troy, New York, to an associate professorship.

A. E. Schild, Carnegie Institute of Technology, to an assistant professorship.

B. M. Stewart, Michigan State College, to an associate professorship.

Walter Strodt, Columbia University, to an assistant professorship.

C. J. Thorne, University of Utah, to an associate professorship.

W. J. Thron, Washington University, to an assistant professorship.

Bradford Tye, Bethany College, to an assistant professorship.

H. J. Zimmerberg, Rutgers University, to an assistant professorship.

H. S. Zuckerman, University of Washington, to an associate professorship.

The following appointments to instructorships are announced: Allegheny College: Mr. E. A. Sturley; Brown University: Dr. A. L. Blakers, Dr. F. M. Stewart; University of California at Los Angeles: Mr. L. J. Paige, Mr. J. D. Swift; University of Chicago: Mr. Edwin Akutowicz, Mr. J. B. Crabtree; Cornell University: Dr. Bryant Tuckerman, Dr. Christine S. Williams, Dr. Bertram Yood; Grinnell College: Mr. R. E. Cross; Harvard University: Mr. W. J. LeVeque, Dr. G. R. MacLane; Haverford College: Mr. D. L. Thomsen; University of Minnesota: Mr. Irwin Stoner; University of Pennsylvania: Mr. D. M. Adelman; Rutgers University: Mr. A. G. Anderson, Mr. A. F. Bartholomay, Dr. Erwin Biser, Dr. R. M. Cohn, Mr. C. D. Firestone, Dr. A. W. Goodman; Syracuse University: Dr. Erik Hemmingsen; University of Tennessee: Mr. R. L. Wilson; Texas Technological College: Mrs. Virginia B. Roberts; Washington University: Dr. C. W. Mathews; Wellesley College: Mrs. Jerry Cowen; Western Illinois State Technological College, Macomb, Illinois: Mr. J. J. Stipanowich; University of Wisconsin: Dr. A. M. Mark; University of Wisconsin at Racine: Mr. M. R. Moore.

Professor Giacomo Albanese of the Normal School of Pisa and the University of São Paulo died June 8, 1947.

Professor Alfred Rosenblatt of the University of San Marcos died

July 8, 1947. He had been a member of the Society since 1914.

Mr. C. C. Carter, Bluffs, Illinois, died May 5, 1947.

Professor Emeritus G. H. Hallett of the University of Pennsylvania died August 13, 1947.

Professor Cora B. Hennel of Indiana University died June 26, 1947. She had been a member of the Society since 1913.

Associate Professor Emeritus B. C. Wong of the University of California died August 19, 1947. He had been a member of the Society since 1922.

ABSTRACTS OF PAPERS

The abstracts below are abstracts of papers presented by title at the Fifty-Third Summer Meeting of the American Mathematical Society. Abstracts of papers presented in person at that meeting will be included in the report of the meeting which will be published in the November issue of this BULLETIN.

Abstracts are numbered serially throughout this volume.

ALGEBRA AND THEORY OF NUMBERS

297. A. A. Albert: *Power-associative rings*. I.

A nonassociative ring is said to be *power-associative* if every quantity of A generates an associative subring. It is proved that if the characteristic of A is zero (and counterexamples are given otherwise) then A is power-associative if and only if $xx^2 = x^2x$, $(x^2x)x = x^2x^2$ for every x of A. If A is commutative, the characteristic need only be prime to 30. If A is any ring in which the equation $2x = a$ has a unique solution x, attach to A a commutative ring $A^{(+)}$ which is the same additive group as A, but whose product $x \cdot y$ is defined in terms of the product xy of A by $2(x \cdot y) = xy + yx$. Similarly attach $A^{(-)}$ defined by the product $(x, y) = xy - yx$. Use the decomposition relative to an idempotent in $A^{(+)}$ to obtain a decomposition relative to an idempotent for any power-associative ring, with more complete results in the case of *flexible* rings, that is, rings such that $x(yx) = (xy)x$. Commutative algebras of shrinkability index two are discussed and a structure theory given for the class of algebras in which $x^2y^2 + (xy)^2 = x(xy^2) + y(yx^2)$. The second part of this paper will be concerned with the structure of those flexible power-associative algebras A such that either $A^{(+)}$ is a semisimple Jordan algebra or $A^{(-)}$ is a semisimple Lie algebra. (Received July 21, 1947.)

298. Richard Bellman and H. N. Shapiro: *On the probability that k integers are relatively prime*.

The authors consider the problem of determining the number of lattice points in the n-dimensional region, $1 \leq x_i \leq m_i$, $i = 1, 2, \cdots, n$, with the property that the greatest common divisor of the n-uple $(x_1, x_2, \cdots, x_n)$ is 1. The m_i are allowed to tend to ∞ independently, and it is shown that, as might be expected on the basis of a simple heuristic argument, the density of these lattice points is $1/\zeta(k)$, where $\zeta(s)$ is the Riemann zeta-function. Various cases of this problem had already been considered by Lehmer and Chowla. The number of lattice points in the above region possessing the property that every 2-uple (x_i, x_j) has greatest common divisor equal to one, $x_i \neq x_j$, is also determined. The method can be extended to yield similar results for spherical, and general classes of symmetrical, regions, but does not seem to be applicable to the general convex region. (Received June 10, 1947.)

299. R. H. Bruck: *An application of loop theory to geometry*. Preliminary report.

If (P) is a property of loops, the loop G will be said to have property (P′) if and only if every loop isotopic to G has property (P). The following properties of multiplicative loop G are considered: (A): $xy=1$ when $yx=1$, where 1 is the unit; (B): $(xx)x=x(xx)$ for every x of G; (C): every element of G generates a group; and it is shown algebraically that the corresponding properties (A′), (B′), (C′) are equivalent. Thus, in geometrical language, a loop has property (C′) if and only if it defines a hexagonal net, or Sechseckgewebe (Blaschke and Bol, *Geometrie der Gewebe*, Berlin, 1938)—where, for present purposes, all continuity assumptions are inappropriate. The result will be applied to simplify and extend some geometrical work of Ruth Moufang (*Zur Struktur der projektiven Geometrie der Ebene*, Math. Ann. vol. 105 (1931) pp. 538–601). (Received July 21, 1947.)

300. Leonard Carlitz and Eckford Cohen: *Cauchy products of divisor functions in $GF[p^n, x]$.*

For given arithmetic functions $\phi(A)$, $\psi(A)$, where $A\in GF[p^n, x]$, the Cauchy product $\zeta=\phi\cdot\psi$ is defined by means of $\sum\phi(A)\psi(B)=\zeta(F)$, where $A+B=F$. Let $r>0$ be fixed. Three cases are considered according as $\deg A=\deg B=r$; $\deg A=r>\deg B$; $\deg A$, $\deg B<r$; these are referred to as C_1, C_2, C_3 respectively. The functions considered in this paper are all of the form $\sum_0^r a_i\delta_i(A)$, where $\delta_i(A)$ denotes the number of divisors of A of degree i. In every case the Cauchy product is also a divisor function. Relative to C_3 the set of divisor functions forms a ring which is a direct sum of fields. Applications are made to the representation of a polynomial in quadratic forms with coefficients in $GF(p^n)$. (Received July 11, 1947.)

301. Eckford Cohen: *Sums of an even number of squares in $GF[p^n, x]$.* II.

In a previous paper (Duke Math. J. vol. 14 (1947)) the author considered the following problem: Let $\alpha_1, \cdots, \alpha_{2s}$ be nonzero elements of $GF(p^n)$, $p\neq 2$, such that $\alpha_1+\cdots+\alpha_l=0$ $(1\leq l\leq 2s)$, and let F be a polynomial of $GF[p^n, x]$ of degree less than $2k$. It was then required to find the number of solutions of $F=\sum_1^{2s}\alpha_i X_i^2$ in polynomials $X_1, \cdots, X_{2s}$ where $X_1, \cdots, X_l$ are primary of degree k and $X_{l+1}, \cdots, X_{2s}$ are arbitrary of degree less than k. The excluded case, namely $l=0$, is treated in the present paper. The result for this case is found to differ considerably from the result obtained in the previous paper. Thus, for $F\neq 0$, the number of solutions is found to be $R_{s-1}(F, \mu)=(p^{ns}-\mu)p^{n(s-1)(2k-1)}\sum_{e=0}^{k-1}\mu^e p^{-ne(s-1)}g_e(F)$, where $g_e(F)=\delta_e(F)-\delta_{2k-e-1}(F)$, and $\delta_i(F)$ is the number of primary divisors of F of degree i. A similar result is obtained for $F=0$. (Received June 1, 1947.)

302. R. P. Dilworth: *An imbedding theorem for semimodular lattices.*

The following theorem is proved: *Let L be a lattice with bounded chains over which an integer-valued semimodular functional ρ is defined. Then L is a sublattice of a semimodular point lattice M with rank function r such that $r(a)=\rho(a)$ for all a in L.* This theorem has as a corollary: *Every finite lattice is a sublattice of a finite semimodular point lattice.* (Received June 27, 1947.)

303. Samuel Eilenberg and Saunders MacLane: *Cohomology and Galois theory.* I. *Symmetry of algebras and Teichmüller's cocycle.*

Teichmüller (Deutsche Mathematik vol. 5 (1940) pp. 138–149) has considered central simple algebras A over a field N which have the property that every automorphism of a finite group Q of automorphisms of N can be extended to an automorphism of A. Each such algebra (termed a "Q-symmetric algebra") determines a 3-dimensional cohomology class of Q with coefficients in the multiplicative group of N. The cohomology classes are characterized as those composed of cocycles which become coboundaries when regarded as cocycles of a larger field $K\supset N$ normal over the field P of elements of N invariant under Q. If G and S are respectively the Galois groups of K over P and N respectively, then any central simple A over N can be represented as a crossed product $A=(f, S, K)$, where f is a 2-cocycle (factor set) of S in K. Such a crossed product is Q-symmetric if and only if there is a function h of two variables in G with values in K which agrees with f on the subgroup S and which has its coboundary δh a function only of arguments in $G/S=Q$. Under these circumstances, δh is the Teichmüller cocycle of A. (Received June 5, 1947.)

304. Saunders MacLane: *Symmetry of algebras over a number field.*

Let K be a finite normal extension of the algebraic number field k, with Galois group Q. A central simple algebra A over K is Q-symmetric if every automorphism of Q can be extended to an automorphism of A. If B is a central simple algebra over the ground field k, its scalar extension B_K is always Q-symmetric in this sense; hence any algebra A similar to a B_K may be termed trivially Q-symmetric. It is shown in this paper that there exist Q-symmetric algebras not trivially Q-symmetric, for suitable fields K. Further, it is proved that the group of Q-symmetric algebra classes, modulo the subgroup of trivially Q-symmetric classes, is cyclic. The order s of this cyclic group is determined as the greatest common divisor of all $S(p)$, where $S(p)$ is the number of factors in K of the prime divisor p of k. The proofs depend on the invariants of an algebra over a number field. (Received June 23, 1947.)

305. B. H. Neumann: *A group connected with ordered semigroups.*

In a fully ordered semigroup in which a product is greater than each factor every well-ordered set of elements (in the given semigroup order) generates a well-ordered semigroup. In this latter every element has only a finite number of factors out of the former. A multiplication is defined between well-ordered sets of elements. With this multiplication they form a group, the null-set being the unit element. The group is commutative if and only if the semigroup is communitative. Its order is at least the cardinal of the continuum. (Received April 23, 1947.)

306. Sam Perlis: *A note on the radical of an ideal.*

If A is an associative algebra or ring and B is an ideal of A, the radical of B is proved to be the intersection of B with the radical of A. The Jacobson definition of the radical is used. This result is applicable several places in the literature where it is proved under special hypotheses. (Received July 25, 1947.)

307. J. O. Reynolds: *On the irreducibility of certain polynomials.*

G. Pólya (Jber. Deutschen Math. Verein. vol. 28 (1919) pp. 31–40) proved the following theorem: If for n integral values of x, the integral polynomial $f(x)$ of degree n has values which are different from zero and, without regard to sign, less than a certain bound which depends only on n, then $f(x)$ is irreducible in the rational domain. This result was improved for positive definite polynomials by H. Ille (Jber. Deutschen Math. Verein. vol. 35 (1926) pp. 204–208), for arbitrary polynomials by Tatuzawa

(Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 253–254) and finally by A. T. Brauer and G. Ehrlich (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 844–856). In this paper a still lower bound is obtained, if the n considered values of $f(x)$ are positive. (Received July 28, 1947.)

308. H. E. Salzer: *An "empirical theorem" which is true for the first 618 cases, but fails in the 619th.*

The statement that integers of the form $10n+6$, $n\geqq 0$, are the sum of four non-negative tetrahedral numbers, that is, $m(m+1)(m+2)/6$, $m\geqq 0$, holds for $n=0, 1, 2, \cdots$ up to $n=617$, fails for $n=618$, and has been verified for $n=619$ up to 2000. Thus this property of the number 618, of being the sole exception to that statement among all numbers ending in 6 which are not greater than 2006, affords an example of an empirical theorem which holds as far as the first 618 cases, before failing in the 619th. (This may encourage those seeking to disprove Fermat's last theorem, since in the latest results to date mentioned by H. S. Vandiver, Amer. Math. Monthly vol. 53 (1946) p. 575, it has been verified for exponents $3+n$, $n=0, 1, 2, \cdots$, up to 615, which are the first 616 cases.) This investigation of numbers $10n+6$ was performed by suitable grouping into pairs of addends, each addend being the sum of two tetrahedrals, and by employing special stencil-like devices for the addition of a number to hundreds of other numbers simultaneously. Every case was checked in two or more ways, involving the use of independently designed stencils. (Received June 10, 1947.)

309. R. D. Schafer: *Note on derivation algebras.*

A derivation D of a non-associative algebra $\mathfrak{A}$ determines a derivation Δ of the (associative) transformation algebra $T(\mathfrak{A})$ as follows: $T\rightarrow T\Delta=[T, D]=TD-DT$ for T in $T(\mathfrak{A})$. Results resembling those for automorphism groups in Bull. Amer. Math. Soc. vol. 53 (1947) pp. 573–583 are obtained. Application is made to derivations of semi-simple (non-associative) algebras over nonmodular fields. (Received July 1, 1947.)

310. P. A. Smith: *The extension of local groups.*

Call a topological local group G extendable if it is locally isomorphic to a topological group. Let N be a closed normal local subgroup of G. It is shown that if G/N and N (together with certain automorphisms of N) are extendable, then so is G. By taking N as the center, one obtains a new proof of the existence of Lie groups in the large. (Received July 3, 1947.)

311. G. E. Wall: *Notes on binomic equations over an E_m-field.* I.

An E_m-field is a field of characteristic 0 over which the mth cyclotomic polynomial is irreducible. A. Capelli has established necessary and sufficient conditions for the irreducibility of a binomic equation over a field of characteristic 0 (cf. Math. Ann. vol. 54 (1901) pp. 602–603). By means of this criterion the following theorem is proved: The polynomial x^m-a, irreducible over the E_m-field R, is reducible over $R(\epsilon)$ ($\epsilon=$primitive mth root of unity) if, and only if, (i) m is even and (ii) $a=b^2P$, where $b\in R$ and P is a product of different prime numbers dividing m (condition (ii) is to be modified if 8 does not divide m). G. Darbi (Annali di Matematica pura ed applicata (4) vol. 4 (1926)) considered x^m-a over the rational field and settled the cases, m odd, $m=2^n$, or m quadratfrei. H. Hopf and H. Schwerdtfeger (not yet published) proved (i) to be necessary over any E_m-field by consideration of the structure of the full linear permutation group. Extending this method, all Galois groups of $x^m - a$ over R, possible under the conditions of the theorem, are established. (Received May 1, 1947.)

312. G. E. Wall: *Notes on binomic equations over an E_m-field.* II.

Let $x^m - a$ be irreducible over the E_m-field R, and reducible over $R(\epsilon)$ (ϵ = primitive mth root of unity), and let $\alpha^m = a$. The form of the coefficient a is determined so that $R(\alpha, \epsilon)$ has a given Galois group $\mathfrak{G}$ over R (out of a class of admissible groups $\mathfrak{G}$ established in part I of this paper). Firstly, using the criteria of G. Darbi for normal binomic equations over an E_m-field (Annali di Matematica pura ed applicata (4) vol. 4 (1926)), the form of a is determined so that $\mathfrak{G}$ is of given order. These results are then applied to certain binomic subfields of $R(\alpha, \epsilon)$ in order to distinguish between groups $\mathfrak{G}$ of equal order. (Received May 1, 1947.)

ANALYSIS

313. R. P. Boas: *Some complete sets of analytic functions.*

In the following theorems are generalized results of Ibragimov (Bull. Acad. Sci. URSS. Sér. Math. vol. 11 (1947) pp. 75–100). Let $f(s)$ be analytic in $|y| < \pi$ and of period 2π. Let η and ζ be positive numbers with $\eta < \pi$, $\zeta \leq \pi - \eta$. Let a set of lower density ζ/π of the Fourier coefficients of positive index of $f(s)$ not vanish. Let $\alpha_n = \beta_n + i\gamma_n$ be a sequence of complex numbers, $0 \leq \beta_n < 2\pi$. Then $\{f(s+\alpha_n)\}$ is complete in $|s| < \zeta$ in the following cases. (1) The set $\{\gamma_n\}$ has a limit point in $|y| < \eta$. (2) The function $f(s)$ is entire, of order ρ and type σ_1, and $2\sigma_1 < \rho \liminf n|\gamma_n|^{1-\rho}$. (Received July 22, 1947.)

314. R. P. Boas and K. Chandrasekharan: *Derivatives of infinite order.*

Let $f(x)$ have derivatives of all orders in (a, b). The following theorems are proved. (1) If $f^{(n)}(x) \to g(x)$ for each x in (a, b), where $g(x)$ is finite, then $g(x) = Ae^x$. (2) If $f(x)$ belongs to a Denjoy-Carleman quasi-analytic class in the open interval (a, b), and $\lim_{n\to\infty} f^{(n)}(x_0) = L$ exists for a single x_0 in (a, b), then $f^{(n)}(x) \to Le^{x-x_0}$ in (a, b). These theorems answer in the affirmative questions raised by V. Ganapathy Iyer (J. Indian Math. Soc. N.S. vol. 8 (1944) pp. 94–108) and remain true if $f^{(n)}(x) \to g(x)$ in a more general sense. If $\{\lambda_n\}$ is a given sequence of constants, the following theorems are proved. (3) Let (*) $\lim_{n\to\infty} f^{(n)}(x)/\lambda_n = g(x)$, $a \leq x \leq b$. If $\liminf |\lambda_{n-1}/\lambda_n| = 0$ and (*) holds uniformly, $g(x) = 0$ in (a, b). If $\liminf |\lambda_{n-1}/\lambda_n| > 0$ and (*) holds dominatedly, $g(x) = Ae^{Bx}$. (4) If $\limsup n^{-1}|\lambda_n|^{1/n} < \infty$ and (*) is true for each x in $a < x < b$, then $g(x) = Ae^{Bx}$. (Received June 26, 1947.)

315. V. F. Cowling: *Some results for factorial series.*

Let $f(s) = \sum_{n=0}^{\infty} a_n n!/s(s+1) \cdots (s+n+1)$ with abscissa of convergence $\tau < \infty$. Let $l-1 < k < l$ where l is integral and positive but otherwise arbitrary. Denote by D the domain in the w-plane $\psi_2 \leq \operatorname{Arg}(w-k) \leq \psi_1$ where $0 < \psi_1 \leq \pi/2$ and $-\pi/2 \leq \psi_2 < 0$. Let $a(w)$ be a function regular in D, with the possible exception of the point at infinity, for which $a(n) = a_n$, $n = 0, 1, \cdots$. Suppose for $w = k + Re^{i\psi}$ in D and $R \geq R_1$ that $|a(k+Re^{i\psi})| \leq R^k \exp(-LR \sin \psi)$ where k is arbitrary and $0 < L < 2\pi$. Then

$f(z)$ may be continued analytically to any point of the plane except the points $z=0, -1, -2, \cdots$. (Received July 28, 1947.)

316. Evelyn Frank: *Continued fraction expansions for meromorphic functions.*

The continued fraction expansion $k_0\gamma_0+k_0(1-\gamma_0\bar{\gamma}_0)z/\bar{\gamma}_0 z-1/k_1\gamma_1+k_1(1-\gamma_1\bar{\gamma}_1)z/\bar{\gamma}_1 z-1/k_2\gamma_2+\cdots$, where the k_p are constants chosen so that $|\gamma_p|\neq 1$, converges uniformly over every bounded closed domain within $|z|<|r_1|<1$ to a meromorphic function $f(z)$ with poles at the points $r_1, r_2, \cdots, r_m$, where $0<|r_1|\leqq|r_2|\leqq\cdots\leqq|r_m|$. Conversely, every such function may be represented in that domain by a continued fraction of this form. The expansion is finite if $f(z)$ is a rational function of a particular form (Frank, Trans. Amer. Math. Soc. vol. 62 (1947)). Furthermore, the expansion may be continued analytically over the entire finite plane with the exception of the poles of $f(z)$, and these continuations converge uniformly to the values of the function in their respective circular domains which contain none of the poles of $f(z)$ and are of radius less than 1. This extends the theory of Schur (J. Reine Angew. Math. vol. 147 (1917)) for analytic functions with moduli less than 1 in the unit circle. Other forms of this continued fraction are found which connect with the preceding theory on such expansions, and certain non-circular convergence regions are shown. (Received July 28, 1947.)

317. K. O. Friedrichs: *Nonlinear hyperbolic differential equations for functions of two independent variables.*

For N functions u^n of x and y, systems of differential equations of the form $\sum_{n=1}^{N}(a^{mn}u^n_x+b^{mn}u^n_y)=g^m$, $m=1, \cdots, N$, are considered, in which the coefficients a, b, g depend on x, y, u. The values of u on a section of the initial line $y=0$ are prescribed such that the system is hyperbolic for them. In a neighborhood of this section, the unique existence of the solution with continuous second derivatives is proved if a, b, g and the initial value of u have continuous second derivatives. If these quantities have higher continuous derivatives, the same is shown to be true for the solution. The result applies to one equation of higher order for one function (an earlier paper of H. Lewy and the author on this subject contains an essential error). The proof consists in establishing sufficiently strong theorems for linear equations so that iterations can be carried out. The method of characteristics is used only for linear equations. (Received July 22, 1947.)

318. Philip Hartman: *Unrestricted solution fields of almost-separable differential equations.*

Let $g(x)$ and $f(t)$ be real continuous functions of a single variable for $-\infty<x<\infty$ and $0\leqq t<\infty$, respectively. Let $x=x(t)$ be a solution, on some t-interval $0\leqq t<\epsilon$, of the differential equation $dx/dt=g(x)+f(t)$. The question as to whether or not the solution $x=x(t)$ can be extended over the entire non-negative t-axis is considered; in the former case, the asymptotic behavior of $x(t)$, as $t\rightarrow\infty$, is also treated. The results obtained for these problems are then applied in a discussion of the asymptotic behavior of $y(t)$ and dy/dt, where $y=y(t)$ is a solution of the second order linear differential equation $d^2y/dt^2-(1+f(t))y=0$ and $f(t)$ is "small" for large values of t. (Received July 21, 1947.)

319. Einar Hille: *Non-oscillation theorems.*

Let $F(x)$ be defined for $x>0$, integrable over every finite interval, real-valued and of constant sign for large x. A necessary and sufficient condition that the equation $y''+F(x)y=0$ have a solution $y(x)$ with $y(x)\to 1$ when $x\to\infty$ is that $xF(x)\in L(1, \infty)$. Let $F(x)>0$ when $x>0$, form $g(x)=x\int_x^\infty F(t)dt$, and let g_* and g^* be the inferior and superior limits of $g(x)$ when $x\to\infty$. A necessary condition that the differential equation have non-oscillatory solutions is that $g_*\leq 1/4$, $g^*\leq 1$, and a sufficient condition is that $g^*<1/4$. These inequalities are sharp. A logarithmic scale of non-oscillatory equations is determined. A new comparison theorem for a class of singular Riccati integral equations is one of the main tools in the investigation. (Received July 29, 1947.)

320. W. H. Ingram: *A generalized Pollard-Moore-Stieltjes integral.*

Let σ_k'' be any subdivision of $[ab]$ which includes among its points of division all points of discontinuity of $f(x)$ of the first kind and of a permissible second kind namely those for which $-\infty<f^U(x-)=\limsup f(x-)-\liminf f(x-)=\liminf f(x+)<+\infty$, $\limsup f(x+)-\liminf f(x+)<+\infty$, at which points, for both kinds, the oscillation exceeds $\omega(k)$ and other points as required such that norm $\sigma_k''<\omega(k)$ norm σ_0''. The limit of the value of the interval function $\sum u_{ki}'' f^U(x_i+)\{g(x_{i+1})-g(x_i)\}$ as $k\to\infty$, $\omega(k)\to 0$, σ_k'' a refinement of σ_{k-1}'', x_i the ith point of σ_k'', $g(x)$ a function of bounded variation, is found to exist. The integral with variable upper limit $\int_a^x f dg$ is a function of bounded variation and, when $dy=y(x+)-y(x)$ unless zero in which case when $dy=(y')^+dx$, $d\int_a^x f dg=f(x+)dg(x)$. Also $\left|\int_a^b f dg-\sum u_i f^U(x_i+)\Delta_i g(x)\right|\leq \Omega_{f/\sigma}v_g(b)$ where $\Omega_{f/\sigma}$ is the maximim of the oscillations of $f(x)$ in the (open) meshes of any arbitrary subdivision σ of $[ab]$ and v_g is the total variation of g. (Received June 16, 1947.)

321. W. H. Ingram: *The Green's matrix for a 1-dimensional boundary value problem.*

Let $Y(x)$ satisfy $dY=dH(x)Y$, det $Y\neq 0$, $H(x)$ of bounded variation. Then Y and Y^{-1} are of bounded variation and the solution of the problem $dY=dH(x)Y+df(x)I$, $LY(a)+RY(b)=0$, in which $df(x, x_0, \delta)$ is a scalar function zero everywhere except over the range $x_0\leq x<x_0+\delta$ where it takes the value dx/δ, is, to within an ϵ, $\Gamma(x, x_0, \delta(\epsilon))=2^{-1}Y(x)\{s(x, x_0, \delta)+(LY(a)+RY(b))^{-1}(LY(a)-RY(b))\}Y^{-1}(\xi_0)$ ξ_0 such that $\int_a^{x>x_0}Y^{-1}df=Y^{-1}(\xi_0)\int_a^{x>x_0}df$, $a\leq x_0<\xi_0\leq x_0+\delta\leq b$; this mean value theorem is either true or becomes true as $\delta\to 0$; $s(x, x_0, \delta)=2\int_a^x df-1$. The Green's matrix is defined to be the limiting form of $\Gamma(x, x_0, \delta)$ as $\delta\to 0$ and satisfies the characterizing equations (i) $G(\xi+, \xi)-G(\xi, \xi)=I$, (ii) $d_xG(x, \xi)=dH(x)G(x, \xi)+2^{-1}d_x$ sign $(x-\xi-)I$, (iii) $LG(a, \xi)+RG(b, \xi)=0$, $a<\xi<b$. The integral of the preceding abstract has application to the problem $dY=dH(x)Y+dF(x)$, $LY(a)+RY(b)=0$, $F(x)$ of bounded variation, and to the problem of Bull. Amer. Math. Soc. Abstract 51-11-219, for example, $Y(x)=\int_a^b G(x, \xi)dF(\xi)$, and is required in integrations involving $d_\xi G(x, \xi)$ which has a discontinuity, as a function of x, of the admissible second kind at $x=\xi$. (Received June 16, 1947.)

322. J. P. LaSalle: *Singular measurable sets and linear functionals.*

A class of measurable sets E is shown to be nonsingular if and only if given E and $0<\lambda<1$ there exists a measurable set $E'\subset E$ with the property that $\mu(E')=\lambda\mu(E)$, where μ is the measure. If one makes use of the above and known theorems for linear topological spaces, a theorem on the existence of a multiplicative-additive continuous functional on the space L^{ω} is obtained, where the integral is over an abstract space and the measure of the space is not assumed to be finite. The result is an extension of a theorem given by Arens (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 931–935) and is similar to a theorem given by Day (Bull. Amer. Math. Soc. vol. 46 (1940) pp. 816–823). The method of proof gives a simple and direct proof of Day's theorem on the existence of linear functionals in the space L^p, $0<p<1$. (Received June 25, 1947.)

323. Morris Marden: *A note on lacunary polynomials.*

In this paper, it is proved that polynomials of the form $f(z)=a_0+a_1z+\cdots+a_pz^p+a_{n_1}z^{n_1}+\cdots+a_{n_k}z^{n_k}$, $a_0a_pa_{n_1}\cdots a_{n_k}\neq 0$, $1\leq p<n_1<\cdots<n_k$, have at least p zeros in or on each of the circles $|z|\leq Ar$ and $|z|\leq Br$ where r is the radius of the circle $|z|=r$ containing all the zeros of the polynomial $F(z)=n_1n_2\cdots n_ka_0+(n_1-1)(n_2-1)\cdots(n_k-1)a_1z+\cdots+(n_1-p)(n_2-p)\cdots(n_k-p)a_pz^p$ and where $A=\csc^k(\pi/2p)$ and $B=\prod_{i=1}^{k}\prod_{j=1}^{p-1}(n_i+j)/(n_i-j)$. These bounds are derived by elementary methods based upon repeated use of the theorem that, if an nth degree polynomial has p zeros in $|z|\leq 1$, its derivative has at least $p-1$ zeros in $|z|\leq\csc\pi/2(n-p+1)=M$ as well as in $|z|\leq\prod_{j=1}^{p-1}(n+j)/(n-j)=N$. The bound M is due to M. Marden (Trans. Amer. Math. Soc. vol. 45 (1939) pp. 335–368) and bound N is due to M. Bernacki (Bull. Soc. Math. France (2) vol. 69 (1945) pp. 197–203). (Received July 16, 1947.)

324. S. Minakshisundaram: *Notes on Fourier expansions.* III.

Let $F(e)$ be an additive function of bounded variation defined on Borel sets in the interval $(0\leq x_1, x_2, \cdots, x_k\leq 2\pi)$ and let $dF(e)\sim\sum C_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)}$, $C_{n_1\cdots n_k}=(1/(2\pi)^k)\int_0^{2\pi}\cdots\int_0^{2\pi}e^{-i(n_1x_1+\cdots+n_kx_k)}dF(e)$. Then (1) the series on the right called the *multiple Fourier-Stieltjes series* is almost everywhere summable $(\nu^2, (k-1)/2+\epsilon)$ for every $\epsilon>0$, that is, $\lim_{R\to\infty}\sum_{\nu^2\leq R^2}(1-\nu^2/R^2)^{(k-1)/2+\epsilon}c_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)}$, $\nu^2=n_1^2+\cdots+n_k^2$ exists almost everywhere, and (2) at a point where the symmetric derivative of $F(e)$ exists the multiple Fourier-Stieltjes series is summable $(\nu^2, (k+1)/2+\epsilon)$. If $k=1$, this is equivalent to a familiar result in simple Fourier-Stieltjes series. (Received July 25, 1947.)

325. Josephine M. Mitchell: *Summability theorems for double orthogonal series whose coefficients satisfy certain conditions.*

Let (I) $\sum_{m,n=1}^{\infty}a_{mn}\phi_{mn}(x, y)$ be the orthogonal development of a function $f(x, y)\in L^2$ on the rectangle $Q(a\leq x\leq b, c\leq y\leq d)$, with respect to the complete orthonormal system $\{\phi_{mn}(x, y)\}$ $(m, n=1, 2, \cdots)$ $(\phi_{mn}(x, y)\in L^2)$. Let $s(m, n)$ be the mnth partial sum and $\sigma(m, n)$ the corresponding $(C, 1, 1)$ partial sum of (I). If the series (II) $\sum_{m,n=1}^{\infty}[\log(m+1)]^{1+\epsilon}a_{mn}^2$ and $\sum_{m,n=1}^{\infty}[\log(n+1)]^{1+\epsilon}a_{mn}^2$ (ϵ any arbitrary positive number) both converge, then series (I) is $(C, 1, 1)$ summable to s almost everywhere in Q, if and only if $\lim_{m,n\to\infty}s(2^m, 2^n)=s$ almost everywhere in Q. Under the weaker hypothesis that $\sum_{m,n=1}^{\infty}a_{mn}^2$ is convergent there exist two increasing sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that $\lim_{i\to\infty}\sigma(m_i, n_i)=s$ almost everywhere in Q, if and only if $\lim_{i\to\infty}s(2^{m_i}, 2^{n_i})=s$ almost everywhere in Q. If series (II) converges, then the $(C, 1, 1)$ summability of series (I) to s implies that $(1/mn)\sum_{\mu,\nu=1}^{m,n}|s_{\mu\nu}-s|$ and $(1/mn)\sum_{\mu,\nu=1}^{m,n}(s_{\mu\nu}-s)^2$ both approach 0 almost everywhere in Q as m and $n\to\infty$. Two properties of the Lebesgue functions, $\int_Q|\sum_{\mu,\nu=1}^{m,n}\phi_{\mu\nu}(x, y)\phi_{\mu\nu}(s, t)|dsdt$, are proved. The methods of proof used in this paper are generalizations of the one-variable proofs. Reference is made to a paper by R. P. Agnew (Proc. London Math. Soc. vol. 33 (1932)). (Received July 28, 1947.)

326. L. A. Ringenberg: *On the theory of the Burkill integral.*

The author is concerned with the regular integrals and derivatives of a function which is defined on a class of plane intervals satisfying a parameter of regularity condition. The results of several papers by Kempisty, Saks, Burkill, and others are coordinated; several converses and generalizations are given. In addition to results stated in terms of monotoneity, continuity, absolute continuity, bounded variation, and so on, several results are given for functions of type A (type of the Lebesgue area) and of type I. The type I property, suggested by Rado, is a simple characterization of a function of intervals which is the Lebesgue integral of its own derivative. (Received April 28, 1947.)

327. M. M. Schiffer: *Faber polynomials in the theory of univalent functions.*

Let D be a domain in the z-plane containing the point at infinity. Let $\xi=f(z)=z+b_0+b_1z^{-1}+\cdots$ be univalent in D and map it upon the domain Δ. The nth Faber polynomial $F_n(t)$ with respect to $f(z)$ is of nth degree in t and satisfies by definition $F_n[f(z)]=z^n+\sum_{m=1}^{\infty}\xi_{nm}z^{-m}$. If the domain Δ undergoes a variation $\xi^*=\xi+a\rho^2(\xi-\xi_0)^{-1}+b\rho^3(\xi-\xi_0)^{-2}+\cdots+\xi^*(\xi)$ univalent in Δ, $\rho>0$, $\xi_0\not\subset\Delta$ (cf. Schiffer, Proc. London Math. Soc. (2) vol. 44 (1938) pp. 432–449) the coefficients ξ_{nm} vary as follows: $\xi_{nm}^*=\xi_{nm}+a\rho^2m^{-1}F_n'(\xi_0)F_m'(\xi_0)+o(\rho^2)$. If $x_{\nu 1}$ $(\nu=1,\cdots,l)$ is an arbitrary set of complex numbers, the quadric $Q=\sum_{n,m=1}^{l}m\xi_{nm}x_nx_m$ is a functional of $f(z)$ (or Δ) with the variation formula $Q^*=Q+a\rho^2(\sum_{n=1}^{l}x_nF_n'(\xi_0))^2+o(\rho^2)$. Functionals with a variation formula of this type, that is, containing a perfect square, lead to extremum problems which are reducible to easily integrable differential equations. Grunsky's inequalities (Math. Zeit. vol. 45 (1939) pp. 29–61) and generalizations are obtained as particular applications. (Received July 14, 1947.)

328. I. J. Schoenberg: *Some analytical aspects of the problem of smoothing.*

The present note contains two parts dealing with two disconnected problems concerning linear transformations of the convolution type (1) $y_n'=\sum_{\nu=-\infty}^{\infty}L_{n-\nu}y_\nu$ $(n=\cdots,-2,-1,0,1,2,\cdots)$. First, the author solves the problem of finding the limiting form of the coefficients of the high order iterates of the transformation (1) which is also assumed to be symmetric, that is, $L_{-\nu}=L_\nu$. This problem was proposed by E. L. De Forest, an early writer on smoothing. Its present solution also helps to determine when a formula (1) may legitimately be called a smoothing formula. Second, the author studies formulas (1) which are variation-diminishing, that is, which have the property that the number of variations of signs in the sequence $\{y_n\}$ is never exceeded by the similar number in the sequence $\{y_n'\}$. Assuming that $L_n\to 0$ exponentially, as $n\to\pm\infty$, it is shown that (1) is variation-diminishing if and only if all minors (of all orders) of the matrix $\|L_{i-j}\|$ $(i,j=0,1,2,\cdots)$ are non-negative. A method of constructing such transformations in terms of their generating functions $F(z)=\sum_{-\infty}^{\infty}L_nz^n$ is also given. The converse proposition, to the effect that this construction furnishes all variation-diminishing transformations, is stated as a conjecture. (Received July 16, 1947.)

329. Walter Strodt: *The resolution into partial fractions of the reciprocal of an exponential polynomial.* Preliminary report.

Let $f(s)=\sum_{j=1}^{n} A_j \exp(\omega_j s)$, with the ω_j a set of complex numbers whose closed convex contains zero. Let $H(s)=1/f(s)$. Let $Z_1, Z_2, \cdots$ be the distinct zeros of $f(s)$. Let $P_m(s)$ be the principal part of $H(s)$ at Z_m. Let $H(B)$ be finite. Then for every positive ϵ there exists a sequence of disjoint sets $\mathfrak{S}_1, \mathfrak{S}_2, \cdots$, whose union is the set $Z_1, Z_2, \cdots$, each $\mathfrak{S}_k$ containing at most $n-1$ points, and the diameter of each $\mathfrak{S}_k$ being less than ϵ, such that if $Q_k(s)$ is the sum over $\mathfrak{S}_k$ of the functions $P_m(s)-P_m(B)$, then $H(s)=H(B)+\sum_{k=1}^{\infty} Q_k(s)$ whenever $H(s)$ is finite. Moreover, for every pair of positive numbers δ_1, δ_2 the sequence $\mathfrak{S}_k$ can be chosen so that $Q_k(s)=(2\pi i)^{-1} \cdot \int_{C_k} H(t)(B-s)(t-s)^{-1}(t-B)^{-1}dt$, whenever $|s-Z_m|>\delta_1'$ $(m=1, 2, \cdots)$, where the C_k are contours of length less than δ_2 on which $H(t)$ is bounded, uniformly in k. This follows from well known theorems, due to C. E. Wilder, Tamarkin, and Pólya, about the zeros of $f(s)$. (Cf. Langer, *On the zeros of exponential sums and integrals*, Bull. Amer. Math. Soc. vol. 37 (1931).) (Received July 29, 1947.)

330. W. J. Thron: *Some properties of continued fractions* $1+d_0 z+K(z/(1+d_n z))$.

In this paper it is shown that for every power series (1) $1+\sum_{n=1}^{\infty} c_n z^n$ there exists one and only one continued fraction (2) $1+d_0 z+K(z/(1+d_n z))$ which corresponds to the given power series (that is, the power series expansion of the nth approximant of (2) agrees with (1) up to and including the term $c_n z^n$). Further, conditions on the sequence $\{d_n\}$ which insure uniform convergence of (2) for: (a) $|z|<d$, (b) $|z|>M$, (c) $\beta_1<\arg z<\beta_2$ are determined. Finally, various necessary conditions for the uniform convergence of (2) in a neighborhood of the origin are derived (for example, $\{d_n/n\}$ must be bounded). (Received June 18, 1947.)

331. J. L. Walsh: *The critical points of linear combinations of harmonic functions.*

Let R be a region bounded by a Jordan curve C, let the function $U(z)$ be bounded and harmonic in R, continuous on C except perhaps in a finite number of points. If the non-euclidean line L in R separates the points of C at which $U(z)$ is positive from the points of C at which $U(z)$ is negative, then no critical point of $U(z)$ lies on L in R. If $U(z)$ is non-negative on C and vanishes on the arc α of C, then the subregion of R bounded by α and by the non-euclidean line joining the end points of α contains no critical points of $U(z)$. This result extends under suitable conditions to critical points of a linear combination with constant coefficients of Green's functions for R and harmonic measures of arcs of C. (Received June 9, 1947.)

332. D. V. Widder and Salomon Bochner: *A homogeneous differential system of infinite order with nonvanishing solution.*

The differential system $(\sin \Pi D)y(x)=\lim_{n\to\infty} \Pi D \prod_{k=1}^{n}(1-k^{-2}D^2)y(x)=0$, $k(\pm\infty)=0$, is homogeneous in the sense that no fundamental solution, e^{ikx} $(k=0, \pm1, \pm2, \cdots)$, of the differential equation satisfies the boundary conditions. If the equation were of finite order it could, as a consequence, have no nonvanishing solution. It is shown that every even derivative of the function $k(x)=e^{-x}/(1+e^{-x})^2$ satisfies the system. Morever, every function $\sum_{k=0}^{\infty} a_k k^{(2k)}(x)$ is a solution if $\sum_{k=0}^{\infty} a_k w^k$ is an even entire function of order one and minimal type. A counter example shows that the word "minimal" cannot be replaced by "normal." (Received June 27, 1947.)

APPLIED MATHEMATICS

333. H. E. Salzer: *Checking and interpolation of functions tabulated at certain irregular logarithmic intervals.*

For functions that are usually represented upon semi-logarithmic graph paper, that is, which behave as polynomials in $\log x$, the problem of checking or interpolation when the x's are in geometric progression is quite simple due to the uniform interval in $\log x$. But in practice functions are often given at some or all of the points 1, 2, 5, 10, 20, 50, 100, 200, 500, 1000 (same as .001, .002, .005, and so on, or .01, .02, and so on). In the present paper coefficients are given which facilitate: (I) checking of such functions when given at some of the more frequently occurring combinations of those points, by obtaining the last divided difference; (II) Lagrangian interpolation according to a generalization of the scheme recently given by W. J. Taylor, Journal of Research, National Bureau of Standards, vol. 35 (1945) pp. 151–155, RP 1667. (Received July 16, 1947.)

334. H. E. Salzer: *Coefficients for expressing the first twenty-four powers in terms of the Legendre polynomials.*

Exact values of the coefficients of $P_m(x)$, the mth Legendre polynomial, in the expression for x^n as a series of Legendre polynomials are tabulated for $n=0, 1, 2, \cdots, 24$. Previous tables due to Byerly or Hobson do not extend beyond $n=8$, and are inadequate for many needs. These coefficients will be useful in approximating a polynomial of high degree (denoted by $f(x)$ after normalization to the interval $[-1, 1]$) by a polynomial of lower degree which will be best in a well known least square sense; that is, for a preassigned r, they will be used to obtain the polynomial $q_r(x)$ of degree not greater than r which minimizes $\int_{-1}^{1}[f(x)-q_r(x)]^2dx$. (Received July 3, 1947.)

335. H. E. Salzer: *Complex interpolation over a square grid, based upon five, six, and seven points.*

Lagrangian coefficients are tabulated for complex interpolation of an analytic function of $Z=x+iy$, which is given over a square grid in the Z-plane. The formulas employed here are based upon the values of the function at five, six, or seven points which are chosen so as to be as close together as possible, at the sacrifice of possible symmetry. (This is a continuation of the tables contained in the article by A. N. Lowan and H. E. Salzer, *Coefficients for interpolation within a square grid in the complex plane*, Journal of Mathematics and Physics vol. 23 (1944), which gives the coefficients for the 3- and 4-point cases.) Denoting the reference point in the lower left-hand corner by Z_0, and h the length of the grid, so that $Z=Z_0+Ph$, where $P=p+iq$, the approximating n-point formulas are of the well known form $\sum L_j^{(n)}(P)f(Z_j)$. Exact values of the coefficients $L_j^{(n)}(P)$ are given for $p=0, .1, .2, \cdots, 1.0$ and $q=0, .1, .2, \cdots, 1.0$. A method for inverse interpolation is indicated, based upon the coefficients of P^m in $L^{(n)}(P)$. (Received July 16, 1947.)

336. H. E. Salzer: *Further remarks on the approximation of numbers as sums of reciprocals.*

The present work consists of three main parts. (1) Comparison of R-expansions with simple continued fractions for rational numbers a/b leads to the analogue of the Euclidean algorithm, with a multiple of the g.c.d. of a and b in place of the g.c.d. One practical result is that for a/b, as a rule, fewer partial fractions are required in the R-expansion than in the s.c.f. (2) Proof that if p/q is an approximation to x obtained by the R-expansion, then the remainder $(x-p/q)<1/q$. (3) Proof that if p/q is an approximation to x obtained by the $\bar{R}$-expansion, then $|x-p/q|<1/2q$ except when $x=3/4$, when the $\leq$ relation may hold. Both theorems are best possible ones. It is shown that when in an R- or $\bar{R}$-expansion the denominator of the $(n+1)$th partial fraction is about k times the minimum value that could arise in an R- or $\bar{R}$-expansion, then not only the closeness of the nth, but of all ensuing approximations p/q will be about $1/kq$ or $1/2kq$ respectively. A critical bibliography is provided, which reviews all work by other authors on R- or $\bar{R}$-expansions. (Received June 18, 1947.)

337. H. E. Salzer: *Polynomials of best approximation in an infinite interval.* Preliminary report.

Chebyshev polynomials $C_n(x)$ are useful for approximating polynomials of high degree in a finite interval $[a, b]$, by polynomials of much lower degree, because of the property that of all polynomials with leading coefficient 1, $C_n(x)$ has the least value of the greatest deviation from 0 in the interval $[-1, 1]$. To approximate functions of the form $e^{-x}p(x)$ and $e^{-x^2}q(x)$, where $p(x)$, $q(x)$ are polynomials, over $[0, \infty]$ and $[-\infty, \infty]$ respectively, it is useful to know: (I) polynomials $P_n(x)$, degree n, leading coefficient 1, such that the greatest absolute value of $e^{-x}P_n(x)$ differs least from 0 in $[0, \infty]$; (II) polynomials $Q_n(x)$, degree n, leading coefficient 1, such that the greatest absolute value of $e^{-x^2}Q_n(x)$ differs least from 0 in $[-\infty, \infty]$. $P_n(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0$ satisfies $2n$ transcendental equations $P_n'(x_i)=P_n(x_i)$, $e^{-x_i}P_n(x_i)=(-1)^i a_n$, $i=1, \cdots, n$, where x_i are the abscissae of the extrema of $e^{-x}P_n(x)$. $Q_n(x)=x^n+b_{n-1}x^{n-1}+\cdots+b_0$ satisfies $2n+1$ equations $Q_n'(x_i)=2x_iQ_n(x_i)$, $e^{-x_i^2}Q_n(x_i)=(-1)^{i-1}e^{-x_1^2}Q_n(x_1)$, $i=1, \cdots, n, n+1$. When suitably normalized, $P_n(x)$ and $Q_n(x)$ are characterized by being tangent alternately to $\pm e^x$ and $\pm e^{x^2}$ respectively. $Q_n(x)=P_{n/2}(x^2)$ for n even, and is an odd function for n odd. (Received July 11, 1947.)

GEOMETRY

338. W. R. Utz: *The properties of geodesics on certain n-dimensional manifolds.* Preliminary report.

Let S denote the interior of an $(n-1)$-dimensional unit sphere in Euclidean n-space and let G denote a properly discontinuous group of homeomorphisms of S onto itself that preserve the hyperbolic metric $\int(dx_i dx_i)^{1/2}/(1-x_ix_i)$. The action of this group is discussed in a manner similar to that of Poincaré (*Théorie des groupes Fuchsiens*, Acta Math. vol. 1 (1882) pp. 1–62) for the case $n=2$. By the identification of congruent points of S under G an n-dimensional manifold, Σ, is secured. An investigation of the geodesics on Σ leads to results concerning geodesics and hyperbolic lines of the same type, asymptotic geodesics and limit geodesics analogous to certain results of Morse (*A fundamental class of geodesics on any closed surface of genus greater than one*, Trans. Amer. Math. Soc. vol. 26 (1924) pp. 25–60)) in the case of certain two-dimensional manifolds. (Received May 24, 1947.)

LOGIC AND FOUNDATIONS

339. David Nelson: *Constructible falsity.*

A definition for constructible truth of number theoretic statements is presented which entails the truth of a statement of the form "Not for all x, $A(x)$" only in case there is an effective method of constructing a natural number n such that "$A(n)$" is false. The definition is similar to that of Kleene (Journal of Symbolic Logic vol. 10 (1945) pp. 109–124) for realizability. Using results of this paper and Nelson (Trans. Amer. Math. Soc. vol. 61 (1947) pp. 307–368) a simply consistent system of number theory satisfying the truth definition is constructed. While some classically acceptable principles which are intuitionistically invalid are reinstated, the system becomes inconsistent upon the adjunction of an axiom schema representing the principle of contradiction. The system contains the systems of classical and intuitionistic number theory on suitable reinterpretation of the logical symbols of those systems. (Received July 28, 1947.)

340. Ira Rosenbaum: *A simple process for obtaining a formula satisfying the nth q-ary truth table of m-valued logic.*

The number, n, of any q-ary truth table of m-valued logic is determinable—by the methods of Bull. Amer. Math. Soc. Abstract 53-5-265—from the relation $n-1=\sum_{j=0}^{c-1}W_{c-j}\cdot m^{c-1-j}$ where $c=m^q$ and $W_i=$ the ith truth-value of the given table minus 1. Divide the summands above into m groups of m^{q-1} summands, and reduce all exponents to the remainders obtained on division of these exponents by m, adding 1 to each subsum obtained. Let the resulting sums be denoted by n_i, $i=1, 2, \cdots, m$. Then the following relation is provable: $F_n^{q(m)}(p_1, \cdots, p_q)=\prod_{i=1}^{m}\{i(p_q)\supset F_{n_{m-i+1}}^{q-1(m)}(p_1, \cdots, p_{q-1})\}$ where the left member of the equivalence denotes the nth q-ary truth-function of m-valued logic, $i(p_q)$ denotes a singulary function of p_q which has the value 1 when p_q has the value i and the value m otherwise. This relation yields a formula—defined in terms of &, $\supset$, singulary functions and $(q-1)$-ary functions—which satisfies the nth q-ary truth table. Recursion on q finally yields a formula with the same property—built only from &, $\supset$, and singulary functions—which is, for q or $m\geqq 3$, more easily obtained and comprehensible than conjunctive or disjunctive normal forms. (Received July 22, 1947.)

STATISTICS AND PROBABILITY

341. H. D. Brunk: *The strong law of large numbers.*

Sufficient conditions for the strong law of large numbers for sequences of independent random variables involving moments of arbitrary even order are obtained using a method based on Kolmogoroff's. These are given for the strong law in a generalized form analogous to Feller's generalized form of the weak law (see Bull. Amer. Math. Soc. vol. 51 (1945) p. 827). Similar sufficient conditions are also given for sequences of independent random variables for which the moments do not exist. (Received May 12, 1947.)

342. J. L. Doob: *Asymptotic properties of Markoff transition probabilities.*

Let $P^{(n)}(x, A)$ be the probability of a transition from a point x into a set A (Markoff process). It is supposed that there is a self-reproducing distribution, that is, a distribution $\Phi(A)$ satisfying $\int P^{(1)}(x, A)\Phi(dG_x)=\Phi(A)$. Various theorems are proved, generalizing results of Doeblin, Kryloff and Bogoliouboff, and Yosida and Kakutani on the asymptotic properties of $P^{(n)}(x, A)$ for large n. For example if the singular component $P_1(x, A, n)$ of $P^{(n)}(x, A)$ measure with respect to $\Phi(A)$ measure goes to 0 when $n\rightarrow\infty$ for all fixed x, the x space can be decomposed into finitely or denumerably many disjunct sets $N, A_1, A_2, \cdots$ such that $\Phi(N)=0$ and (*) $P^{(n)}(x, A)\rightarrow Q(x, A)$ (Cesàro convergence) for all x; here $Q(x, A)$ is a probability measure in A for fixed x, and $Q(x, A)=Q_j(A)$ is independent of x for x in $A\subset A_j$. Each ergodic set A_j can be further subdivided into a finite number d_j of sets through which the system governed by the given transition probabilities runs cyclically; if $d_j=1$ the limit in (*) is to be taken in the ordinary sense. Previously used hypotheses are equivalent to supposing that $P_1(x, A, n)\rightarrow 0$ uniformly in x; (*) is then also uniform in x, and there are at most finitely many A_j's. (Received May 3, 1947.)

343. J. L. Doob: *Renewal theory from the point of view of the theory of probability.*

The renewal problem is considered as a probability problem, leading as usual to the integral equation (*) $U(t)=F(t)+\int_0^t U(t-s)dF(s)$ where $F(s)$ is the probability of death before age s and $U(t)$ is the expected number of deaths in the interval $(0, t)$. The properties of $U(t)$ satisfying (*) are deduced from a probability analysis of the stochastic process involved. For example if $\Delta=\int_0^\infty s\,dF(s)$, Feller's theorem that $U(t)/t\rightarrow 1/\Delta$ is deduced as a consequence of the law of large numbers. Using the theory of Markoff processes it is shown that if $\Delta<\infty$ and if some convolution of $F(s)$ is not a singular function, $\lim_{t\to\infty}[U(t+h)-U(t)]=h/\Delta$ for all h. (Received May 3, 1947.)

344. T. E. Harris: *Some theorems on the Bernoullian multiplicative process.*

A single entity may have j descendents with probability p_j, $j=0, 1, 2, \cdots$. Each "first generation" entity has then the same procreative probabilities, and so on. Let $f(s)=p_0+p_1s+\cdots$. If z_n is the number of entities in the nth generation, it is known that $P(z_n=j)$ is given by the coefficient of s^j in the nth iterate $f[f\cdots(f)]=f_n(s)$. Let $Ez_1=x$, $1<x<\infty$. Conditions are given insuring that as $n\rightarrow\infty$ the cumulative distribution of the variate z_n/x^n approaches a limit function which is absolutely continuous except for a possible single jump. Let $g(u)$ be the corresponding frequency function. If $f(s)$ is a polynomial of degree k, let $q=\log_e k/(\log_e k-1)$. Otherwise $q=1$. Then $g(u)\cdot\exp\{u^{q+\epsilon}\}$ is [is not] summable $(0, \infty)$ according as ϵ is negative [positive]. Behavior of $g(u)$ near $u=0$ is also considered. Special cases are considered where $g(u)=\text{constant}\cdot u^{1/m-1}e^{-u/m}$, m a positive integer. Maximum likelihood estimates for the parameters $p_0, p_1, \cdots$ and x are obtained as functions of n successive values $z_1, z_2, \cdots, z_n$. Consistency in a certain sense is proved. A specialized method is given for finding the moment-generating function of the variate N, the smallest value of n such that $z_n=0$. (Received July 21, 1947.)

345. A. M. Mark: *Some probability limit theorems.*

Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables, each having mean 0 and standard deviation 1 and let $S_k=X_1+X_2+\cdots+X_k$. It is shown that the limiting distributions of $n^{-1/2}\min(S_{N+1}, S_{N+2}, \cdots, S_n)$ and $n^{-2}\sum_{j=1}^{n}P(j/n)S_j^2$ exist and are independent of the distribution of the X's when

$N=[\alpha n]$, $0<\alpha<1$, and $p(t)$ is a suitably restricted function defined over $0\leqq t\leqq 1$. A similar limit theorem is proved for the distribution of $n^{-2}\sum_{j=1}^{n}S_{nj}^2$ where $S_{nj}=X_{n1}+X_{n2}+\cdots+X_{nj}$ and $X_{n1}, X_{n2}, \cdots, X_{nn}$ are independent, identically distributed random variables each having mean μ_n and standard deviation 1, if $\lim_{n\to\infty} n^{1/2}\mu_n$ is finite. In each case the limiting distribution is first shown to be independent of the distribution of the X's and is then calculated by a convenient choice of a particular distribution of the X's. This method was first used by Erdös and Kac (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 292–302). (Received July 28, 1947.)

346. Kenneth May: *Probabilities of certain election results.*

Consider n election districts in each of which m votes are cast for one of two parties, the district votes following independent uniform distributions. The author discusses the probability $P(m, n)$ that either party gain a majority of the districts with a minority of the total vote. The problem is connected with that of enumerating the compositions of a number into n parts $n\geqq 0$, $n\leqq m$, and such that more than $n/2$ parts are greater than $m/2$. Taking m and n odd to avoid ties, results are stated in terms of $\mu=(m+1)/2$ and $\nu=(n+1)/2$. Utilizing the generating function $(1-s^{\mu})^n(1-s)^{-n}$, the probability is found to be $2^{1-n}\mu^{-n}\sum\sum(-1)^s C_{n,\nu+i}C_{n,s}C_{n+\mu\nu-\mu-\nu-(i+s)\mu,n}$ where the summation extends over i and s satisfying $\mu\nu-\nu-\mu-(i+s)\mu\geqq 0$. In particular, $P(m, 3)=(\mu^2-1)/8\mu^2$. For large n, utilizing Bernstein's central limit theorem, there results $P(m, \infty)=1/2-(1/\pi)\arctan\{\mu 3^{1/2}/(\mu^2-1)^{1/2}\}$. For large m, with the assistance of Fourier integrals, it is shown that $P(\infty, n)=2^{1-n}(n!)^{-1}\sum\sum(-1)^s C_{n,s}C_{n,s+r}r^n$ where the summation is taken over r and s satisfying $\nu-1-r-s\geqq 0$. From $P(3, 3)=3/32$, with increasing m and n, the probability rapidly approaches $P(\infty, \infty)=1/6$. (Received May 21, 1947.)

347. H. E. Robbins: *Some remarks on the inequality of Tchebychef.*

Let C_n denote the class of random variables X such that $X=X_1+\cdots+X_n$ where the X_i are independent and equidistributed and $E(X_i)=0$, $E(X_i^2)=1/n$. Let $\phi_n(t)$ denote the upper bound of $P[|X|\geqq t]$, where X runs over the class C_n. It is shown that $\phi_n(t)<t^{-2}$ for any $n>1$ and sufficiently large t. Thus Tchebychef's inequality can be improved for random variables in C_n, even though, as t becomes infinite, $\phi_n(t)\sim t^{-2}$. (Received July 28, 1947.)

348. Abraham Wald: *On the distribution of the maximum of successive cumulative sums of independent but not identically distributed chance variables.*

For each positive integer N, let $X_{N1}, \cdots, X_{NN}$ be a set of independently distributed chance variables. Put $S_{Ni}=X_{N1}+\cdots+X_{Ni}$ and $M_N=\max(S_{N1}, \cdots, S_{NN})$. It is shown that under some weak restrictions the limiting distribution of $M_N/N^{1/2}$ depends only on the first two moments of X_{Ni}. The exact distribution of M_N is obtained when X_{Ni} can take only the values $+1$ and -1. Lower and upper limits for M_N are given which yield particularly simple lower and upper limits for the probability distribution of M_N when the X's are symmetrically distributed around zero. (Received June 26, 1947.)

349. J. E. Walsh: *Loss of information in t-tests with unbalanced samples.*

Consider two normal populations $N(a_1, \sigma_1^2)$ and $N(a_2, \sigma_2^2)$, where σ_1/σ_2 (ratio of the standard deviations) has a known value C. If the equality of the means, $a_1 = a_2$, is to be tested by a t-test (one-sided or symmetrical) using n_1 sample values from $N(a_1, \sigma_1^2)$ and n_2 values from $N(a_2, \sigma_2^2)$ ($n_1 + n_2 = n$, fixed), it is shown that this experiment is most powerful when $n_1/n_2 = \sigma_1/\sigma_2$ (integer considerations neglected). The t-tests satisfying this condition are called balanced. Thus information is lost by not using a balanced experiment. A quantitative measure of the information lost by using given values of n_1 and n_2 is determined by the total sample size m ($m_1 + m_2 = m$) of the balanced t-test (same significance level) having approximately the same power. Then $n - m$ sample values are wasted by using (n_1, n_2) rather than (m_1, m_2), that is, only $100m/n\%$ of the information obtainable per observation is used by (n_1, n_2). A symmetrical t-test with significance level 2α has the same value of m as a one-sided t-test with significance level α. For one-sided t-tests with significance level α: $m \doteq 2^{-1}(B + (B^2 - 8A)^{1/2})$, where $B = 2 + A + K_\alpha^2/2$, $A = (C+1)^2[1 - K_\alpha^2/2(n-2)] \cdot [C^2/n_1 + 1/n_2]^{-1}$, and K_α is the standardized normal deviate exceeded with probability α. This approximation to m is valid for $m \geqq 5$ if $\alpha = .05$, $m \geqq 6$ if $\alpha = .025$, $m \geqq 7$ if $\alpha = .01$, $m \geqq 8$ if $\alpha = .005$. (Received July 16, 1947.)

350. J. E. Walsh: *Some significance tests for the median which are valid under very general conditions.* Preliminary report.

Consider n independent values drawn from populations satisfying only: (1) Each population has a unique median. (2) The median has the same value ϕ for each population. (3) Each population is symmetrical. (4) Each population is continuous. (No two of the values are necessarily drawn from the same population.) Significance tests are derived for ϕ. These tests are based on order statistics of certain combinations of order statistics, each combination being either a single order statistic of the n values or one-half the sum of two order statistics. The tests are reasonably efficient if the values represent a sample from a normal population. The significance levels are of the form $r/2^n$ ($r = 1, \cdots, 2^n - 1$). Each value of r can be obtained for some one-sided test. The major disadvantage of these tests is the limited number of suitable significance levels for small n. This disadvantage is partially eliminated by the development of tests having a specified significance level if the values are a sample from a normal population and a significance level bounded near this value if only (1)–(4) hold. Applications of these tests furnish generalized results for the Behrens-Fisher problem, certain large "sample" cases, quality control, slippage tests, the sign test, and situations where some of the n values are dependent. (Received July 16, 1947.)

TOPOLOGY

351. S. S. Chern: *On the characteristic ring of a differentiable manifold.*

Let M be a differentiable manifold of dimension n, which may be finite or infinite, orientable or non-orientable, and let $H(n, N)$ be the Grassmann manifold of all the linear spaces of dimension n through a fixed point O of a Euclidean space E^{n+N} of dimension $n+N$, $N \geqq n+2$. By imbedding M in E^{n+N} and constructing through O the linear spaces parallel to the tangent linear spaces of M, a mapping of M into $H(n, N)$ is obtained. This mapping induces a ring homomorphism of the cohomology ring of $H(n, N)$ into the cohomology ring of M, whose image is called the characteristic ring $C(M)$ of M. Take as coefficient ring the ring of residue classes mod 2. Formulas are established which give the cup product of two elements of $C(M)$ as a linear combination of elements of $C(M)$. Consequences of these formulas are: (1) The Stiefel-Whitney classes W^k ($k=1, \cdots, n$) form a basis of the characteristic ring under ring operations; (2) Necessary conditions are obtained for M to be imbeddable in E^{n+1} or E^{n+2}; (3) Projective spaces of even dimension n, when n is not of the form $n=2(2^k-1)$, $k\geqq 1$, cannot be imbedded in E^{n+2}. (Received June 23, 1947.)

352. S. S. Chern: *On the common zeros of a set of scalar densities on a complex analytic manifold and the automorphic forms in several complex variables.*

On a compact complex analytic manifold of dimension n a scalar density f of weight k is a geometric quantity transformed according to the law: $f^*=f\cdot J^k$, where J is the Jacobian. On the locus of common zeros of a set of scalar densities $f_1, \cdots, f_m$, $m\leqq n$, can in general be defined a cycle, called the cycle of common zeros and denoted by $[f_1, \cdots, f_m]$. Then when both sides have sense, $[f_1, \cdots, f_l]\cdot[f_{l+1}, \cdots, f_{l+m}]=[f_1, \cdots, f_{l+m}]$, where the left-hand side denotes the intersection cycles of the two cycles in the sense of algebraic topology. The homology classes of the cycles $[f_1, \cdots, f_m]$ are identified with the characteristic classes which the author introduced in his study of Hermitian manifolds. These results can be applied to Siegel's theory of automorphic functions in several complex variables, with the conclusion that the cycle of common zeros of a set of automorphic forms with respect to a discontinuous group Δ in Siegel's half-space depends only on the analytic structure of the fundamental domain F of Δ, if F is compact. In particular, the number of common zeros of n automorphic forms is equal to the Euler-Poincaré characteristic of F. (Received June 23, 1947.)

353. H. S. M. Coxeter: *Configurations and maps.*

The regular maps on a torus (cf. Brahana, Amer. J. Math. vol. 48 (1926) pp. 225–240) are found to be of three types: a map of b^2+c^2 quadrangles, one of b^2+bc+c^2 hexagons, and one of $2(b^2+bc+c^2)$ triangles. Alternate faces of the last map form a triple system which can sometimes be interpreted as an interesting configuration. When $b=2$ and $c=1$ this becomes Fano's finite projective geometry $PG(2, 2)$; when $b=3$ and $c=0$, the Pappus configuration. Similarly, alternate faces of the map of 24 triangles on the unorientable surface of characteristic -3 (Duke Math. J. vol. 10 (1943) p. 298, Fig. 11) yield the triple system formed by the collinear sets among the nine inflexions of the general cubic curve in the complex projective plane. (Received July 26, 1947.)

354. W. H. Gottschalk: *Recursive properties of transformation groups.* II.

Let T be a locally compact topological group which acts as a transformation group on a topological space X. Let there be distinguished in T certain sets, called *admissible*, which satisfy this condition: If A is an admissible set and if B is a set in T such that $A\subset BK$ for some compact set K in T, then B is an admissible set. A subgroup R of T is said to be *recursive* at $x\in X$ provided that to each neighborhood U of x there corresponds an admissible set A such that $A\subset R$ and $xA\subset U$. It is proved that if T is recursive at $x\in X$ and if S is a relatively dense invariant subgroup of T, then S is recursive at x. (Received June 30, 1947.)

355. S. W. Hahn: *Universal spaces under strong homeomorphisms.* Preliminary report.

Two subsets X and Y of a Euclidean n-space E_n are said to be ***strongly homeomorphic*** if there exists a homeomorphism f of E_n onto itself such that $f(X)=Y$. A subset Y_k of E_n will be called ***universal for k-dimensional subsets of*** E_n if: (1) dim $Y_k=k$ and (2) for every X in E_n with dim $X=k$, X is strongly homeomorphic to a subset of Y_k. It is proved that for $0\leq k<n-1$ there is no universal Y_k. (Received April 25, 1947.)

356. Deane Montgomery: *Analytic parameters in three-dimensional groups.*

It is shown that every connected locally Euclidean three-dimensional topological group must be a Lie group. (Received May 22, 1947.)

357. Deane Montgomery: *Connected one-dimensional groups.*

Let G be a locally compact connected one-dimensional topological group whose toplogy is separable metric. If G is not compact then G is isomorphic to the group of real numbers under addition. (Received May 22, 1947.)

358. Paul Olum: *Pseudomanifolds and their covering spaces.*

Let K be an n-dimensional pseudomanifold and $\bar{K}$ its universal covering space. The present paper studies the structure of $\bar{K}$ in relation to that of K, particularly with regard to questions of orientation. This leads to an explicit determination of a certain n-dimensional cohomology group with operators in K over an arbitrary coefficient group. (This is essentially the nth cohomology group of K with local coefficients, in the sense of Steenrod, for an arbitrary system of local coefficient groups.) On the basis of this result, a classification is given of the homotopy classes of mappings of K into a topological space T, where the homotopy groups $\pi_r(T)$ vanish for $1<r<n$. (See the abstract *Homology with operators and mapping theory*, to be published in November.) The results are applied in particular to the mappings of an n-pseudomanifold into projective n-space. (Received July 25, 1947.)

359. Pierre Samuel: *Ultrafilters and compactification of uniform spaces.*

The purpose of the author is to give a purely topological (that is, not using real numbers) proof of the fact that every separated uniform space S is a subspace of a compact Hausdorff space. This compact space $\bar{S}$ is constructed as an identification space of the Boolean space of all ultrafilters over S. $\bar{S}$ contains the completion S^* of S as a subspace, and is identical with S^* if, and only if, S is precompact. The Čech, Wallman and Alexandroff compactifications are studied as particular cases of this construction. A locally compact space is characterized as a space for which there exists a minimal uniform structure compatible with its topology. A characterization of normal sequentially compact spaces is also given. A purely topological characterization of uniformizable spaces is given in terms of mappings into compact spaces, and its equivalence with complete regularity is proved. (Cf. Bourbaki, *Topologie générale*, Paris, 1940, chap. 2.) (Received May 12, 1947.)

360. P. A. White: *Regular transformations on generalised manifolds*

It is shown that the image of a sphere-like orientable generalized-n-manifold under a $(n-1)$-regular transformation with the additional property that the inverse of each point is an lc_{n-1} is an orientable generalized-n-manifold. Other more general theorems concerning the above transformations are also studied. (Received March 17, 1947.)

361. J. W. T. Youngs: *Homeomorphic approximations to monotone mappings.*

In this paper the author proves the following theorem. If $m: M \rightrightarrows M$ (= from M onto M) is a monotone mapping where M is a closed 2-manifold or a 2-manifold with boundary, and $\epsilon > 0$, then there is a homeomorphism $h: M \rightrightarrows M$ such that $\rho\{m(x), h(x)\} < \epsilon$, $x \in M$. (Received July 24, 1947.)

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PRIMITIVE RECURSIVE FUNCTIONS

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1. **Definition of recursive functions.** In this paper, we shall consider certain reductions in the recursion scheme for defining primitive recursive functions. Hereafter, we shall refer to such functions simply as recursive functions.[1] In §1, we define what is meant by a recursive function, and also define some recursive functions which will be used. The statement of the principal results of the paper will be found in §2.

By a number, we shall mean one of the natural numbers 0, 1, 2, 3, $\cdots$. We shall consider functions of any number of variables, each variable ranging over all numbers, and the values of the function being numbers. Small letters will denote variables assuming numerical values, and capital letters will denote functions. In the case of a function of one variable, we shall usually write Fx instead of $F(x)$.

A function will be called *recursive* if it can be obtained from certain initial functions by repeated substitution and recursion.

The initial functions are the following:

The identity functions; that is, for every n and k with $1 \leqq k \leqq n$, the function I_{nk} defined by

$$I_{nk}(x_1, \cdots, x_n) = x_k.$$

The zero functions; that is, for every $n \geqq 0$, the function O_n defined by

$$O_n(x_1, \cdots, x_n) = 0.$$

The successor function; that is, the function S of one variable, such that Sx is the next number after x.

The substitution rule is the following: If $A_1, \cdots, A_m$ are known functions of n variables, and B is a known function of m variables,

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[1] Various types of recursive functions play a fundamental role in mathematical logic. Primitive recursive functions were used by K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme* I, Monatshefte für Mathematik und Physik vol. 38 (1931) pp. 173–198; for the definition, see pp. 179–180. For a discussion of general recursive functions (which may be considered to be the computable functions), see S. C. Kleene, *General recursive functions of natural numbers*, Math. Ann. vol. 112 (1936) pp. 727–742. For some other types of recursion, see the paper by R. Péter cited in footnote 2. In the present paper, we consider only the simplest type of recursive function, the primitive recursive function, and no knowledge of other papers is assumed.

then a function F of n variables may be defined by

$$F(x_1, \cdots, x_n) = B(A_1(x_1, \cdots, x_n), \cdots, A_m(x_1, \cdots, x_n)).$$

It is not excluded that m or n should be zero.

In particular, we can obtain by substitution the functions

$$SO_n(x_1, \cdots, x_n), SSO_n(x_1, \cdots, x_n), \cdots,$$

that is, all constant functions. Also, if we put O_m for B, and any functions of n variables for $A_1, \cdots, A_m$, then F is the function O_n. Thus it was not necessary to assume all the zero functions as initial functions; we could in fact have taken just O_0.

Furthermore, using the identity functions, any desired permutation of variables, or introduction of extra variables, may be accomplished. For example, given a function B of two variables, suppose it is required to define a function F of three variables, such that

$$F(x, y, z) = B(z, x).$$

This is seen to be included in our substitution rule when written in the form

$$F(x, y, z) = B(I_{33}(x, y, z), I_{31}(x, y, z)).$$

Finally, we must tell what is meant by recursion. If A is a known function of n variables, and B a known function of $n+2$ variables, then a function F of $n+1$ variables may be defined by the recursion scheme

$$F(u_1, \cdots, u_n, 0) = A(u_1, \cdots, u_n),$$
$$F(u_1, \cdots, u_n, Sx) = B(u_1, \cdots, u_n, x, F(u_1, \cdots, u_n, x)).$$

The recursion is said to be with respect to x; the other variables $u_1, \cdots, u_n$ are called parameters. In case $n=0$, we may replace $A(u_1, \cdots, u_n)$ by a number a. This completes the definition of what is meant by a recursive function.

Familiar examples of recursive definitions with one parameter are those of $u+x$, ux, and u^x, which must be made in this order. The definitions are:

$u+x$: $u+0=u$, $u+Sx=S(u+x)$; that is, $Au=u$, $B(u, x, y)=Sy$.
ux: $u\cdot 0=0$, $u\cdot Sx=ux+u$; that is, $Au=0$, $B(u, x, y)=y+u$.
u^x: $u^0=1$, $u^{Sx}=u^x\cdot u$; that is, $Au=1$, $B(u, x, y)=yu$.

We shall make much use of the function 0^x, and also of sgn $x=0^{0^x}$.

We shall also use predecessor and difference, defined by

$$Px: \quad P0 = 0, \qquad PSx = x;$$

$$u \dot{-} x: \quad u \dot{-} 0 = u, \qquad u \dot{-} Sx = P(u \dot{-} x).$$

Notice that $u \dot{-} x = 0$ if $u < x$. We also use the absolute difference

$$|u - x| = (u \dot{-} x) + (x \dot{-} u).$$

The notation $u - x$, without dot or vertical bars, will always be used in an ambiguous sense, to stand for any recursive function $F(u, x)$ which is equal to $u - x$ for $u \geqq x$, regardless of its value when $u < x$. Any such function will be called a difference function. In particular, $u \dot{-} x$ and $|u - x|$ are difference functions.

The functions Dx and Tx, giving the remainders when x is divided by 2 or by 3, are defined by

$$Dx: \quad D0 = 0, \qquad DSx = 0^{Dx};$$

$$Tx: \quad T0 = 0, \qquad TSx = 0^{Tx} + 2 \cdot 0^{|Tx - 1|}.$$

We may also describe D as the characteristic function of odd numbers. Two further recursive definitions are

$$Fx = [x/2]: \quad F0 = 0, \qquad FSx = Fx + Dx;$$

$$Fx = [x^{1/2}]: \quad F0 = 0, \qquad FSx = Fx + 0^{(SFx)^2 - Sx}.$$

In the last definition, FSx is obtained from Fx by adding 1 when the next square after x is the next number after x, but adding 0 when the next square is larger than the next number. We may now obtain by substitution the excess over a square, and the characteristic function of squares:

$$Ex = x - [x^{1/2}]^2, \qquad Qx = 0^{Ex}.$$

All of the symbols defined in this section will be used with the same meaning throughout the paper. The letters J, K, L will have a special meaning defined in §§3 and 4. On the other hand, the symbols A, B, F will be reserved for general functions, and may have a different meaning each time they are used.

2. Statement of principal results.

We shall consider certain more special recursion schemes than the one which occurs in the definition of recursive functions. One restriction is to limit the number of parameters. Another restriction is to suppose that B does not depend on all of its variables. If $B(u_1, \cdots, u_n, x, y)$ does not depend on the parameters $u_1, u_2 \cdots, u_n$, we shall call the recursion scheme *iteration*. (If $n = 0$, this is no restriction.) If $B(u_1, \cdots, u_n, x, y)$ does not depend on x, we shall speak of *pure recursion*, and if it depends only on

y, of *pure iteration*. Thus with one parameter, we have the four recursion schemes:

Recursion: $F(u, 0) = Au, \quad F(u, Sx) = B(u, x, F(u, x)).$
Pure Recursion: $F(u, 0) = Au, \quad F(u, Sx) = B(u, F(u, x)).$
Iteration: $F(u, 0) = Au, \quad F(u, Sx) = B(x, F(u, x)).$
Pure Iteration: $F(u, 0) = Au, \quad F(u, Sx) = BF(u, x).$

It should be noticed, here and below, that the more special schemes are always actually included in the more general ones, since (by using the identity functions) a function of certain variables may always be considered as a function of more variables.

The last two schemes are no less general, if taken in the form:

Iteration: $F(u, 0) = u, \quad F(u, Sx) = B(x, F(u, x)).$
Pure Iteration: $F(u, 0) = u, \quad F(u, Sx) = BF(u, x).$

For it is easily seen that if $F(u, x)$ is defined in this way, then the previously defined function is simply $F(Au, x)$.

The function $F(u, x)$ obtained by the last scheme (pure iteration) is simply the result of applying the function B x times to u. Hence we write

$$F(u, x) = B^x u.$$

For example, $u + x = S^x u$.

With no parameter, we have but two recursion schemes, since there is no distinction between recursion and iteration. These are:

Recursion: $F0 = a, \quad FSx = B(x, Fx).$
Pure Recursion: $F0 = a, \quad FSx = BFx.$

Since $n = 0$, the function $A(u_1, \cdots, u_n)$ is replaced by the number a. The pure recursion scheme defines the function $Fx = B^x a$. Unlike $B^x u$ above, this is a function of the one variable x, since a is a given number.

We also consider the somewhat more special schemes where only the value 0 is allowed for a:

Recursion: $F0 = 0, \quad FSx = B(x, Fx).$
Pure Recursion: $F0 = 0, \quad FSx = BFx.$

Even the last and most special recursion scheme, $Fx = B^x 0$, will be seen to be adequate to define all recursive functions, if two functions are adjoined to the initial functions.

For each degree of specialization of the recursion scheme, we shall ask what recursive functions need to be adjoined to the initial functions, in order that all recursive functions can be defined. It is understood that we always keep the identity, zero, and successor functions as initial functions, and that the substitution rule is unchanged. It will be shown that the functions given in the following table are sufficient to adjoin to the initial functions.

	One Parameter		No Parameter
	Recursion	Iteration	
Mixed	—	—	$u+x$, Q; or $\|u-x\|$
Pure	P	Q	$u+x$, E; or $\|u-x\|$, Q

In the table, the word mixed is used as the opposite of pure, and indicates that B may depend on x. The dash in the cases of (mixed) recursion and (mixed) iteration indicates that no adjunctions are necessary. Two alternatives are given in the cases with no parameter. They are both valid even in the special case $a=0$. It is undersood that $u+x$ in the table indicates that we are to adjoin to the initial functions that function F of two variables for which $F(u, x)=u+x$, and similarly for $|u-x|$.

Rózsa Péter has shown that recursion with one parameter is adequate.[2] She has also shown that recursion with no parameter is sufficient to define all recursive functions, if three recursive functions (u^x, the xth prime p_x, and the exponent of the highest power of p_x which divides u) are adjoined to the initial functions.[3] We extend these results by showing that even iteration with one parameter is adequate, that simpler adjunctions can be made in the case of recursion with no parameter, and by discussing the various types of pure recursion.

Whether all the adjunctions made are actually necessary, I do not know. However, it is shown in §8 that in the case of recursion with no parameter, some function of more than one variable must be adjoined. In the case of pure recursion with no parameter, more extensive results are obtained, which show in particular that $u+x$ and Q would not be sufficient to adjoin.

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[2] R. Péter *Über den Zusammenhang der verschiedenen Begriffe der rekursiven Funktion*, Math. Ann. vol. 110 (1934) pp. 612–632, especially pp. 619–620.

[3] R. Péter, *Konstruktion nichtrekursiver Funktionen*, Math. Ann. vol. 111 (1935) pp. 42–60, especially pp. 45–48.

In §3, we shall make the reduction to the various schemes with one parameter, and in §4 to schemes with no parameter, by making certain adjunctions to the initial functions. The lemma of §5 makes an essential reduction in the number of functions adjoined; but some of the more difficult definitions are reserved for §6, where the proof of the results given in the above table is completed. In §7, it is shown how all recursive functions of one variable may be defined without introducing functions of more than one variable. The necessity of the given adjunctions is discussed in §8.

3. Reduction to schemes with one parameter. All of the reductions make use of the idea of associating an ordered pair of numbers with a single number. We must establish a one-to-one correspondence between all pairs of numbers and some numbers. This requires three functions $J(u, v)$, Kx, Lx, satisfying the equations

$$KJ(u, v) = u, \qquad LJ(u, v) = v.$$

If we have in addition

$$J(Kx, Lx) = x,$$

then a one-to-one correspondence is established between all pairs of numbers and all numbers. However, this condition is not needed, and we shall not assume it.

We shall assume that some suitable functions J, K, L are adjoined to the initial functions. A possible set of functions is

$$J(u, v) = (u + v)^2 + u, \qquad Kx = Ex, \qquad Lx = [x^{1/2}] - Ex.$$

These functions are all expressible in terms of

$$u + x, \qquad u - x, \qquad x^2, \qquad [x^{1/2}],$$

where $u-x$ denotes any difference function. Thus it would be sufficient to adjoin these functions to the initial functions.

As a first reduction in the recursion scheme, we show that the number of parameters can be reduced to one. It is sufficient to show how to reduce the number of parameters by one, when there are two or more to begin with. In fact, it is sufficient to show how to reduce the number of parameters from two to one, since additional parameters could be carried through unchanged.

The given scheme has the form

$$F(u, v, 0) = A(u, v), \qquad F(u, v, Sx) = B(u, v, x, F(u, v, x)).$$

It is seen that the function

$$F'(u, x) = F(Ku, Lu, x)$$

is defined by the recursion scheme with one parameter

$$F'(u, 0) = A'u, \qquad F'(u, Sx) = B'(u, x, F'(u, x)),$$

where

$$A'u = A(Ku, Lu), \qquad B'(u, x, y) = B(Ku, Lu, x, y).$$

Finally, we may put

$$F(u, v, x) = F'(J(u, v), x).$$

Thus we have shown that recursion with one parameter,

$$F(u, 0) = Au, \qquad F(u, Sx) = B(u, x, F(u, x))$$

is adequate. It remains to show how the u and x may be eliminated from $B(u, x, y)$. Let us first eliminate the u. (The opposite order could equally well be used.) We see that the function

$$F'(u, x) = J(u, F(u, x))$$

is defined by the iteration with one parameter

$$F'(u, 0) = A'u, \qquad F'(u, Sx) = B'(x, F'(u, x)),$$

where

$$A'u = J(u, Au), \qquad B'(x, y) = J(Ky, B(Ky, x, Ly)).$$

Finally,

$$F(u, x) = LF'(u, x).$$

Thus iteration with one parameter,

$$F(u, 0) = Au, \qquad F(u, Sx) = B(x, F(u, x))$$

is also sufficient. Now the function

$$F'(u, x) = J(x, F(u, x))$$

may be defined by

$$F'(u, 0) = A'u, \qquad F'(u, Sx) = B'F'(u, x),$$

where

$$A'u = J(0, Au), \qquad B'y = J(SKy, B(Ky, Ly)).$$

Finally,

$$F(u, x) = LF'(u, x).$$

Hence pure iteration with one parameter,

$$F(u, 0) = Au, \qquad F(u, Sx) = BF(u, x),$$

is sufficient to define all recursive functions. The further reduction of replacing Au by u was discussed in §2.

4. **Elimination of the last parameter.** We shall now show how to eliminate the last parameter. For this purpose, we shall suppose that the pairing functions which we have adjoined to the initial functions have certain additional properties, namely:

$J(0, 0) = 0$, hence $K(0) = 0$ and $L(0) = 0$.

If $LSx \neq 0$, then $KSx = Kx$ and $LSx = SLx$.

An example of suitable functions is

$$J(u, v) = ((u + v)^2 + u)^2 + v, \qquad Kx = E[x^{1/2}], \qquad Lx = Ex,$$

as is easily verified. These functions can be defined in terms of

$$u + x, \qquad u \dot{-} x, \qquad x^2, \qquad [x^{1/2}],$$

as was the case for the functions J, K, L mentioned in §3.

We now discuss the meaning of the conditions imposed. If we think of all pairs (u, v) arranged in a table, where u denotes the column number and v denotes the row number, then the pair (Kx, Lx) traverses the table in such a way that it starts at the top of column 0 and descends a certain number of steps; then it starts at the top of another column and descends a certain number of steps; and this process is repeated. Since we have pairing functions, every position in the table is traversed at some time. It is clear in fact that the equations $Kx = u$, $Lx = v$ have infinitely many solutions for x, one of which is called $J(u, v)$. A more precise description of the way the table is traced out for the particular K and L mentioned above could easily be given.

We showed in §3 that iteration with one parameter is adequate to define all recursive functions. This may be taken in the form

$$F(u, 0) = u, \qquad F(u, Sx) = B(x, F(u, x)).$$

The still more reduced scheme of pure iteration might be used, but there would be no advantage in this. We see that

$$F'x = F(Kx, Lx)$$

can be defined by

$$F'0 = 0, \qquad F'Sx = B'(x, F'x),$$

where

$$B'(x, y) = \begin{cases} KSx & \text{if } LSx = 0, \\ B(Lx, y) & \text{if } LSx \neq 0. \end{cases}$$

In obtaining this result, we have taken account of the special assumptions concerning the pairing functions made at the beginning of this section. If we had not imposed these conditions, we should not have an expression for $F'Sx$ in terms of x and $F'x$, but should have obtained a generalized type of recursion (with no parameter), which would require a further reduction. This less direct method was followed by R. Péter. We have finally

$$F(u, x) = F'J(u, x).$$

In this proof, we have obtained $B'(x, y)$ by piecing together two known functions. The definition may be written explicitly as

$$B'(x, y) = 0^{LSx} \cdot KSx + \operatorname{sgn} LSx \cdot B(Lx, y).$$

This can be obtained by substitution if

$$u + x, \qquad u \cdot 0^{x}$$

are included in the initial functions.

We have thus shown that recursion with no parameter,

$$F0 = 0, \qquad FSx = B(x, Fx),$$

is adequate to define all recursive functions, if suitable functions are adjoined to the initial functions.

The x can be eliminated in the same way as in §3. We see that

$$F'x = J(x, Fx)$$

can be defined by

$$F'0 = 0, \qquad F'Sx = B'F'x,$$

where

$$B'y = J(SKy, B(Ky, Ly)).$$

Finally,

$$Fx = LF'x.$$

Thus we have reached the very simplest recursion scheme, pure recursion with no parameter, starting from 0, that is,

$$F0 = 0, \qquad FSx = BFx,$$

which defines the function $Fx = B^x 0$.

To make this reduction, it is sufficient to adjoin

$$u + x, \qquad u - x, \qquad x^2, \qquad [x^{1/2}], \qquad u \cdot 0^x$$

to the initial functions. Instead of $u \cdot 0^x$, we may of course adjoin the two functions ux and 0^x. Since

$$ux = [\{(u + x)^2 - u^2 - x^2\}/2],$$

we may also say that the functions

$$u + x, \qquad u - x, \qquad x^2, \qquad 0^x, \qquad [x/2], \qquad [x^{1/2}]$$

are sufficient to adjoin.

5. Sufficiency of adjoining certain functions.

LEMMA. *All recursive functions are definable by any of the recursion schemes considered, if the functions shown in the following table are adjoined to the initial functions.*

	One Parameter		No Parameter
	Recursion	Iteration	
Mixed	—	—	$u+x$, $u-x$
Pure	P, Q	P, Q	$u+x$, $u-x$, Q

PROOF. It will be sufficient to show that the functions listed at the end of §4 can be defined in all cases.

$u+x$. This function is given in the cases with no parameter. Otherwise, we may define $u+x = S^x u$ by pure iteration.

$u-x$. A difference function is given in the cases with no parameter. Otherwise, P is given in the pure cases, and may be defined by $P0 = 0$, $PSx = x$ in the mixed cases; hence we may define the two difference functions

$$u \dot{-} x = P^x u, \qquad |u - x| = (u \dot{-} x) + (x \dot{-} u).$$

x^2 (mixed cases). We may define $Fx = x^2$ by

$$F0 = 0, \qquad FSx = Fx + 2x + 1.$$

Here $2x$ means $x+x$, and similarly below.

x^2 (pure cases). We first define the function

$$Fx = x + 2[x^{1/2}].$$

Notice that

$$FSx = SFx + 2QSx.$$

Now the following are equivalent: Sx is a square; x is of the form n^2+2n; Fx is of the form n^2+4n; $Fx+4$ is a square. Thus

$$QSx = Q(Fx + 4).$$

Hence we may put

$$Fx = B^x0 \quad \text{with } By = Sy + 2Q(y + 4).$$

Finally, we see that

$$x^2 = (SF)^x0.$$

This definition makes use of Q and $u+x$ but not of P.

0^x. We may define

$$0^x = B^x1 \qquad \text{with } By = 0,$$

except in the case with no parameter and with $a=0$. In this case, we first define

$$\operatorname{sgn} x = B^x0 \qquad \text{with } By = 1,$$

and then put $0^x=1-\operatorname{sgn} x$.

$[x/2]$ (mixed cases). As in §1, we first define

$$Dx = B^x0 \qquad \text{with } By = 0^y,$$

and then define $Fx=[x/2]$ by

$$F0 = 0, \qquad FSx = Fx + Dx.$$

$[x/2]$ (pure cases). Here we first define

$$Tx = B^x0 \qquad \text{with } By = 0^y + 2\cdot 0^{(y-1)+(1-y)}.$$

This is nearly the same definition given in §1; we observe that the exponent $(y-1)+(1-y)$ vanishes if and only if $y=1$, no matter what difference function is used. Now if we define

$$Fx = B^x0 \qquad \text{with } By = Sy + Ty,$$

we see that

$$[x/2] = Fx - x.$$

$[x^{1/2}]$ (mixed cases). We define $Fx=[x^{1/2}]$ as in §1 by

$$F0 = 0, \qquad FSx = Fx + 0^{(SFx)^2 - Sx}.$$

$[x^{1/2}]$ (pure cases). In defining x^2, we first defined

$$Fx = x + 2[x^{1/2}].$$

We may put

$$[x^{1/2}] = [(Fx - x)/2].$$

6. Proof of the principal results.

THEOREM. *All recursive functions are definable by any of the recursion schemes considered, if the functions shown in the table in §2 are adjoined to the initial functions.*

PROOF. This theorem may be reduced to the lemma of §5, by making the definitions (1)–(5) below. In carrying out the proof, definitions given in the proof of the lemma are referred to when they are applicable.

(1) Define Q in terms of P by pure recursion with one parameter. We may use the definitions of $u+x$, $|u-x|$, and 0^x from §5. Now define ux by the pure recursion

$$u \cdot 0 = 0, \qquad u \cdot Sx = ux + u,$$

and put $x^2 = xx$. We next define a function $F(u, x)$ by the pure recursion

$$F(u, 0) = 0, \qquad F(u, Sx) = B(u, F(u, x)),$$

where

$$B(u, y) = Sy + 0^{|y^2 - u|}.$$

That is, $F(u, Sx)$ is obtained from $F(u, x)$ by adding 1 unless $\{F(u, x)\}^2 = u$, in which case we add 2. This can happen only if $x^2 = u$. We thus see that

$$F(u, x) = \begin{cases} x & \text{if } x^2 \leqq u, \\ x + Qu & \text{if } x^2 > u. \end{cases}$$

Hence we may define

$$Qu = F(u, Su) - Su.$$

(2) Define P in terms of Q by pure iteration with one parameter. We may define $u+x$ and x^2 as in §5. Next define

$$u \cdot 0^x = B^x u \qquad \text{with } By = 0;$$

in particular, $0^x = 1\cdot 0^x$. After defining $Dx = B^x 0$ with $By = 0^y$, we can obtain the product $Dx\cdot Qx$ from $u\cdot 0^v$ by putting $u = Dx$ and $v = 0^{Qx}$. Notice that $Dx\cdot Qx$ is the characteristic function of odd squares. Let

$$By = SSy\cdot 0^{Dy\cdot Qy} = \begin{cases} 0 & \text{if } y \text{ is an odd square.} \\ y+2 & \text{otherwise.} \end{cases}$$

Thus the function B has the effect of increasing any number by 2, except for an odd square, which is replaced by 0. Now consider the function

$$Fx = B^x(x^2 + x + 1).$$

If x is even and positive, then $x/2$ additions of 2 are carried out to reach the next odd square x^2+2x+1. The next application of B gives 0, and the remaining $(x/2)-1$ steps produce $x-2$. Thus

$$Fx = x - 2 \qquad \text{if } x \text{ is even and positive.}$$

It is now easy to define Px. We have in fact

$$Px = 0^{0^x}\cdot 0^{Dx}\cdot SFx + Dx\cdot FSx.$$

The first term can evidently be obtained by repeated substitution in $u\cdot 0^v$, and the second term can be obtained by substituting $u = FSx$ and $v = 0^{Dx}$. The correctness of the formula is easily checked by considering separately the cases $x = 0$, x even and positive, and x odd.

(3) Define $u - x$ and Q in terms of $u+x$ and E by pure recursion with no parameter.

We first define $\operatorname{sgn} x = B^x 0$ with $By = 1$, and then put

$$0^x = E(2 + 2 \operatorname{sgn} x), \qquad Qx = 0^{Ex}.$$

We may then define x^2 as in §5. Now

$$E((u + x)^2 + 3u + x + 1) = u - x \qquad \text{if } u \geqq x,$$

since the preceding square is $(u+x)^2+2u+2x+1$. We have thus defined both Q and a difference function.

(4) Define $u-x$ in terms of $u+x$ and Q by recursion with no parameter.

Again define $\operatorname{sgn} x = B^x 0$ with $By = 1$, and then put

$$0^x = QS \operatorname{sgn} x, \qquad Dx = B^x 0 \text{ with } By = 0^y.$$

We may define x^2 as in §5; the simpler definition for the mixed cases may now be used. Define $Fx = [x^{1/2}]$ by

$$F0 = 0, \qquad FSx = Fx + QSx,$$

and Px by $P0=0$, $PSx=x$. Then define a new function Fx by

$$F0 = 0, \qquad FSx = P(Fx + 2D[x^{1/2}]).$$

We have

$$FSx = \begin{cases} PFx & \text{if } [x^{1/2}] \text{ is even,} \\ SFx & \text{if } [x^{1/2}] \text{ is odd.} \end{cases}$$

Hence

$$Fx = Ex \qquad \text{if } [x^{1/2}] \text{ is odd,}$$

so that

$$F((2u + 2x)^2 + 5u + 3x + 1) = u - x \qquad \text{if } u \geq x.$$

(5) Define $u+x$ in terms of $|u-x|$ by pure recursion with no parameter. (Using this in place of (3) and (4) gives the second alternative of the table in §2 for both types of recursion with no parameter.)

Since

$$u + x = F(u, x) - \{(F(u, x) - u) - x\},$$

for any function F with $F(u, x) \geq u+x$, we have only to define some such function. We first define

$$2x = (SS)^x 0, \qquad 2x + 1 = S(2x),$$
$$2^x - 1 = B^x 0 \text{ with } By = 2y + 1, \qquad 2^x = S(2^x - 1).$$

We may then obtain the function

$$F(u, x) = |2^{4u} - 2^{4x+2}|$$

by substitution. Now

$$F(u, x) = |2^{2u} - 2^{2x+1}| \cdot (2^{2u} + 2^{2x+1}).$$

Since the first factor is not 0, and the second factor is more than $u+x$, we have

$$F(u, x) > u + x.$$

7. Recursive functions of one variable. It is easy to see that *all recursive functions of more than one variable can be obtained from recursive functions of one variable, the function* $u+x$, *and the various identity functions, by substitution alone.* To see this, we shall make use of the pairing functions of §3,

$$J(u, v) = (u + v)^2 + u, \qquad Kx = Ex, \qquad Lx = [x^{1/2}] - Ex.$$

Suppose now that the recursive function $F(u, v)$ is given. If we let

$$F'x = F(Kx, Lx),$$

then $F'x$ is a recursive function of one variable. But

$$F(u, v) = F'J(u, v) = F'((u + v)^2 + u).$$

Thus $F(u, v)$ is obtained by substitution from the four functions

$$F'x, \qquad x^2, \qquad u + v, \qquad I_{21}(u, v),$$

of which the first two are recursive functions of one variable. Thus the proof is complete so far as functions of two variables are concerned. In the case of a function of more variables, a similar argument serves to reduce the number of variables by one, so that repeated use of the argument leads to the desired result.

The interest of this result is considerably enhanced by the fact that it is possible to define all recursive functions of one variable without making use of functions of more than one variable in the process. The third theorem below shows that this is the case.

THEOREM 1. *The functions of one variable which are obtainable from certain initial functions by substitution and recursion of the form* $Fx = B^x a$ *are also obtainable if substitution is not allowed except for defining functions of one variable.*

PROOF. Let us call the functions thus obtainable from the given initial functions when substitution is used only for defining functions of one variable recursive$_1$, and functions obtainable by recursion and unrestricted substitution recursive$_2$. It will be sufficient to consider the class of functions obtainable by unrestricted substitution from recursive$_1$ functions, and to show:

(1) All functions of one variable so obtained are recursive$_1$.

(2) All recursive$_2$ functions are obtained.

PROOF OF (1). In defining a function of one variable by successive substitutions, if the "innermost" substitutions are made first, only functions of one variable need be defined by substitution in the process. But a function of one variable defined by substitution from recursive$_1$ functions is recursive$_1$.

PROOF OF (2). Our family of functions includes all the initial functions and is closed with respect to substitution. Also, recursion of the form $Fx = B^x a$ is allowed if B is recursive$_1$, and hence by (1) for all functions B of one variable in our family. Thus the family includes all recursive$_2$ functions.

THEOREM 2. *If the initial functions include the identity and zero functions, and the function $u+x$, but no other functions of more than one variable, then all functions of one variable obtainable by substitution and recursion of the form $Fx=B^{\bullet}a$ are also obtainable by repeated use of the three formulas*

$$Fx = Ax + Bx, \qquad Fx = BAx, \qquad Fx = B^{\bullet}a$$

to define F when A and B are known functions of one variable.

PROOF. By the preceding theorem, it is sufficient to use substitution to define only functions of one variable. Now substitution of functions of one variable into I_{nk} gives one of the same functions; and such substitution into O_n gives O_1, which is one of the initial functions. Hence only substitution into $u+x$ and into a function of one variable need be considered. In this way, we construct the functions $Ax+Bx$ and BAx.

Remark. Theorems 1 and 2 are also valid if we allow only recursion of the form $Fx=B^{\bullet}0$.

THEOREM 3. *All recursive functions of one variable can be obtained by starting with the two functions S and E, and repeatedly using any of the formulas*

$$Fx = Ax + Bx, \qquad Fx = BAx, \qquad Fx = B^{\bullet}0$$

to construct a new function from known functions A and B.

PROOF. The identity and zero functions of one variable can be defined by

$$I_{11}x = S^{\bullet}0, \qquad O_1x = I_{11}^{\bullet}0.$$

Thus by Theorem 2, we can obtain all the functions of one variable which we could get by substitution and recursion $Fx=B^{\bullet}0$ from all the identity and zero functions, $u+x$, S, and E. But by the theorem of §6, all recursive functions can be so obtained.

Remarks. We could also prove a theorem similar to Theorem 2, with $|u-x|$ in place of $u+x$, and $|Ax-Bx|$ in place of $Ax+Bx$. Using this, a theorem analogous to Theorem 3 could be proved, with Q in place of E, and $|Ax-Bx|$ in place of $Ax+Bx$.

8. Possibility of further improvements. Were all the adjunctions made in the Theorem of §6 necessary? For example, in the case of pure recursion with one parameter, we adjoined P to the initial functions. This adjunction is necessary if and only if P itself cannot be defined. We are thus led to the interesting but apparently unsolved problem: *Can predecessor be defined by substitution and pure recursion from the identity, zero, and successor functions?* In a similar way, we may ask whether Q can be defined by pure iteration.

In the case of recursion with no parameter, with or without $a=0$, some adjunction to the initial functions is necessary. Indeed, some function of more than one variable must be adjoined. For otherwise, each of the initial functions would depend on at most one of its variables, since the only functions of more than one variable which are given are the identity and zero functions. Now substitution applied to functions, each depending on at most one of its variables, can lead only to such a function; and our recursion scheme leads to a function of one variable. Thus no function depending on more than one variable can be defined; in particular, $u+x$ cannot be defined.

There remains the possibility that all recursive functions of one variable might be definable with no adjunctions. Finally, if it were possible to define Q by adjoining only $u+x$, then the adjunction of $u+x$ would be sufficient for the definition of all recursive functions.

We come now to the simplest recursion scheme, pure recursion with no parameter, with or without $a=0$. We know that it is sufficient to adjoin $u+x$ and E, or $|u-x|$ and Q, to the initial functions. We shall show that it is not sufficient to adjoin $u+x$ and Q, and in particular that P could not then be defined. This result is included in the more general theorem:

THEOREM. *Suppose that to the identity, zero, and successor functions, we adjoin the function $u+x$, and any number of functions of one variable each of which has one or the other of the two properties:*

$$Fx \geq x, \qquad Fx \text{ is bounded.}$$

Then by repeated substitution and pure recursion with no parameter, no function of one variable not having one or the other of these two properties can be defined.

PROOF. According to Theorem 2 of §7, this reduces to the following lemma.

LEMMA. *From functions A and B, each having one or the other of the two properties of the theorem, only functions having one of these properties can be defined by the three formulas*

$$Fx = Ax + Bx, \qquad Fx = BAx, \qquad Fx = B^x a.$$

PROOF. (1) If $Ax \geq x$ or $Bx \geq x$, then $Ax+Bx \geq x$. If both functions are bounded, then so also is their sum.

A PROOF OF TWO FUNDAMENTAL THEOREMS ON LINEAR TRANSFORMATIONS IN HILBERT SPACE, WITHOUT USE OF THE AXIOM OF CHOICE

I. BARSOTTI

The definitions and notations used in this paper may be found in the volume *Linear transformations in Hilbert space* by M. H. Stone,[1] which work I shall quote as S.

Many of the consequences of S, chaps. 4–9, are based on the existence of transformations of a particular kind called projections (S, Definition 2.16), and such existence is a consequence of S, Theorem 1.23, whose proof is based on S, Theorem 1.18. The last one is a particular case of a Hausdorff theorem[2] on abstract spaces, and its proof depends upon the axiom of choice. I shall show, without use of that axiom, that "every closed linear manifold $\mathfrak{M}$ in the Hilbert space $\mathfrak{H}$ has an orthogonal complement $\mathfrak{H}\ominus\mathfrak{M}$." The consistency of S, Definition 2.16 will thus be ensured.

Another theorem based on the axiom of choice is S, Theorem 2.25. This one I shall prove as well without the aid of the axiom.

It is necessary to make clear the meaning of the word "closed" applied to a subset $\mathfrak{M}$ of the Hilbert space $\mathfrak{H}$.

Let f be a point of $\mathfrak{H}$. f is said to be a *point of accumulation* of $\mathfrak{M}$ if for every real number $\epsilon>0$ there exists a point $g\neq f$ of $\mathfrak{M}$ such that $|g-f|<\epsilon$. $\mathfrak{M}$ is said to be *closed* when every point of accumulation of $\mathfrak{M}$ is a point of $\mathfrak{M}$.

An alternative definition is the following: f is a point of accumulation of $\mathfrak{M}$ if there exists an infinite sequence $\{g_i\}$ of points of $\mathfrak{M}$ such that $g_i\neq f$ $(i=1, 2, \cdots)$, $\lim_{i\to\infty} g_i=f$; $\mathfrak{M}$ is said to be closed when every point of accumulation of $\mathfrak{M}$ is a point of $\mathfrak{M}$.

It is well known that, if the axiom of choice is accepted, these two definitions are equivalent; if not, a subset $\mathfrak{M}$ closed according to the first definition is also closed according to the second one, but the converse is not necessarily true. Therefore, we shall assume the first meaning of the word "closed" to be the correct one.

In the logical development of the theory, Theorems 1.3 to 1.14 of S can be stated and proved: it must be remarked that in the proof of S, Theorem 1.13, a well determined orthonormal set $\{\phi_n\}$ is found. Therefore, when a sequence $\{f_n\}$ is given, it is possible to

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[1] Amer. Math. Soc. Colloquium Publications, vol. 15, 1932.

[2] Hausdorff, *Grundzüge der Mengenlehre*, Theorem VIII, p. 273.

choose, among the generally infinitely many orthonormal sets which determine the same linear manifold and the same closed linear manifold as $\{f_n\}$, a particular one, which will be called the orthonormal set *attached* to $\{f_n\}$. Let $\mathfrak{M}$ now be a closed linear manifold in $\mathfrak{H}$, $\mathfrak{S}$ a relatively open subset of $\mathfrak{M}$, in brief an *O-set* in $\mathfrak{M}$, that is, such that for every f of $\mathfrak{S}$ there exists a real $\epsilon>0$ such that if g is in $\mathfrak{M}$ and $|g-f|\leqq\epsilon$, then g belongs to $\mathfrak{S}$. The set consisting of the points of $\mathfrak{S}$ and of its points of accumulation is the closure of an O-set of $\mathfrak{M}$, in brief a *C-set* in $\mathfrak{M}$, and is still a subset of $\mathfrak{M}$.

The *sphere* with center $\mathfrak{S}$ and radius $r>0$ is the set consisting of the points g of $\mathfrak{H}$ such that $|g-f|<r$ for every f of $\mathfrak{S}$, whenever such set is not empty. The same definition holds when $\mathfrak{S}$ is a C-set in $\mathfrak{M}$; the set consisting of the points of sphere and of its points of accumulation will be named a *closed sphere.*

Recalling the definition of diameter of a set in a metric space,[3] it is easy to prove the following lemma.

(1) LEMMA. *Let $\mathfrak{S}$ be an O-set or a C-set in the closed linear manifold $\mathfrak{M}$. Then a necessary and sufficient condition that a closed sphere of center $\mathfrak{S}$ and radius r exist is that $r\geqq d$, where d is the diameter of $\mathfrak{S}$; and if d_1 is the diameter of the sphere, then $d\leqq d_1\leqq 2r$. A sphere (a closed sphere) is an O-set (a C-set) in $\mathfrak{H}$.*

(2) LEMMA. *Let $\{\mathfrak{S}_m\}$ be a sequence of C-sets in a closed linear manifold $\mathfrak{M}$, such that $\mathfrak{S}_{m+1}\subseteq\mathfrak{S}_m$ $(m=1, 2, \cdots)$. If $d_m>0$ is the diameter of $\mathfrak{S}_m$ and $\lim_{m\to\infty}d_m=0$, then there exists one (and not more than one) point belonging to every $\mathfrak{S}_m$.*[4]

PROOF. Let $\mathfrak{S}_m^{(1)}$ be the closed sphere with centre $\mathfrak{S}_m$ and radius d_m (1), and $\mathfrak{S}_m^{(2)}$ the intersection of $\mathfrak{S}_1^{(1)}, \mathfrak{S}_2^{(1)}, \cdots, \mathfrak{S}_m^{(1)}$. Then $\mathfrak{S}_m^{(2)}$ is a non-empty C-set in $\mathfrak{H}$, since $\mathfrak{S}_m\subseteq\mathfrak{S}_m^{(2)}$, and the relation $\mathfrak{S}_{m+1}^{(2)}\subseteq\mathfrak{S}_m^{(2)}$ is satisfied for $m=1, 2, \cdots$.

Let $\{f_m\}$ be a denumerably infinite set of points dense in $\mathfrak{H}$ (whose existence is asserted by S, Postulate D, p. 3) fixed once for all. If $\{f_m^{(1)}\}$ is the sequence of the elements of $\{f_m\}$ belonging to $\mathfrak{S}_1^{(1)}$, $\{f_m^{(1)}\}$ is surely not empty, and may be easily shown to be dense in $\mathfrak{S}_1^{(1)}=\mathfrak{S}_1^{(2)}$.

The sequence $\{f_m^{(2)}\}$ formed with the elements of $\{f_m\}$ belonging to $\mathfrak{S}_2^{(2)}$ is also not empty, is dense in $\mathfrak{S}_2^{(2)}$, and is a sub-sequence of $\{f_m^{(1)}\}$;

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[3] Kuratowski, *Topologie*, vol. 1, Warsaw, 1933, chap. 15, III, p. 85.

[4] This is a particular case of a theorem stated by Saks (*Theory of the integral*, 1937, chap. 2, Theorem 9.1, p. 54) for more general spaces, with the aid of the axiom of choice.

and the same properties are valid for the sequence $\{f_m^{(n)}\}$ chosen with respect to $\mathfrak{S}_m^{(n)}$ $(n=1, 2, \cdots)$. Let $f_{m(n)}$ be the first element, in the ordering of $\{f_m\}$, which belongs to $\{f_m^{(n)}\}$. If $\epsilon>0$ is given, there exists an integer M such that $d_M<\epsilon/2$, and if $n_1, n_2>M$, it follows that

$$\left| f_{m(n_1)} - f_{m(n_2)} \right| \leqq 2d_M < \epsilon.$$

Therefore, for S, Postulate E, p. 3, there exists an f such that $\lim_{n\to\infty} f_{m(n)}=f$. Since, for $n>M$, $f_{m(n)}$ belongs to $\mathfrak{S}_M^{(1)}$, and $\mathfrak{S}_M^{(1)}$ is closed, f belongs to $\mathfrak{S}_M^{(1)}$ as well, and then it belongs to every $\mathfrak{S}_m^{(1)}$.

When g is an arbitrarily chosen element of $\mathfrak{S}_M$, we have

$$\left| g - f \right| \leqq 2d_M < \epsilon,$$

and therefore f is a point of accumulation of $\mathfrak{M}$ according to our definition. $\mathfrak{M}$ being closed, we conclude that f belongs to $\mathfrak{M}$, and consequently to every $\mathfrak{S}_m$. The uniqueness of f is easily proved, Q.E.D.

(3) THEOREM. *Let $\mathfrak{M}$ be a closed linear manifold, f a point of $\mathfrak{H}$. Then there exists a (unique) point g_0 of $\mathfrak{M}$ such that $|f-g_0|<|f-g|$ for every $g\neq g_0$ of $\mathfrak{M}$.*

PROOF. When f belongs to $\mathfrak{M}$, it is $g_0=f$. Otherwise, let $d>0$ be the lower bound of the values of $|f-g|$ when g ranges over $\mathfrak{M}$, and let $\mathfrak{S}_m^{(0)}$ be the (non-empty) set consisting of the elements g of $\mathfrak{M}$ such that $d\leqq|g-f|<d+1/m$ $(m=1, 2, \cdots)$. When g_1 belongs to $\mathfrak{S}_m^{(0)}$ and g_2 to $\mathfrak{M}$, it follows from $|g_1-g_2|<\epsilon$ that

$$\left| g_2 - f \right| \leqq \left| g_2 - g_1 \right| + \left| g_1 - f \right| < \left| g_1 - f \right| + \epsilon,$$

so that if we choose $\epsilon<d+1/m-|g_1-f|$, the relation $|g_2-f|<d+1/m$ holds, showing that $\mathfrak{S}_m^{(0)}$ is a 0-set in $\mathfrak{M}$. Therefore the set $\mathfrak{S}_m$ consisting of the points g of $\mathfrak{M}$ for which the relation

$$d \leqq \left| g - f \right| \leqq d + 1/m \qquad (m = 1, 2, \cdots)$$

holds, is a C-set in $\mathfrak{M}$, and we have, of course, $\mathfrak{S}_{m+1}\subseteq\mathfrak{S}_m$ $(m=1, 2, \cdots)$. Let now g_1, g_2 be distinct elements of $\mathfrak{S}_m$. We have:[5] $|g_2-f|^2=|g_1-f|^2+|g_2-g_1|^2+2\mathfrak{R}[(g_1-f)\times(g_2-g_1)]\leqq(d+1/m)^2$, where $\mathfrak{R}[x]$ is the real part of x, and also $-|g_1-f|^2\leqq-d^2$. This inequality, added to the former one, gives $|g_2-g_1|^2+2\mathfrak{R}[(g_1-f)\times(g_2-g_1)]\leqq(1/m)(2d+1/m)$. From this, if $\mathfrak{R}[(g_1-f)\times(g_2-g_1)]\geqq 0$, it follows that $|g_2-g_1|\leqq((1/m)(2d+1/m))^{1/2}$. If $\mathfrak{R}[(g_1-f)\times(g_2-g_1)]<0$ but $\mathfrak{R}[(g_2-f)\times(g_1-g_2)]\geqq 0$, we may reach the same result by interchanging g_1 and g_2.

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[5] I am using $f\times g$ instead of (f, g).

Let now $\Re[(g_1-f)\times(g_2-g_1)]<0$ and $\Re[(g_2-f)\times(g_1-g_2)]<0$, and put $g_3=g_1+\alpha(g_2-g_1)$, where

$$0<\alpha=\frac{-\Re[(g_1-f)\times(g_2-g_1)]}{|g_2-g_1|^2}.$$

We may also put

$$0<\beta=\frac{-\Re[(g_2-f)\times(g_1-g_2)]}{|g_1-g_2|^2},$$

and so $\alpha+\beta=1$, $g_3=g_2+\beta(g_1-g_2)$. From the comparison of these two expressions obtained for g_3, we are enabled to say that every equality or inequality involving α, β, g_1, g_2, g_3 will still remain true after having interchanged α with β, g_1 with g_2. Now, we have:

$$\begin{aligned}|g_3-f|^2&=|g_1-f|^2+\alpha^2|g_2-g_1|^2+2\alpha\Re[(g_1-f)\times(g_2-g_1)]\\&=|g_1-f|^2+\frac{\Re^2[(g_1-f)\times(g_2-g_1)]}{|g_2-g_1|^2}\\&\qquad-2\frac{\Re^2[(g_1-f)\times(g_2-g_1)]}{|g_2-g_1|^2}\\&=|g_1-f|^2-\frac{\Re^2[(g_1-f)\times(g_2-g_1)]}{|g_2-g_1|^2}\\&<|g_1-f|^2\leqq\left(d+\frac{1}{m}\right)^2,\end{aligned}$$

and therefore g_3 belongs to $\mathfrak{S}_m$. Moreover:

$$\begin{aligned}&\Re[(g_3-f)\times(g_1-g_3)]\\&\quad=\Re[(g_1-f)\times(g_1-g_3)]+\alpha\Re[(g_2-g_1)\times(g_1-g_3)]\\&\quad=\alpha\Re[(g_1-f)\times(g_1-g_2)]+\alpha^2\Re[(g_2-g_1)\times(g_1-g_2)]\\&\quad=\alpha\{-\Re[(g_1-f)\times(g_2-g_1)]+\Re[(g_1-f)\times(g_2-g_1)]\}=0,\end{aligned}$$

so that we may apply to g_1 and g_3 the result obtained above. Thus $|g_3-g_1|\leqq((1/m)(2d+1/m))^{1/2}$, and interchanging g_1 with g_2, we have $|g_3-g_2|\leqq((1/m)(2d+1/m))^{1/2}$. Therefore, under any circumstance, $|g_1-g_2|\leqq 2((1/m)(2d+1/m))^{1/2}$.

We have now shown that the diameter of $\mathfrak{S}_m$ converges to 0 when $m\rightarrow\infty$, and that the requirements of Lemma (2) are fulfilled for the sequence $\{\mathfrak{S}_m\}$. The lemma gives an element g_0 belonging to every $\mathfrak{S}_m$, that is, such that $|g_0-f|\leqq d+1/m$ for $m=1, 2, \cdots$. Hence

$|g_0-f|=d$. It may easily be shown that $|g-f|>d$ if $g\neq g_0$, Q.E.D.

(4) THEOREM. *A closed linear manifold is separable.*

PROOF. Let $\mathfrak{M}$ be a closed linear manifold, $\{f_n\}$ a sequence dense in $\mathfrak{H}$. For (3) it is possible to find, for each f_n, a point $g_0^{(n)}$ in $\mathfrak{M}$ such that $|f_n-g_0^{(n)}|<|f_n-g|$ for every $g\neq g_0^{(n)}$ of $\mathfrak{M}$. We want to show that $\{g_0^{(n)}\}$ is dense in $\mathfrak{M}$. In fact, for every g in $\mathfrak{M}$ and $\epsilon>0$, there exists an integer n such that $|g-f_n|<\epsilon/2$. Then $|g_0^{(n)}-f_n|\leq|g-f_n|<\epsilon/2$, $|g_0^{(n)}-g|\leq|g_0^{(n)}-f_n|+|f_n-g|<\epsilon$, Q.E.D.

When the closed linear manifold $\mathfrak{M}$ is given, and $\{g_0^{(n)}\}$ is built according to the proof of (4), let $\{\phi_n\}$ be the orthonormal set attached to $\{g_0^{(n)}\}$. $\{\phi_n\}$ determines the closed linear manifold $\mathfrak{M}$, and will be said to be *attached* to $\mathfrak{M}$.

Now we are able to prove the fundamental theorem:

(5) THEOREM. *Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be closed linear manifolds, and $\mathfrak{M}_1\subset\mathfrak{M}_2$; then the orthogonal complement $\mathfrak{M}_3=\mathfrak{M}_2\ominus\mathfrak{M}_1$ (that is, the set of all the elements of $\mathfrak{M}_2$ which are orthogonal to every element of $\mathfrak{M}_1$) is a non-empty closed linear manifold. Furthermore, if f_2 is a point of $\mathfrak{M}_2$, then $f_2=f_1+f_3$, with f_1 in $\mathfrak{M}_1$, f_3 in $\mathfrak{M}_3$, uniquely determined.*

PROOF. Let $\{\phi_n\}$ be the orthonormal set attached to $\mathfrak{M}_1$, and

$$f_1=\sum_1^\infty{}_\alpha(f_2\times\phi_\alpha)\phi_\alpha;$$

according to S, Theorem 1.7, $f_3=f_2-f_1$ is orthogonal to $\mathfrak{M}_1$, hence an element of $\mathfrak{M}_3$. The other parts of the proof may be omitted, as they are very simple.

Thus the first aim pointed out in the beginning of this paper is reached. We may add, in case $\mathfrak{M}_2=\mathfrak{H}$ in (5), that f_1 is the *projection of $f_2=f$* on $\mathfrak{M}_1$, and that the element g_0 of (3) is the projection of f on $\mathfrak{M}$.

We now want to prove the following theorem.

(6) THEOREM. *If H is a symmetric transformation whose domain is $\mathfrak{H}$, then H is bounded and therefore continuous.*

PROOF. (I) The proof consists of a deeper development of the one given in S, p. 59. Let $f_1, \cdots, f_m$ be arbitrary elements of $\mathfrak{H}$, $\mathfrak{M}_1$ the closed linear manifold determined by $f_1, \cdots, f_m$, $Hf_1, \cdots, Hf_m$, $H^2f_1, \cdots, H^2f_m$, and $\mathfrak{M}_2=\mathfrak{H}\ominus\mathfrak{M}_1$. If f is an arbitrary element of $\mathfrak{M}_2$, we have $f\times f_i=f\times Hf_i=Hf\times f_i=f\times H^2f_i=Hf\times Hf_i=0$ $(i=1, 2, \cdots, m)$. Let $\{f_n^{(1)}\}$ be a sequence dense in $\mathfrak{H}$, and g_n the projection of $f_n^{(1)}$ on $\mathfrak{M}_2$, and let $g_{n(m)}$ be the first element of $\{g_n\}$ satisfying the relation $|Hg_{n(m)}| \geqq m|g_{n(m)}|$, when such an element exists. Then $\phi_{m+1} = g_{n(m)}/|g_{n(m)}|$ is normal, and ϕ_{m+1}, $H\phi_{m+1}$ are orthogonal to $f_1, \cdots, f_m, Hf_1, \cdots, Hf_m$.

(II) If an arbitrary normal element ϕ_1 is chosen, let us put $m=1$, $f_1=\phi_1$ in (I), and determine, if possible, an element ϕ_2 with the method outlined in (I). Then put $m=2, f_1=\phi_1, f_2=\phi_2$, and determine a ϕ_3, and so on, as far as possible. Two cases may occur, which we are going to discuss separately.

Case A: the process may be carried on indefinitely. When this is the case, the elements which have been determined form an orthonormal set $\{\phi_n\}$ with cardinal number $\aleph_0$, and such that

$$H\phi_i \times H\phi_k = 0 \quad \text{if} \quad i \neq k, \quad |H\phi_k| \geqq k.$$

If we set

$$k_n = \sum_{1}^{n}{}_{\alpha} \frac{1}{\alpha}\phi_\alpha,$$

the sequence $\{k_n\}$ converges to some k, according to S, Theorem 1.6, and we have:

$$|Hk|^2 = k \times H^2k = \lim_{n\to\infty} (k_n \times H^2k),$$

$$k_n \times H^2k = H^2k_n \times k = \sum_{1}^{n}{}_{\alpha} \sum_{1}^{\infty}{}_{\beta} \frac{1}{\alpha\beta} H^2\phi_\alpha \times \phi_\beta$$

$$= \sum_{1}^{n}{}_{\alpha} \sum_{1}^{\infty}{}_{\beta} \frac{1}{\alpha\beta} H\phi_\alpha \times H\phi_\beta = \sum_{1}^{n}{}_{\alpha} \frac{1}{\alpha^2} |H\phi_\alpha|^2 \geqq n,$$

so that $|Hk|^2 \geqq n$ for every n. This is absurd.

Case B: the process cannot be carried on indefinitely. If this is the case, there exists an orthonormal set $\phi_1, \cdots, \phi_m$ which determines a closed linear manifold $\mathfrak{M}_1$ such that, if $\mathfrak{M}_2 = \mathfrak{H} \ominus \mathfrak{M}_1$, the projections $g_n^{(2)}$ $(n=1, 2, \cdots)$ of the elements $f_n^{(1)}$ on $\mathfrak{M}_2$ satisfy the limitations $|Hg_n^{(2)}| < m|g_n^{(2)}|$. If f is an arbitrary element of $\mathfrak{M}_1$, we have

$$f = x_1\phi_1 + \cdots + x_m\phi_m, \qquad |f|^2 = |x_1|^2 + \cdots + |x_m|^2,$$

$$|Hf|^2 = \sum_{1}^{m}{}_{\alpha,\beta}\, x_\alpha \bar{x}_\beta H\phi_\alpha \times H\phi_\beta \leqq P \sum_{1}^{m}{}_{\alpha,\beta} |x_\alpha|\,|x_\beta|$$

$$\leqq \frac{1}{2} P \sum_{1}^{m}{}_{\alpha,\beta} (|x_\alpha|^2 + |x_\beta|^2) = \frac{m+1}{2} P |f|^2,$$

where P is a suitable positive number. If $g_n^{(1)}$ is the projection of $f_n^{(1)}$ on $\mathfrak{M}_1$, we have

$$f_n^{(1)} = g_n^{(1)} + g_n^{(2)}, \qquad |f_n^{(1)}|^2 = |g_n^{(1)}|^2 + |g_n^{(2)}|^2,$$
$$|g_n^{(1)}| \leq |f_n^{(1)}|, \qquad |g_n^{(2)}| \leq |f_n^{(1)}|,$$

and

$$\begin{aligned}
|Hf_n^{(1)} \times f_n^{(1)}| &= \left| \sum_{1}^{2}{}_{\alpha,\beta} Hg_n^{(\alpha)} \times g_n^{(\beta)} \right| \leq \sum_{1}^{2}{}_{\alpha,\beta} |Hg_n^{(\alpha)} \times g_n^{(\beta)}| \\
&\leq \sum_{1}^{2}{}_{\alpha,\beta} |Hg_n^{(\alpha)}|\,|g_n^{(\beta)}| \leq \sum_{1}^{2}{}_{\alpha,\beta} K_\alpha |g_n^{(\alpha)}|\,|g_n^{(\beta)}| \\
&\leq \frac{1}{2}(K_1 + K_2) \sum_{1}^{2}{}_{\alpha,\beta} (|g_n^{(\alpha)}|^2 + |g_n^{(\beta)}|^2) \\
&= \frac{3}{2}(K_1 + K_2)|f_n^{(1)}|^2,
\end{aligned}$$

where $K_1 = ((m+1)/2)P$, $K_2 = m$. If we set $3(K_1+K_2)/2 = K$, the last inequality may be written $Hf_n^{(1)} \times f_n^{(1)}| \leq K|f_n^{(1)}|^2$.

If we repeat the process replacing H with H^2, and introducing a new constant Q^2, we obtain:

$$|H^2 f_n^{(1)} \times f_n^{(1)}| = |Hf_n^{(1)} \times Hf_n^{(1)}| \leq Q^2 |f_n^{(1)}|^2,$$
$$|Hf_n^{(1)}| \leq Q|f_n^{(1)}| \qquad (Q > 0).$$

Now let f be an arbitrary element of $\mathfrak{H}$. We have $|f_n^{(1)} \times Hf| = |Hf_n^{(1)} \times f| \leq |Hf_n^{(1)}|\ |f| \leq Q|f_n^{(1)}|\ |f|$, and if g is another arbitrary element, and $\epsilon > 0$, there exists an integer n such that $|g - f_n^{(1)}| < \epsilon$. It follows that

$$\begin{aligned}
|g \times Hf| &\leq |f_n^{(1)} \times Hf| + |(g - f_n^{(1)}) \times Hf| \\
&\leq Q|f_n^{(1)}|\,|f| + |g - f_n^{(1)}|\,|Hf| \\
&< Q|f_n^{(1)}|\,|f| + \epsilon|Hf| \\
&\leq Q|g|\,|f| + Q|f_n^{(1)} - g|\,|f| + \epsilon|Hf| \\
&< Q|g|\,|f| + \epsilon(Q|f| + |Hf|),
\end{aligned}$$

and thus that $|g \times Hf| \leq Q|g|\,|f|$. With this result, and by S, Theorems 2.21, 2.22, the proof is completed, Q.E.D.

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CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$

R. P. BAMBAH and S. CHOWLA

Introduction. With Ramanujan we define $\tau(n)$ by

$$\sum_{1}^{\infty} \tau(n)x^n = x\prod_{1}^{\infty}(1 - x^n)^{24} \qquad (|x| < 1).$$

Write $\sigma_k(n)$ for the sum of the kth powers of the divisors of n; $\sigma(n)=\sigma_1(n)$. It is known that[1]

$$\tau(n) \equiv n\sigma(n) \pmod 5,$$
$$\tau(n) \equiv \sigma(n) \pmod 3 \qquad \text{if } (n, 3) = 1.$$

The object of this note is to give proofs of the much stronger results:

$$\text{(A)} \qquad \tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3}$$

when n is prime to 5;

$$\text{(B)} \qquad \tau(n) \equiv (n^2 + k)\sigma_7(n) \pmod{3^4}$$

when n is prime to 3 and where $k=0$ if $n\equiv 1(3)$, $k=9$ if $n\equiv 2(3)$.

1. Some lemmas.

LEMMA 1. *We have*

$$\sum u\sigma_3(u)\sigma_5(v) \equiv \sum \sigma(u)\sigma(v) - P(n) \pmod 5$$

where

$$P(n) = \sum_{u\equiv 0 \pmod 5} \sigma(u)\sigma(v)$$

where $u+v=n$; $u, v\geqq 1$ *in all three sums* ($\sum$).

PROOF. We have

$$(1) \qquad u\sigma_3(u)\sigma_5(v) \equiv 0 \pmod 5 \quad \text{when} \quad u \equiv 0(5);$$

when $(u, 5)=1$ we have

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[1] The first of these is proved in Hardy's *Ramanujan* (Cambridge, 1940); the second by Gupta in J. Indian Math. Soc. vol. 9 (1945) pp. 59–60. In what follows we refer to Ramanujan's *Collected papers* (Cambridge, 1927) by the letters RCP. We have also proved that $\tau(n)\equiv\sigma_{11}(n) \pmod{2^8}$ if n is odd; this result has been accepted for publication in J. London Math. Soc.

$$u\sigma_3(u) = u\sum_{d|u} d^3 \equiv u\sum_{d|u} d^{-1} = \sum_{d|u} \frac{u}{d} = \sigma(u),$$

so that

$$u\sigma_3(u) \equiv \sigma(u) \pmod 5 \qquad \text{when } (u, 5) = 1. \tag{2}$$

Similarly

$$\sigma_5(v) \equiv \sigma(v) \pmod 5. \tag{3}$$

From (1), (2), (3):

$$\sum u\sigma_3(u)\sigma_5(v) \equiv \sum_{(u,5)=1} \sigma(u)\sigma(v) \equiv \sum \sigma(u)\sigma(v) - P(n) \pmod 5.$$

LEMMA 2. *If* $(n, 5)=1$ *we have*

$$\sum uv\sigma_3(u)\sigma_3(v) \equiv \sum \sigma(u)\sigma(v) - 2P(n) \pmod 5$$

where, as in Lemma 1, *the conditions*

$$u + v = n, \qquad u, v \geqq 1$$

are understood in every $\sum$.

PROOF. If u or $v \equiv 0(5)$, $uv\sigma_3(u)\sigma_3(v) \equiv 0(5)$. From this and (1) we get since $(n, 5)=1$,

$$\sum uv\sigma_3(u)\sigma_3(v) \equiv \sum_{(u,5)=1,\,(v,5)=1} \sigma(u)\sigma(v)$$

$$\equiv \sum \sigma(u)\sigma(v) - 2P(n) \pmod 5,$$

the desired result.

2. **Proof of (A).** Write, for x numerically less than unity,

$$P = 1 - 24\sum_1^\infty \sigma(n)x^n,$$

$$Q = 1 + 240\sum_1^\infty \sigma_3(n)x^n,$$

$$R = 1 - 504\sum_1^\infty \sigma_5(n)x^n.$$

Then (44), p. 144 of RCP, is

$$Q^3 - R^2 = 1728\sum_1^\infty \tau(n)x^n \tag{4}$$

and we deduce from relations 5 and 2, Table II, p. 142 of RCP, that

$$
\begin{aligned}
1584\sum_{1}^{\infty} n\sigma_9(n)x^n
&= 3(Q^3 - R^2) - 5R(PQ - R)\\
&= 5184\sum_{1}^{\infty}\tau(n)x^n - 5\left(1 - 504\sum_{1}^{\infty}\sigma_5(n)x^n\right)(PQ - R)\\
&= 5184\sum_{1}^{\infty}\tau(n)x^n - 5\left(1 - 504\sum_{1}^{\infty}\sigma_5(n)x^n\right)720\sum_{1}^{\infty} n\sigma_3(n)x^n.
\end{aligned}
$$

Comparing coefficients of x^n and using Lemma 1 we have

(5) $1584n\sigma_9(n) = 5184\tau(n) - 3600n\sigma_3(n) + 5\cdot 504\cdot 720\sum u\sigma_3(u)\sigma_5(v)$

(where $u+v=n$ $(u, v \geq 1)$ in the $\sum$ sum),

$$
(6)\qquad
\begin{aligned}
84n\sigma_9(n) \equiv{}& 59\tau(n) + 25n\sigma_3(n)\\
&+ 25\sum\sigma(u)\sigma(v) - 25P(n) \pmod{125}.
\end{aligned}
$$

Again, relations 4, Table III, and 2, Table II, p. 142 of RCP give us

$$
(6')\qquad
\begin{aligned}
8640\sum_{1}^{\infty} n^2\sigma_7(n)x^n &= 5(Q^3 - R^2) + 9(PQ - R)^2\\
&= 8640\sum_{1}^{\infty}\tau(n)x^n + 9\cdot 720^2\left\{\sum_{1}^{\infty} n\sigma_3(n)x^n\right\}^2.
\end{aligned}
$$

Comparing the coefficients of x^n here we get

(7) $$n^2\sigma_7(n) = \tau(n) + 135\cdot 4\sum u\sigma_3(u)v\sigma_3(v)$$

or

(7′) $$15n^2\sigma_7(n) \equiv 15\tau(n) - 25\sum uv\sigma_3(u)\sigma_3(v) \pmod{125}.$$

From (7′) and Lemma 2 we get

(8) $$15n^2\sigma_7(n) \equiv 15\tau(n) - 25\sum\sigma(u)\sigma(v) + 50P(n) \pmod{125}.$$

Eliminating $P(n)$ from (6) and (8) we get

$$
\begin{aligned}
168n\sigma_9(n) + 15n^2\sigma_7(n) \equiv{}& 133\tau(n) + 50n\sigma_3(n)\\
&+ 25\sum\sigma(u)\sigma(v) \pmod{125},
\end{aligned}
$$

or

$$
(9)\qquad
\begin{aligned}
8\tau(n) \equiv{}& 43n\sigma_9(n) + 15n^2\sigma_7(n) - 50n\sigma_3(n)\\
&- 25\sum\sigma(u)\sigma(v) \pmod{125}.
\end{aligned}
$$

Again (relation 1, Table IV, p. 146 of RCP)

$$\sum \sigma(u)\sigma(v) = \frac{5\sigma_3(n) - 5n\sigma(n)}{12} - \frac{(n-1)\sigma(n)}{12} \equiv 2(n-1)\sigma(n) \pmod 5. \tag{10}$$

From (9) and (10) we obtain

$$\begin{aligned} 8\tau(n) \equiv{} & 43n\sigma_9(n) + 15n^2\sigma_7(n) \\ & - 50n\sigma_3(n) - 50(n-1)\sigma(n) \pmod{5^3}. \end{aligned}$$

Hence, multiplying by 47,

$$\begin{aligned} \tau(n) &\equiv 21n\sigma_9(n) - 45n^2\sigma_7(n) + 25n\sigma_3(n) \\ &\qquad + 25(n-1)\sigma(n) \pmod{5^3} \\ &\equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3} \end{aligned} \tag{11}$$

for

$$\begin{aligned} 25n\sigma_9(n) - 50n^2\sigma_7(n) + 25n\sigma_3(n) + 25(n-1)\sigma(n) &= 50\{n\sigma_9(n) - n^2\sigma_7(n)\} \\ &\quad + 25n\{\sigma(n) - \sigma_9(n)\} + 25\{n\sigma_3(n) - \sigma(n)\} \\ &\equiv 0 \pmod{125}, \end{aligned}$$

since the terms inside each set of braces are a multiple of 5 provided $(n, 5) = 1$. Thus (A) is proved by (11).

3. **Proof of (B).** We shall need the following results:

$$\sigma(3t + 2) \equiv 0 \ (3), \tag{12}$$

$$\sigma_3(3t + 2) \equiv 0 \pmod 9 \tag{13}$$

where t is 0 or a positive integer. To prove (13) we observe that to every divisor $3m+1$ of $3t+2$, there corresponds another $3n+2 = (3t+2)/(3m+1)$, and

$$(3m + 1)^3 + (3n + 2)^3 \equiv 0(9);$$

while (12) is proved still more simply. We next prove the following lemma.

LEMMA 3. *If* $n \equiv 1(3)$, *we have*

$$\sum uv\sigma_3(u)\sigma_3(v) \equiv 0(3)$$

where (in the summation $\sum$*)* $u+v=n$ *and* $u, v \geq 1$.

PROOF. Since $n \equiv 1(3)$ and $u+v=n$, we have the 3 cases:

$$u \equiv 0(3), \qquad v \equiv 1(3),$$
$$u \equiv 1(3), \qquad v \equiv 0(3),$$
$$u \equiv 2(3), \qquad v \equiv 2(3),$$

so that $uv\sigma_3(u)\sigma_3(v) \equiv 0(3)$ in each case on account of (13). Hence the lemma is proved.

LEMMA 4. *If* $n \equiv 2(3)$, *we have*

$$\sum_{u+v=n,\, u,v\geqq 1} uv\sigma_3(u)\sigma_3(v) \equiv \frac{\sigma_7(n) - \sigma_3(n)}{120} \pmod{3}.$$

PROOF. If $u+v=n$, $n\equiv 2(3)$, we have 3 cases:

(i) $u \equiv 0(3), \qquad v \equiv 2(3),$

(ii) $u \equiv 2(3), \qquad v \equiv 0(3),$

(iii) $u \equiv 1(3), \qquad v \equiv 1(3).$

In the first two cases

$$uv\sigma_3(u)\sigma_3(v) \equiv 0 \pmod{3};$$

while in the third case

$$uv\sigma_3(u)\sigma_3(v) \equiv \sigma_3(u)\sigma_3(v) \pmod{3}.$$

Hence we have (in the sums $u+v=n$; u, $v\geqq 1$), using (13),

$$\sum uv\sigma_3(u)\sigma_3(v) \equiv \sum_{u,v\equiv 1(3)} \sigma_3(u)\sigma_3(v) \pmod{3}$$

$$\equiv \sum \sigma_3(u)\sigma_3(v) \equiv \frac{\sigma_7(n) - \sigma_3(n)}{120} \pmod{3}$$

since (relation 3, Table IV of RCP, p. 146)

$$\sum \sigma_3(u)\sigma_3(v) = \frac{\sigma_7(n) - \sigma_3(n)}{120} \tag{14}$$

where, in the $\sum$, $u+v=n$ $(u, v\geqq 1)$.

We are now ready to prove (B). Comparing the coefficients of x^n in (6′) we obtain

$$27.320n^2\sigma_7(n) = 27.320\tau(n) + 27^2\cdot 80^2 \sum_{u+v=n,\, u,v\geqq 1} uv\sigma_3(u)\sigma_3(v). \tag{15}$$

We, therefore, have

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}. \tag{16}$$

Case 1. $n \equiv 1(3)$. In this case (15) and Lemma 3 give

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^4}. \tag{17}$$

Case 2. $n \equiv 2 \pmod 3$. In this case (15) and Lemma 4 give

$$\tau(n) \equiv n^2\sigma_7(n) - \frac{27\cdot 20}{120}\left\{\sigma_7(n) - \sigma_3(n)\right\} \pmod{3^4}$$

or

$$\begin{aligned}\tau(n) &\equiv (n^2 + 36)\sigma_7(n) \pmod{3^4}\\ &\equiv (n^2 + 9)\sigma_7(n) \pmod{3^4}\end{aligned} \tag{18}$$

since, when $n \equiv 2(3)$, we have

$$\sigma_7(n) \equiv \sigma(n) \equiv 0(3),$$
$$\sigma_3(n) \equiv 0(9),$$

from (12) and (13).

(17) and (18) together give (B).

Mordell proved Ramanijan's conjecture

$$\tau(mn) = \tau(m)\tau(n) \qquad \text{if } (m, n) = 1.$$

From this result or directly we can prove that

$$\tau(n) \equiv 16n\sigma_9(n) \pmod{5^3} \qquad \text{if } n \equiv 0(5), \tag{C}$$

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^4} \qquad \text{if } n \equiv 0(3). \tag{D}$$

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ADDITION TO MY NOTE ON SEMI-SIMPLE RINGS

OSCAR GOLDMAN

In my Bulletin note on semi-simple rings[1] I made use of the following definition of the radical of a ring which I attributed to C. Chevalley: "The radical of a ring A is the intersection of the annihilators of all simple A-modules." Recently N. Jacobson has called my attention to the fact that the radical thus defined coincides with the one considered by him,[2] and that Chevalley's statement can easily be shown to be equivalent with the following characterization of the radical by Jacobson:[2,3] "If A is not a radical ring, then the radical of A is the intersection of all the primitive ideals contained in A."

To see the relation between the two statements, we need to make use of Jacobson's characterization of a primitive ideal as a proper ideal B such that the factor ring A/B is isomorphic with a simple ring of endomorphisms. From this it is clear that B is primitive if, and only if, B is proper and is the annihilator of a simple A-module. If the word "proper" is dropped from the definition of a primitive ideal, then the concept of primitive ideal is equivalent to that of annihilator of a simple A-module. Hence Chevalley's definition is essentially the same as Jacobson's characterization.[4]

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[1] *A characterization of semi-simple rings*, Bull. Amer. Math. Soc. vol. 52 (1946) p. 1021.

[2] N. Jacobson, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. vol. 67 (1945) p. 301.

[3] N. Jacobson, *A topology for the set of primitive ideals in an arbitrary ring*, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) p. 333.

[4] It is to be noted that, as a result of this equivalence, Theorems I and II of my note become superfluous. See Theorems V, IX, and XXV in footnote 2 above.

A NOTE ON THE SCHMIDT-REMAK THEOREM

FRED KIOKEMEISTER

Let G be a group with operator domain Ω. We shall say that G satisfies the modified maximal condition for Ω-subgroups if the chain $H_1 \subset H_2 \subset \cdots \subset H \neq G$ is finite whenever $H_1, H_2, \cdots, H$ are Ω-subgroups of G.

Let $A_1, A_2, \cdots$ be a countable set of groups. The direct product of $A_1, A_2, \cdots$ will be defined to be the set of elements $(a_1, a_2, \cdots)$ where a_i is an element of A_i for $i = 1, 2, \cdots$, and where but a finite number of the a_i are not the identity elements of the groups in which they lie. A product in the group is defined by the usual componentwise composition of two elements. This group will have the symbol $A_1 \times A_2 \times \cdots$.

The following theorem is in a sense a generalization of the Schmidt-Remak theorem.

THEOREM. *Let G be a group with operator domain Ω, and let Ω contain the inner automorphisms of G. Let $G = A_1 \times A_2 \times \cdots$ where each of the Ω-subgroups A_i is directly indecomposable, and each satisfies the minimal condition and the modified maximal condition for Ω-subgroups. Then if $G = B_1 \times B_2 \times \cdots$ is a second direct product decomposition of G into indecomposable factors, the number of factors will be the same as the number of the A_i. Further the A_i may be so rearranged that $A_i \cong B_i$, and for any j*

$$G = B_1 \times B_2 \times \cdots \times B_j \times A_{j+1} \times A_{j+2} \times \cdots .$$

A proof of the theorem can be based on any standard proof of the Schmidt-Remak theorem such as that given by Jacobson[1] or by Zassenhaus[2] with but slight changes in the two fundamental lemmas.

We state the following lemmas for a group G with operator domain Ω, and we assume that for G and Ω:

(1) Ω contains all inner automorphisms of G.

(2) G satisfies the minimal condition and the modified maximal condition for Ω-subgroups.

(3) G is indecomposable into the direct product of Ω-subgroups.

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[1] Nathan Jacobson, *The theory of rings*, Mathematical Surveys, vol. 2, New York, 1943.

[2] H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Leipzig, 1937.

LEMMA 1. *Let α be an Ω-operator of G. If there exists in G an element h not equal to the identity of G such that $h^{\alpha}=h$, then α is an automorphism of G.*

This lemma follows by the usual arguments. It is only necessary to note that the fixed point h is sufficient to guarantee that the union of the kernels of the operators $\alpha, \alpha^2, \cdots$ is not G, and that the modified maximal condition then yields that this union is the kernel of some α^k.

LEMMA 2. *Let $\alpha_1, \alpha_2, \cdots$ be addible Ω-operators such that if g is an element of G, then there exists an integer $N(g)$ such that $g^{\alpha_i}=e$, the identity element of G, for all $i>N(g)$. If $\alpha=\alpha_1+\alpha_2+\cdots$ is an automorphism of G then, for some k, α_k is an automorphism of G.*

Let g be an element of G, $g\neq e$. Let $\beta_1=\alpha_1+\alpha_2+\cdots+\alpha_N$, $\beta_2=\alpha_{N+1}+\alpha_{N+2}+\cdots$ where $N=N(g)$. Thus $\alpha=\beta_1+\beta_2$ and $g^{\beta_2}=e$. We may assume that α is the identity operator. Then $g=g^{\alpha}=g^{\beta_1}g^{\beta_2}=g^{\beta_1}$. The group G and the operator β_1 satisfy the conditions of Lemma 1, and β_1 is an automorphism of G.

Similarly let $\gamma=\alpha_1+\alpha_2+\cdots+\alpha_{N-1}$. Then $\beta_1=\gamma+\alpha_N$. We may assume that β_1 is the identity operator. If α_N is not an automorphism of G, the kernel of α_N must contain an element $h\neq e$, since G satisfies the minimal condition. Again we may show that γ is an automorphism of G. A repetition of this argument establishes the lemma.

By reference to Lemma 2 the cited proofs of the Schmidt-Remak theorem can be made to yield the following: To each B_i there corresponds a group A_{α_i} where α_i is a positive integral subscript such that $\alpha_i=\alpha_k$ implies $i=k$ and A_{α_i} is operator isomorphic with B_i for all i. Further

$$G = B_1 \times B_2 \times \cdots \times B_j \times A_{\beta_1} \times A_{\beta_2} \times \cdots$$

where $\beta_n\neq\alpha_i$ for any n or i, and where the set of integers $\{\alpha_1, \alpha_2, \cdots, \alpha_j, \beta_1, \beta_2, \cdots\}$ is the set of all positive integers. Let A_m contain the element $g\neq e$. Then for some M, g is an element of the group $B_1\times B_2\times\cdots\times B_M$, and since

$$(B_1 \times B_2 \times \cdots \times B_M) \cap (A_{\beta_1} \times A_{\beta_2} \times \cdots) = e,$$

$m\neq\beta_k$ for all k. Thus for some i, $1\leq i\leq M$, we have $m=\alpha_i$, and the set of integers $\{\alpha_1, \alpha_2, \cdots\}$ includes all subscripts. There then exists a reordering of these subscripts such that $\alpha_i=i$.

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ALGEBRAS AND THEIR SUBALGEBRAS

A. H. DIAMOND AND J. C. C. McKINSEY

There exist sets of postulates for Boolean algebras which consist entirely of universal sentences[1] in three variables. From this the following statement clearly follows: If an algebra has the property that every subalgebra generated by three elements is a Boolean algebra, then the algebra is itself a Boolean algebra.

The question arises, whether the above statement can be strengthened to read: If an algebra has the property that every subalgebra generated by two elements is a Boolean algebra, then the algebra is itself a Boolean algebra. This question can be answered in the negative by the following theorem:

THEOREM 1. *There exists an algebra* Γ *which is not a Boolean algebra, but such that every subalgebra of* Γ *with two generators is a Boolean algebra.*

PROOF. If x and y are any entities, then by "$\{x, y\}$" we shall mean the set whose only members are x and y; and by "$\langle x, y\rangle$" we shall mean the ordered couple whose first member is x and whose second member is y.

Let H be the intersection of all sets K which satisfy the following conditions:

(i) $2\in K$, $3\in K$, and $4\in K$;

(ii) If x and y are any distinct elements of K, neither of which belongs to the other, then $\{x, y\}\in K$.

Let H_1 be the class which contains all the members of H, and in addition the number 5. (Thus H_1 has just one more member than has H—namely 5.)

Let G be the class of all ordered couples $\langle\alpha, \beta\rangle$ where α is any member of H_1, and β is either 0 or 1.

We wish now to define a unary operation $-$, and two binary operations $+$ and $\cdot$, over the set G, in such a way that the system $\Gamma=\langle G, +, \cdot, -\rangle$ will not be a Boolean algebra, but every subalgebra of Γ with two generators will be a Boolean algebra.

If $\langle\alpha, \beta\rangle$ is any member of G, then we set

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[1] By a universal sentence we mean here a sentential function without any bound variables ("for all x," "there exists an x," and so on)—or a sentence obtained from such a sentential function by prefixing universal quantifiers. We do not impose the condition that a Boolean algebra must contain at least two elements.

$$-\langle \alpha, \beta\rangle = \langle \alpha, 1-\beta\rangle,$$

where the minus sign in the right member denotes ordinary arithmetical subtraction.

If α and β are distinct members of H such that $\alpha \notin \beta$ and $\beta \notin \alpha$, then we set

$$\begin{aligned}
&\langle \alpha, 0\rangle + \langle \beta, 0\rangle = \langle \{\alpha, \beta\}, 0\rangle,\\
&\langle \alpha, 1\rangle + \langle \beta, 1\rangle = \langle 5, 1\rangle,\\
&\langle \alpha, 1\rangle + \langle \beta, 0\rangle = \langle \beta, 0\rangle + \langle \alpha, 1\rangle = \langle \alpha, 1\rangle.
\end{aligned}$$

If α and β are distinct members of H such that $\beta = \{\alpha, \gamma\}$ for some γ, then we set

$$\begin{aligned}
&\langle \alpha, 0\rangle + \langle \beta, 0\rangle = \langle \beta, 0\rangle + \langle \alpha, 0\rangle = \langle \beta, 0\rangle,\\
&\langle \alpha, 1\rangle + \langle \beta, 1\rangle = \langle \beta, 1\rangle + \langle \alpha, 1\rangle = \langle \alpha, 1\rangle,\\
&\langle \alpha, 1\rangle + \langle \beta, 0\rangle = \langle \beta, 0\rangle + \langle \alpha, 1\rangle = \langle 5, 1\rangle,\\
&\langle \alpha, 0\rangle + \langle \beta, 1\rangle = \langle \beta, 1\rangle + \langle \alpha, 0\rangle = \langle \gamma, 1\rangle.
\end{aligned}$$

If α is any member of H_1, we set

$$\begin{aligned}
&\langle \alpha, 0\rangle + \langle \alpha, 0\rangle = \langle \alpha, 0\rangle,\\
&\langle \alpha, 1\rangle + \langle \alpha, 1\rangle = \langle \alpha, 1\rangle,\\
&\langle \alpha, 1\rangle + \langle \alpha, 0\rangle = \langle \alpha, 0\rangle + \langle \alpha, 1\rangle = \langle 5, 1\rangle.
\end{aligned}$$

If, finally, $\langle \alpha, \beta\rangle$ is any member of G, then we set

$$\begin{aligned}
&\langle \alpha, \beta\rangle + \langle 5, 0\rangle = \langle 5, 0\rangle + \langle \alpha, \beta\rangle = \langle \alpha, \beta\rangle,\\
&\langle \alpha, \beta\rangle + \langle 5, 1\rangle = \langle 5, 1\rangle + \langle \alpha, \beta\rangle = \langle 5, 1\rangle.
\end{aligned}$$

We define the operation $\cdot$ by means of De Morgan's law, as follows:

$$x\ y = -(-x + -y).$$

From the way in which the system $\Gamma = \langle G, +, \cdot, -\rangle$ has been defined, it is easily seen that every subalgebra with two generators is a Boolean algebra. Thus, for instance, if $\langle \alpha_1, \beta_1\rangle$ and $\langle \alpha_2, \beta_2\rangle$ are such that $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq 5 \neq \alpha_2$, $\alpha_1 \notin \alpha_2$, and $\alpha_2 \notin \alpha_1$, then $\langle \alpha_1, \beta_1\rangle$ and $\langle \alpha_2, \beta_2\rangle$ generate a Boolean algebra with the eight elements: $\langle \alpha_1, 0\rangle$, $\langle \alpha_2, 0\rangle$, $\langle \{\alpha_1, \alpha_2\}, 0\rangle$, $\langle 5, 0\rangle$, $\langle \alpha_1, 1\rangle$, $\langle \alpha_2, 1\rangle$, $\langle \{\alpha_1, \alpha_2\}, 1\rangle$, $\langle 5, 1\rangle$. In other cases, the Boolean algebra generated by $\langle \alpha_1, \beta_1\rangle$ and $\langle \alpha_2, \beta_2\rangle$ may have fewer than eight elements.

On the other hand, the system $\langle G, +, \cdot, -\rangle$ is not itself a

Boolean algebra, since it does not satisfy the associative law. Consider, for instance, the elements $\langle 2, 0\rangle$, $\langle 3, 0\rangle$, and $\langle 4, 0\rangle$; we have $(\langle 2, 0\rangle+\langle 3, 0\rangle)+\langle 4, 0\rangle=\langle\{\{2, 3\}, 4\}, 0\rangle$, while $\langle 2, 0\rangle+(\langle 3, 0\rangle+\langle 4, 0\rangle)=\langle\{2, \{3, 4\}\}, 0\rangle$, and $\{\{2, 3\}, 4\}$ is not the same as $\{2, \{3, 4\}\}$.

COROLLARY. *It is not possible to give a set of postulates for Boolean algebra where each postulate is a universal sentence containing only two variables.*

One can establish analogous theorems for a number of other kinds of algebras. Thus, for example, suppose we define the operation $\circ$ in terms of the operations of Γ as follows:

$$x \circ y = (x \cdot - y) \quad (- x \cdot y).$$

Then it is easily shown that the system $\langle G, \circ \rangle$ is not a group, but that every subalgebra of $\langle G, \circ \rangle$ with two generators is an Abelian group. Thus it follows that one cannot give a set of postulates for groups (or for Abelian groups) where each postulate is a universal sentence with only two variables. A similar result follows for rings, since every subalgebra with two generators of the system $\langle G, \circ, \cdot\rangle$ is a ring, while $\langle G, \circ, \cdot\rangle$ is not itself a ring.

Since every Boolean algebra is a distributive lattice, and since every lattice satisfies the associative law for addition, we see that there exists a system $\Delta=\langle G, +, \cdot\rangle$, which is not a lattice, but such that every subalgebra of Δ with two generators is a distributive lattice. Thus it is not possible to give a set of postulates for lattices where each postulate is a universal sentence with only two variables. The same is clearly true also of distributive lattices, and of modular lattices.

Similar results are easily seen to hold, finally, for closure algebras, and for Brouwerian algebras.[2]

Our theorem also finds application in connection with the axiomatization of the sentential calculus. It is well known that every Boolean algebra can be regarded as a matrix for the classical (two-valued) sentential calculus; thus every subalgebra with two generators of the system Γ of Theorem 1 is a matrix for the classical sentential calculus. Moreover, the formula

$$[p \vee (q \vee r)] \rightarrow [(p \vee q) \vee r]$$

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[2] For definitions of these algebras, see J. C. C. McKinsey and Alfred Tarski, *The algebra of topology*, Ann. of Math. vol. 45 (1944) pp. 141–191, and *On closed elements in closure algebras*, Ann. of Math. vol. 47 (1946) pp. 122–162.

is provable in the classical sentential calculus; hence Γ is not itself a matrix for the classical sentential calculus. From these two facts it is easily seen to follow that there cannot be a set of axioms for the classical sentential calculus, where each axiom contains only two variables.

The result obtained in the preceding paragraph can obviously be strengthened: If A is a set of tautologies, which includes the tautology

$$[p \vee (q \vee r)] \rightarrow [(p \vee q) \vee r],$$

then A cannot be derived from a set of axioms, each of which involves only two variables.[3] From this it is seen to follow that every set of axioms for the Heyting calculus, or for any one of the Łukasiewicz-Tarski calculi, must contain at least one axiom which involves three or more variables. A similar result is easily obtained for the Lewis systems S1, · · · , S5.

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[3] It can be shown in a similar way that the tautology $p \rightarrow [q \rightarrow (r \rightarrow p)]$ is not derivable from tautologies each of which involves only two variables. This result was originally discovered by M. Wajsberg, and was stated without proof by J. Łukasiewicz and A. Tarski in Satz 15 of *Untersuchungen über den Aussagenkalkül*, Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie, Classe III vol. 23 (1930) pp. 1–21.

NONLINEAR NETWORKS. IIa

R. J. DUFFIN

A network is a collection of conducting wires and batteries arbitrarily interconnected. Kirchhoff [1, 6][1] gave a topological-type proof that the currents in the wires are uniquely determined for wires obeying Ohm's law. (Ohm's law is a linear law stating that current and potential drop are proportional.) If the wires obey a nonlinear law, more than one distribution of current is in general possible. For some engineering application a multiplicity of states is desirable as, for example, in counting circuits and oscillators. For other applications it is essential that only one state be possible. It is seldom intuitively evident, however, whether or not a given nonlinear network will have multiple states. Hence, it appears that a qualitative mathematical treatment of nonlinear networks should be of some practical importance [5].

A large class of conductors used in engineering are such that the current through the conductor and the potential drop across the conductor are nondecreasing functions of one another. Such conductors we shall term *quasi-linear*. Examples are: selenium, copper oxide, silicon carbide (thyrite), and thermionic rectifiers [9]. The main result of this note is the proof that *a network of quasi-linear conductors has a stable state of currents, and this state is unique.*

A stable state of currents in a network must satisfy Kirchhoff's laws, which simply are statements of the conservation of electricity and the single valuedness of the potential function. Maxwell [8] discovered two concise ways of expressing these laws: the junction equations and the mesh equations. More or less as a digression we shall show that the mesh equations may be put in the same functional form as the junction equations if and only if the network is planar.

The formulation of mechanical analogs to electric networks has received considerable attention in the literature because of the transfer of techniques suggested by the analogy. We discuss here a different type of analog which we call an *elastic network*. An elastic network is a collection of springs connected to each other at junction points. Forces are applied to the junction points to hold the network in a stretched condition. A tennis net is an example. Electric networks are analogous to one-dimensional elastic networks. Planar or spatial

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[1] Numbers in brackets refer to the references cited at the end of the paper.

elastic networks are more general than electric networks and may give rise to nonlinear problems even for springs obeying Hooke's law.

1. Maxwell's junction equations. In order to simplify notation and to avoid ambiguity the following trivial restrictions shall be imposed on the configuration of the networks considered. A *proper network* is a set of $n+1$ junction points $(0, 1, \cdots, n)$, $n\geq 1$, connected by conducting wires such that each wire connects exactly two distinct junction points and no more than one wire directly connects the same two junction points.

The conductivity function $g(x)$ of a wire is the experimentally determined relation between the current w through the wire and the potential drop x across it, $w=g(x)$. If a wire obeys Ohm's law, $g(x)=kx$ where k is a positive constant, the conductivity.

The conductivity function of the wire connecting junction points i and j will be designated as $g_{ij}(x)$ and is such that if w_{ij} is the current flowing from i to j and v_i and v_j are the potentials of these junctions then

$$w_{ij} = g_{ij}(v_i - v_j). \tag{1}$$

By the conservation of electricity, $w_{ij}= -w_{ji}$; so $g_{ij}(x) = -g_{ji}(-x)$. If there is no wire directly connecting i and j then $g_{ij}(x)\equiv 0$.

Suppose currents u_i flow into the junction points from an external source. If a stable state of such a network exists, no electricity may pile up at a junction point so

$$\begin{array}{lllllll} u_0 = & 0 & + g_{01}(v_0 - v_1) & + g_{02}(v_0 - v_2) & + \cdots & + g_{0n}(v_0 - v_n), \\ u_1 = & g_{10}(v_1 - v_0) + & 0 & + g_{12}(v_1 - v_2) & + \cdots & + g_{1n}(v_1 - v_n), \\ \multicolumn{6}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,} \\ u_n = & g_{n0}(v_n - v_0) & + g_{n1}(v_n - v_1) & + g_{n2}(v_n - v_2) & + \cdots + & 0. \end{array} \tag{2}$$

These are Maxwell's junction equations. Since $g_{ij}(x)+g_{ji}(-x)\equiv 0$ it follows that $\sum_{i=0}^{n} u_i\equiv 0$, and so at least one of the equations is dedependent. Moreover, because the potentials v_i enter only as paired differences, one of them may be given an arbitrary value.

If equations (2) are solvable for the potentials, then relations (1) determine the distribution of current in the wires when the currents entering the junction are given.

It should be noted that the definition given of the conductivity function is broad enough to allow batteries to be included in the network. Thus, if for one value of x the relation $g_{ij}(x)=0$ holds, then $-x$ may be regarded as the value of the potential jump of a battery inserted in the wire connecting points i and j. Thus, for an isolated network excited by batteries equations (2) become

$$0 = \sum_{j=0}^{n} g_{ij}(v_i - v_j), \qquad i = 0, 1, \cdots, n. \tag{3}$$

In a non-proper network two or more wires may directly connect the same junction points. This case may be reduced to a proper network since the conductivity function for wires connected in parallel is the sum of the separate functions. The class of functions to be considered in what follows has the property that the sum of two functions also belongs to the class; hence, there is no essential restriction in considering only proper networks.

2. Existence and uniqueness conditions. The junction equations will now be considered abstractly. However, all variables and constants shall be assumed real.

THEOREM 1. *Equations* (3) *have at least one solution in the variables* v_i *if*:

(a) $g_{ij}(x) = -g_{ji}(-x)$.

(b) *The functions* $g_{ij}(x)$ *are continuous for all* x.

(c) *For each pair* (i, j) *either* $g_{ij}(x) \equiv 0$ *or* $\int_0^x g_{ij}(t)dt \to +\infty$ *as* $x \to \pm\infty$.

PROOF. We shall say that a set of functions $\{g_{ij}(x)\}$, $i, j = 0, 1, \cdots, n$, satisfies the *chain condition* if for each integer i there exists an ordered sequence of integers $i, a, b, c, \cdots, e, f$ (dependent on i) such that no function of the sequence $g_{ia}, g_{ab}, g_{bc}, \cdots, g_{ef}, g_{f0}$ vanishes indentically. First, suppose that the chain condition is satisfied and define $G_{ij}(x) = \int_0^x g_{ij}(t)dt$. Then by conditions (a) and (c) it is clear that there is an unbounded increasing function $h(x)$ such that either $G_{ij}(x) \equiv 0$ or $G_{ij}(x) \geqq h(|x|)$. It may be assumed, moreover, that $h(x)$ is independent of i and j and that $h(0)$ is negative.

Let $v_0 = 0$ and define $\psi(v_1, v_2, \cdots, v_n) = \sum_{i=0}^{n}\sum_{j=0}^{n} G_{ij}(v_i - v_j)$. Consider the value of ψ on an n-dimensional cube whose corners are at the points $(\pm l, \pm l, \cdots, \pm l)$, $l > 0$. For some value of i it follows that $v_i = \pm l$. Hence, $(v_i - v_a) + (v_a - v_b) + \cdots + (v_e - v_f) + (v_f - v_0) = (v_i - v_0) = \pm l$. All integers of the chain sequence $i, a, b, \cdots, f, 0$ may be assumed distinct; so there are at most n adjacent pairs. Thus, for some adjacent pair, say (d, e), it follows that $|v_d - v_e| \geqq l/n$ and $G_{de}(v_d - v_e) \geqq h(l/n)$. An estimate of ψ may be made from this single term, $\psi \geqq h(l/n) + (n+1)^2 h(0)$. For l sufficiently large this shows that ψ is everywhere greater on the surface of the cube than it is at the center. In the interior, ψ must have a minimum point and at this point

$$0 = 2^{-1}\partial\psi/\partial v_k = \sum_{j=0}^{n} g_{kj}(v_k - v_j), \qquad k = 1, 2, \cdots, n.$$

These are precisely equations (3) except for the equation $k=0$, which is dependent on the others.

If the chain condition is not satisfied, equations (3) are split into two sets. The first set includes the equation 0 and all equations i such that a nonvanishing chain $g_{ia}, g_{ab}, \cdots, g_{ef}, g_{f0}$ does exist. If g_{is} appears in the first set as a nonvanishing function so also does g_{si}, because the chain $g_{si}, g_{ia}, g_{ab}, \cdots, g_{ef}, g_{f0}$ is nonvanishing. This shows that the equations and variables appearing in the first set have the same labels. The second set, being what is left over, also has the same labels on the equations and the variables appearing. Clearly both sets have the same form as equations (3) but are of lower dimension than n. This splitting process is repeated on the second set, and so on. Finally, there results a division into a number of mutually exclusive sets, and each set either satisfies the chain condition or vanishes identically. This completes the proof.

In the network language the chain condition means that there is a chain of wires connecting every pair of junction points. In other words, the network is one piece.

THEOREM 2. *Equations* (2) *have a solution in the variables* v_i *for any choice of the constants* u_i *such that* $\sum_{i=0}^{n} u_i = 0$ *if*:

(a) $g_{ij}(x) = -g_{ji}(-x)$.

(b) *The function* $g_{ij}(x)$ *are continuous for all* x.

(c) *For each pair* (i, j) *either* $g_{ij}(x) \equiv 0$ *or* $g_{ij}(x) \to +\infty\,(-\infty)$ *as* $x \to +\infty\,(-\infty)$.

(d) *The chain condition is satisfied.*

PROOF. Define $\psi_1 = \sum_{i=0}^{n}\sum_{j=0}^{n} G_{ij}(v_i - v_j) - 2\sum_{i=0}^{n} u_i v_i$ $(v_0 = 0)$. Either $G_{ij}(x) \equiv 0$ or $G_{ij}(x) \geqq h(|x|)$, where now $h(x)$ is an increasing function such that $h(x)/x \to +\infty$ as $x \to +\infty$. Proceeding as in the proof of Theorem 1 gives the inequality

$$\psi_1 \geqq h(l/n) + (n+1)h(0) - l\sum_{i=1}^{n} |u_i| \to +\infty \qquad \text{as } l \to +\infty.$$

This shows that ψ_1 has a minimum at a point inside a large cube; hence at this point

$$0 = 2^{-1}\partial\psi_1/\partial v_k = \sum_{j=0}^{n} g_{kj}(v_k - v_j) - u_k, \qquad k = 1, 2, \cdots, n.$$

This completes the proof.

THEOREM 3. *Equations* (2) *may not have more than one solution in the variables* v_i, $i=1, 2, \cdots, n$, *with* $v_0=0$ *if*:

(a) $g_{ij}(x) = -g_{ji}(-x)$.

(b) *For each pair* (i, j) *either* $g_{ij}(x) \equiv 0$ *or* $g_{ij}(x)$ *is an increasing function for all* x.

(c) *The chain condition is satisfied.*

PROOF. Let $u_i = \sum_{j=0}^{n} g_{ij}(v_i - v_j)$ and $u_i' = \sum_{j=0}^{n} g_{ij}(v_i' - v_j')$ then

$$\begin{aligned}
\sum_{i=0}^{n} (u_i - u_i')(v_i - v_i') &= \sum_{i=0}^{n}\sum_{j=0}^{n} \{g_{ij}(v_i - v_j) - g_{ij}(v_i' - v_j')\}(v_i - v_i') \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} \{g_{ji}(v_j - v_i) - g_{ji}(v_j' - v_i')\}(v_j - v_j') \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} \{g_{ij}(v_i - v_j) - g_{ij}(v_i' - v_j')\}(v_j' - v_j) \\
&= 2^{-1}\sum_{i=0}^{n}\sum_{j=0}^{n} \{g_{ij}(v_i - v_j) - g_{ij}(v_i' - v_j')\}\{(v_i - v_j) - (v_i' - v_j')\}.
\end{aligned}$$

If $u_i = u_i'$, the left side vanishes. Each term of the last series on the right side is non-negative, because g_{ij} is a nondecreasing function; hence,

$$\{g_{ij}(v_i - v_j) - g_{ij}(v_i' - v_j')\}\{(v_i - v_j) - (v_i' - v_j')\} = 0.$$

Under the assumption that g_{ij} is an increasing function, these factors must vanish together, so $\{(v_i - v_j) - (v_i' - v_j')\} = 0$. For a chain sequence,

$$\{(v_i - v_a) - (v_i' - v_a')\} + \{(v_a - v_b) - (v_a' - v_b')\} + \cdots \\ + \{(v_f - v_0) - (v_f' - v_0')\} = v_i - v_i'.$$

Because each expression in braces vanishes, $v_i = v_i'$, and the proof is completed.

3. **Maxwell's mesh equations.** In actual engineering applications the junction equations are seldom employed but rather the alternative method of the mesh equations is used. A direct proof of the equivalence of the two methods does not seem to be available in the literature (we shall not consider this question either).

We now derive the mesh equations in the special case of an isolated network which can be diagramed in a plane without crossed wires. Let the regions (meshes) formed be numbered from 0 to m (including the exterior region). Let the resistance function be defined as the inverse of the conductivity function $g(x)$, and let $r_{ij}(w)$ be the sum of the resistance functions of the wires common to region i and region j. Let w_i be a cyclic current flowing around region i. The sign of w_i is determined by a clockwise convention for the interior regions and by a counterclockwise convention for the exterior regions. The potential is a single-valued function, so the net change in potential around the region i vanishes. Thus

$$0 = \sum_{j=0}^{m} r_{ij}(w_i - w_j), \qquad i = 0, 1, \cdots, m. \tag{4}$$

These are the mesh equations for a planar network. After the fictitious currents w_i are found, the actual currents are given immediately by the expressions $w_i - w_j$. Obviously, equations (4) are of the same form in the variables w_i as equations (3) are in the variables v_i.

It can be shown that any distribution of current which satisfies the conservation of electricity could be realized by assigning suitable values to a set of n of the cyclic currents w_i. Such a set is called a *complete set* [1, 6].

To write the mesh equations for a non-planar network, some complete set of cyclic currents is selected and a similar procedure is followed [3]. However, a theorem of MacLane [7] on "graphs" implies that some wire of a non-planar network must be traversed by at least three of the cyclic currents. This proves the following theorem.

THEOREM 4. *The mesh equations for an isolated network can be expressed in the same form as the junction equations if and only if the network is planar.*

To form mesh equations for a non-isolated network, it is simply necessary to add fixed non-cyclic currents which enter at one junction, follow some path, and leave at another junction.

4. Elastic networks.

A proper elastic network is a set of $n+1$ junction points $(0, 1, \cdots, n)$, $n \geq 1$, connected by springs such that each spring connects exactly two distinct junction points and no more than one spring directly connects the same two junction points. The junction points are the end points of the springs. The force function $f(x)$ of a spring is the force required to stretch a spring so that the distance between its end points is x. Let $f_{ij}(x) = f_{ji}(x)$ be the force function of the spring connecting junction points i and j. If r_i is the position vector of the ith junction point, the components of force which the junction i exerts on the spring (i, j) are

$$w^x_{ij} = f_{ij}(|r_i - r_j|)\,\frac{x_i - x_j}{|r_i - r_j|},$$

$$w^y_{ij} = f_{ij}(|r_i - r_j|)\,\frac{y_i - y_j}{|r_i - r_j|},$$

$$w^z_{ij} = f_{ij}(|r_i - r_j|)\,\frac{z_i - z_j}{|r_i - r_j|}.$$

It is convenient to express these relations in the single vector equation $w_{ij} = g_{ij}(r_i - r_j)$, where $g_{ij}(r)$ is a vector function whose magnitude is $|f_{ij}(|r|)|$ and whose direction in parallel to that of r. Clearly $g_{ij}(r) = -g_{ij}(-r)$ if $r \neq 0$, and to have this relation hold for $r = 0$ it must be assumed that $f_{ij}(0) = 0$.

If u_i is the total force applied to the ith junction from outside, the equations of equilibrium are

$$u_i = \sum_{j=0}^{n} g_{ij}(r_i - r_j), \qquad i = 0, 1, \cdots, n. \tag{5}$$

These are analogous to Maxwell's junction equations (2) but express the conservation of force rather than the conservation of electricity.

We shall say that a spring is *quasi-linear* if its force function $f(x)$ is a continuous, increasing, and unbounded function and if $f(0) = 0$.

THEOREM 5. *If ν of the junctions ($\nu \geq 1$) of a one piece network of quasi-linear springs are held fixed and given forces are applied to the other junctions then there is an equilibrium configuration, and this configuration is unique.*

PROOF. We may suppose that the junctions $(0, 1, \cdots, \nu - 1)$ are held fixed and that $r_0 = 0$. Define $G_{ij}(x) = \int_0^x f_{ij}(t)dt$ and

$$\psi = \sum_{i=0}^{n}\sum_{j=0}^{n} G_{ij}(|r_i - r_j|) - 2\sum_{i=\nu}^{n} u_i \cdot r_i.$$

(Note that $G_{ij}(x)$ is the potential energy of the spring.) Then ψ is a function of $3(n - \nu + 1)$ variables. As before, on a $3(n - \nu + 1)$-dimensional cube of edge $2l$, one of the variables takes the value $\pm l$; for definiteness, suppose that it is the x coordinates of the radius vector r_i. Then, considering a chain sequence $i, a, b, \cdots, f, 0$,

$$(x_i - x_a) + (x_a - x_b) + \cdots + (x_e - x_f) + (x_f - x_0) = \pm l.$$

For some adjacent pair, say (d, e), we have $|r_d - r_e| \geq |x_d - x_e| \geq l/n$. Then, as before, it follows that ψ is everywhere greater on a sufficiently large cube than it is at the center. Setting the partial derivatives of ψ with respect to the coordinates equal to zero at a minimum point gives exactly equations (5) from r to n. This proves the existence of a solution.

As in the proof of Theorem 3, one obtains the identity

$$\tag{6}\begin{aligned}&\sum_{i=0}^{n}(u_i-u_i')\cdot(r_i-r_i')\\&=2^{-1}\sum_{i=0}^{n}\sum_{j=0}^{n}\{g_{ij}(r_i-r_j)-g_{ij}(r_i'-r_j')\}\cdot\{(r_i-r_j)-(r_i'-r_j')\}.\end{aligned}$$

The expression on the left is clearly zero because either $(u_i-u_i')=o$ or $(r_i-r_i')=o$. Each term on the right is of the form $\{g_{ij}(p)-g_{ij}(p')\}\cdot\{p-p'\}$, and $g_{ij}(p)=cp$ where c is a positive scalar, $|p|c=f_{ij}(|p|)$. Thus,

$$\tag{7}\begin{aligned}\{g_{ij}(p)-g_{ij}(p')\}\cdot\{p-p'\}&=\{cp-c'p'\}\cdot\{p-p'\}\\&=c|p|^2-(c+c')p\cdot p'+c'|p'|^2\\&\geqq c|p|^2-(c+c')|p||p'|+c'|p'|^2\\&=\{c|p|-c'|p'|\}\{|p|-|p'|\}\\&=\{f_{ij}(|p|)-f_{ij}(|p'|)\}\{|p|-|p'|\}\geqq 0.\end{aligned}$$

Thus, each term on the right side of equation (6) is non-negative, and because the sum is zero each term must vanish. It follows that relation (7) is actually an equality and that $\{f_{ij}(|p|)-f_{ij}(|p'|)\}\{|p|-|p'|\}=0$. Thus, $|p|=|p'|$ if $f_{ij}(x)\neq 0$. But the vectors p and p' are parallel otherwise, $p\cdot p'<|p||p'|$; $|p|\neq 0$ and relation (7) could not be an equality: hence $p=p'$ if $f_{ij}(x)\neq 0$.

The remainder of the uniqueness proof parallels the proof of Theorem 3. Note that a similar theorem is valid for an electric network of quasi-linear conductors.

The analogy between elastic and electric networks developed here suggests a method which, though somewhat abstract, is nevertheless quite practical. The idea is simply to set up the analog of Maxwell's mesh equations for elastic networks or even for more complicated problems in statics such as trusses. Without the analogy it would appear difficult to invent such a method, as the concept of using fictitious circulating forces in the meshes, for variables, is rather fantastic.

The writer has built a mechanical model of a quasi-linear electric network (a wheatstone bridge circuit) by constraining the junctions of an elastic network to move along vertical rods. Hanging weights on the junctions gives "applied currents." It is clear that, even with springs obeying Hooke's law, a nonlinear problem arises.

The discrete boundary value problems discussed here suggest analogous considerations for continuous media. In a later note such nonlinear Dirichlet problems will be treated.

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A BOUND FOR THE ERROR IN COMPUTING THE BESSEL FUNCTIONS OF THE FIRST KIND BY RECURRENCE

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It is sometimes useful to obtain the value of a Bessel function of the first kind of a fairly high integral order. Inasmuch as $J_0(x)$ and $J_1(x)$ are relatively easy to compute, even apart from the existence of numerous tables of $J_0(x)$ and $J_1(x)$ to considerable accuracy, one obvious method of attack is to use the recurrence relation,

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \tag{1}$$

the starting values being the already available values of $J_0(x)$ and $J_1(x)$. In using this formula, however, it is necessary to have a bound for the error in $J_n(x)$ caused by errors in $J_0(x)$ and $J_1(x)$. It is the purpose of this note to furnish such a bound.

Let $e_n(x)$ represent the difference between the value of $J_n(x)$ and the approximate value obtained by applying (1) to values which differ from $J_0(x)$ and $J_1(x)$ by errors $e_0(x)$ and $e_1(x)$, respectively. Since $e_n(x)$ is additive to the approximate value and since both the approximate and true values satisfy the recurrence formula, $e_n(x)$ must also satisfy the recurrence formula.

If exact values of $e_0(x)$ and $e_1(x)$ were known, an exact value of $e_n(x)$ could be obtained by the recurrence relation (1). However, by the nature of the problem bounds for the absolute values of $e_0(x)$ and $e_1(x)$, and in some cases their signs, are the only information available. These bounds cannot be used as starting values for recurrence to find a bound to the absolute value of $e_n(x)$ since in the relation (1) a substitution of the bounds on the right-hand side may yield a result whose absolute value is well within the bound for the absolute value of the left-hand side.

In the relation due to Lommel and Hankel,[1]

$$2/\pi x = Y_n(x)J_{n+1}(x) - Y_{n+1}(x)J_n(x), \tag{2}$$

$Y_n(x)$ being the customary Bessel function of the second kind of order n, setting $n=0$, multiplying both sides by $(\pi x/2)e_0(x)$, adding the null term $(\pi x/2)e_1(x)Y_0(x)J_0(x) - (\pi x/2)e_1(x)Y_0(x)J_0(x)$ and rearranging the terms give

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[1] Watson, G. N., *A treatise on the theory of Bessel functions*, 2d ed., 1944, p. 77.

$$e_0(x) = (\pi x/2)\,Y_0(x)\left[e_0(x)J_1(x) - e_1(x)J_0(x)\right] \qquad (3)$$
$$+ (\pi x/2)\,J_0(x)\left[e_1(x)Y_0(x) - e_0(x)Y_1(x)\right].$$

Setting $n=0$ in (2), multiplying by $(\pi x/2)e_1(x)$, adding the null term $(\pi x/2)e_0(x)\,Y_1(x)J_1(x)-(\pi x/2)e_0(x)\,Y_1(x)J_1(x)$ to the right-hand side and rearranging give

$$e_1(x) = (\pi x/2)\,Y_1(x)\left[e_0(x)J_1(x) - e_1(x)J_0(x)\right] \qquad (4)$$
$$+ (\pi x/2)\,J_1(x)\left[e_1(x)Y_0(x) - e_0(x)Y_1(x)\right].$$

Since $e_n(x)$ satisfies the recurrence formula (1) as do $J_n(x)$ and $Y_n(x)$, it follows by complete induction that

$$e_n(x) = (\pi x/2)\,Y({}_nx)\left[e_0(x)J_1(x) - e_1(x)J_0(x)\right] \qquad (5)$$
$$+ (\pi x/2)\,J_n(x)\left[e_1(x)Y_0(x) - e_0(x)Y_1(x)\right].$$

Since $Y_n(x)$ for fixed x increases without any limit independent of n as n increases without limit, so does $e_n(x)$ unless the coefficient of $Y_n(x)$ vanishes. The values of $e_0(x)$ and $e_1(x)$ are the remainders in truncating series for $J_0(x)$ and $J_1(x)$, respectively, or the errors in obtaining these functions by asymptotic expansions, numerical quadratures or step-by-step integrations and therefore nothing is known of them except bounds for their absolute values and, in some cases, their signs. Hence, it can hardly be expected that the coefficient of $Y_n(x)$ vanishes; and almost always the error in $J_n(x)$ exceeds any bound independent of n provided n is large enough.

The function $(\pi x/2)\,Y_n(x)$ is the sum of a negative function of x with decreasing absolute value and a function whose absolute value is bounded by $\pi x/2+1$, for[2]

$$\pi Y_n(x) = -\sum_{m=0}^{<n/2} \frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{-n+2m}$$
$$+\int_0^{\pi} \sin\,(x\sin\theta - n\theta)\,d\theta \qquad (6)$$
$$+\,2(-1)^{n+1}\int_0^{\infty} e^{-nt-x\sinh t}\,dt,$$

the first integral being less in absolute value than π and the second less in absolute value than $2\int_0^{\infty} e^{-nt-xt}dt$ or $2/(n+x)$. Hence, $(\pi x/2)J_n(x)\left[e_1(x)\,Y_0(x)-e_0(x)\,Y_1(x)\right]$ is in absolute value less than $(\pi x+3)\left|J_n(x)\right|\,\max\left\{\left|e_0\right|,\left|e_1\right|\right\}$, the bound being obtained by setting

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[2] Ibid. pp. 340–342.

$n=0$ and $n=1$ in (6), determining bounds for $|Y_0(x)|$ and $|Y_1(x)|$, and using these bounds to obtain a bound for $|e_1(x)Y_0(x)-e_0(x)Y_1(x)|$.

The formula (6) is very difficult to use as a basis for estimation of $Y_n(x)$ if $n>10$. If tables of $Y_0(x)$ and $Y_1(x)$ are available, $Y_n(x)$ may be computed by use of the recurrence formula, or if tables of $Y_n(x)$ are available these may be used. It is unlikely that tables of $Y_n(x)$ will be available and not tables of $J_n(x)$ for $n>1$ and another procedure is necessary in this case. If only $J_0(x)$ and $J_1(x)$ are available, it is still possible by means of doubling the computation to obtain an auxiliary function $E_n(x)$, defined as the value obtained by recurring on 1 and -1 as starting values. Then from (5)

$$(7)\quad E_n(x) = (\pi x/2)\,Y_n(x)\,[J_1(x)+J_0(x)] - (\pi x/2)\,J_n(x)\,[Y_0(x)+Y_1(x)].$$

Then by use of the estimate

$$(8)\qquad (\pi x/2)\,|J_n(x)\,[Y_0(x)+Y_1(x)]| < (\pi x+3)\,|J_n(x)|$$

and by solving (7) for $Y_n(x)$, there results

$$(9)\qquad \frac{\pi x}{2}\,Y_n(x) = \frac{E_n(x)}{J_1(x)+J_0(x)} + \rho_n(x),$$

with

$$(10)\qquad |\rho_n(x)| < \frac{(\pi x+3)\,|J_n(x)|}{|J_1(x)+J_0(x)|}.$$

Hence, for all $x\geqq x_0>0$, denoting by $\phi(x)$ the polynomial in $1/x$ which appears in the expansion of $(\pi x/2)Y_n(x)$,

$$
\begin{aligned}
|e_n(x)| &\leqq (\pi x/2)\,|Y_n(x)|\,|e_1(x)J_0(x) - e_0(x)J_1(x)| \\
&\qquad + (\pi x+3)\cdot|J_n(x)|\cdot\max(|e_0|,\,|e_1|) \\
&\leqq \left[\left\{|\phi(x)| + \frac{\pi x}{2} + 1\right\}\{|J_0(x)| + |J_1(x)|\}\right. \\
(11)\qquad &\qquad \left. + (\pi x+3)\cdot|J_n(x)|\right]\cdot\max(|e_0|,\,|e_1|) \\
&\leqq \left[\left\{|\phi(x_0)| + \frac{\pi x}{2} + 1\right\}\{|J_0(x)| + |J_1(x)|\}\right. \\
&\qquad \left. + (\pi x+3)\,|J_n(x)|\right]\cdot\max(|e_0|,\,|e_1|).
\end{aligned}
$$

However,

$$\frac{\pi x}{2} Y_n(x_0) = \phi(x_0) + r(x_0) \tag{12}$$

or

$$|\phi(x_0)| \leq \frac{\pi x_0}{2}|Y_n(x_0)| + \frac{\pi x_0}{2} + 1. \tag{13}$$

Finally,

$$|\phi(x_0)| \leq \left\{\frac{|E_n(x_0)|}{|J_1(x_0) + J_0(x_0)|} + \frac{(\pi x_0 + 3)|J_n(x_0)|}{|J_1(x_0) + J_0(x_0)|} + \frac{\pi x_0}{2} + 1\right\} \tag{14}$$

or

$$\begin{aligned} |e_n(x)| \leq \Bigg[\Bigg\{ & \frac{|E_n(x_0)|}{|J_1(x_0) + J_0(x_0)|} + \frac{(\pi x_0 + 3)|J_n(x_0)|}{|J_1(x_0) + J_0(x_0)|} \\ & + \frac{\pi(x_0 + x)}{2} + 2\Bigg\} \\ & \cdot \{|J_0(x)| + |J_1(x)|\} + (\pi x + 3)\cdot|J_n(x)|\Bigg] \cdot \max(|e_0|, |e_1|). \end{aligned} \tag{15}$$

This bound will be almost exact for $e_n(x_0)$. To make it as good as possible, the value of x_0 should be taken as large as possible consistent with the values of x for which $J_n(x)$ is to be tabulated. In case approximations to $J_n(x)$ and $J_{n-1}(x)$ are available for $n>1$, similar considerations will enable one to determine a bound to the error in $J_k(x)$ with $k>n$, obtained by recurrence on $J_n(x)$ and $J_{n-1}(x)$.

NAVY DEPARTMENT

AN ELECTRO-MECHANICAL INVESTIGATION OF THE RIEMANN ZETA FUNCTION IN THE CRITICAL STRIP

BALTH. VAN DER POL

The theoretical basis underlying the following electromechanical investigations concerns a simple integral representation of the Riemann zeta function in the critical strip. To the right of the critical strip ($\operatorname{Re} s > 1$) we have the usual expression:

$$(1) \qquad \Gamma(s)\cdot\zeta(s) = \int_0^\infty \frac{1}{e^x - 1}\, x^{s-1}dx, \qquad \operatorname{Re} s > 1.$$

Subtraction from (1) of

$$(2) \qquad \Gamma(s-1)\cdot\alpha^{1-s} = \int_0^\infty e^{-\alpha x}x^{s-2}dx, \qquad \operatorname{Re} s > 1,\ \operatorname{Re} \alpha > 0,$$

yields

$$(3) \quad \Gamma(s)\cdot\left\{\zeta(s) - \frac{\alpha^{1-s}}{1-s}\right\} = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{e^{-\alpha x}}{x}\right) x^{s-1}dx, \quad \operatorname{Re} s > 0.$$

In the integrand of (3) the simple pole at $x=0$ of $(e^x-1)^{-1}$ is compensated by the simple pole with the same residue of $x^{-1}e^{-\alpha x}$. Therefore (3) already converges in the wider band $\operatorname{Re} s > 0$. If we now restrict s to the critical strip

$$0 < \operatorname{Re} s < 1$$

we can let $\alpha \to 0$ in (3), so that we obtain the basic integral

$$(4) \qquad \Gamma(s)\cdot\zeta(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1}dx, \qquad 0 < \operatorname{Re} s < 1,$$

this being the analytical continuation of (1).

The representation (4) of the zeta-function in the critical strip enables us to derive the functional equation of the zeta-function in an extremely simple way. To this end we make use of Legendre's relation[1]

$$(5) \quad 2\int_0^\infty \sin xt \frac{1}{e^{2\pi t}-1}\,dt = \frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} = \frac{1}{2}\coth\frac{x}{2} - \frac{1}{x},$$

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[1] See, for example, Whittaker-Watson, *Modern analysis*, 4th ed., p. 122.

which is valid for all real values of x; furthermore both sides of (5) represent an odd function of x. We now restrict x to positive values and subtract

$$2\int_0^\infty \sin xt\cdot\frac{dt}{2\pi t}=\frac{1}{2}$$

from (5), so that (5) assumes the self-reciprocal form

$$2\int_0^\infty \sin xt\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)dt=\frac{1}{e^x-1}-\frac{1}{x},\qquad x>0. \tag{6}$$

Substitution of (6) in (4) yields

$$\Gamma(s)\cdot\zeta(s)=\int_0^\infty x^{s-1}\left\{2\int_0^\infty \sin xt\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)dt\right\}dx$$

which, with the substitutions $2\pi t=\tau$ and $x=2\pi\xi/\tau$, becomes

$$\Gamma(s)\cdot\zeta(s)=\frac{(2\pi)^s}{\pi}\int_0^\infty\frac{\sin\xi}{\xi^{1-s}}d\xi\cdot\int_0^\infty\left(\frac{1}{e^\tau-1}-\frac{1}{\tau}\right)\tau^{-s}d\tau. \tag{7}$$

The latter integral has the form of (4), s being replaced by $1-s$, while the former integral is of a well known type. Hence we obtain

$$\begin{aligned}\Gamma(s)\cdot\zeta(s)&=\frac{(2\pi)^s}{\pi}\int_0^\infty\frac{\sin\xi}{\xi^{1-s}}d\xi\cdot\Gamma(1-s)\cdot\zeta(1-s)\\&=\frac{1}{2}\,\frac{(2\pi)^s}{\Gamma(1-s)\cdot\cos(\pi s/2)}\cdot\Gamma(1-s)\zeta(1-s)\end{aligned}$$

or

$$\cos\frac{\pi s}{2}\cdot\Gamma(s)\cdot\zeta(s)=\frac{1}{2}(2\pi)^s\zeta(1-s).$$

This is the functional equation, which, by analytical continuation, is valid in the whole s-plane.

Returning to (4), we notice that this equality can be written as

$$\begin{aligned}\Gamma(s)\zeta(s)&=\int_0^\infty\left(\sum_1^\infty e^{-\nu x}-\int_0^\infty e^{-\nu x}d\nu\right)x^{s-1}dx\qquad(0<\operatorname{Re}s<1)\\&=\int_0^\infty\int_0^\infty e^{-ux}d([u]-u)\cdot x^{s-1}dx.\end{aligned}\tag{8}$$

It is evident from (8) that the zeta-function in the critical strip is generated by the difference of a *sum* and an *integral*. It therefore seems not unlikely that the typical difficulties associated with a study of this function in the critical strip are closely related to this fact.

We now integrate the central Stieltjes integral by parts, as

$$
\begin{aligned}
\text{(8a)}\qquad \int_0^\infty e^{-ux}d([u]-u) &= e^{-ux}([u]-u)\Big|_0^\infty + x\int_0^\infty ([u]-u)e^{-ux}du \\
&= x\int_0^\infty ([u]-u)e^{-ux}du.
\end{aligned}
$$

Substitution of (8a) in (8) and writing $x=\xi/u$ yields

$$
\Gamma(s)\cdot\zeta(s) = \int_0^\infty\int_0^\infty \frac{[u]-u}{u^{s+1}}du\cdot e^{-\xi}\xi^s d\xi = \Gamma(s+1)\int_0^\infty \frac{[u]-u}{u^{s+1}}du,
$$

or

$$
\text{(9)}\qquad \frac{1}{s}\zeta(s) = \int_0^\infty \frac{[u]-u}{u^{s+1}}du, \qquad 0<\operatorname{Re} s<1.
$$

Finally, calling $u=e^x$ and taking $s=1/2+it$, we obtain from (9) as an expression for $\zeta(s)$ on the critical line $\operatorname{Re} s=1/2$:

$$
\text{(10)}\qquad \frac{\zeta(1/2+it)}{1/2+it} = \int_{-\infty}^{\infty}\left\{e^{-x/2}[e^x]-e^{x/2}\right\}\cdot e^{-itx}dx.
$$

With (10) the investigation of the zeta-function on the critical line is reduced to a problem in the real domain which can be attacked with physical methods. It is therefore on (10), which has the form of a Fourier transform, that we based our experiments.

The function

$$
\text{(11)}\qquad y(x) = e^{x/2} - e^{-x/2}[e^x],
$$

of which

$$
-\frac{\zeta(1/2+it)}{1/2+it}
$$

is the Fourier transform, is represented in Fig. 1. (See insert opposite p. 980.) It is a sawtooth-function, of which the height of the teeth varies exponentially, the width of the nth tooth being given by $\log((n+1)/n)$. Further we have

$$
0 \leq y(x) \leq e^{-|x|/2}, \qquad -\infty < x < +\infty,
$$

which ensures the convergence of (10) for all real values of t.

Thus the experimental problem consisted of constructing an electro-mechanical machine giving the Fourier integral of the function $y(x)$. This however was found to be impracticable and therefore we resorted to obtaining experimentally the modulus of the Fourier series for $y(x)$ modified in such a manner that $y(x)$ was taken to extend only from $x_1 = -9.00$ to $x_2 = +9.00$, being zero outside these limits. The remaining function was thus repeated indefinitely. It may be noted that the amplitude of $y(x_1)$ and $y(x_2)$ at these limits is only $\exp(-9/2) = .011$, so that the error involved is certainly small. The problem thus having been made periodical, it was easy to introduce rotating machinery. For this purpose the function (cut off in this way) was very carefully drawn on paper. This was done radially along the circumference of a circle having a radius of $a = 13$ cm. (the transformation being $r - a = y(\alpha\zeta)$). Thereupon it was cut out as accurately as possible. The resulting paper disc is represented in Fig. 2. Thus the first tooth had a width $(\log 2 - \log 1)$ corresponding to to about 1/26 of the circumference of the circle. The last tooth which it was practicable to cut out was the 35th with a width of $(\log 36 - \log 35)$, corresponding to about 1/621 part of the circumference. This paper disc was clamped perfectly flat between two aluminum discs, the paper teeth extending well outside the metal discs. This assembly was mounted centrally on the shaft of a synchronous motor. The motor was driven at a frequency of 25.018 revolutions per second by an amplifier behind an electrically controlled tuning fork of 1000,72 c/s, the frequency of the current from the tuning fork being demultiplied with the aid of relaxation oscillations in the ratio of 40:1. This tuning fork drive ensured a much greater precision in the angular speed of the synchronous motor than could be obtained by using the mains.

With the aid of cylindrical lenses a beam of light was projected onto the paper teeth in a direction parallel to the axis of the motor. The cross section of the beam was a long, very narrow rectangle, with the longer side oriented radially. The light thus passed by the teeth of the disc fell upon a photocell, causing an electric current which was an exact replica of the form of the teeth of the disc and therefore of the modified function. This current was amplified and a strong sinusoidal current was superimposed thereon, the frequency of which was made to vary very slowly in an approximately linear manner from 0 c/s to 15000 c/s in about 12 hours, the law of this variation having been previously determined. Further, an effectively quadratic push-pull detector (selenium cells) was inserted in the circuit, producing difference-tones between (a) all the harmonics produced by the rotating disc and (b) the applied sinusoidal current. These difference-tones were fed to a very sharply tuned electromechanical vibrator with a natural period of 7.00 c/s and a logarithmic decrement of .0143. The variation of the frequency of the applied sinusoidal current was necessarily slow, in order to be sure that the current through the mechanical vibrator went through "quasi-stationary" states.[1] Thus, when the applied sinusoidal current had a frequency exactly 7.00 c/s greater or smaller than one of the harmonics generated by the teeth on the disc, a difference-tone was produced which excited the mechanical vibrator. The oscillations of the latter were translated, in a separate circuit, as an electric current which was rectified (giving the modulus of the function) and which was registered by a recording milliammeter. The recording paper of the latter moved with a speed which was also controlled by the same tuning fork. Hence the meter recorded a series of spectral lines situated in pairs, one of each pair being 7 c/s to the left, and the other 7 c/s to the right of each harmonic present in the function produced by the rotating disc (fig. 2), which, itself, represents the function $y(x)$ of (11) (made periodical), the modulus of whose Fourier-transform is

$$\left|\frac{\zeta(1/2+it)}{1/2+it}\right|. \tag{12}$$

Fig. 3 is a direct unretouched reproduction of the record thus obtained of (12). From the accurately known constants of the apparatus it was possible to mark the values of t shown at the top of the record. The abscissae marked at the bottom with a | represent the 29 accurately known zeros of $\zeta(1/2+it)$, which were taken from Jahnke-Emde, ***Funktionen tafeln*** (zweite, neubearbeite Auflage) p. 324.[2] It is seen that all these 29 zeros appear in this record either as very deep minima (namely, the zeros no. 1, 2,,3, 4, 5, 6, 7, 11, 12, 13, 16, 18, 20, 21, 23, 27, 29), or in any case as less deep, but none the less very decided minima (namely, the zeros no. 9, 14, 15). It is further seen that the value of the recorded zeros coincide with those calculated with an accuracy better than 1%.

In order that the recording for the larger values should not be too small, the sensitivity of the apparatus was increased 4-fold from $t=35$

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[1] Balth. van der Pol, Journal of the Institution of Electrical Engineering (London) vol. 93 (1946) p. 153.

[2] For the number of zeros on Re $s=1/2$ up to $t=1468$ see E. C. Titchmarsh, Proc. Roy. Soc. London Ser. A vol. 151 (1935) p. 234 and vol. 157 (1936) p. 261.

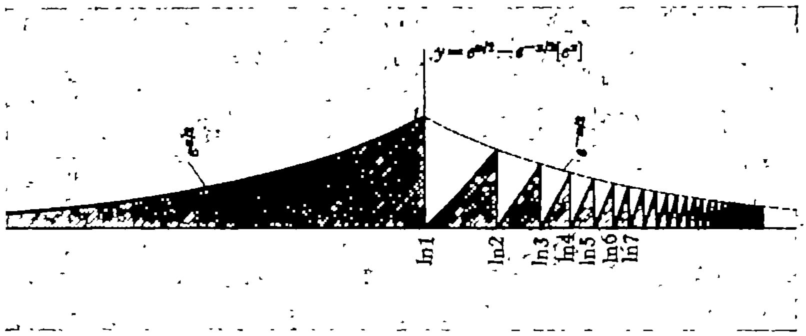

FIG. 1. A graph of the sawtooth function $y(x) = e^{x/2} - e^{-x/2}[e^x]$.

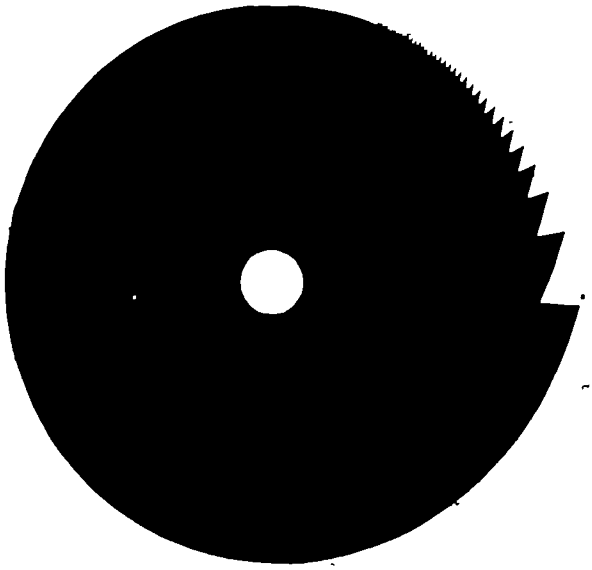

FIG. 2. Paper disc, showing the sawtooth function $y(x)$ for the range $-9 < x < +9$.

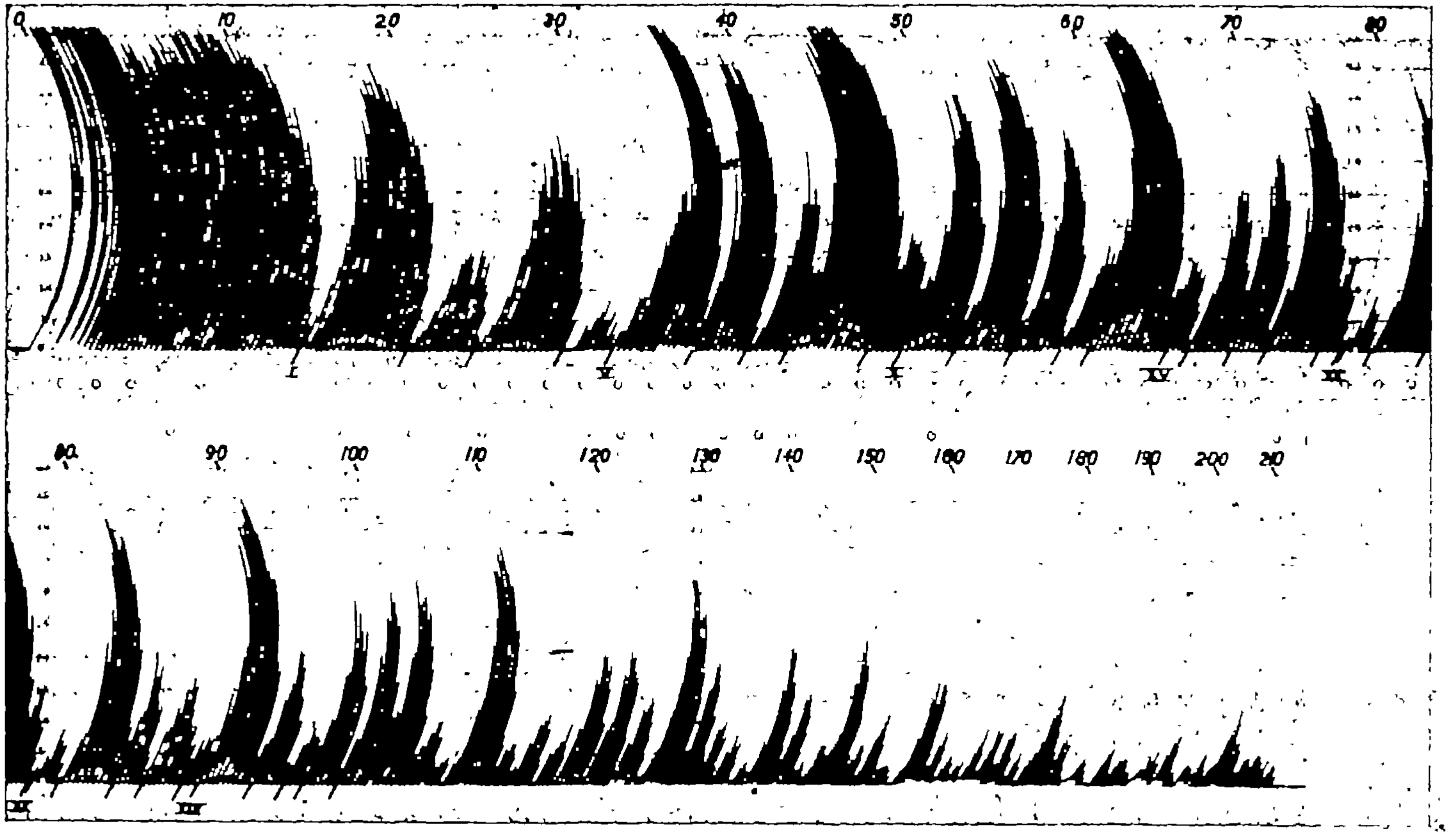

FIG. 3. Record of $|\zeta(1/2+it)/(1/2+it)|$ as produced electromechanically showing o.a. minima, the first 29 of which (marked |) correspond to the 29 known zeros of the zeta-function on the critical line.

onwards, as indicated in the record, so that for $t>35$, the recorded amplitude is 4 times larger than in the range $t<35$.

The first 29 zeros of $\zeta(1/2+it)$ are to be found in the range $0<t<100$. However it was practically possible to extend the recording up to $t=210$. Up to this point the record shows a total of 73 minima which most probably may be interpreted as zeros, the more so as all the pronounced minima of the envelope of the curve for $t<100$ correspond to known zeros.

As to the modulus, attention may be drawn to the fact that the record does not show $|\zeta(1/2+it)|$, but shows this function divided by $|1/2+it|$, see (12).

It is of interest to remark that our record shows that there seems to be no simple relation between the difference between successive zeros and the height of the maximum between them. For example, the height of the maximum between the 20th and 21st zero is considerably smaller than that between the 17th and 18th, although the interval between these pairs of zeros is not very different. The experiments carried out so far, and in which harmonics beyond the 600th were measured (the highest harmonic recorded corresponded about to the width of the last tooth cut in the paper), were performed with relatively limited means, and it is felt that many improvements could still be applied, such as:

(a) assuring a still more constant speed of revolution of the synchronous motor;

(b) determining the effect on the record of the number of teeth cut in the paper disc, and so on.

Therefore, with the modern technical means available, the present method of exploring the behavior of the zeta-function in the critical strip (which is closely related to Riemann's conjecture), seems capable of improvement with regard to precision. Further, a considerable extension with regard to the number of zeros recorded, including those, if any, outside the line Re $s=1/2$, seems possible by this method.

Finally I wish to thank Mr. C. C. J. Addink for the great care and skill with which he performed the experiments.

Note added in proof (October 6, 1947). The less deep minima as in Fig. 3 were later on found much deeper when in further experiments the angular frequency was kept more homogeneous.

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ON THE SUM OF THE RELATIVE EXTREMA OF $|f(z)|$ ON THE UNIT CIRCLE

ROBERT BREUSCH

Summary and introduction. Dr. Erdős has made the following conjecture: "Let $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$, with $z_\nu=r_\nu e^{i\theta_\nu}$, $r_\nu\leqq 1$, $f(z)\not\equiv z^n$. Let $z_\mu=e^{i\theta_\mu}$ ($\mu=1, 2, \cdots k$) be the points on the circumference of the unit circle, for which $|f(z)|$ is a relative extremum. Then $\sum_{\mu=1}^{k}|f(z_\mu)|\leqq 2^n$, and the equal sign applies only if $f(z)=(z-e^{i\theta_0})^n$." It will be proved that this is correct for large values of n, but incorrect for small ones.

With $z=e^{i\theta}$,

$$|f(z)|^2=\prod_{\nu=1}^{n}|e^{i\theta}-r_\nu e^{i\theta_\nu}|^2=\prod_{\nu=1}^{n}[1+r_\nu^2-2r_\nu\cos(\theta-\theta_\nu)],$$

or, with $t=\tan(\theta/2)$,

$$|f(z)|^2=F(t)=(1+t^2)^{-n}$$
$$\cdot\prod_{\nu=1}^{n}[(1+r_\nu^2+2r_\nu\cos\theta_\nu)t^2-4r_\nu\sin\theta_\nu t+(1+r_\nu^2-2r_\nu\cos\theta_\nu)],$$

$$F(t)=\frac{At^{2n}+\cdots}{(1+t^2)^n},$$

$$F'(t)=\frac{(1+t^2)(2nAt^{2n-1}+\cdots)-2nt(At^{2n}+\cdots)}{(1+t^2)^{n+1}}=\frac{Bt^{2n}+\cdots}{(1+t^2)^{n+1}}.$$

It may be assumed for the moment that $|f(e^{i\theta})|$ is not an extremum for $\theta=\pi$, that is, $t=\infty$; otherwise the coordinate system may be rotated. Thus it is seen that the number of relative extrema of $|f(z)|$ on the unit circle cannot exceed $2n$.

If $|f(e^{i\theta})|$ is less than $2^n/2n$ for every real value of θ, then $\sum_{\mu=1}^{k}|f(e^{i\theta_\mu})|$ is obviously less than 2^n.

Assume now that $|f(z)|$ has a relative extremum at $z=1$, and that $|f(1)|\geqq 2^n/2n$. The proof then proceeds in the following steps: It will be shown, that for large values of n:

(1) All but $o(n)$ roots must be in a region R_1 of the unit circle close to $z=-1$.

(2) There is an arc T of the circumference of the unit circle, close to $z=+1$, such that the sum of all the extreme values of $|f(e^{i\theta})|$ with $e^{i\theta}$ not on T is $O(2^n/n)$.

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(3) The total number of extrema on T is $O(\log n)$.

(4) The sum of all the extreme values is not greater than 2^n.

PROOFS. (1) Divide the unit circle and its interior into 3 parts $R_1R_2R_3$, where R_1 is outside of and on a circle with radius $2-n^{-1/2}$ and center at 1, R_2 is inside of this circle, but outside of and on a circle with radius $2^{1/2}$ and center at 1, and R_3 is the rest.

The number of roots in R_2 plus R_3 must be *less than* $n^{3/4}$, otherwise $|f(1)| \leqq 2^{n-n^{3/4}}\cdot(2-n^{-1/2})^{n^{3/4}} = 2^n(1-1/2n^{1/2})^{n^{3/4}} < 2^n/e^{n^{1/4}/2} < 2^n/2n$, for n large enough. Similarly the number of roots in R_3 must be *less than* $3 \log n$.

For points $z=r\cdot e^{i\theta}$ in R_1, $\cos((\pi-\theta)/2) \geqq (2-n^{-1/2})/2 = 1-n^{-1/2}/2$. Consequently for such points, θ differs from π by less than $3\cdot n^{-1/4}$, for n large enough.

(2) Let T be the arc of the circumference of the unit circle with the midpoint at $z=1$ and central angle $12\cdot n^{-1/4}$. If $z=e^{i\theta}$ lies on the arc of the unit circle complementary to T then the distance from the point z to any point of R_1 is less than $2-n^{-1/2}$, so

$$|f(z)| < (2-n^{-1/2})^{n-n^{3/4}}\cdot 2^{n^{3/4}} = 2^n\left(1-\frac{1}{2n^{1/2}}\right)^{n-n^{3/4}}$$

$$= O\left(\frac{2^n}{e^{n^{1/4}}}\right) < \frac{2^n}{4n^2},$$

for n large enough. Therefore the sum of all the extreme values of $|f(e^{i\theta})|$ with $e^{i\theta}$ not on T is less than $2n\cdot 2^n/4n^2 = 2^n/2n$. Along T, $|t| = |\tan(\theta/2)| < 4\cdot n^{-1/4}$.

(3) $|f(z)|^2 = F(t) = \prod_{\nu=1}^{n}(1+r_\nu^2+2r_\nu \cos \theta_\nu)\cdot\prod_{\lambda=1}^{2n}(t-t_\lambda)\cdot(1+t^2)^{-n}$, where $t_{2\nu-1}$ and $t_{2\nu}$ are the roots of $(1+r_\nu^2+2r_\nu\cos\theta_\nu)t^2-4r_\nu \sin\theta_\nu\cdot t+(1+r_\nu^2-2r_\nu\cos\theta_\nu)=0$. This equation has real coefficients, and its discriminant is $-4(1-r_\nu^2)^2 \leqq 0$. Therefore

$$|t_{2\nu-1}| = |t_{2\nu}| = \left(\frac{1+r_\nu^2-2r_\nu\cos\theta_\nu}{1+r_\nu^2+2r_\nu\cos\theta_\nu}\right)^{1/2} = \left|\frac{z_\nu-1}{z_\nu+1}\right|.$$

For z_ν in R_1, $|t_\lambda| > 1/3n^{-1/4} = n^{1/4}/3$. For z_ν in R_2, $|t_\lambda| \geqq 1$. The relative extrema of $|f(z)|$ for which $f(z)=0$ do not contribute to $\sum_{\mu=1}^{k}|f(z_\mu)|$. All the other extrema are characterized by

$$G(t) \equiv \frac{F'(t)}{F(t)} = \sum_{\lambda=1}^{2n}\frac{1}{t-t_\lambda} - \frac{2nt}{1+t^2} = 0.$$

To find an upper bound for the number of those extrema on T, consider $G(t)$ as a function of the complex variable t and integrate

$G'(t)/G(t)$ along a circle with center O and radius p, where p is a number between $1/4$ and $1/2$ such that for $|t| = p$, $|t-t_\lambda| > 1/50 \log n$. This is possible, because the number of roots z_ν in R_3 is less than $3 \log n$, and therefore the number of $|t_\lambda| < 1$ is less than $6 \log n$. Along this circle

$$\left| \sum_{\lambda=1}^{2n} \frac{1}{t-t_\lambda} \right| \leqq \sum_{\lambda=1}^{2n} \frac{1}{|t-t_\lambda|}$$

$$= \sum_{z_\nu \text{ in } R_1} \frac{1}{|t-t_\lambda|} + \sum_{z_\nu \text{ in } R_2} \frac{1}{|t-t_\lambda|} + \sum_{z_\nu \text{ in } R_3} \frac{1}{|t-t_\lambda|}$$

$$< 2n \frac{1}{n^{1/4}/3 - 1/2} + 2n^{3/4} \cdot \frac{1}{1-1/2}$$

$$+ 6 \log n \cdot \frac{1}{1/50 \log n} = o(n),$$

and

$$\left| \frac{2nt}{1+t^2} \right| \geqq \frac{2n \cdot 1/4}{1+1/4} > \frac{n}{3}.$$

Therefore

$$|G(t)| = \left| \sum_{\lambda=1}^{2n} \frac{1}{t-t_\lambda} - \frac{2nt}{1+t^2} \right| > \frac{n}{3} - o(n) > \frac{n}{4}$$

for n large enough,

$$|G'(t)| = \left| \sum_{\lambda=1}^{2n} \frac{-1}{(t-t_\lambda)^2} - \frac{2n}{1+t^2} + \frac{4nt^2}{(1+t^2)^2} \right|$$

$$< 2n \frac{1}{(n^{1/4}/3 - 1/2)^2} + 2n^{3/4} \cdot \frac{1}{(1-1/2)^2}$$

$$+ 6 \log n \cdot \frac{1}{(1/50 \log n)^2} + \frac{2n}{1-1/4} + \frac{n}{(1-1/4)^2} < 5n$$

for n large enough.

Thus $G'(t)/G(t) < 20$, and $|(1/2\pi i) \cdot \int_O G'(t)/G(t) dt| < (1/2\pi) \cdot 2\pi p \cdot 20 \leqq 10$. But $(1/2\pi i) \cdot \int_O G'(t)/G(t) dt =$ number of roots$-$number of poles of $G(t)$, and the number of poles in this circle is not greater than $6 \log n$. Therefore the number of roots of $G(t)$, or the number of relative extrema of $|f(z)|$ in $|\tan(\theta/2)| < 1/4$ (and consequently the number of those on T), is less than $7 \cdot \log n$.

(4) If a circle K with radius $10\cdot n^{-1/4}$ around $z=1$ contains in its interior any roots z_ν, then for every point of T, $|f(z)|$ is less than $16\cdot 2^{n-1}/n^{1/4}$, and the sum of all the extrema on T is less than $7\cdot\log n\cdot 16\cdot 2^{n-1}/n^{1/4}<2^n/2$ for n large enough. The sum of all the extrema on the unit circle will be less than $2^n/2+2^n/2n<2^n$.

If K contains no roots in its interior, then for all the roots z_ν in R_2,

$$|t_\lambda| = \left|\frac{z_\nu-1}{z_\nu+1}\right| > \frac{10\cdot n^{-1/4}}{2} = 5n^{-1/4}.$$

For every point on T, $|t| = |\tan(\theta/2)|$ is less than $4\cdot n^{-1/4}$, and

$$\left|\sum_{\lambda=1}^{2n}\frac{-1}{(t-t_\lambda)^2}\right| \leq \sum_{\lambda=1}^{2n}\frac{1}{|t-t_\lambda|^2} < 2n\cdot\frac{1}{(n^{1/4}/3-4n^{-1/4})^2}$$

$$+\,2n^{3/4}\cdot\frac{1}{(1-4n^{-1/4})^2}+\frac{6\log n}{(5n^{-1/4}-4n^{-1/4})^2} = o(n);$$

$$\frac{4nt^2}{(1+t^2)^2} = o(n),$$

and

$$G'(t) = \sum_{\lambda=1}^{2n}\frac{-1}{(t-t_\lambda)^2}-\frac{2n}{1+t^2}+\frac{4nt^2}{(1+t^2)^2} = -\frac{2n}{1+t^2}+o(n),$$

or: $G'(t)$ is negative everywhere on T. Consequently $G(t)$ can vanish not more than once on T, and $|f(z)|$ has on T no other extremum except at $z=1$. In this case, $\sum_{\mu=1}^{k}|f(z_\mu)| \leqq |f(1)|+2^n/2n$.

If R_2 or R_3 contains any roots z_ν, then $|f(1)|<2^{n-1}(2-n^{-1/2})$, and $\sum_{\mu=1}^{k}|f(z_\mu)|<2^n-2^{n-1}/n^{1/2}+2^{n-1}/n<2^n$.

If every root z_ν is in R_1, then $|z_\nu+1|<3\cdot n^{-1/4}$ for $\nu=1, 2, \cdots, n$. Call "q" the largest of the values $|z_\nu+1|$. Then $|\theta_\nu-\pi|\leqq\sin^{-1}q<2q$, for every value of ν. In this case, $|f(z)|$ increases steadily as θ changes from $\pi-2q$ to $6n^{-1/4}$, or from $\pi+2q$ to $2\pi-6n^{-1/4}$, and $|f(e^{i\theta})|$ can have no relative extremum except $|f(1)|$ outside of the arc $\pi-2q<\theta<\pi+2q$. Each extremum on this arc is less than $(3q)^n$, $|f(1)|$ is less than $2^{n-1}(4-q^2)^{1/2}<2^n-2^{n-3}q^2$, and

$$\sum_{\mu=1}^{k}|f(e^{i\theta_\mu})| < 2^n-2^{n-3}q^2+2n(3q)^n$$

$$= 2^n-2^{n-3}q^2\cdot\left[1-36n\left(\frac{3q}{2}\right)^{n-2}\right].$$

Since $q<3n^{-1/4}$, the expression in the parenthesis is positive for n large enough, and $\sum_{\mu=1}^{k}|f(e^{i\theta_\mu})|<2^n$, unless $q=0$. But $q=0$ means that every $z_\nu=-1$, and $f(z)=(z+1)^n$.

The theorem has thus been proved for large values of n.

(5) For small values of n the theorem is incorrect. For example, if $f(z)=(z+1)\cdot(z-1/2)$, then $|f(e^{i\theta})|$ has relative extrema for $\theta=0, \pi$, arc cos $(1/8)$, and the sum of the values of $|f(e^{i\theta})|$ at these places is $4.18>2^2$.

$f(z)=(z+1)^2\cdot(z-1/3)$ is another example in which $\sum|f(z_\mu)|=8.14>2^3$.

AMHERST COLLEGE

ON LOCALLY SIMPLE CURVES

G. T. WHYBURN

1. **Introduction.** A continuous transformation $f(A)=B$ is *locally simple* provided that for each $x\in A$ there is a neighborhood U of x such that $U\cdot f^{-1}(y)$ contains at most one point for each $y\in B$. For the case where A is a circle and B is planar, such mappings have been studied recently by Morse and Heins [1][1] and used effectively in investigating meromorphic and other functions by topological methods.

In this paper topological characterizations will be obtained for those continua (compact connected metric spaces) which admit locally simple representations on the circle, that is, which are the image of the circle under some locally simple mapping.

It is clear that if A is compact and metric, a mapping (continuous) $f(A)=B$ is locally simple if and only if there exists an $e>0$ such that any subset of A of diameter not greater than e maps topologically onto its image under f or, equivalently, for any $y\in B$, any two distinct points of $f^{-1}(y)$ are at a distance greater than e apart.

A simple arc ab in a continuum M will be called *doubly extensible* in M provided M contains a simple arc a_1abb_1 such that every point of ab is an interior point of a_1abb_1. A continuum homeomorphic with the letter θ or with the figure 8 is called a *θ-curve* or an *8-curve* respectively; one homeomorphic with the sum of two disjoint circles plus a segment joining them with just an end point in each will be called a

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

dumbbell curve; and one homeomorphic with a circumference plus a radius will be called a *lariat curve*. A set is *cyclic* if it is connected and has no cut point.

2. Imbedding results.

(2.1) THEOREM. *In a locally connected continuum M any doubly extensible arc ab is contained in either a* (a) *simple closed curve,* (b) *θ-curve,* (c) *8-curve,* (d) *dumbbell curve,* (e) *lariat curve, or* (f) *maximal arc in M, where all end points of curves* (e) *or* (f) *are end points of M.*

PROOF. Let a_1abb_1 be an arc in M containing ab in its interior. If a is not a cut point of M, let uv be an arc in $M-a$ such that $u\in a_1a$, $v\in ab_1$ and $uv\cdot a_1b_1=u+v$. Let $C_a=uv+$arc uav of a_1b_1. If $v\in bb_1$, C_a is a simple closed curve containing ab. Hence we may suppose $v\in ab-a-b$. Similarly if b is not a cut point, we construct an arc rs in $M-b$ such that $s\in bb_1$, $r\in ab-a-b$ and let $C_b=rs+$arc rbs of a_1b_1.

If a is a cut point of M, let R be a component of $M-a$ not containing $ab-a$. If $R+a$ contains a simple closed curve N_a, let C_a be N_a+an arc aq in $R+a$, where $aq\cdot N_a=q$, and $aq=a$ if $a\in C_a$. Otherwise let C_a be an arc ae in $R+a$ joining a and an end point e of M. Construct C_b similarly if b is a cut point.

Now if neither a nor b is a cut point, the open arcs $uv-u-v$ and $rs-r-s$ may intersect. In this case their sum$+u+s$ contains an arc us intersecting a_1b_1 in just $u+s$, and clearly $us+uabs$ (of a_1b_1) is a simple closed curve containing ab. If these arcs are disjoint, let $C=C_a+C_b+ab$. Then C contains ab and is a θ-curve if r precedes v on ab, an 8-curve if $v=r$, and a dumbbell curve if r follows v on ab.

In case at least one of the points a and b is a cut point, the sets C_a and C_b are disjoint and the set $C=C_a+C_b+ab$ is a dumbbell curve, a simple arc, or a lariat curve and all end points of C are end points of M.

(2.11) COROLLARY. *If M has no end points, any doubly extensible arc in M is contained in a curve of one of the types* (a)–(d) *in M.*

(2.2) THEOREM. *Every doubly extensible arc ab in a cyclic locally connected continuum C is contained in a cyclic graph in C.*

For let a_1abb_1 be an arc in C containing ab in its interior. Let S be the set of all $x\in ab$ such that there exists a cyclic graph G_x in C intersecting ab_1 in an arc ay where $y\in xb_1-x$. Then S is nonempty since $M-a$ contains an arc wz intersecting a_1b_1 in just w and z and so that $w\in a_1a$, $z\in ab_1$. Also S is open in ab because if $x\in S$ and G_x and ay are determined by x, clearly they serve also for any $x'\in ay-y$. But S

is also closed in ab. For let p be a limit point of S and suppose p not in S. Let uv be an arc in $C-p$ with $u\in ap$, $v\in pb_1$ and $uv\cdot ab_1=u+v$. Then there is a point $x\in S\cdot(up-u)$. Let G_x be the corresponding cyclic graph with $G_x\cdot ab_1=ay$. Then $y\in up$, as otherwise $p\in S$. But then if q is the last point of G_x on uv, $G_x+ypv+qv$ is a cyclic graph in C intersecting ab_1 in just av and $v\in pb_1-p$.

Thus S is both open and closed in ab and hence $S=ab$. Accordingly there exists a cyclic graph G_b intersecting ab_1 in ay, where $y\in bb_1-b$, and hence containing ab.

Since every arc in a cyclic graph, or in any graph without end points, obviously is doubly extensible in that graph, we have

(2.21) COROLLARY. *In a cyclic locally connected continuum C, a simple arc is doubly extensible if and only if it is imbeddable in a cyclic graph in C.*

A simple example of a cyclic locally connected continuum containing a non-extensible arc is afforded by the right triangle with vertices (0, 0), (1, 0), and (1, 1) together with an infinite sequence of vertical segments joining the hypotenuse and base and converging to the origin. Then any simple arc in this continuum containing infinitely many of the vertical segments fails to be doubly extensible.

3. Locally simple representations on the circle.

We begin with an addition theorem.

(3.1) THEOREM. *If the continua C_1 and C_2 have locally simple representations on the circle and $C_1\cdot C_2\neq 0$, then C_1+C_2 has a locally simple representation on the circle.*

PROOF. Let $p\in C_1-C_1\cdot C_2$, $q\in C_2-C_1\cdot C_2$, and let $f_1(z)$ and $f_2(z)$ be locally simple mappings of the circle $|z|=1$ onto C_1 and C_2 respectively so chosen that $f_1(1)=p$, $f_2(1)=q$. Let θ_1 be the least value of θ in the range $\pi/2<\theta<\pi$ such that $f_1(e^{i\theta})=w\in C_2$ and let θ_2 be the least value of θ in the range $3\pi/2<\theta<2\pi$ such that $f_2(e^{i\theta})=s\in C_1$. Let α and β be positive or 0 angles less than $\pi/2$ such that

$$f_2[e^{i(\theta_1-\alpha)}]=w, \qquad f_1[e^{i(\theta_2-\beta)}]=s.$$

Now let us define $f(z)$ for $z=e^{i\theta}$, $0\leqq\theta\leqq 2\pi$, as follows:

$$\begin{aligned} f(z) &= f_1(e^{i\theta}) && (\text{for } 0\leqq\theta\leqq\theta_1)\\ &= f_2(e^{i\phi}) && (\text{for } \theta_1\leqq\theta\leqq\theta_2, \text{ where } \phi=\theta-\alpha(\theta_2-\theta)/(\theta_2-\theta_1))\\ &= f_1(e^{i\psi}) && (\text{for } \theta_2\leqq\theta\leqq 2\pi, \text{ where } \psi=\theta-\beta(2\pi-\theta)/(2\pi-\theta_2)). \end{aligned}$$

Then f is continuous since the definitions agree at the places where they overlap, namely, the values θ_1, θ_2 and 2π or 0 for θ. Also since f_1 and f_2 are locally simple and the image set under f of the arc given by $\pi/2 \leqq \theta \leqq \theta_1$ has just w in common with C_2 whereas $f(z) \in C_2$ for $\theta_1 \leqq \theta \leqq 3\pi/2$ and a similar situation holds in the neighborhood of θ_2, it follows that f is locally simple. Finally, since f agrees with f_1 over an interval of θ of length greater than $\pi/2$ and also with f_2 over a similar interval, it follows that f maps the circle $|z| = 1$ onto $C_1 + C_2$.

(3.11) Corollary. *Any continuum which is the sum of a finite number of continua each of which has a locally simple representation on the circle itself has a locally simple representation on the circle.*

(3.2) Theorem. *In order that a continuum M have a locally simple representation on the circle it is necessary and sufficient that it be the sum of a finite number of doubly extensible arcs.*

For suppose the mapping $f(C) = M$ is locally simple, where C is a circle. Then for some $e > 0$, every arc of C of diameter less than $3e$ maps topologically onto its image in M. Thus if we subdivide C into a finite number of arcs of diameter less than e, the image of each of these arcs is a doubly extensible simple arc in M and the sum of these images is M itself.

On the other hand, suppose M is the sum of a finite number of double extensible arcs $\alpha_1, \alpha_2, \cdots, \alpha_n$. Since then no point of M can be an end point, by (2.11) each α_i is contained in either a simple closed curve, a θ-curve, an 8-curve, or a dumbbell curve in M. Thus each α_i is contained in either a dumbbell curve or in the sum of at most two simple closed curves in M. Accordingly, we can write $M = \sum_1^n \alpha_i = \sum_1^m C_i$, where each C_i is either a simple closed curve or a dumbell curve. Since clearly each C_i has a locally simple representation on the circle, it follows by (3.11) that so also has M.

(3.3) Theorem. *In order that a continuum have a locally simple representation on the circle it is necessary and sufficient that it be the sum of a finite number of simple closed curves and dumbbell curves.*

For if M satisfies this condition, it has a locally simple representation on the circle by (3.11), since each of the component curves has such a representation; and if M has such a representation, by (3.2) and its proof, M is the sum of a finite number of curves of the required type.

(3.31) *A connected graph has a locally simple representation on the circle if and only if it has no end points.*

(3.4) Theorem. *A cyclic continuum M has a locally simple repre-*

sentation on the circle if and only if it is the sum of a finite number of simple closed curves.

For if M is such a sum, it has a locally simple representation on the circle by (3.11). On the other hand, if M has this type of representation, by (3.2) it is the sum of a finite number of doubly extensible arcs. Then since each such arc lies in a cyclic graph in M by (2.2), M is the sum of an equal number of cyclic graphs. But each cyclic graph is the sum of a finite number of simple closed curves, because each edge lies on a simple closed curve in the graph. Accordingly, M is the sum of a finite number of simple closed curves.

Example. It is now clear that the continuum described at the end of §2 admits no locally simple representation on the circle, since it is not a finite sum of simple close curves or of extensible arcs.

4. Representations of boundary curves.

A *boundary curve* is a locally connected continuum every true cyclic element of which is a a simple closed curve, or equivalently, a locally connected continuum which is homeomorphic with the boundary of a plane region. For this and other terms used in §§4 and 5, the reader is referred to the author's book [2].

(4.1) THEOREM. *The following properties are equivalent for locally connected continua M:*

(i) *M has only a finite number of nodes and no end points;*

(ii) *there exists a $d>0$ such that each node of M is of diameter not less than d;*

(iii) *there exists an $e>0$ such that each component of the complement in M of an arbitrary point of M is of diameter not less than e.*

First, (i) implies (ii), since every degenerate node is an end point. To show that (ii) implies (iii), we have only to take $e=d$. Then since the closure of any component of the complement in M of a point of M contains at least one node of M, (iii) is satisfied. Finally, if (iii) holds, M can have neither an end point nor an infinity of nodes, because in either case we could find a cut point p of M and a component of $M-p$ of diameter less than any given positive number since the nondegenerate nodes can form a null-sequence at most.

(4.2) THEOREM. *If a continuum M has a locally simple representation on the circle, it has the properties listed in* (4.1).

For by (3.3), $M=\sum_1^n C_i+\sum_1^m D_j$, where each C_i is a simple closed curve and each D_j is a dumbbell curve. Let C_{1j} and C_{2j} be the two simple closed curves in D_j for each $j\leq m$, and set $e=\min\ [\delta(C_i),$

$\delta(C_{1j})$, $\delta(C_{2j})]$ for all $i \leqq n$ and $j \leqq m$. Then for any $p \in M$ and any component R of $M-p$, we must have $\delta(R) \geqq e$, because $\bar{R}$ contains either some C_i, some C_{1j}, or some C_{2j}. Thus (iii) holds.

(4.3) THEOREM. *Let B be any boundary curve having one of the properties in* (4.1). *Then any light non-alternating mapping $f(C)=B$ of a circle C onto B is locally simple.*

For by property (iii), there exists an $e>0$ such that for any $p \in B$, each component of $B-p$ is of diameter not less than e. Let $\delta>0$ be chosen so that the image under f of any subset of C of diameter less than δ is of diameter less than e. Then on any arc of C of diameter less than δ, f must be 1-1. For otherwise there would exist a point p of B and an arc xy of C with $\delta(xy)<\delta$ and $f(x)=f(y)=p$, $f(xy-x-y) \subset B-p$. But this is impossible, because $f(xy-x-y)$ is a component of $B-p$ since f is non-alternating, and $\delta[f(xy)]<e$. Accordingly, f is locally simple.

(4.4) THEOREM. *A boundary curve has a locally simple representation on the circle if and only if it has the properties in* (4.1).

This is a direct consequence of (4.2), (4.3), and the fact that any boundary curve has light non-alternating representations on the circle.

5. **Cyclic connectedness.** We conclude by showing how the imbedding result (2.2) affords a simple approach to the cyclic connectedness theorem. First we give two additional elementary results.

(5.1) *Any two points of a locally connected continuum M which are interior points of arcs of M* (thus actually any two non-end points) *lie together on a doubly extensible arc in M.*

For let a and b be two such points and let α and β be disjoint arcs in M having a and b respectively as interior points. Let a_1b_1 be any arc in M with $a_1b_1 \cdot \alpha = a_1$, $a_1b_1 \cdot \beta = b_1$. Then clearly the arc $ab = a_1b_1 + a_1a$ (of α) $+ b_1b$ (of β) is doubly extensible in M.

(5.2) *Any two points a and b of a cyclic graph G lie together on a simple closed curve in G.*

For let S be the set of all points x in G such that some simple closed curve C_x in G contains both a and x. Then S is closed. For if S contains a point x of an open edge E of G, the simple closed curve C_x must contain $\bar{E}$ so that $S \supset \bar{E}$. However, S is also open in G. For as just seen, S contains every open edge of G which it intersects; and if a vertex x of G is in S and C_x is a simple closed curve in G containing $a+x$ and E is an open edge of G having x as an end point but not intersecting C_x, E lies in an arc xy of G with $xy \cdot C_x = x + y$. Since then $xy +$ arc xay of C_x is a simple closed curve containing E, we have $E \subset S$ and thus x is interior to S. Hence S is both open and closed in G and accordingly $S = G$.

Clearly (5.1) and (5.2) together with (2.2) give:

(5.3) *Any two points of a cyclic locally connected continuum C which are interior point of arcs of C lie together on a simple closed curve in C.*

Now if we make use of the fact, somewhat harder to prove, that *every point* of a cyclic locally connected continuum C is interior to some arc in C, we can omit the clause "which are interior points of arcs of C" from (5.3) and we have the cyclic connectedness theorem.

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UNIVERSITY OF VIRGINIA

CONSTRAINED MOTION UPON A SURFACE UNDER A GENERALIZED FIELD OF FORCE

JOHN DeCICCO

1. **Introduction.** We shall present here some theorems concerning the trajectories of a particle moving in a given smooth surface when acted upon only by a generalized field of force and free of the pressure of the surface.

Kasner has developed the differential geometry of the dynamical trajectories of general positional fields of force on a plane, in space, and on a surface, in his Princeton Colloquium Lectures.[1] Recently Kasner and DeCicco have introduced the concept of *generalized* fields of force which depend not only upon the position of the point but also upon the direction through the point. Positional fields may be described as *isotropic* and generalized fields as *anisotropic*.[2]

Consider a generalized field of force defined in a certain region of ordinary space.[3] Let Σ be a smooth surface in this particular region. The trajectories described by a particle constrained to move in Σ and acted upon only by this generalized field of force are ∞^3 in number, just as is the case for a plane.

In general curvilinear coordinates (u, v) of Σ, these trajectories are defined by a differential equation of the form

$$\text{(F)} \qquad v''' = F(u, v, v') + G(u, v, v')v'' + H(u, v, v')v''^2.$$

Conversely any differential equation of this form which is satisfied by the totality of geodesics of Σ represents the trajectories of a particle constrained to move in Σ, which is acted upon only by a generalized field of force.

Kasner has proved that in the (u, v)-plane, the curves defined by a differential equation of third order of the form (F) may be characterized in the following manner.[4] The locus of the foci of the osculating parabolas constructed to the ∞^1 trajectories passing

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[1] Kasner, *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 3, 1913, 1934 (referred to as the Princeton Colloquium). Also see series of papers in Trans. Amer. Math. Soc. vols. 7–11 (1906–1910).

[2] Kasner and DeCicco, *A generalized theory of dynamical trajectories*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 23–38.

[3] Kasner and DeCicco, *Generalized dynamical trajectories in space*, Duke Math. J. vol. 4 (1943) pp. 733–742.

[4] See the Princeton Colloquium, pp. 104–107.

through a given lineal-element is a special quartic curve.

If the special quartic curve degenerates into a circle passing through the point of the given lineal-element, then Σ is a developable surface, which is represented on the (u, v)-plane by an unrolling of Σ followed by an affinity. In this event, the trajectories are represented as those of a positional field of force in the (u, v)-plane.

More generally, if the motion on the surface Σ is pictured in the (u, v)-plane by a motion due to a positional field together with a resistance depending on the velocity components and acting in the direction of the motion, then Σ is of constant curvature and the representation is geodesic.

For any generalized field of force on a surface Σ, the ∞^1 trajectories starting from a given lineal-element of Σ have osculating spheres at the common point, whose centers lie on a general conic in the plane normal to the element. This conic passes through the center of curvature of the normal section of the surface Σ determined by the given lineal-element. If the element is in one of the principal directions of the surface Σ, the conic touches the normal to Σ.

Although Kasner first obtained the preceding property for a positional field of force, it is found to be valid also for generalized fields.

After discussing the differential equation of third order defining generalized dynamical trajectories on the surface Σ, we shall compare the ∞^1 lines of force and the ∞^2 rest trajectories. Let a particle which is initially at rest start from a point O. The rest trajectory will be tangent to the line of force at O. We study the ratio ρ of the departure of the rest trajectory to that of the line of force at O. In the positional case, Kasner showed that $\rho = 1/3$ in the case of first order contact. For the generalized case of the plane, Kasner and DeCicco obtained a more general formula. In the present paper, we shall prove that this more general formula is valid also for a surface Σ.

2. The acceleration vector. Let a particle be constrained to move in a surface Σ which is defined parametrically by the equations

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v). \tag{1}$$

Of course, at least two of these three functions of (u, v), are functionally independent in a certain region of the (u, v)-plane.

The rectangular components of the velocity vector are given by the expressions

$$\dot{x} = x_u\dot{u} + x_v\dot{v}, \qquad \dot{y} = y_u\dot{u} + y_v\dot{v}, \qquad \dot{z} = z_u\dot{u} + z_v\dot{v}. \tag{2}$$

The speed ω is defined by the formula

$$\omega^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2. \tag{3}$$

The rectangular components of the acceleration vector are

$$\begin{aligned} \ddot{x} &= x_u\ddot{u} + x_v\ddot{v} + x_{uu}\dot{u}^2 + 2x_{uv}\dot{u}\dot{v} + x_{vv}\dot{v}^2, \\ \ddot{y} &= y_u\ddot{u} + y_v\ddot{v} + y_{uu}\dot{u}^2 + 2y_{uv}\dot{u}\dot{v} + y_{vv}\dot{v}^2, \\ \ddot{z} &= z_u\ddot{u} + z_v\ddot{v} + z_{uu}\dot{u}^2 + 2z_{uv}\dot{u}\dot{v} + z_{vv}\dot{v}^2. \end{aligned} \tag{4}$$

The length a_m of the projection of the acceleration vector along any direction $(\delta u, \delta v)$ on Σ is found to satisfy the equation

$$\begin{aligned} &[E\delta u^2 + 2F\delta u\delta v + G\delta v^2]^{1/2}a_m \\ &= \left[E\ddot{u} + F\ddot{v} + \frac{E_u}{2}\dot{u}^2 + E_v\dot{u}\dot{v} + \left(F_v - \frac{G_u}{2}\right)\dot{v}^2\right]\delta u \\ &\quad + \left[F\ddot{u} + G\ddot{v} + \left(F_u - \frac{E_v}{2}\right)\dot{u}^2 + G_u\dot{u}\dot{v} + \frac{G_v}{2}\dot{v}^2\right]\delta v. \end{aligned} \tag{5}$$

Before proceeding further with our work, let us recall that the geodesic curvature k of a curve on the surface Σ is

$$k = \frac{H[v'' - nv'^3 - (2m - \nu)v'^2 - (l - 2\mu)v' + \lambda]}{[E + 2Fv' + Gv'^2]^{3/2}}, \tag{6}$$

where

$$\begin{aligned} l &= \frac{1}{2H^2}(GE_u - 2FF_u + FE_v), & \lambda &= \frac{1}{2H^2}(2EE_u - EE_v - FE_u), \\ m &= \frac{1}{2H^2}(GE_v - FG_u), & \mu &= \frac{1}{2H^2}(EG_u - FE_v), \\ n &= \frac{1}{2H^2}(2GF_v - GG_u - FG_v), & \nu &= \frac{1}{2H^2}(EG_v - 2FF_v + FG_u). \end{aligned} \tag{7}$$

Of course, $H = (EG - F^2)^{1/2}$.

Let a_1 denote the component of the acceleration vector parallel to the direction defined by $F\delta u + G\delta v = 0$, and let a_2 denote the component of the acceleration vector parallel to the direction defined by $E\delta u + F\delta v = 0$. By (5) and (7), we obtain

$$\begin{aligned} a_1 &= \frac{1}{G^{1/2}H}[\ddot{u} + l\dot{u}^2 + 2m\dot{u}\dot{v} + n\dot{v}^2], \\ a_2 &= \frac{1}{E^{1/2}H}[\ddot{v} + \lambda\dot{u}^2 + 2\mu\dot{u}\dot{v} + \nu\dot{v}^2]. \end{aligned} \tag{8}$$

If a_t is the tangential component and a_n the normal component on the surface Σ of the acceleration vector, then from (5) and (6) can be deduced the well known formulas

$$a_t = \dot v = v\frac{dv}{ds}, \qquad a_n = kv^2 = \frac{v^2}{r}\cdot \tag{9}$$

3. **The differential equations of the generalized dynamical trajectories on the surface Σ.** Let $G^{-1/2}H^{-1}\phi(u, v, v')$ denote the component of the generalized force vector parallel to the direction defined by $F\delta u + G\delta v = 0$, and let $E^{-1/2}H^{-1}\psi(u, v, v')$ denote the component of the force vector parallel to the direction defined by $E\delta u + F\delta v = 0$. By (8), it is found that *the differential equations of motion of a particle of unit mass, constrained to move in the surface Σ and acted upon only by a generalized field of force, are*[5]

$$\ddot u = \phi - l\dot u^2 - 2m\dot u\dot v - n\dot v^2,\ \ddot v = \psi - \lambda\dot u^2 - 2\mu\dot u\dot v - \nu\dot v^2. \tag{10}$$

It is desirable to find ϕ and ψ in terms of the original force vector defined by the rectangular components $X(u, v, v')$, $Y(u, v, v')$, $Z(u, v, v')$. The values are

$$\begin{aligned} \phi &= G(Xx_u + Yy_u + Zz_u) - F(Xx_v + Yy_v + Zz_v), \\ \psi &= -F(Xx_u + Yy_u + Zz_u) + E(Xx_v + Yy_v + Zz_v). \end{aligned} \tag{11}$$

Substituting (10) into (5) where $\delta u : \delta v = \dot u : \dot v$, we find that the tangential component T of the force vector along a trajectory is

$$T = \frac{(E\phi + F\psi) + (F\phi + G\psi)v'}{(E + 2Fv' + Gv'^2)^{1/2}}\cdot \tag{12}$$

Substituting the normal direction $\delta v/\delta u = -(E + Fv')/(F + Gv')$ into (5), we discover the normal component N of the force vector to be

$$N = \frac{H(\psi - v'\phi)}{(E + 2Fv' + Gv'^2)^{1/2}}\cdot \tag{13}$$

From (6), (9), and (13), it is seen that the square of the speed ω is

$$\omega^2 = \frac{(\psi - v'\phi)(E + 2Fv' + Gv'^2)}{v'' - nv'^3 - (2m - \nu)v'^2 - (l - 2\mu)v' + \lambda}\cdot \tag{14}$$

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[5] These could have been easily obtained by means of the Lagrangian equations. However, we prefer to derive the differential equations of motion from fundamental principles. For an analogous discussion, see Whittaker, *Analytical dynamics*, Dover Publications, 1944, pp. 99–109.

The direction of the projected force vector on the surface Σ is the one which yields the maximum value of T in (12). For this, it is necessary and sufficient that $N=0$. *Thus the lines of force on the surface* Σ *(if they exist) are given by the differential equation of first order*

$$\psi(u, v, v') - v'\phi(u, v, v') = 0. \tag{15}$$

Substituting this into (12), the magnitude a of the projected force vector is

$$a^2 = E\phi^2 + 2F\phi\psi + G\psi^2. \tag{16}$$

From (12), (13), and (16), we find

$$a^2 = T^2 + N^2. \tag{17}$$

We proceed to eliminate the time t from the equations (10) representing the trajectories on the surface Σ. For this purpose, it is found useful to introduce the quantity Δ defined by the equation

$$\Delta = v'' - nv'^3 - (2m - \nu)v'^2 - (l - 2\mu)v' + \lambda. \tag{18}$$

Thus $\Delta=0$ only along the geodesics of Σ.

Upon substituting (10) into the formulas

$$\dot{v} = v'\dot{u}, \qquad \ddot{v} = v''\dot{u}^2 + v'\ddot{u}, \tag{19}$$

we find

$$\dot{u}^2 = (\psi - v'\phi)/\Delta\,\cdot \tag{20}$$

Differentiating this with respect to u and eliminating $\dot{u}$, we find the equation of third order

$$\begin{aligned}(\psi - v'\phi)\left[\frac{d\Delta}{du} - 2\Delta(l + 2mv' + nv'^2)\right]&\\ = -\,2\phi\Delta^2 + \Delta\frac{d}{du}(\psi - v'\phi).&\end{aligned} \tag{21}$$

Thus the differential equation of third order representing the ∞^3 *dynamical trajectories on the surface* Σ *is*

$$\begin{aligned}(\psi - v'\phi)\frac{d\Delta}{du} = {}&(\psi_{v'} - v'\phi_{v'} - 3\phi)\Delta^2\\ &+ [(\psi_{v'} - v'\phi_{v'} - \phi)\{nv'^3 + (2m - \nu)v'^2 + (l - 2\mu)v' - \lambda\}\\ &+ 2(\psi - v'\phi)(l + 2mv' + nv'^2)\\ &+ \psi_u + v'(\psi_v - \phi_u) - v'^2\phi_v]\Delta.\end{aligned} \tag{22}$$

THEOREM 1. *Any differential equation of third order of the form*

$$\frac{d\Delta}{du} = G(u, v, v')\Delta + H(u, v, v')\Delta^2, \tag{23}$$

where Δ *is given by* (18), *represents a dynamical family of trajectories of a generalized field of force, constrained to move on a surface* Σ.

The remarkable similarity between this differential equation and that of the type (G) which has been studied extensively by Kasner should be noticed. Of course, this reduces to the G-type when Σ is developable and the mapping on the (u, v)-plane is an unrolling of Σ followed by an affinity.

In the case of a positional field of force, the functions G and H appearing in (23) must be restricted further.

We prove the above theorem. First note that there is a one-to-one correspondence between the functions (N, T) and the functions (ϕ, ψ). This correspondence is given by (12) and (13).

Let P and Q be defined by the expressions

$$P = \frac{2\acute{T}}{(E + 2Fv' + Gv'^2)^{1/2}}, \qquad Q = \frac{N}{H}(E + 2Fv' + Gv'^2)^{1/2}. \tag{24}$$

Thus if (ϕ, ψ) are known, the quantities (P, Q) can be calculated, and conversely.

If we use the equations (9) and (24), it is found that the differential equation of the dynamical family may be written in the form

$$P = \frac{d}{du}\left(\frac{Q}{\Delta}\right). \tag{25}$$

The expansion of this is

$$\frac{d\Delta}{du} = (Q_u + v'Q_v + v''Q_{v'})\frac{\Delta}{Q} - \frac{P\Delta^2}{Q}. \tag{26}$$

Now $(Q_u+v'Q_v)/Q$ can represent any function of (u, v, v') say $R(u, v, v')$. Finally let

$$\begin{aligned} G &= R + \frac{Q_{v'}}{Q}\left[nv'^3 + (2m - \nu)v'^2 + (l - 2\mu)v' - \lambda\right], \\ H &= \frac{Q_{v'} - P}{Q}. \end{aligned} \tag{27}$$

From these equations, it is seen that G and H can be any pair of arbitrary functions of (u, v, v').

Substituting (27) into (26) and making use of (18), we obtain the equation (23). This completes our proof of Theorem 1.

THEOREM 2. *In the auxiliary (u, v)-plane, any differential equation of third order of the type* (F) *which is satisfied by the ∞^3 geodesics of the surface Σ can represent the dynamical trajectories on Σ when acted upon only by a generalized field of force.*

This result is an immediate consequence of Theorem 1. For upon expanding the differential equation (23) by means of (18), we find the type (F). This must be satisfied by the ∞^3 geodesics of Σ since $\Delta=0$ is a solution of (23). Conversely, if $\Delta=0$ is a solution of a differential equation of third order of the type (F), then it must be written in the form (23). Thus Theorem 2 follows.

4. The generalization upon a surface Σ of Kasner's theorem concerning one-third of the curvatures. In this section, we shall study, in the motion of a particle constrained to move on a surface Σ when acted upon only by a generalized field of force, the ratio ρ of the departure of the rest trajectory to that of the line of force at a given point O. For the positional case, Kasner showed that $\rho=1/3$. In the generalized type on a plane,[6] Kasner and DeCicco obtained a more general formula for ρ in terms of the angular rate λ. We shall show that this more general formula is valid also upon a surface Σ.[7]

At this point, we shall define the angular rate λ which is associated to any lineal-element E on the surface Σ for the motion of a particle constrained to move on Σ when acted upon by a generalized field of force. As the lineal-element E rotates about its point O, the projection F on Σ of the corresponding force vector also rotates about O. The *angular rate* λ is the instantaneous rate of change of the inclination of F with respect to the inclination of E. It is given by the formula

$$\lambda=\frac{(E+2Fv'+Gv'^2)(\phi\psi_{v'}-\psi\phi_{v'})}{E\phi^2+2F\phi\psi+G\psi^2}. \tag{28}$$

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[6] In space there is, in general, no analogous formula for generalized fields of force. See DeCicco, *Extensions of certain dynamical theorems of Halphen and Kasner*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 736–744.

[7] For a generalization to acceleration fields of higher order, see Kasner and Mittleman, *A general theorem on the initial curvature of dynamical trajectories*, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 48–52. Also *Extended theorems in dynamics*, Science vol. 95 (1942) pp. 249–250. Also Kasner and Fialkow, Trans. Amer. Math. Soc. vol. 41 (1937). Fialkow, *Initial motion in fields of force*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 495–501.

For any ordinary positional field of force, $\lambda=0$. However, there are also certain generalized fields of force for which the angular rate $\lambda=0$. These are such that the direction of the force vector at any point 0 does not vary, but its magnitude may.

Since along any line of force $\psi=v'\phi$, it is found that along any lineal-element E of a line of force, the angular rate λ is

$$\lambda=\frac{\psi_{v'}-v'\phi_{v'}}{\phi}. \tag{29}$$

Thus the condition that the angular rate $\lambda\neq 1$ is equivalent to stating that there exists a function $v'=f(u, v)$ which satisfies the differential equation (15) identically. Thus, in general there are ∞^1 lines of force.

A *rest trajectory* is defined as the path described by a particle, constrained to move in the surface Σ when acted upon only by a generalized field of force, which starts at rest from a given point O.

From the equations (19) together with the initial conditions $\dot{u}=0$, $\dot{v}=0$, it is deduced that a rest trajectory is initially tangent to the line of force through the given point O.

Since, on any rest trajectory, there is only one such point O where the particle is at rest, there are ∞^2 rest trajectories on a surface Σ.

At the point O, let E_0 be the lineal-element of the line of force. The rest trajectory described by a particle initially at rest at O will be tangent to the line of force along E_0.

In the present paper, we shall assume that the geodesic curvature of the rest trajectory is not zero at the point O. The case of higher order contact with E_0 will be discussed elsewhere. Since our rest trajectory is tangent to the line of force at O, the first derivative must satisfy (15). Therefore from this and (22), it is found that the second derivative v'' of the rest trajectory at 0 is given by the expression

$$\begin{aligned}&(\psi_{v'}-v'\phi_{v'}-3\phi)\Delta\\ &\quad+(\psi_{v'}-v'\phi_{v'}-\phi)\left\{nv'^3+(2m-\nu)v'^2+(l-2\mu)v'-\lambda\right\}\\ &\quad+\psi_u+(\psi_v-\phi_u)v'-\phi_v v'^2=0.\end{aligned} \tag{30}$$

If $\bar{\Delta}$ denotes the analogous expression (18) for the line of force, we find, by differentiating (15), the following expression

$$\begin{aligned}&(\psi_{v'}-v'\phi_{v'}-\phi)\bar{\Delta}\\ &\quad+(\psi_{v'}-v'\phi_{v'}-\phi)\left\{nv'^3+(2m-\nu)v'^2+(l-2\mu)v'-\lambda\right\}\\ &\quad+\psi_u+(\psi_v-\phi_u)v'-\phi_v v'^2=0.\end{aligned} \tag{31}$$

Subtracting the above two equations and using the formula (29) wherein $\phi \neq 0$ since the force vector by assumption is not zero, we obtain

$$(3 - \lambda)\Delta = (1 - \lambda)\bar{\Delta}. \tag{32}$$

As $\rho = \Delta/\bar{\Delta}$ is the ratio of the geodesic curvature of the rest trajectory to that of the line of force, we have proved the following result.

THEOREM 3. *The ratio ρ of the geodesic curvatures at E_0 of the rest trajectory and the line of force is given by the formula*

$$\rho = (1 - \lambda)/(3 - \lambda). \tag{33}$$

By this result, it is seen that if $\rho = 1/3$, then the angular rate $\lambda = 0$ at the lineal-element E_0.

THEOREM 4. *All generalized fields of force for which the ratio of departures $\rho = 1/3$ must be such that the eliminant with respect to v' of the equations*

$$\psi - v'\phi = 0, \qquad \psi_{v'} - v'\phi_{v'} = 0, \tag{34}$$

be identically zero.

This general class of generalized fields of force includes the positional fields as special cases.

ILLINOIS INSTITUTE OF TECHNOLOGY

ON THE NOTION OF RECURRENCE IN DISCRETE STOCHASTIC PROCESSES

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1. Introduction. It is the purpose of this note to discuss "statistical" versions of the "Wiederkehrsatz" and the Poincaré cycle and to relate these versions to the ones encountered in dynamical considerations.[2] Although the content of the note is entirely elementary and in part known it is hoped that it will help elucidate some of the basic notions of statistical physics.

2. Recurrence and mean recurrence time in a class of discrete stochastic processes. Let $x_1, x_2, \cdots$ be a sequence of random variables each capable of assuming the values $a_1, a_2, \cdots$. We shall say that the sequence $x_1, x_2, \cdots$ is a stationary process if: (a) for each j the probability

$$\text{Prob. } \{x_n = a_j\}$$

is independent of n; (b) for each set of values $a_{s_1}, a_{s_2}, \cdots, a_{s_r}$ the probability

$$\text{Prob. } \{x_{k_1} = a_{s_1}, x_{k_2} = a_{s_2}, \cdots, x_{k_r} = a_{s_r}\}$$

depends only on the differences $|k_i - k_j|$.

Let

$$\text{Prob. } \{x_n = a_j\} = W_1(a_j) = W(a_j)$$

and

$$\text{Prob. } \{x_1 = a_{s_1}, x_2 = a_{s_2}, \cdots, x_r = a_{s_r}\} = W_r(a_{s_1}, a_{s_2}, \cdots, a_{s_r}).$$

We shall use the notation $\bar{a}_k$ to denote any a_j different from a_k. For instance,

$$\text{Prob. } \{x_1 = a_1, x_2 \neq a_1\} = W_2(a_1, \bar{a}_1),$$

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[1] John Simon Guggenheim Memorial Fellow.

[2] That the theory of stationary stochastic processes is mathematically equivalent with an "ergodic" theory (to which one is also led by dynamical considerations) was clearly recognized by Doob in 1934. See J. L. Doob, *Stochastic processes and statistics*, Proc. Nat. Acad. Sci. U.S.A. vol. 20 (1934) pp. 376–379. The analogies discussed in the present paper are but particular cases of Doob's general equivalence principle. However, since the motivations underlying the statistical and the dynamical points of view are of a somewhat different physical character it seemed desirable to treat both cases separately.

where it is understood that

$$W_2(a_1, \bar{a}_2) = \sum_{r \neq 2} W_2(a_1, a_r).$$

We shall assume that $W(a_j) \neq 0$ for all j.

We also define the conditional probabilities $P_k(a_{s_1} \mid a_{s_2}, \cdots, a_{s_k})$ by the usual formula

$$P_k(a_{s_1} \mid a_{s_2}, \cdots, a_{s_k}) = \frac{W_k(a_{s_1}, a_{s_2}, \cdots, a_{s_k})}{W(a_{s_1})}. \tag{1}$$

To simplify the notation we shall use a without a subscript to denote a chosen and then fixed a_j.

We now established the following, almost trivial, formula

$$\begin{aligned} W_{k+2}(a, \bar{a}, \cdots, \bar{a}, a) &= W_k(\bar{a}, \bar{a}, \cdots, \bar{a}) - 2W_{k+1}(\bar{a}, \bar{a}, \cdots, \bar{a}) \\ &\quad + W_{k+2}(\bar{a}, \bar{a}, \cdots, \bar{a}), \qquad k \geq 0, \end{aligned} \tag{2}$$

where it is understood that $W_0 = 1$. In fact, we have

$$\begin{aligned} W_{k+2}(a, \bar{a}, \cdots, \bar{a}, a) &= W_{k+1}(a, \bar{a}, \cdots, \bar{a}) - W_{k+2}(a, \bar{a}, \cdots, \bar{a}), \\ W_{k+1}(a, \bar{a}, \cdots, \bar{a}) &= W_k(\bar{a}, \bar{a}, \cdots, \bar{a}) - W_{k+1}(\bar{a}, \bar{a}, \cdots, \bar{a}), \\ W_{k+2}(a, \bar{a}, \cdots, \bar{a}) &= W_{k+1}(\bar{a}, \bar{a}, \cdots, \bar{a}) - W_{k+2}(\bar{a}, \bar{a}, \cdots, \bar{a}), \end{aligned}$$

and formula (2) follows at once if one also notices that

$$W_1(a) = 1 - W_1(\bar{a}) = W_0 - W_1(\bar{a}).$$

THEOREM 1. *For each* a

$$\sum_{k=0}^{\infty} P_{k+2}(a \mid \bar{a}, \cdots, \bar{a}, a) = 1. \tag{3}$$

In view of the definition (1) of conditional probabilities it is sufficient to prove that

$$\sum_{k=0}^{\infty} W_{k+2}(a, \bar{a}, \cdots, \bar{a}, a) = W(a). \tag{4}$$

Using (2) we obtain

$$\begin{aligned} \sum_{k=0}^{n} W_{k+2}(a, \bar{a}, \cdots, \bar{a}, a) &= 1 - W(\bar{a}) - (w_{n+1} - w_{n+2}) \\ &= W(a) - (w_{n+1} - w_{n+2}) \end{aligned} \tag{5}$$

where from now on we shall use the abbreviation

$$w_s = W_s(\bar{a}, \bar{a}, \cdots, \bar{a}).$$

Since $w_{n+1} \geq w_{n+2}$ and the w's are non-negative we deduce that $\lim_{n\to\infty} w_n$ exists and hence

$$\lim_{n\to\infty} (w_{n+1} - w_{n+2}) = 0.$$

This together with (5) implies (4) and hence (3). The interpretation of (3) is almost immediate.

In fact, $P_{k+2}(a \mid \bar{a}, \cdots, \bar{a}, a)$ is the probability that if the "state" a was observed at the beginning, then it will be observed again, for the first time, after $k+1$ steps. Consequently,

$$\sum_{k=0}^{\infty} P_{k+2}(a \mid \bar{a}, \cdots, \bar{a}, a)$$

is the probability that if a was observed at the beginning it will eventually be observed again (the events of observing a for the first time after 1, 2, 3, $\cdots$ steps are obviously mutually exclusive).

Theorem 1 can thus be restated as follows: *with probability* 1 *each state is bound to recur.*

This is the "statistical" version of the "Wiederkehrsatz" and it will be seen in the next section that it implies the classical "Wiederkehrsatz."

If we think of the random variable x_n as being associated with the observation of the system at time $n\tau$ we can define the mean recurrence time of the state a as

$$\theta_a = \sum_{k=0}^{\infty} (k+1)\tau P_{k+2}(a \mid \bar{a}, \cdots, \bar{a}, a). \tag{6}$$

This is the statistical analogue of Poincaré's cycle. In this connection we have the following theorem.

THEOREM 2. *If*

$$\lim_{n\to\infty} W_n(\bar{a}, \bar{a}, \cdots, \bar{a}) = 0 \tag{7}$$

then

$$\theta_a = \frac{\tau}{W(a)}. \tag{8}$$

It will be evident from the proof that (7) is also necessary for the validity of (8).

It is sufficient to prove that

$$(9) \qquad \sum_{k=0}^{\infty}(k+1)W_{k+2}(a, \bar{a}, \cdots, \bar{a}, a) = 1.$$

Using (2) we have

$$\sum_{k=0}^{n-1}(k+1)W_{k+2}(a, \bar{a}, \cdots, \bar{a}, a) = 1 - \left\{(n+1)w_n - nw_{n+1}\right\}.$$

Now,

$$(n+1)w_n - nw_{n+1} = n(w_n - w_{n+1}) + w_n$$

and since we have assumed that $w_n \to 0$ it is sufficient to prove that

$$\lim_{n\to\infty} n(w_n - w_{n+1}) = 0.$$

Since the partial sums of (9) form a nondecreasing sequence it follows that $n(w_n - w_{n+1}) + w_n$ is a nonincreasing sequence. Since furthermore its terms are non-negative we deduce that

$$\lim_{n\to\infty}\left\{n(w_n - w_{n+1}) + w_n\right\}$$

exists.

Moreover, $\lim_{n\to\infty} w_n$ exists (this is true regardless of the assumption (7)) and hence $\lim_{n\to\infty} n(w_n - w_{n+1})$ exists.

Now, the series of non-negative terms

$$\sum_{n=1}^{\infty}(w_n - w_{n+1})$$

converges (since $\lim_{n\to\infty} w_n$ exists) and hence (since $\lim_{n\to\infty} n(w_n - w_{n+1})$ exists)

$$\lim_{n\to\infty} n(w_n - w_{n+1}) = 0.$$

This completes the proof of (8).

3. **The classical formulations.** Let Ω be a set in which a Lebesque measures has been established, and assume for the sake of simplicity that the measure of Ω is 1. Let ω denote an element of Ω and let T be a one-to-one measure-preserving transformation of Ω into itself. Let furthermore A be a measurable subset of Ω. The classical "Wiederkehrsatz" can now be stated as follows:

THEOREM 1′. *For almost every ω in A there exists an $n \geqq 1$ such that $T^n\omega$ is again in A.*

Let $f(\omega)$ be the characteristic function of the set A and consider the sequence of measurable functions

$$f(\omega),\ f(T\omega),\ f(T^2\omega),\ \cdots.$$

These functions can be considered as random variables which form a stationary stochastic process. Each random variable can assume only values 1 and 0 with respective probabilities $|A|$ and $1-|A|$.

Note that $W_{k+2}(1, 0, \cdots, 0, 1)$ is nothing but the measure of the set of those ω in A which return to A for the first time after $k+1$ steps (that is, $T^{k+1}\omega$ is the first iteration to fall into A).

Formula (4) thus asserts that the measure of the set of those ω which eventually return to A is equal to the measure of A, and this is equivalent to the statement of Theorem 1′. It should be emphasized that if we were to rewrite the proof of Theorem 1 in the terminology of Theorem 1′ the resulting proof would not differ essentially from the orthodox one.[3]

We are led to a more interesting situation in trying to reformulate Theorem 2. The interest lies in the fact that condition (7) is essentially equivalent to the condition that T is metrically transitive.

For $\omega \in A$ let $n(\omega)$ denote the first $n \geqq 1$ for which $T^n\omega \in A$. Thus $n(\omega)$ is the Poincaré cycle of ω relative to the set A. We have now the following theorem.

THEOREM 2′. *If T is metrically transitive and $|A|>0$ then,*

$$\int_A n(\omega)d\mu = 1. \tag{10}$$

If we define the *average Poincaré cycle* relative to A as

$$\bar{n}_A = \frac{1}{|A|}\int_A n(\omega)d\mu$$

we can restate (10) in the form

$$\bar{n}_A = \frac{1}{|A|}$$

which is more reminiscent of (8).

It is clear that all one has to prove is that metric transitivity of T implies condition (7) which in the present terminology can be written as

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[3] See C. Carathéodory, *Über den Wiederkehrsatz von Poincaré*, Berliner Berichte (1919) pp. 580–584.

$$\lim_{n\to\infty}\int_\Omega \prod_{k=0}^{n}(1-f(T^k\omega))d\mu = 0.$$

Let B denote the set whose characteristic function is

$$\prod_{k=0}^{\infty}(1-f(T^k\omega)).$$

We must prove that $|B|=0$.

Consider the sets T^nB. Since the characteristic function of T^nB is

$$\prod_{k=n}^{\infty}(1-f(T^k\omega))$$

we see that the sets T^nB form a nondescending sequence of sets all of which include B. Moreover, $|T^nB|=|B|$.

Let the set C be the limit of the sets T^nB, that is,

$$C = \lim_{n\to\infty} T^nB.$$

Clearly $TC=C$ and hence, by metric transitivity, $|C|$ is either 0 or 1. Since $|C|=|B|$ we have that either $|B|=0$ or $|B|=1$. If $|B|=1$ then noticing that B is contained in the complement of A we would have $|A|=0$, contrary to the assumption. Hence $|B|=0$ and the proof of Theorem 2′ is completed.

An even shorter proof of the fact that $|B|=0$ can be given by appealing to Birkhoff's ergodic theorem. In fact, this theorem asserts (T being metrically transitive) that for almost all ω we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(T^k\omega) = |A| > 0.$$

If $\omega\in B$ then $f(T^k\omega)=0$ and hence for all $\omega\in B$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(T^k\omega) = 0.$$

Thus $|B|=0$.

It should be noted that if T is not metrically transitive then a set A can be found such that $|A|>0$ and

$$\int_A n(\omega)d\mu < 1.$$

In fact, let D be such that $0<|D|<1$ and $TD=D$. Put $A=\Omega-D$ and note that in this case

$$\lim_{n\to\infty}\int_{\Omega}\prod_{k=0}^{n}(1-f(T^{k}\omega))d\mu=|D|.$$

This implies that

$$\int_{A} n(\omega)d\mu=1-|D|<1.$$

4. **Smoluchowski's definition of the mean recurrence time.** In his important researches concerning the question of irreversibility in thermodynamics, Smoluchowski[4] used a different definition of the mean recurrence time. He defines the mean recurrence time of the state a by means of the formula

$$\theta_{a}^{*}=\tau\frac{1-W(a)}{W_{2}(\bar{a},a)}. \tag{11}$$

In order to justify this definition he considered an actual sequence of observations which consists of a's and $\bar{a}$'s. He then defines the *observed* mean recurrence time as τ times the ratio of the number of $\bar{a}$'s to the number of unbroken blocks of $\bar{a}$'s. For instance, from the sequence of fifteen observations

$$aa\bar{a}a\bar{a}\bar{a}aa\bar{a}\bar{a}\bar{a}a\bar{a}aa$$

he would find the observed mean recurrence time to be $7\tau/4$. Denoting by $k_{n}(\bar{a})$ the number of $\bar{a}$'s in a sequence of n observations and by $l_{n}(\bar{a})$ the number of unbroken blocks of $\bar{a}$'s in that sequence we see that the observed mean recurrence time is

$$\tau\frac{k_{n}(\bar{a})}{l_{n}(\bar{a})}.$$

Let furthermore $k_{n}(\bar{a}, a)$ be the number of pairs of adjacent observations of the form $(\bar{a}, a)$. Then,

$$l_{n}(\bar{a})=k_{n}(\bar{a},a)$$

if the sequence ends with a, and

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[4] M. v. Smoluchowski, *Drei Vorträge über Diffusion, Brownsche Molekularbewegungen und Koagulation von Kolloidteilchen*, Physikalische Zeitschrift vol. 17 (1916) pp. 557–571 and 585–599, in particular pp. 564–568.

The author was also fortunate to have had access to unpublished notes of Professor G. E. Uhlenbeck, where many aspects of Smoluchowski's work were thoroughly analyzed. He is also indebted to Professor Uhlenbeck for many stimulating discussions.

$$l_n(\bar{a}) = k_n(\bar{a}, a) + 1$$

if the sequence ends with $\bar{a}$.

If, in the limit as $n \to \infty$, we replace (following Smoluchowski) the frequencies

$$\frac{k_n(\bar{a})}{n} \quad \text{and} \quad \frac{k_n(\bar{a}, a)}{n}$$

by the corresponding probabilities

$$W(\bar{a}) = 1 - W(a) \quad \text{and} \quad W_2(\bar{a}, a) = W_2(a, \bar{a})$$

we can replace (again in the limit as $n \to \infty$) the ratio $k_n(\bar{a})/l_n(\bar{a})$ by

$$\frac{1 - W(a)}{W_2(\bar{a}, a)}.$$

Smoluchowski considered the process of replacing limits of frequencies by probabilities as self-evident. Actually he tacitly assumed the validity of certain laws of large numbers. Fortunately, Smoluchowski applied his definition mainly to the process of fluctuations of concentration for which the necessary laws of large numbers can easily be established.

Smoluchowski's definition can be also introduced in an "a priori" fashion by setting

$$\theta_a^* = \sum_{k=0}^{\infty} (k+1)\tau P_{k+3}(a, \bar{a} \mid \bar{a}, \cdots, \bar{a}, a). \tag{12}$$

It can then be shown using (2) and (7) that

$$\theta_a^* = \tau \frac{1 - W(a)}{W_2(a, \bar{a})} = \tau \frac{1 - W(a)}{W_2(\bar{a}, a)}. \tag{13}$$

It is well to point out that in accepting this definition we agree not to count as a recurrence the event when a state is followed by itself.

By giving an "a priori" definition we have not eliminated the necessity for laws of large numbers. In facts, laws of large numbers are indispensable whenever we want to check predictions based on "a priori" results through calculations performed on actual sequences of observations.

The advantage of Smoluchowski's definition can be best appreciated if we consider the situation in which the random variables x_n are derived from a continuous stochastic process $x(t)$. In fact, in most applications

$$x_n = x(n\tau).$$

In many cases the limit, as τ approaches 0, of

$$\frac{\tau}{W_2(a, \bar{a})}$$

exists and is different from 0 (note that now $W_2(a, \bar{a})$ depends on τ). Thus it is often possible to define the mean recurrence time in the limit as observations are taken continuously.

If we use the definition of §2 we always get in the limit as $\tau \to 0$ the trivial answer 0. The reason for this is quite obvious, inasmuch as for small τ the probability of a state being followed by itself is very close to 1 and consequently too much weight is attached to what after all is only a fake recurrence.

CORNELL UNIVERSITY

ON THE NUMBER OF POSITIVE SUMS OF INDEPENDENT RANDOM VARIABLES

P. ERDŐS AND M. KAC[1]

1. Introduction. In a recent paper[2] the authors have introduced a method for proving certain limit theorems of the theory of probability. The purpose of the present note is to apply this general method in order to establish the following:

THEOREM. *Let* $X_1, X_2, \cdots$ *be independent random variables each having mean* 0 *and variance* 1 *and such that the central limit theorem is applicable. Let* $s_k = X_1 + X_2 + \cdots + X_k$ *and let* N_n *denote the number of* s_k*'s,* $1 \leqq k \leqq n$*, which are positive. Then,*

$$\lim \text{Prob.} \left\{ \frac{N_n}{n} < \alpha \right\} = \frac{2}{\pi} \text{arc sin } \alpha^{1/2}, \qquad 0 \leqq \alpha \leqq 1.$$

This theorem has been proved in the binomial case

$$\text{Prob. } \{X_j = 1\} = \text{Prob. } \{X_j = -1\} = 1/2$$

by P. Lévy[3] who also indicated that it is true for more general random variables.

However, the authors felt that their proof is of independent interest, especially since it follows an already established pattern.[2]

2. The invariance principle. We first prove the following:

If the theorem can be established for *one particular* sequence of independent random variables $Y_1, Y_2, \cdots$ satisfying the conditions of the theorem then the conclusion of the theorem holds for *all* sequences of independent random variables which satisfy the conditions of the theorem. In other words, if the limiting distribution exists it is independent of the distributions of the individual X's.

Let

$$\psi(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leqq 0, \end{cases}$$

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[1] This paper was written while both authors were John Simon Guggenheim Memorial Fellows.

[2] P. Erdős and M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 292–302.

[3] P. Lévy, *Sur certains processus stochastiques homogènes*, Compositio Math. vol. 7 (1939) pp. 283–339, in particular, Corollaire 2 on pp. 303–304.

and note that $N_n = \sum_{r=1}^{n} \psi(s_r)$. Let now

$$n_j = \left[j\,\frac{n}{k} \right], \qquad j = 0, 1, \cdots, k,$$

and put

$$D_n = \frac{1}{n} \sum_{i=1}^{k} \sum_{r=n_{i-1}+1}^{n_i} \left(\psi(s_{n_i}) - \psi(s_r)\right).$$

We have

$$E(|D_n|) \leq \frac{1}{n} \sum_{i=1}^{k} \sum_{r=n_{i-1}+1}^{n_i} E(|\psi(s_{n_i}) - \psi(s_r)|)$$

and we wish now to estimate $E(|\psi(s_{n_i}) - \psi(s_r)|)$ for $n_{i-1}+1 \leqq r \leqq n_i$. Notice that

$$\begin{aligned} E(|\psi(s_{n_i}) - \psi(s_r)|) = {} & \text{Prob. } \{s_{n_i} > 0, s_r \leqq 0\} \\ & + \text{Prob. } \{s_{n_i} \leqq 0, s_r > 0\} \end{aligned}$$

and that for $\epsilon > 0$

$$\begin{aligned} \text{Prob. } \{s_{n_i} > 0, s_r \leqq 0\} = {} & \text{Prob. } \{s_{n_i} \geqq \epsilon n_i^{1/2}, s_r \leqq 0\} \\ & + \text{Prob. } \{0 < s_{n_i} < \epsilon n_i^{1/2}, s_r \leqq 0\} \\ \leqq {} & \text{Prob. } \{s_{n_i} - s_r \geqq \epsilon n_i^{1/2}\} \\ & + \text{Prob. } \{0 < s_{n_i} < \epsilon n_i^{1/2}\}. \end{aligned}$$

By Chebysheff's inequality

$$\text{Prob. } \{s_{n_i} - s_r \geqq \epsilon n_i^{1/2}\} \leqq \frac{n_i - r}{\epsilon^2 n_i}$$

and hence

$$\text{Prob. } \{s_{n_i} > 0, s_r \leqq 0\} \leqq \frac{n_i - r}{\epsilon^2 n_i} + \text{Prob. } \{0 < s_{n_i} < \epsilon n_i^{1/2}\}.$$

In exactly the same way we obtain

$$\text{Prob. } \{s_{n_i} \leqq 0, s_r > 0\} \leqq \frac{n_i - r}{\epsilon^2 n_i} + \text{Prob. } \{-\epsilon n_i^{1/2} < s_{n_i} \leqq 0\}$$

and thus (for $n_{i-1}+1 \leqq r \leqq n_i$)

$$E(|\psi(s_{n_i}) - \psi(s_r)|) \leqq 2\frac{n_i - r}{\epsilon^2 n_i} + \text{Prob.}\left\{-\epsilon n_i^{1/2} < s_{n_i} < \epsilon n_i^{1/2}\right\}.$$

Finally,

$$\begin{aligned} E(|D_n|) &\leqq \frac{2}{n\epsilon^2}\sum_{i=1}^{k}\frac{1}{n_i}\sum_{r=n_{i-1}+1}^{n_i}(n_i - r) \\ &\quad + \frac{1}{n}\sum_{i=1}^{k}(n_i - n_{i-1})\,\text{Prob.}\left\{-\epsilon n_i^{1/2} < s_{n_i} < \epsilon n_i^{1/2}\right\} \\ &= \frac{1}{n\epsilon^2}\sum_{i=1}^{k}\frac{(n_i - n_{i-1})(n_i - n_{i-1} - 1)}{n_i} \\ &\quad + \frac{1}{n}\sum_{i=1}^{k}(n_i - n_{i-1})\,\text{Prob.}\left\{-\epsilon n_i^{1/2} < s_{n_i} < \epsilon n_i^{1/2}\right\} \\ &= R(n, \epsilon, k). \end{aligned}$$

We note that, letting $n \to \infty$ while keeping k and ϵ fixed, we get

$$\lim_{n\to\infty} R(n, \epsilon, k) = \frac{1}{k\epsilon^2}\sum_{i=1}^{k}\frac{1}{i} + (2\pi)^{-1/2}\int_{-\epsilon}^{\epsilon} e^{-u^2/2}du < \frac{1 + \log k}{k\epsilon^2} + \epsilon$$

(using the central limit theorem). Let $\delta > 0$; we have

$$\text{Prob.}\left\{|D_n| \geqq \delta\right\} \leqq R(n, \epsilon, k)/\delta$$

or, recalling the definition of D_n,

$$\text{Prob.}\left\{\left|\frac{1}{n}\sum_{r=1}^{n}\psi(s_r) - \frac{1}{n}\sum_{i=1}^{k}(n_i - n_{i-1})\psi(s_{n_i})\right| \geqq \delta\right\} \leqq \frac{R(n, \epsilon, k)}{\delta}.$$

We have furthermore

$$\begin{aligned} \text{Prob.}\left\{\frac{1}{n}\sum_{r=1}^{n}\psi(s_r) < \alpha\right\} &= \text{Prob.}\left\{\frac{1}{n}\sum_{r=1}^{n}\psi(s_r) < \alpha,\ |D_n| < \delta\right\} \\ &\quad + \text{Prob.}\left\{\frac{1}{n}\sum_{r=1}^{n}\psi(s_r) < \alpha,\ |D_n| \geqq \delta\right\} \\ &\leqq \text{Prob.}\left\{\frac{1}{n}\sum_{i=1}^{k}(n_i - n_{i-1})\psi(s_{n_i}) < \alpha + \delta\right\} \\ &\quad + \frac{R(n, k, \epsilon)}{\delta}. \end{aligned}$$

In the same way we obtain

$$\begin{aligned}&\text{Prob.}\left\{\frac{1}{n}\sum_{r=1}^{n}\psi(s_r)<\alpha\right\}\\ &\qquad \geqq \text{Prob.}\left\{\frac{1}{n}\sum_{i=1}^{k}(n_i-n_{i-1})\psi(s_{n_i})<\alpha-\delta\right\}-\frac{R(n,\epsilon,k)}{\delta}.\end{aligned}$$

Combining the above inequalities we have

$$\begin{aligned}&\text{Prob.}\left\{\frac{1}{n}\sum_{i=1}^{k}(n_i-n_{i-1})\psi(s_{n_i})<\alpha-\delta\right\}-\frac{R(n,\epsilon,k)}{\delta}\\ &\qquad\leqq \text{Prob.}\left\{\frac{N_n}{n}<\alpha\right\}\\ &\qquad\leqq \text{Prob.}\left\{\frac{1}{n}\sum_{i=1}^{k}(n_i-n_{i-1})\psi(s_{n_i})<\alpha+\delta\right\}+\frac{R(n,\epsilon,k)}{\delta}.\end{aligned}\tag{1}$$

Let now $G_1, G_2, \cdots$ be independent, normally distributed random variables each having mean 0 and variance 1 and let $R_j=G_1+\cdots+G_j$. It then follows, almost immediately, from the multidimensional central limit theorem that

$$\begin{aligned}&\lim_{n\to\infty}\text{Prob.}\left\{\frac{1}{n}\sum_{i=1}^{k}(n_i-n_{i-1})\psi(s_{n_i})<\beta\right\}\\ &\qquad\qquad = \text{Prob.}\left\{\frac{1}{k}\sum_{i=1}^{k}\psi(R_i)<\rho\right\}=p_k(\beta).\end{aligned}$$

If in (1) we let $n\to\infty$ while keeping k, ϵ and δ fixed we obtain

$$\begin{aligned}\rho_k(\alpha-\delta)-\frac{1+\log k}{k\epsilon^2\delta}-\frac{\epsilon}{\delta}&\leqq \liminf_{n\to\infty}\text{Prob.}\left\{\frac{N_n}{n}<\alpha\right\}\\ &\leqq \limsup_{n\to\infty}\text{Prob.}\left\{\frac{N_n}{n}<\alpha\right\}\\ &\leqq \rho_k(\alpha+\delta)+\frac{1+\log k}{k\epsilon^2\delta}+\frac{\epsilon}{\delta}.\end{aligned}\tag{2}$$

Notice that the above inequality holds for all random variables satisfying the conditions of our theorem. Thus if for some particular sequence $Y_1, Y_2, \cdots$ we can prove that

$$\lim_{n\to\infty}\text{Prob.}\left\{\frac{N_n}{n}<\alpha\right\}=\frac{2}{\pi}\text{arc sin }\alpha^{1/2}=\rho(\alpha),\qquad 0\leqq\alpha\leqq 1,$$

then, making use of (2), with α replaced by $\alpha-\delta$ and by $\alpha+\delta$ we get at once

$$\rho(\alpha-\delta)-\frac{1+\log k}{k\epsilon^2\delta}-\frac{\epsilon}{\delta}\leqq \rho_k(\alpha)\leqq \rho(\alpha+\delta)+\frac{1+\log k}{k\epsilon^2\delta}+\frac{\epsilon}{\delta}\cdot$$

Making use of (2) again we are led to

$$\begin{aligned}\rho(\alpha-2\delta)-2\,\frac{1+\log k}{k\epsilon^2\delta}-\frac{2\epsilon}{\delta}&\leqq \liminf_{n\to\infty}\text{Prob.}\left\{\frac{N_n}{n}<\alpha\right\}\\&\leqq \limsup_{n\to\infty}\text{Prob.}\left\{\frac{N_n}{n}<\alpha\right\}\\&\leqq \rho(\alpha+2\delta)+2\,\frac{1+\log k}{k\epsilon^2\delta}+\frac{2\epsilon}{\delta}\cdot\end{aligned}$$

Since this inequality holds for all k, ϵ and δ we can put, for instance,

$$\delta=k^{-1/10},\qquad \epsilon=k^{-1/5}$$

and obtain, by letting $k\to\infty$,

$$\lim_{n\to\infty}\text{Prob.}\left\{\frac{N_n}{n}<\alpha\right\}=\rho(\alpha).$$

Use has been made of the fact that $\rho(\alpha)$ is continuous for all α. The invariance principle stated at the beginning of this section has thus been proved.

3. Explicit calculations for a particular sequence of random variables.

Having established the invariance principle we could appeal to P. Lévy's result concerning the binomial case and thus complete the proof of the general theorem. We prefer, however, to work with another sequence of independent random variables because we feel that the calculations are of some independent interest and because they again emphasize the usefulness of integral equations in proving limit theorems.[4]

We consider independent random variables $Y_1, Y_2, \cdots$ such that

$$\text{Prob.}\left\{Y_j<u\right\}=\frac{1}{2}\int_{-\infty}^{u}e^{-|y|}dy,\qquad j=1,2,\cdots,$$

and we set $s_j=Y_1+\cdots+Y_j$.

It should be noted that although $E(Y_j^2)\neq 1$ we are justified in using the Y's inasmuch as N_n is independent of the variance of the Y's as long as the variance is different from 0. In fact, N_n remains unaltered if one multiplies all the Y's by the same positive constant.

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[4] See M. Kac, *On the average of a certain Wiener functional and a related limit theorem in calculus of probability*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 401–414.

We are going to calculate

$$\phi_n(u) = E(\exp(uN_n)) = E\left(\exp\left(u\sum_{j=1}^{n}\psi(s_j)\right)\right).$$

Setting

$$p(y) = 2^{-1}\exp(-|y|)$$

we have

$$\phi_n(u) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(u\sum_{j=1}^{n}\psi\left(\sum_{r=1}^{j}y_j\right)\right) \cdot p(y_1)\cdots p(y_n)dy_1\cdots dy_n$$

or, introducing the variables

$$\begin{aligned} s_1 &= y_1, \\ s_2 &= y_1 + y_2, \\ &\cdots\cdots\cdots\cdots, \\ s_n &= y_1 + y_2 + \cdots + y_n, \end{aligned}$$

$$\phi_n(u) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(u\sum_{j=1}^{n}\psi(s_j)\right) \cdot p(s_1)p(s_2 - s_1)\cdots p(s_n - s_{n-1})ds_1ds_2\cdots ds_n.$$

We also introduce the auxiliary functions

$$F_n(u, s_n) = \int_{-\infty}^{\infty}\cdots\int_{\infty}^{-\infty}\exp\left(u\sum_{j=1}^{n}\psi(s_j)\right) \cdot p(s_1)p(s_2 - s_1)\cdots p(s_n - s_{n-1})ds_1\cdots ds_{n-1}, \quad n > 1,$$

and

$$F_1(u, s_1) = \exp(u\psi(s_1))p(s_1).$$

We have, for $n > 1$,

$$F_n(u, s_n) = \exp(u\psi(s_n))\int_{-\infty}^{\infty}p(s_n - s_{n-1})F_{n-1}(u, s_{n-1})ds_{n-1}$$

or, in more convenient notation,

$$F_n(u, s) = \exp(u\psi(s)) \int_{-\infty}^{\infty} p(s - t)F_{n-1}(u, t)dt. \tag{3}$$

We also have

$$\phi_n(u) = \int_{-\infty}^{\infty} F_n(u, s)ds \tag{4}$$

and

$$|F_n(u, s)| \leqq \exp(\operatorname{Re} u) \max_{-\infty<t<\infty} |F_{n-1}(u, t)|.$$

Since

$$|F_1(u, s)| \leqq \exp(\operatorname{Re} u)$$

we get

$$|F_n(u, s)| \leqq \exp(n \operatorname{Re} u).$$

Thus the series

$$G(u, s; z) = \sum_{n=1}^{\infty} F_n(u, s)z^{n-1}$$

converges for sufficiently small $|z|$.

Using (3), we obtain, almost immediately,

$$\frac{G(u, s; z) - \exp(u\psi(s))p(s)}{z} = \exp(u\psi(s)) \int_{-\infty}^{\infty} p(s - t)G(u, t; z)dt$$

or

$$\begin{aligned} &\exp(-u\psi(s))G(u, s; z) - 2^{-1}\exp(-|s|) \\ &\qquad = \frac{z}{2}\int_{-\infty}^{\infty} \exp(-|s - t|)G(u, t; z)dt. \end{aligned} \tag{5}$$

For $s>0$ we have

$$\begin{aligned} e^{-us}G(u, s; z) - 2^{-1}e^{-s} = {}& \frac{z}{2} e^{-s} \int_{-\infty}^{s} e^{t}G(u, t; z)dt \\ &+ \frac{z}{2} e^{s} \int_{s}^{\infty} e^{-t}G(u, t; z)dt. \end{aligned} \tag{6}$$

It is easily seen that, for sufficiently small $|z|$, $G(u, t; z)$ is an absolutely integrable function of t.

Differentiating (6) twice with respect to s we are readily led to

$$\frac{d^2G}{ds^2} + (ze^u - 1)G = 0.$$

In the same way we get for $s<0$

$$\frac{d^2G}{ds^2} + (z - 1)G = 0.$$

Thus we get

$$G = A \exp ((1 - ze^u)^{1/2}s) + B \exp (-(1 - ze^u)^{1/2}s), \qquad s > 0,$$

and

$$G = C \exp ((1 - z)^{1/2}s) + D \exp (-(1 - z)^{1/2}s), \qquad s < 0,$$

where A, B, C, D are functions of u and z. Since G is an integrable function of s we must have $A=D=0$.

Thus

$$G = B \exp (-(1 - ze^u)^{1/2}s), \qquad s > 0, \tag{7}$$

$$G = C \exp ((1 - z)^{1/2}s), \qquad s < 0. \tag{8}$$

From the absolute integrability of G and the integral equation (5) it follows that

$$\exp (-u\psi(s))G(u, s; z)$$

is continuous for all s and, in particular, for $s=0$. This observation yields

$$Be^{-u} = C. \tag{9}$$

Rewriting (6) in the form

$$e^{-u}G(s) - 2^{-1}e^{-s} = \frac{z}{2}e^{-s}\int_{-\infty}^{0} e^{t}G(t)dt + \frac{z}{2}e^{-s}\int_{0}^{s} e^{t}G(t)dt$$
$$+ \frac{z}{2}e^{s}\int_{s}^{\infty} e^{-t}G(t)dt$$

and substituting expressions (7) and (8) we get after a few elementary transformations

$$\frac{z}{1 + (1 - z)^{1/2}}C - \frac{z}{1 - (1 - ze^u)^{1/2}}B = -1.$$

This together with (9) yields

$$B = \frac{(1-z)^{1/2} - (1-ze^{u})^{1/2}}{z(e^{u}-1)} e^{u}, \qquad C = \frac{(1-z)^{1/2} - (1-ze^{u})^{1/2}}{z(e^{u}-1)}$$

and thus G is completely determined.[5]

In particular, we get

$$\int_{-\infty}^{\infty} G(u, s; z)ds$$

$$= \frac{(1-z)^{1/2} - (1-ze^{u})^{1/2}}{z(e^{u}-1)} \left[\frac{e^{u}}{(1-ze^{u})^{1/2}} + \frac{1}{(1-z)^{1/2}} \right].$$

On the other hand (see (4))

$$\int_{-\infty}^{\infty} G(u, s; z)ds = \sum_{n=1}^{\infty} \phi_n(u) z^{n-1}$$

and hence $\phi_n(u)$ is the coefficient of z^{n-1} in the power series expansion of

$$\frac{(1-z)^{1/2} - (1-ze^{u})^{1/2}}{z(e^{u}-1)} \left[\frac{e^{u}}{(1-ze^{u})^{1/2}} + \frac{1}{(1-z)^{1/2}} \right].$$

A simple computation gives finally

$$E(\exp (uN_n) = \phi_n(u)$$

$$= \frac{1}{e^{u}-1} \sum_{k+l=n} C_{1/2,k} C_{-1/2,l} (-1)^{k+l} (1-e^{ku})(1+e^{(l+1)u}).$$

Setting

$$u = \frac{i\xi}{n}$$

we obtain the characteristic function of N_n/n.

For large k and l we have

$$C_{1/2,k} \sim (-1)^{k-1} \frac{1}{2k-1} \frac{1}{(\pi k)^{1/2}}, \qquad C_{-1/2,l} \sim (-1)^{l} \frac{1}{(\pi l)^{1/2}},$$

and it is easily seen that, in the limit $n \to \infty$, we can use these asymptotic expressions in the sum.

---

[5] Having found an explicit expression for G it is not too difficult to find explicit expressions for $F_n(u, s)$ and verify the recursive relation (3). Since this recursive relation together with the formula for $F_1(u, s)$ determine G uniquely one might substitute the verification process for the derivation of the formula for G.

Thus

$$\lim_{n\to\infty} E\left(\exp\left(\frac{i\xi}{n} N_n\right)\right)$$

$$= \lim_{n\to\infty} \frac{1}{\pi(1-e^{i\xi/n})} \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{(k(n-k))^{1/2}} \cdot (1-e^{i\xi k/n})(1+e^{i\xi(1-k/n+1/n)})$$

$$= \frac{1}{2\pi i\xi}\int_0^1 \frac{(1-e^{i\xi x})(1+e^{i\xi(1-x)})}{x(x(1-x))^{1/2}}\,dx = \frac{1}{\pi}\int_0^1 e^{i\xi x}\frac{dx}{(x(1-x))^{1/2}}$$

$$= \int_0^1 e^{i\xi x} d\left(\frac{2}{\pi}\operatorname{arc\,sin} x^{1/2}\right).$$

We complete the proof by appealing to the continuity theorem for Fourier-Stieltjes transforms.

SYRACUSE UNIVERSITY AND
CORNELL UNIVERSITY

NUMERICAL INVERTING OF MATRICES OF HIGH ORDER

JOHN VON NEUMANN AND H. H. GOLDSTINE

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PREFACE

The purpose of this paper is to derive rigorous error estimates in connection with the inverting of matrices of high order. The reasons for taking up this subject at this time in such considerable detail are essentially these: First, the rather widespread revival of mathematical interest in numerical methods, and the introduction of new procedures and devices which make it both possible and necessary to perform operations of this type on matrices of much higher orders than was even remotely practical in the past. Second, the fact that considerably diverging opinions are now current as to the extremely high or extremely low precisions which are required when inverting matrices of orders $n \geqq 10$. (Cf. in this connection footnotes 10, 11, 12 below.)

It has been our aim to provide a rigorous discussion of this rather involved problem in estimation. Our estimates of errors have furthermore been carried out in strict observance of these two rules, which seem to us to be essential: To produce no numbers (final or intermediate) that lie outside a given finite interval (for which we chose $-1, 1$), and to treat these numbers solely as aggregates of a fixed number of digits, given in advance.

The reader will find a complete enumeration and interpretation of our results in Chapter VII, especially in §§7.7, 7.8. He may find it convenient to consult these first. We conclude there, for example, matrices of the orders 15, 50, 150 can usually be inverted with a (relative) precision of 8, 10, 12 decimal digits less, respectively, than the number of digits carried throughout. By "usually" we mean that if a plausible statistic of matrices is assumed, then these estimates hold with the exception of a low probability minority. These general estimates are based on rigorous individual estimates, valid for all matrices (cf. §§7.4–7.5). If we had been willing to use a probability treatment for individual matrices, too, our estimates could have been improved by several decimal digits (cf. §2.3).

We made no effort to obtain numerically optimal estimates, but we believe that our estimates are optimal at least as far as the practical orders of magnitude are concerned and with respect to the over-all mathematical method used and the principles indicated above.

This work has been made possible by the generous support of the Office of Naval Research, under Contract N7onr-388. Earlier related work will be published elsewhere by V. Bargmann, D. Montgomery and one of us. (Cf. the references in footnote 24 below.)

CHAPTER I. THE SOURCES OF ERRORS IN A COMPUTATION

1.1. **The sources of errors.** When a problem in pure or in applied mathematics is "solved" by numerical computation, errors, that is, deviations of the numerical "solution" obtained from the true, rigorous one, are unavoidable. Such a "solution" is therefore meaningless, unless there is an estimate of the total error in the above sense.

Such estimates have to be obtained by a combination of several different methods, because the errors that are involved are aggregates of several different kinds of contributory, primary errors. These primary errors are so different from each other in their origin and character, that the methods by which they have to be estimated must differ widely from each other. A discussion of the subject may, therefore, advantageously begin with an analysis of the main kinds of primary errors, or rather of the sources from which they spring.

This analysis of the sources of errors should be objective and strict inasmuch as completeness is concerned, but when it comes to the defining, classifying, and separating of the sources, a certain subjectiveness and arbitrariness is unavoidable. With these reservations, the following enumeration and classification of sources of errors seems to be adequate and reasonable.

(A) The mathematical formulation that is chosen to represent the underlying problem may represent it only with certain idealizations, simplifications, neglections. This is even conceivable in pure mathematics, when the numerical calculation is effected in order to obtain a preliminary orientation over the underlying problem. It will, however, be the rule and not the exception in applied mathematics, where these things are hardly avoidable in a mathematical representation. This complex is further closely related to the methodological observation that a mathematical formulation necessarily represents only a (more or less explicit) theory of some phase of reality, and not reality itself.

(B) Even if the mathematical formulation according to (A) is not questioned, that is, if the theoretical description which it represents and the idealizations, simplifications, and neglections which it involves are accepted as final (and not viewed as sources of errors), this further point remains: The description according to (A) may involve parameters, the values of which have to be derived directly or indirectly (that is, through other theories or calculations) from observations. These parameters will be affected with errors, and these underlying errors will cause errors in the result of our calculation.

(C) Now let (A), (B) (mathematical formulation and observational data) go unquestioned. The next stumbling block is this: The mathematical formulation of (A) will in general involve transcendental operations (for example, functions like sin or log, operations like integration or differentiation, and so on) and implicit definitions (for example, solutions of algebraical or transcendental equations, proper value problems of various kinds, and so on). In order to be approached by numerical calculation, these have to be replaced by elementary processes (involving only those elementary arithmetical operations which the computer can handle directly) and explicit definitions, which correspond to a finite, constructive procedure that resolves itself into a linear sequence of steps.[1]

---

[1] This applies directly to all digital computing schemes: Digital computing by human operators, by "hand" and by semi-automatic "desk" machines, also computing by the large modern fully automatic, "self-sequenced," computing machines. Fundamentally, however, it applies equally to those "analogy" machines which can perform certain operations directly, that are "transcendental" or "implicit" from the digital point of view. Thus for machines of the genus of the "differential analyser" differentiating, integrating and solving certain (essentially implicit) differential equations are elementary, explicit operations. While a digital procedure must replace a total

Similarly every convergent, limiting process, which in its strict mathematical form is infinite, must in a numerical computation be broken off at some finite stage, where the approximation to the limiting value is known to have reached a level that is considered to be satisfactory. It would be easy to give further examples.

All these replacements are, as stated, approximative, and so the *strict* mathematical statement of (A) is now replaced by an *approximate* one. This constitutes our third source of errors.

(D) Finally, let not only (A), (B), but even the approximation process of (C) pass unchallenged. There still remains this limitation: No computing procedure or device can perform the operations which are its "elementary" operations (or, at least, all of them) rigorously and faultlessly. This point is most important, and is best discussed separately for digital and for "analogy" procedures or devices.

The case of the analogy devices is immediate and clear: An analogy device which represents two numbers x and y by two physical quantities $\mathfrak{x}$ and $\mathfrak{y}$ will form the sum $x+y$ or the product xy as two physical quantities $\mathfrak{x}\oplus\mathfrak{y}$ or $\mathfrak{x}\otimes\mathfrak{y}$. Yet $\mathfrak{x}\oplus\mathfrak{y}$ or $\mathfrak{x}\otimes\mathfrak{y}$ will unavoidably be affected with the (more or less) random "noise" of the computing instrument, that is, with the errors and imperfections inherent in any physical, engineering embodiment of a mathematical principle. Hence $\mathfrak{x}\oplus\mathfrak{y}$ and $\mathfrak{x}\otimes\mathfrak{y}$ will correspond not to the true $x+y$ and xy, but to certain $x+y+\epsilon^{(s)}$ and $xy+\epsilon^{(p)}$, where $\epsilon^{(s)}$, $\epsilon^{(p)}$ are (more or less) random, "noise" variables, of which only the probable (and possibly the maximum) size is known in advance. Also, $\epsilon^{(s)}$, $\epsilon^{(p)}$ assume new and (usually in the main) independent values with every new execution of an operation $+$ or $\times$. The same goes for the other operations which the device can perform, any or all of $-$, $/$, $\sqrt{\ }$, $\int$, d/dx, and possibly others.

---

differential equation by finite difference equations (to make it elementary) and possibly use iterative, trial-and-error methods (to make it explicit), the "differential analyser" may be able to treat such a problem "directly." But for a partial differential equation (where a digital procedure requires the same circumventory measures as in the above case of a total differential equation), the "differential analyser" can give its "direct" treatment only to one (independent) variable, while on the other (independent) variable or variables it will have to resort to finite-difference and possibly iteration and trial-and-error methods, very much like a digital procedure has to on all variables.

Thus the differences are only in degree (number of processes that rate as "elementary" and "explicit") but not in kind. Such differences, by the way, exist even among digital devices: Thus one may treat square rooting as an "elementary," "explicit" process, and another one not, and so on.

For digital procedures or devices, we must note first that they represent (continuous, real) numbers x by finite digital aggregates $\bar{x}=(\alpha_1, \cdots, \alpha_s)$ where each $\alpha_\sigma=0, 1, \cdots, \beta-1$. Here $\beta(=2, 3, \cdots)$ is supposed to be the *basis* of the digital representation,[2] s the number of *places* used, while we need not for the moment pay attention to the position t of the β-adic point.[3] Now for two s-place numbers $\bar{x}$, $\bar{y}$ the sum and the difference $\bar{x}\pm\bar{y}$ are again s-place numbers,[4] but their product $\bar{x}\bar{y}$ is not. $\bar{x}\bar{y}$ is, of course, a $2s$-place number. One might carry along this $\bar{x}\bar{y}$ in subsequent computations as a $2s$-place number, but later multiplications will increase the number of places further and further, if such a scheme is being followed consistently. With any practical procedure or device a line has to be drawn somewhere, that is, a maximum number of places in a number x set. In our discussion we might as well assume then that s has already reached this maximum. Hence for two s-place numbers $\bar{x}$, $\bar{y}$ the true $2s$-place product $\bar{x}\bar{y}$ must be replaced by some s-place approximation. Let us denote this s-place approximation by $\bar{x}\times\bar{y}$, and call $\bar{x}\bar{y}$ the *true* and $\bar{x}\times\bar{y}$ the *pseudo-product.*

The same observations apply to the quotient $\bar{x}/\bar{y}$, except that the true $\bar{x}/\bar{y}$ will in general have infinitely many places, and not only $2s$. We call $\bar{x}/\bar{y}$ the *true* quotient, and a suitable s-place approximation $\bar{x}\div\bar{y}$ the *pseudo-quotient.*[5] Similar pseudo-operations should be introduced for other operations if they are "elementary" for the device under consideration (for example, the square root), but we need not consider these matters here.

The transition from the true operations to their pseudo-operations is effected by any one of the familiar methods of *round off.*[6] Thus the true $\bar{x}\bar{y}$, $\bar{x}/\bar{y}$ are replaced by the pseudo $\bar{x}\times\bar{y}=\bar{x}\bar{y}+\eta^{(p)}$, $\bar{x}\div\bar{y}=(\bar{x}/\bar{y})+\eta^{(q)}$, with the *round off errors* $\eta^{(p)}$, $\eta^{(q)}$.

There is a good deal of similarity between these $\eta^{(p)}$, $\eta^{(q)}$ and the $\epsilon^{(s)}$, $\epsilon^{(p)}$ that we encountered above (for analogy devices): While the

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[2] The most probable choices are $\beta=10$ and $\beta=2$.

[3] This is between t and $t+1$ $(t=0, 1, \cdots, s)$: $\bar{x}=(\alpha_1, \cdots, \alpha_s)=\sum_{\sigma=1}^{s}\beta^{t-\sigma}\alpha_\sigma$.

[4] Unless they exceed the permissible limits of size $\bar{x}+\bar{y}\geq\beta^t$ or $\bar{x}-\bar{y}<0$. Regarding this, cf. footnote 17. We shall not discuss this complication here, nor the connected one, that the digital aggregates have to be provided with a sign. The latter point is harmless, and both are irrelevant at this point, cf., however, (a) in 2.1, particularly (2.1).

[5] We omit again discussions of size at this point. Cf footnote 4 and its references.

[6] The simplest method consists of omitting all digits beyond place s. A more elaborate one required adding $\beta/2$ units of place $s+1$ first, and then (having effected the carries which are thus caused) omitting as above. There exist still other procedures.

η are strictly very complicated but uniquely defined number theoretical functions (of $\bar{x}$, $\bar{y}$), yet our ignorance of their true nature is such that we best treat them as random variables. We know their average and maximum sizes: with the usual round off method (the second one in footnote 6), the maximum of $|\eta|$ is $\beta^{-s}/2$ (that is, $\beta\ \beta^{-(s+1)}/2$ in the sense of loc. cit.), and if we assume η to be equidistributed in $-\beta^{-s}/2, +\beta^{-s}/2$ then

$$\text{(1.1)} \qquad \text{Mean } (\eta) = 0,$$

$$\text{(1.2)} \qquad \text{Dispersion } (\eta) = (\text{Mean } (\eta^2))^{1/2} = (\beta^{-2s}/12)^{1/2} = .29\beta^{-s}.$$

Finally, $\eta^{(\times)}$, $\eta^{(\div)}$ assume new and (usually in the main) independent values with every new execution of an operation × or ÷.

Thus the "digital" η are in many essential ways "noise" variables, just as the "analogy" ϵ.

These *noise variables* or *round off variables* ϵ and η, which are injected into the computation every time when an "elementary" operation is performed (excepting ± in the digital case) constitute our fourth and last source of errors.

1.2. **Discussion and interpretation of the errors (A)–(D). Stability.** The errors described in (A) are the *errors due to the theory*. While their role is clearly most important, their analysis and estimation should not be considered part of the mathematical or of the computational phase of the problem, but of the underlying subject, in which the problem originates. There can be little doubt regarding this methodological position. We will therefore not concern ourselves here with (A) any further.

The errors described in (B) are essentially the *errors due to observation*. To this extent they are, strictly construed, again no concern of the mathematician. However, their influence on the result is the thing that really matters. In this way, their analysis requires an analysis of this question: What are the limits of the change of the result, caused by changes of the parameters (data) of the problem within given limits? This is the question of the *continuity of the result as a function of the parameters of the problem*, or, somewhat more loosely worded, of the *mathematical stability of the problem*. This question of continuity or stability is actually not the subject matter of this paper, but it has some influence on it (cf. the discussion in §1.3), and we can therefore not let it slip out of sight completely.

The errors described in (C) are those which are most conspicuous as *errors of approximation* or *truncation*. Most discussions in "approximation mathematics" are devoted to analysis and estimation of these errors: Numerical methods to obtain approximate solutions of algebraical equations by iterative and interpolation processes, numerical methods to evaluate definite integrals, stepwise (finite-difference) methods to integrate differential equations correctly to varying "orders in the differential," and so on. Just because these errors have formed the subject of the major part of the existing literature, we shall not consider them here to any great extent. In fact, we have selected the specific problem of this paper in such a way that this source of errors has no direct part in it. (Cf. §1.5.)

There is, however, one phase of this part of the subject about which a little more should be said, just in view of the distribution of emphases in our present work: This is its relation to the question of *stability*, to which we have already referred in our above remarks on (B). Let us therefore consider this matter, before we go on to the discussion of (D).

1.3. Analysis of stability. The results of Courant, Friedrichs and Lewy.

The point is this: (B) dealt with the continuity or stability of (A), that is, of its result when viewed as a function of the parameters. This applies not only to the errors in the values of those parameters due to the causes mentioned in (B) (observational), but also to any other perturbations which may affect the values of any of the parameters which enter into the mathematical formulation of (A). Such perturbations equally affect quantities which are usually not interpreted as parameters at all, because they are not of observational origin. (This aspect of the matter will be relevant in connection with (D), cf. below in §1.4.)

Now (C) replaces the (strict) mathematical problem of (A) by a different one (the approximate problem). The considerations of (C) must establish that the problem of (C) differs quantitatively but little from the problem of (A). This does, however, not guarantee necessarily that the continuity or stability of (A) implies that of (C) as well. (Cf. below.) Yet, the actual computation deals with the problem of (C), and not with the problem of (A); consequently it is the continuity or stability of the former (of which the latter is a limiting case) that is really required.

That the stability of the strict problem need not imply that of an arbitrarily close approximant was made particularly clear by some important results of R. Courant, K. Friedrichs, and H. Lewy.[7] They showed, among other things, that although the partial differential equation

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[7] *Über die partiellen Differenzengleichungen der Mathematischen Physik*, Math. Ann. vol. 100 (1927) pp. 32–74.

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(F\left(\frac{\partial y}{\partial x}\right)\right) \tag{1.3}$$

is usually stable,[8] its stepwise, finite-difference approximant

$$\begin{aligned} &\frac{y(t + \Delta t, x) - 2y(t, x) + y(t - \Delta t, x)}{\Delta t^2} \\ &\qquad = \frac{1}{\Delta x}\left\{F\left(\frac{y(t, x + \Delta x) - y(t, x)}{\Delta x}\right) - F\left(\frac{y(t, x) - y(t, x - \Delta x)}{\Delta x}\right)\right\} \end{aligned} \tag{1.4}$$

need not be, no matter how small Δt, Δx are.[9] The necessary and sufficient condition for the stability of (1.4) is

$$\Delta x \geqq c\Delta t \qquad \text{where } c = \left(\frac{dF(v)}{dv}\right)^{1/2} \tag{1.5}$$

in the entire domain of integration.

Thus (C) requires an extension of the stability considerations of (B) from the original, strict problem of (A) to the secondary, approximant problem of (C).

1.4. Analysis of "noise" and round off errors and their relation to high speed computing. We now come to the errors described in (D). As we saw, they are due to the inner "noise level" of the numerical computing procedure or device—in the digital case this means: to the round off errors.

These differ from the perturbations of the apparent or the hidden parameters of the problem, to which we referred in §1.3, in this significant respect: Those perturbations will cause a parameter to deviate from its ideal value, but this deviation takes place only once, and is then valid with a constant value throughout the entire problem. The perturbations of (D), on the other hand, take place anew, and essentially independently, every time an "elementary" operation is performed. (Cf. the discussion in §1.1.) They form, therefore, a constantly renewed source of contaminations, and are likely to be more dangerous than the single perturbations of §1.3 (that is, of (B)). Their influence increases with the number of "elementary" operations that have to be performed. They are therefore especially important in long computations, involving many such operations. Such long computations will undoubtedly be normal for very high speed computing devices.[10] It is therefore just for the highest speed devices that the source (D) will prove to be most important. We propose to concentrate on it in this paper.

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[8] This is the Lagrangean form of the equation of motion of a one-dimensional, compressible, isentropic, nonviscous, nonconductive flow. It need not be linear, that is, it may go beyond the "acoustic" approximation.

[9] This approximant is correct up to second order terms in the differentials Δt, Δx, and it is the one that is most frequently used in numerical work.

The errors which the source of (D) is continuously injecting into a computation will be individually small, but appear in large numbers. The decisive factor that controls their effect is therefore a continuity or stability phenomenon of the type discussed in §1.3 above. And it is the stability of the approximant procedure of (C), and not of the strict procedure of (A), which matters—just as we saw in §1.3. For this reason stability discussions in the sense of §1.3 should play an important part in this phase of the problem.

1.5. **The purpose of this paper. Reasons for the selection of its problem.** On the basis of what has been stated so far we can define the purpose of this paper. We wish to analyze the stability of a computational procedure in the sense of (D), that is, with respect to the "inner noise" of the computation—in the digital situation: with respect to the round off errors. We shall attempt to isolate the phase of the problem that we want to analyze from all other, obscuring influences as much as possible. We shall therefore select a problem in which the difficulties due to (D) are a maximum and all others are a minimum. In other words, we shall choose a problem which is strictly "elementary," that is, where no transcendental or limiting processes occur, and where the result is defined by purely algebraical formulae. On the other hand the problem should lead with ease to very large numbers of "elementary" operations. This points towards problems with a high order iterative character. Finally, it should be of inherently low, or rather insecure, stability. Errors committed

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[10] Fully automatic electronic computing machines which multiply two real numbers (full size digital aggregates) in 10^{-4} to 10^{-3} seconds, and which are sufficiently well organized to be able to have a duty-cycle of 1/10 to 1/5 with respect to multiplication, will probably come into use in a not too distant future. Single problems consuming 2 to 20 hours on such a machine should be the norm.

Taking average figures: $3 \cdot 10^{-4}$ second multiplier, 1/7 duty cycle and a 6 hour problem, gives 10^7 multiplications for a single problem. This number may serve as an orientation regarding the orders of magnitude that are likely to be involved. For more specific figures in the problem of matrix inversion cf. the remarks at the end of §7.8.

(that is, noise introduced) in an earlier stage of the computation should be exposed to a possibility of considerable amplification by the subsequent operations of the computation.

For this purpose the problem of solving n (simultaneous) linear equations in n variables seems very appropriate, when n assumes large values.[11] Besides this problem will be a very important one when the fast digital machines referred to in footnote 10 become available; those machines will create a prima facie possibility to attack a wide variety of important problems that require matrix manipulations, and in particular inversions, for unusually large values of n.[12]

1.6. **Factors which influence the errors (A)–(D). Selection of the elimination method.** It should be noted that the four error sources (A)–(D) show an increasing dependence on procedural detail: (A) depends only on the strict mathematical statement of the problem, and this is still true of (B), although observational elements begin to appear. (C) introduces the dependence on the mathematical approximations used. (D), finally, depends even on the actual algorithm according to which the equations of (C) are processed: The order in which they are taken up, whether an expression $(a+b)c$ is formed in this order or as $ac+bc$, whether an expression ab/c is formed in this order or as $(a/c)b$ or $a(b/c)$ or $(a/c^{1/2})(b/c^{1/2})$ (regarding a set of such alternatives cf. §6.1), and so on.

Since we wish to study the role of (D) in the problem of matrix inversion, it is necessary to decide which of the several available algorithms is to be used. We select the well known ***elimination method***, or rather a new variant of it that we shall develop, because we conclude, from the results that we shall obtain, that this method is superior to the other known methods. (For the details of the procedure cf. the preliminary discussion of §§5.1, 5.2; the more specific procedures at the end of §6.1, especially (6.3), (6.4); the first part of §6.9; and the final discussion together with formally complete references in §7.6. Regarding the value of the method, cf. §§7.7, 7.8.)

1.7. **Comparison between "analogy" and digital computing methods.** We conclude this chapter with a general remark regarding the comparison between digital and "analogy" machines, from the point of view of the "noise variables" of (D) in §1.1.

We have noted the fact that these two categories of devices do not differ very essentially in that respect, where one might prima facie look for the main difference: we mean the circumstance that "analogy" machines are undoubtedly "approximate" in their effecting the "elementary" operations, while the digital devices might be viewed as rigorous. This is not so, or at least not so in the sense in which it really matters: "Analogy" devices are, of course, affected in their "elementary" operations by a genuine, physical "noise." In digital devices, on the other hand, the round off errors are unavoidable by the intrinsic nature of things, and they play exactly the same role as the true "noise" in an "analogy" device. It is therefore best to talk of "noise" in both cases. In digital devices this "noise" affects only multiplication and division, but not addition and subtraction—but this circumstance does not cause a very important differentiation from the "analogy" devices.

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[11] The difficulties of present day numerical methods in the problem of matrix-inversion begin to assume very serious dimensions when n increases beyond 10.

[12] We anticipate that $n \sim 100$ will become manageable. Cf. the end of §7.8.

The circumstance which is important is that the "noise level" in a digital device can be made much lower than in an "analogy" device. For s-place, base β numbers it is $\sim .29\beta^{-s}$. (Cf. (1.2). This is, of course, relative to the maximum numerical size allowed.) A typical situation is $\beta = 10$, $s = 10$;[13] that is, the dispersion of the "noise variable" η is $\sim 3 \cdot 10^{-11}$. Even the best "analogy" devices that are possible with present techniques have dispersions greater than or equal to 10^{-5} for their "noise variable" ϵ (again relative to the maximum size allowed).

In addition, a conventional "analogy" device which is built for extreme precision is naturally working in an area of "decreasing returns" for precision: Cutting the size of the dispersion of ϵ by an additional factor of, say, 2 gets the more difficult, the smaller this dispersion is already. In a digital machine, on the other hand, cutting the size of the dispersion of η by an additional factor 2 (or 10) is equivalent to building the machine with one more binary (or decimal) digit, and this addendum gets percentually less when the number of digits increases, that is, when the attained dispersion η decreases.

Thus the digital procedure may be best viewed as the most effective means yet discovered to reduce the "inner noise level" of computing. This aspect becomes increasingly important as the rate at which this "noise" is injected into the computation increases, that is, as the computations assume larger sizes (consist of greater numbers of "elementary" operations), and the machines which carry them out get faster.

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[13] All existing machines (or almost all) are decimal, that is, have $\beta = 10$. With rare exceptions $s = 7$ to 10, for example, on the familiar "desk" machines $s = 8$ or 10. The "Mark I" computer at Harvard University has $s = 11$ or 23.

Non-decimal machines of the future are likely to adhere, at least at first, to the same standard: for example, $\beta = 2$, $s = 30$ to 40.

Chapter II. Round off errors and ordinary algebraical processes

2.1. Digital numbers, pseudo-operations. Conventions regarding their nature, size and use: (a), (b). In Chapter I, and more particularly in §§1.5 and 1.6, we defined our purpose in this paper: We wish to determine the precision and the stability of the familiar *elimination method* for the *inversion of matrices* of order n, when n is large, with the primary emphasis on the effects of the "inner noise" of the digital computing procedure caused by the round off errors, that is, we want to determine how many (base β) places have to be carried in order to obtain significant results (meeting some specified standard of precision) in inverting a matrix of order n by the elimination method. We should thus obtain a lower limit for the number s of (base β) places in terms of the matrix order n. In doing this, we are prepared to accept as standard even such a variant of the elimination method which may not be among the commonly used ones, provided that it permits us to derive more favorable estimates of precision—that is, lower limits of s in terms of n. (This will actually happen, cf. the references at the end of §1.6.)

The main tools of our analysis will therefore be real members $\bar{x}$ which are represented by s-place, base β, digital aggregates in the sense of (D) in §1.1. We shall call them *digital numbers*, to distinguish them from the *ordinary* (real) *numbers*, which will also play a certain role in the discussions. When we deal with digital numbers, we shall observe certain rigid conventions, which facilitate an unequivocal and rigorous treatment, and which seem to us to be simple and reasonable, both in manipulation and in interpretation. It will, furthermore, always be permissible to view (in an appropriate part of the discussion) a number which was introduced as a digital number as an ordinary real number. To the extent to which we do this, the conventions in question will not apply.

We now enumerate these conventions:

(a) A *digital number* $\bar{x}$ is an s-place, base β, digital aggregate with sign:[14]

$$\bar{x} = \epsilon(\alpha_1, \cdots, \alpha_s);$$

$$\epsilon = \left\{\begin{matrix} +, \text{ that is, } +1 \\ -, \text{ that is, } -1 \end{matrix}\right| ; \qquad \alpha_1, \cdots, \alpha_s = 0, 1, \cdots, \beta - 1. \tag{2.1}$$

The *sum* and the *difference* have their ordinary meaning, and will be denoted by $\bar{x} \pm \bar{y}$. The *product* and the *quotient*, on the other hand, will be rounded off to s places (cf. (D) in §1.1), and the quantities which result in this way will be called *pseudo-product* and *pseudo-quotient*, and will be denoted by $\bar{x} \times \bar{y}$ and $\bar{x} \div \bar{y}$. In addition to these *pseudo-operations*, others could be introduced, for other "elementary" operations (for example, for the square root), too. This, however, does not seem necessary for our present purposes.

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[14] This is our first step beyond the limitations of footnote 4.

Occasionally a digital number $\bar{x}$ has to be multiplied by an integer $l\ (=0, \pm 1, \pm 2, \cdots): l\bar{x}$. This should be thought of in terms of repeated additions or subtractions, and it is therefore not a pseudo-operation and involves no round offs.

(b) The position of the β-adic point in representing $\bar{x}$ was already referred to before (cf. (D) in §1.1, particularly footnote 3). It seems to us simplest to fix it at the extreme left (that is, $t=0$ in the notations of loc. cit. above). Any other position can be made equivalent to this one by the use of appropriate *scale factors*.[15] Besides, other positions of the β-adic point are of advantage only in relation to particular and very specifically characterized problems or situations, while the position at the extreme left permits a considerable uniformity in discussing very general situations. Finally, this positioning has the effect that the maximum size of any digital number is 1, so that absolute and relative error sizes[16] coincide, which simplifies and clarifies all assessments. This positioning requires, of course, a careful and continuing check on all number sizes which develop in the course of the computation, and the introduction of scale factors when they threaten to grow out of range.[17] It should be noted, however, that this would be equally necessary for any other, fixed positioning of the β-adic point.[18]

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[15] These scale factors are of considerable importance. They are usually integer factors, most conveniently powers β^p ($p=0, \pm 1, \pm 2, \cdots$) of the base β. (Cf. in this respect the further analysis of §2.5.) Their main purpose is to keep the numbers resulting from intermediate operations within the operating range of the machine (cf. footnote 17 below), and also to avoid that they get crowded into a small segment of this interval (usually near to 0) with an attendant loss of "significant digits," that is, of ultimate precision.

They are by no means characteristic of digital machines. They are equally necessary in "analogy" machines. Thus, in differential analyzers appropriate gears are essential to insure that no integrator runs off its wheel, and that none should be limited systematically to insignificant movements, and so on.

For a proper appreciation of the importance of these scale factors it should be realized that no computing scheme or estimation of errors and of validity in a computing scheme is complete without a precise accounting for their role We shall have to do with them again subsequently: 2^{p_ν} in, §6.4; 2^{r_1}, 2^{q}, 2^{q_1} in §6.7; 2^{q_1} in §6.10; 2^{p}, $2^{p'}$ in §7.3.

[16] Relative to the maximum number size.

[17] Owing to this positioning all $|\bar{x}| \leq 1$, $|\bar{y}| \leq 1$, cf. below. Hence automatically $|\bar{z}| \leq 1$ for $\bar{z} = \bar{x} \times \bar{y}$, but not necessarily for $\bar{z} = \bar{x} \pm \bar{y}$ or $\bar{z} = \bar{x} \div \bar{y}$. For $\bar{z} = \bar{x} \pm \bar{y}$ a scale factor β^{-1} will always be adequate, for $\bar{z} = \bar{x} \div \bar{y}$ a scale factor β^{-n} with any $n=1$,

This choice of the position of the β-adic point permits us to expand (2.1) to

$$\begin{aligned} &\bar{x} = \epsilon(\alpha_1, \cdots, \alpha_s) = \epsilon \sum_{\nu=1}^{s} \beta^{-\nu}\alpha_\nu; \\ (2.1') \quad & \epsilon = \left\{\begin{matrix} +, \text{ that is, } +1 \\ -, \text{ that is, } -1 \end{matrix}\right| ; \qquad \alpha_1, \cdots, \alpha_s = 0, 1, \cdots, \beta - 1, \end{aligned}$$

and to assert that

(2.2) a digital number $\bar{x}$ lies necessarily in the interval $-1, 1$.

2.2. **Ordinary real numbers, true operations. Precision of data. Conventions regarding these: (c), (d).** To these convention-setting remarks (a), (b) we add in a more discursive sense:

(c) We shall also use *ordinary real* numbers x. We shall even reinterpret, whenever it is convenient and for any appropriate part of the discussion, a number, which was introduced as a digital number, as an ordinary real number.

Ordinary real numbers are subject to no restrictions in size, and to them the *true operations* $x \pm y$, xy, x/y, and so on, apply.

(d) The parameters of our problem (that is, the elements of the matrix to be inverted) will usually be introduced as digital numbers. The question arises, as to what ordinary real numbers they replace.[19] The effects that these replacements, that is, errors, in the parameters have on the result are properly the subject of (B) and not of (D). It is therefore justified to view them separately, and to discuss (D) itself under the assumption that the (digital) parameter values are strictly correct. Regarding (C) cf. also §1.3.

2.3. **Estimates concerning the round off errors.** Two further remarks regarding the technique and character of the round off:

2, · · · may be called for. (Our first reference to these possibilities was made in footnote 4.)

[18] We shall not discuss here the possibilities of a movable and self-adjusting, "floating" β-adic point. From the point of view of the precision of the calculation they do not differ from those of the continuous size-check-and-scale-factor procedure, to which we propose to adhere. Indeed, these two procedures bear to each other simply the relationship of automatic vs. mathematically conscious application of the same arithmetical principles.

[19] Possibly, but not necessarily, by round off. Cf., for example, the discussion of §7.5.

(e) We pointed out in (D) that the round off errors[10] η behave, as far as is known at present, essentially like independent random variables, although they are actually uniquely defined number-theoretical functions. Taking the probabilistic view of η we have (by (1.1), (1.2))

$$\text{(2.3)} \qquad \text{Mean } (\eta) = 0, \qquad \text{Dispersion } (\eta) = .29\cdot\beta^{-s};$$

taking the strict view, on the other hand, we can only assert that

$$\text{(2.4)} \qquad \text{Max } (|\eta|) = .5\cdot\beta^{-s}.$$

This discrepancy becomes even more significant when we deal with a sum of, say, m such quantities $\eta_1, \cdots, \eta_m$: Probabilistically we may infer from (2.3) that

$$\text{(2.5)} \qquad \text{Mean}\left(\sum_{l=1}^{m}\eta_l\right) = 0, \quad \text{Dispersion}\left(\sum_{l=1}^{m}\eta_l\right) = .29\cdot m^{1/2}\cdot\beta^{-s}$$

while strictly we can infer from (2.4) only that

$$\text{(2.6)} \qquad \text{Max}\left(\left|\sum_{l=1}^{m}\eta_l\right|\right) \leqq .5\cdot m\cdot\beta^{-s}.$$

The estimate (2.6) is inferior to the estimate (2.5) by a factor $.5m/.29m^{1/2} = 1.7m^{1/2}$!

This creates a strong temptation to use *probabilistic estimates* instead of *strict estimates*, especially because expressions of the form

$$\text{(2.7.a)} \qquad \sum_{l=1}^{m}\bar{x}_l\bar{y}_l,$$

which give rise to round off errors

$$\text{(2.7.b)} \quad \sum_{l=1}^{m}\bar{x}_l\bar{y}_l - \sum_{l=1}^{m}\bar{x}_l\times\bar{y}_l = \sum_{l=1}^{m}(\bar{x}_l\bar{y}_l - \bar{x}_l\times\bar{y}_l) = \sum_{l=1}^{m}\eta_l$$

of the type in question, will be particularly frequent in our deductions. We shall, nevertheless, adhere to strict estimates throughout this paper (with some specified exceptions in §3.5).

(f) There is an alternative method which reduces the total round off error in the situations (2.7.a)–(2.7.b), and which deserves consideration. In fact, it effects an even greater reduction of the round off error in question than the probabilistic view of (e), and it does so on the basis of strict estimates. It requires, however, an actual change in the computing technique—but this is a change to which most existing and most planned digital computing devices lend themselves readily.

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[10] We mean $\eta^{(\times)} = \bar{x}\bar{y} - \bar{x}\times\bar{y}$ and $\eta^{(\div)} = (\bar{x}/\bar{y}) - (\bar{x}\div\bar{y})$. The considerations which follow are primarily significant for $\eta^{(\times)}$.

This method may be described as follows:

In multiplying two s-place numbers, most computing machines do actually form the true $2s$-place product, and the rounding off to s-places is a separate operation, which may (and usually is) effected subsequently, but which can also be omitted. The s-place character of the machine finds its expression at a different point: The machine can accept s-place factors only, that is, it cannot form the product of two $2s$-place numbers (neither the $2s$-place pseudo-product, nor the $4s$-place true product). In addition, it can accept s-place addends (or minuends and subtrahends) only. It is easy, however, to use such a machine to add or to subtract $2s$-place numbers, but it would be considerably more involved to use it to obtain products of $2s$-place numbers.

It is therefore usually quite feasible and convenient to do this: Maintain the definition of digital numbers as s-place aggregates, that is, maintain (a)–(b) in (2.1). When the situation (2.7.a)–(2.7.b) arises, that is, when an expression (2.7.a) has to be computed, then do not form in the conventional way

$$(2.7.\mathrm{a}') \qquad \sum_{l=1}^{m} \bar{x}_l \times \bar{y}_l,$$

that is, do not round off each term of (2.7.a) separately to s places. Instead, form the true $2s$-place products $\bar{x}_l\bar{y}_l$ of the s-place factors $\bar{x}_l$, $\bar{y}_l$, form their sum $\sum_{l=1}^{m}$ correctly to $2s$-places, and then (at the end) round off to s places. The result is a digital number in the original sense, that is, s-place, to be denoted by

$$(2.7.\mathrm{a}'') \qquad \sum_{l=1}^{m}{}^{*} \bar{x}_l\bar{y}_l.$$

This (2.7.a″) is a much better approximant of (2.7a) than (2.7.a′). Indeed, for the latter we have only the estimate

$$(2.7.\mathrm{b}') \qquad \left| \sum_{l=1}^{m} \bar{x}_l\bar{y}_l - \sum_{l=1}^{m} \bar{x}_l \times \bar{y}_l \right| \leqq \frac{m\beta^{-s}}{2}$$

while for the former clearly

$$(2.7.\mathrm{b}'') \qquad \left| \sum_{l=1}^{m} \bar{x}_l\bar{y}_l - \sum_{l=1}^{m}{}^{*} \bar{x}_l\bar{y}_l \right| \leqq \frac{\beta^{-s}}{2}.$$

Thus, the estimate (2.7.b′) is inferior to the estimate (2.7.b″) by a factor m. Note that both estimates are strict (not probabilistic).

This is the ***double precision procedure.*** There are several places in this paper where it could be used to improve our estimates. We shall, however, not use it in this paper (with some specified exceptions in §3.5).

2.4. The approximative rules of algebra for pseudo-operations. The pseudo-operations with which we shall have to work affect the ordinary laws of algebra in a manner which deserves some comment.

The laws to which we refer are these: Distributivity, commutativity, and associativity of multiplication, and the inverse relation between multiplication and division. When we replace true multiplication and division by pseudo-multiplication and division, then all of these, with the sole exception of the commutative law of multiplication, cease to be strictly valid. They are replaced by inequalities involving the round off error $\beta^{-s}/2$.

The basic inequalities in this field are

$$|\bar a \times \bar b - \bar a\bar b| \leq \beta^{-s}/2, \tag{2.8}$$

$$|\bar a \div \bar b - \bar a/\bar b| \leq \beta^{-s}/2. \tag{2.9}$$

From these we derive further inequalities as follows:

$$\text{(2.8) implies } |(\bar a + \bar b) \times \bar c - (\bar a \times \bar c + \bar b \times \bar c)| \leq 3\beta^{-s}/2.$$

However, the left-hand side is an integer multiple of β^{-s}, hence

$$|(\bar a + \bar b) \times \bar c - (\bar a \times \bar c + \bar b \times \bar c)| \leq \beta^{-s}. \tag{2.10}$$

We mentioned already

$$\bar a \times \bar b = \bar b \times \bar a. \tag{2.11}$$

Next

$$\bar a \times (\bar b \times \bar c) - \bar a\bar b\bar c = (\bar a \times (\bar b \times \bar c) - \bar a(\bar b \times \bar c)) + \bar a(\bar b \times \bar c - \bar b\bar c),$$

hence

$$|\bar a \times (\bar b \times \bar c) - \bar a\bar b\bar c| \leq (1 + |\bar a|)\beta^{-s}/2 \leq \beta^{-s}. \tag{2.12}$$

Interchanging $\bar a$, $\bar c$ and adding gives

$$\begin{aligned} &|\bar a \times (\bar b \times \bar c) - (\bar a \times \bar b) \times \bar c| \\ &\qquad \leq (2 + |\bar a| + |\bar c|)\beta^{-s}/2 \leq 2\beta^{-s}. \end{aligned} \tag{2.13}$$

In addition, if either $|\bar a| \neq 1$ or $|\bar c| \neq 1$, then this is less than $2\beta^{-s}$, and since the left-hand side is an integer multiple of β^{-s}, it is necessarily less than or equal to β^{-s}. For $|\bar{a}| = |\bar{c}| = 1$, that is, $\bar{a}, \bar{c} = \pm 1$, the left-hand side is clearly 0. Hence always

$$|\bar{a} \times (\bar{b} \times \bar{c}) - (\bar{a} \times \bar{b}) \times \bar{c}| \leqq \beta^{-s}. \tag{2.13'}$$

Finally

$$(\bar{a} \div \bar{b}) \times \bar{b} - \bar{a} = ((\bar{a} \div \bar{b}) \times \bar{b} - (\bar{a} \div \bar{b})\bar{b}) + (\bar{a} \div \bar{b} - \bar{a}/\bar{b})\bar{b},$$

hence

$$|(\bar{a} \div \bar{b}) \times \bar{b} - \bar{a}| \leqq (1 + |\bar{b}|)\beta^{-s}/2 \leqq \beta^{-s}. \tag{2.14}$$

Again, if $|\bar{b}| \neq 1$, then this is less than β^{-s}, and since the left-hand side is an integer multiple of β^{-s} it is necessarily equal to 0. For $|\bar{b}| = 1$, that is, $\bar{b} = \pm 1$, the left-hand side is clearly 0. Hence always

$$(\bar{a} \div \bar{b}) \times \bar{b} = \bar{a}. \tag{2.14'}$$

On the other hand

$$(\bar{a} \times \bar{b}) \div \bar{b} - \bar{a} = ((\bar{a} \times \bar{b}) \div \bar{b} - (\bar{a} \times \bar{b})/\bar{b}) + (\bar{a} \times \bar{b} - \bar{a}\bar{b})/\bar{b},$$

hence

$$|(\bar{a} \times \bar{b}) \div \bar{b} - \bar{a}| \leqq (1 + |\bar{b}|^{-1})\beta^{-s}/2 \leqq |\bar{b}|^{-1}\beta^{-s}. \tag{2.15}$$

Note how unfavorably (2.15) compares with (2.14'), or even with (2.14), especially when $\bar{b} \ll 1$. Distinctions of this type will play an important role in our work, and they are worth emphasizing, since they are not at all in the spirit of ordinary algebra.

2.5. **Scaling by iterated halving.** The pseudo-operations that we have discussed so far are probably adequate for our work. It is nevertheless convenient to introduce an additional one. It must be said that both the need for this operation and the optimality of the form in which we introduce it are less cogently established than their equivalents for the pseudo-operations considered so far. The second point is particularly relevant: Better ways of defining and manipulating a new pseudo-operation with essentially the same potentialities may be found. At present, however, the procedure that we propose to follow seems reasonable and adequate.

The operation in question is needed in order to facilitate the manipulation of the scale factors mentioned in (b) in **2.1**. If an increase in the size of a (digital) number $\bar{a}$ is wanted, we can multiply it with an integer l $(=2, 3, \cdots)$: $l\bar{a}$. This is not a pseudo-operation (cf. the end of (a) in §2.1). In order to be able to effect a decrease in size, it is desirable to be able to perform the inverse operation: Division by an integer l $(=2, 3, \cdots)$. This is necessarily a pseudo-operation: $\bar{a} \div l$. (Since l does not lie in the interval $-1, 1$, it is not a digital number in the sense of (a) in §2.1.) It will be important to arrange this operation so that it can be iterated with as little extra complication as possible, since scale factors in a calculation are likely to be introduced successively and take cumulative effect. This indicates the desirability of having the "associative law"

$$(\bar{a} \div l) \div m = \bar{a} \div lm. \tag{2.16}$$

It also suggests that it might be sufficient to use only those l which are powers of a fixed integer γ ($=2, 3, \cdots$):

$$l = \gamma^p \qquad (p = 0, 1, 2, \cdots). \tag{2.17}$$

We can then obtain (2.16) with ease, by defining $\bar{a} \div \gamma^p$ not as the result of *a single division of $\bar{a}$ by $l = \gamma^p$*, but of *a p times iterated division of $\bar{a}$ by γ*. We shall adhere to this definition throughout what follows:

$$\bar{a} \div \gamma^p = (\cdots((\bar{a} \div \gamma) \div \gamma)\cdots) \div \gamma \qquad (p \text{ times}). \tag{2.18}$$

In the choice of γ two considerations intervene. First, the smaller γ, the more precise, that is, the less wasteful, the adjustments of scale will be that we base on it. (Cf., for example, the relationship of (6.50.a) and (6.50.b).) Since $\gamma = 2, 3, \cdots$, this suggests the choice $\gamma = 2$. Second, it simplifies things somewhat if we put γ equal to the base of our digital system: $\gamma = \beta$. Indeed, in this case $\bar{a} \div \gamma$ is merely a shift of $\bar{a}$ by one place to the right. (Or, equivalently, a shift of the β-adic point by one place to the left. We prefer the first formulation, in view of the convention regarding the position of the β-adic point, formulated in (b) in §2.1.)

Thus we have two competing choices: $\gamma = 2$ and $\gamma = \beta$. For $\beta = 2$, that is, in the binary system, the two coincide. Indeed, this seems to be one of the major arguments in favor of the use of the binary system in high speed, automatic computing. It seems preferable, however, to make here no assumptions concerning β, but to dispose of γ only. After taking all factors into account, it seems to us that the choice

$$\gamma = 2 \tag{2.19}$$

is preferable for all β, and we shall therefore use (2.19) throughout what follows.

We conclude with two estimates.

Clearly

$$\left| \bar{a} \div 2 - \bar{a}/2 \right| \leq \beta^{-s}/2. \tag{2.20}$$

The formula

$$\bar{a} \div 2^p - \bar{a}/2^p = \sum_{q=1}^{p} ((\bar{a} \div 2^{q-1}) \div 2 - (\bar{a} \div 2^{q-1})/2)/2^{p-q}$$

now gives

$$\begin{aligned} |\bar{a} \div 2^p - \bar{a}/2^p| &\leqq \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-1}}\right)\frac{\beta^{-s}}{2} \\ &= \left(2 - \frac{1}{2^{p-1}}\right)\frac{\beta^{-s}}{2} = \left(1 - \frac{1}{2^p}\right)\beta^{-s}. \end{aligned}$$

From this we infer first:

$$|\bar{a} \div 2^p - \bar{a}/2^p| < \beta^{-s}. \tag{2.21}$$

Second, if $|\bar{a} - b| \leqq k\beta^{-s}$, then in view of the formula

$$\bar{a} \div 2^p - b/2^p = (\bar{a} \div 2^p - \bar{a}/2^p) + (\bar{a} - b)/2^p$$

we infer

$$\begin{aligned} |\bar{a} \div 2^p - b/2^p| &\leqq \left(1 - \frac{1}{2^p}\right)\beta^{-s} + \frac{k}{2^p}\beta^{-s} = \left(1 + \frac{k-1}{2^p}\right)\beta^{-s} \\ &\leqq (1 + \operatorname{Max}(0, k-1))\,\beta^{-s} \\ &= \operatorname{Max}(1, k)\,\beta^{-s}, \end{aligned}$$

that is,

$$\begin{gathered} |\bar{a} - b| \leqq k\beta^{-s} \quad \text{implies} \\ |\bar{a} \div 2^p - b/2^p| \leqq \operatorname{Max}(1, k)\,\beta^{-s}. \end{gathered} \tag{2.22}$$

CHAPTER III. ELEMENTARY MATRIX RELATIONS

3.1. The elementary vector and matrix operations. Since our discussions will center on *n*th order matrices

$$A = (a_{ij}),\ B = (b_{ij}),\ \cdots \qquad (i, j = 1, \cdots, n),$$

we have to introduce matrix notations. It will also be convenient to be able to refer to *n*th order vectors: $\xi = (x_i)$, $\eta = (y_i)$, $\cdots$ $(i = 1, \cdots, n)$. At first we shall discuss these in terms of ordinary real numbers (and true operations) only, but in §3.5 we shall introduce digital numbers (and pseudo-operations), too.

We use, of course, the *sum* and the *scalar product* for vectors and for matrices: $\xi + \eta = (x_i + y_i)$, $a\xi = (ax_i)$, $A + B = (a_{ij} + b_{ij})$, $aA = (aa_{ij})$. We fix the conventions for the *application* of a matrix to a vector: $A\xi = \eta$ with $\sum_{j=1}^{n} a_{ij}x_j = y_i$ and for the *matrix product*: $AB = C$ with

$\sum_{k=1}^{n} a_{ik}b_{kj} = c_{ij}$, so as to have the mixed associative law: $A(B\xi) = (AB)\xi$.

We need further:

For vectors: The ***inner product*** $(\xi, \eta) = \sum_{i=1}^{n} x_i y_i$, and the ***norm*** $|\xi| \geqq 0$ with $|\xi|^2 = (\xi, \xi) = \sum_{i=1}^{n} x_i^2$.

For matrices: The ***transposed*** matrix $A^* = (a_{ji})$, the determinant $D(A)$, and the ***trace*** $t(A) = \sum_{i=1}^{n} a_{ii}$. Clearly $t(AB) = t(BA) = \sum_{i,j=1}^{n} a_{ij}b_{ji}$; the ***norm*** $N(A) \geqq 0$ with $(N(A))^2 = t(A^*A) = t(AA^*) = \sum_{i,j=1}^{n} a_{ij}^2$; also the (***upper***) ***bound*** $|A|$ and the ***lower bound*** $|A|_l$, which will be defined further below.

The properties of these entities are too well known to require much discussion. We shall only touch briefly on those which link the most crtical ones: $|A|$, $|A|_l$ and $N(A)$.

3.2. **Properties of $|A|$, $|A|_l$ and $N(A)$.** We begin with $|A|$, $|A|_l$. We define:

$$|A| = \operatorname*{Max}_{|\xi|=1} |A\xi|, \tag{3.1.a}$$

$$|A|_l = \operatorname*{Min}_{|\xi|=1} |A\xi|. \tag{3.1.b}$$

It follows immediately, that

(3.2.a) $|A|$ is the smallest c for which $|A\xi| \leqq c|\xi|$ holds for all ξ,

(3.2.b) $|A|_l$ is the largest c for which $|A\xi| \geqq c|\xi|$ holds for all ξ.

Clearly

$$|A| \geqq |A|_l \geqq 0. \tag{3.3}$$

Also:

(3.4) $|A| > 0$ is equivalent to $A \neq 0$.

(3.5) $|A|_l > 0$ is equivalent to this:

(3.5.a) $A\xi = \eta$ is a one-to-one mapping of all vectors ξ on all vectors η, that is to this:

(3.5.b) A^{-1} exists.

This is, of course, equivalent to

$$D(A) \neq 0, \tag{3.5.c}$$

and is termed the ***nonsingularity*** of A.

For a nonsingular A we have further:

$$|A^{-1}| = |A|_l^{-1}, \tag{3.5.d}$$

$$|A^{-1}|_l = |A|^{-1}. \tag{3.5.e}$$

Other obvious relations are:

$$|aA| = |a|\,|A|, \tag{3.6.a}$$

$$|aA|_l = |a|\,|A|_l, \tag{3.6.b}$$

$$|I| = |I|_l = 1, \tag{3.7}$$

where I is the unit matrix:

$$I = (\delta_{ij}), \qquad \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{3.7.a}$$

Further

$$|A + B| \begin{cases} \leq |A| + |B|, \\ \geq \big||A| - |B|\big|, \end{cases} \tag{3.8.a}$$

$$|A + B|_l \begin{cases} \leq |A|_l + |B|, \\ \geq |A|_l - |B| \end{cases} \tag{3.8.b}$$

(and the same with A, B interchanged),

$$|A|_l\,|B|_l \leq |AB|_l \leq \left\{\begin{matrix} |A|\,|B|_l \\ |A|_l\,|B| \end{matrix}\right\} \leq |AB| \leq |A|\,|B|. \tag{3.9}$$

Next

$$|A| \text{ is the smallest } c \text{ for which } |(A\xi, \eta)| \leq c\,|\xi|\,|\eta|. \tag{3.10}$$

To see this, it suffices to show that for any given c the validity of $|A\xi| \leq c|\xi|$ for all ξ is equivalent to the validity of $|(A\xi, \eta)| \leq c|\xi|\,|\eta|$ for all ξ, η. Now the former implies $|(A\xi, \eta)| \leq |A\xi|\,|\eta| \leq c|\xi|\,|\eta|$, that is, it implies the latter; and the latter implies (with $\eta = A\xi$) $|A\xi|^2 = |(A\xi, A\xi)| \leq c|\xi|\,|A\xi|$, hence $|A\xi| \leq c|\xi|$ (this obtains by division by $|A\xi|$ when $|A\xi| > 0$, otherwise it is obvious, since $|A\xi| = 0$), that is, it implies the former.

Since $(A^*\xi, \eta) = (A\eta, \xi)$, therefore (3.10) implies

$$|A| = |A^*|. \tag{3.11.a}$$

Since $(A^*)^{-1}$ exists if and only if A^{-1} exists and is then $= (A^{-1})^*$, therefore (3.5) on one hand and (3.5.d), (3.5.e), (3.11.a) on the other give

$$|A|_l = |A^*|_l. \tag{3.11.b}$$

Next

$$(3.12.a) \qquad |A^*A| = |A|^2.$$

Indeed: $\leqq$ follows from (3.9), (3.11.a). $\geqq$ obtains by using (3.10) for A^*A and (3.2.a) for A: $|A\xi|^2 = (A\xi, A\xi) = (A^*A\xi, \xi) \leqq |A^*A|\,|\xi|^2$, $|A\xi| \leqq (|A^*A|)^{1/2}|\xi|$, hence $|A| \leqq (|A^*A|)^{1/2}$, $|A^*A| \geqq |A|^2$.

If A^{-1} exists, then $(A^*A)^{-1}$ exists and equals $A^{-1}(A^{-1})^*$; if $(A^*A)^{-1}$ exists then A^{-1} exists and equals $(A^*A)^{-1}A^*$. Hence $(A^*A)^{-1}$ exists if and only if A^{-1} exists and is then $A^{-1}(A^{-1})^*$. Therefore, (3.5) on one hand and (3.5.d), (3.5.e), (3.11.b), (3.12.a) on the other give

$$(3.12.b) \qquad |A^*A|_l = |A|_l^2.$$

We now pass to the consideration of $N(A)$.

Clearly

$$(3.13) \qquad N(A) = N(A^*),$$

and

$$(3.14) \qquad N(aA) = |a|\,N(A).$$

For $A = (a_{ij})$ fix $j = 1, \cdots, n$ and view $i = 1, \cdots, n$ as a vector index, then $A^{[j]} = (a_{ij})$ $(i = 1, \cdots, n)$ defines a vector $A^{[j]}$. Clearly $(N(A))^2 = \sum_{j=1}^n |A^{[j]}|^2$. Now $(A+B)^{[j]} = A^{[j]} + B^{[j]}$, hence

$$\begin{aligned} N(A + B) &= \left(\sum_{j=1}^n |A^{[j]} + B^{[j]}|^2 \right)^{1/2} \\ &\leqq \left(\sum_{j=1}^n (|A^{[j]}| + |B^{[j]}|)^2 \right)^{1/2} \\ &\leqq \left(\sum_{j=1}^n |A^{[j]}|^2 \right)^{1/2} + \left(\sum_{j=1}^n |B^{[j]}|^2 \right)^{1/2} \\ &\leqq N(A) + N(B), \end{aligned}$$

that is

$$(3.15) \qquad N(A + B) \leqq N(A) + N(B).$$

Furthermore, $(AB)^{[j]} = A(B^{[j]})$, hence

$$\begin{aligned} N(AB) &= \left(\sum_{j=1}^n |A(B^{[j]})|^2 \right)^{1/2} \leqq \left(\sum_{j=1}^n (|A|\,|B^{[j]}|)^2 \right)^{1/2} \\ &= |A| \left(\sum_{j=1}^n |B^{[j]}|^2 \right)^{1/2} = |A|\,N(B), \end{aligned}$$

that is,

$$N(AB) \leqq |A|\, N(B). \tag{3.16.a}$$

Applying (3.16.a) to B^*, A^* (in place of A, B) and using $(AB)^* = B^*A^*$ as well as (3.11.a), (3.13) gives

$$N(AB) \leqq |B|\, N(A). \tag{3.16.b}$$

Given a vector $\xi = (x_i)$ $(i = 1, \cdots, n)$ define a matrix

$$\xi^{\sim} = (x_{ij}) \qquad x_{ij} = \begin{cases} x_i & \text{for } j = 1, \\ 0 & \text{for } j \neq 1. \end{cases}$$

Then $N(\xi^{\sim}) = |\xi|$, $(A\xi)^{\sim} = A(\xi^{\sim})$. Hence (3.16.b) gives $|A\xi| \leqq N(A)|\xi|$, that is, $|A| \leqq N(A)$. Combining this with (3.16.a) with $B = I$ (note that $N(I) = n^{1/2}$) gives

$$|A| \leqq N(A) \leqq n^{1/2}|A|. \tag{3.17.a}$$

Both estimates of (3.17.a) are optimal: The second $\leqq$ becomes $=$ for $A = I = (\delta_{ij})$, the first $\leqq$ becomes $=$ for $A = (1)$.

Consider the vectors $I^{(k)} = (\delta_{ik})$ $(i = 1, \cdots, n)$. $|I^{(k)}| = 1$, $(AI^{(j)}, I^{(i)}) = a_{ij}$ hence (3.10) gives $|a_{ij}| \leqq |A|$. Again $(N(A))^2 = \sum_{i,j=1}^{n} a_{ij}^2 \leqq n^2 \operatorname{Max}_{i,j=1,\cdots,n} a_{ij}^2$, hence (by 3.17.a) (first $\leqq$) $|A| \leqq n \operatorname{Max}_{i,j=1,\cdots,n} |a_{ij}|$. Thus

$$\operatorname*{Max}_{i,j=1,\cdots,n} |a_{ij}| \leqq |A| \leqq n \operatorname*{Max}_{i,j=1,\cdots,n} |a_{ij}|. \tag{3.17.b}$$

Both estimates of (3.17.b) are optimal: The first $\leqq$ becomes $=$ for $A = I$; the second $\leqq$ becomes $=$ for $A = (1)$.

3.3. **Symmetry and definiteness.** We recall further the definitions of *symmetry* and of *definiteness*[21] for matrices. A is *symmetric* if

$$A = A^*, \quad \text{that is, if} \quad a_{ij} = a_{ji} \qquad (i, j = 1, \cdots, n). \tag{3.18}$$

A is *definite* if it is symmetric and if

$$(A\xi, \xi) \geqq 0 \qquad \text{for all } \xi. \tag{3.19}$$

We note:

$$A^*A \text{ is always definite.} \tag{3.20}$$

Indeed: $(A^*A)^* = A^*A^{**} = A^*A$, and $(A^*A\xi, \xi) = (A\xi, A\xi) = |A\xi|^2 \geqq 0$.

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[21] Our present concept of definiteness corresponds to what is usually known as "non-negative semi-definiteness."

We define the *proper values* $\lambda_1, \cdots, \lambda_n$ of a matrix A (with multiplicities) as usual: They are the roots (with multiplicities) of the nth order polynomial $D(\lambda I - A)$. We shall only use them when A is symmetric; in this case they are all real. For a symmetric A definiteness is equivalent to

$$\lambda_i \geqq 0 \qquad \text{for all } i = 1, \cdots, n. \tag{3.21.a}$$

In this case we make it a convention to arrange the proper values in a monotone nonincreasing sequence:

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0. \tag{3.21.b}$$

For a definite A

$$|A| = \lambda = \lambda_1, \tag{3.22.a}$$

$$|A|_l = \mu = \lambda_n \tag{3.22.b}$$

and therefore (using (3.5))

$$A \text{ is non-singular if and only if } \lambda_n > 0. \tag{3.22.c}$$

Further

$$D(A) = \prod_{i=1}^{n} \lambda_i, \tag{3.23}$$

$$t(A) = \sum_{i=1}^{n} \lambda_i,^{22} \tag{3.24}$$

and by applying (3.24) to $A^*A = A^2$, whose proper values are $\lambda_1^2, \cdots, \lambda_n^2$,

$$N(A) = \left(\sum_{i=1}^{n} \lambda_i^2 \right)^{1/2}.^{23} \tag{3.25}$$

3.4. **Diagonality and semi-diagonality.** To conclude, we refer to the classes of *diagonal*, *upper semi-diagonal* and *lower semi-diagonal* matrices. A matrix $A = (a_{ij})$ belongs to these classes if $a_{ij} = 0$ whenever $i \neq j$, or whenever $i > j$, or whenever $i < j$, respectively. Denote these three classes by $\mathcal{C}_0$, $\mathcal{C}_+$, $\mathcal{C}_-$, respectively. For $\mathcal{C} = \mathcal{C}_0$ or $\mathcal{C}_+$ or $\mathcal{C}_-$ define $\mathcal{C}' = \mathcal{C}_0$ or $\mathcal{C}_-$ or $\mathcal{C}_+$, respectively. Now the following facts are well known:

(3.26) Let A, B belong to $\mathcal{C}$. Then aA, $A \pm B$, AB and (if it exists) A^{-1} belong to $\mathcal{C}$, while A^* belongs to $\mathcal{C}'$. A^{-1} exists if and only if all

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[22] (3.23), (3.24) hold, of course, for all matrices A.

[23] Here $A^* = A$ is being used; (3.25) holds only for symmetric matrices A.

diagonal elements of A are unequal to 0. In all these procedures the diagonal elements of A behave as if they formed a diagonal matrix by themselves.

(3.27) For $A=(a_{ij})=(a_i\delta_{ij})$ in $\mathcal{C}_0$ the diagonal elements $a_1, \cdots, a_n$ are the proper values of A (not necessarily monotone) and

$$|A| = \operatorname*{Max}_{i=1,\cdots,n} |a_i|,$$

$$|A|_l = \operatorname*{Min}_{i=1,\cdots,n} |a_i|.$$

These two relations are not valid in $\mathcal{C}_+$ and $\mathcal{C}_-$.

For an $A=(a_{ij})=(a_i\delta_{ij})$ in $\mathcal{C}_0$ the formation of A^{-1} is trivial: $A^{-1}=(a_i^{-1}\delta_{ij})$. For an $A=(a_{ij})$ in $\mathcal{C}_+$ or $\mathcal{C}_-$, A^{-1} still obtains by a fairly simple and explicit algorithm. We shall see subsequently (cf. the end of §4.3) that this is one of the two salient points of the elimination method.

3.5. **Pseudo-operations for matrices and vectors. The relevant estimates.** We now pass to the pseudo-operations for matrices and vectors. We shall actually need the *matrix pseudo-product*, and it is quite convenient, essentially for the purpose of illustration, to introduce the (vector) *inner pseudo-product*, too. Besides, we shall discuss each one of these in two forms: ordinary precision (cf. (a) in §2.1) and double precision (cf. (f) in §2.3).

In (a) in §2.1 we introduced digital numbers $\bar{x}$, which could, however, also be viewed as ordinary real numbers, cf. (c) in §2.2. We introduce now, in the same sense, *digital matrices* $\bar{A}=(\bar{a}_{ij})$, $\bar{B}=(\bar{b}_{ij})$, $\cdots$ $(i,j=1,\cdots,n)$ and digital vectors $\bar{\xi}=(\bar{x}_i)$, $\bar{\eta}=(\bar{y}_i)$, $\cdots$ $(i=1,\cdots,n)$—the relevant fact being that the $\bar{a}_{ij}$, $\bar{b}_{ij}$, $\cdots$, $\bar{x}_i$, $\bar{y}_i$, $\cdots$ are digital numbers. As indicated above, we introduce only two pseudo-operations, but each in two forms:

The (*ordinary precision*) ***inner pseudo-product***: $(\bar{\xi}\bigcirc\bar{\eta})=\sum_{i=1}^{n}\bar{x}_i\times\bar{y}_i$; the ***double precision inner pseudo-product***: $(\bar{\xi}\bigcirc\bigcirc\bar{\eta})=\sum_{i=1}^{*n}\bar{x}_i\bar{y}_i$; the (*ordinary precision*) *matrix pseudo-product*: $\bar{A}\times\bar{B}=\bar{C}$ with $\bar{c}_{ij}=\sum_{k=1}^{n}\bar{a}_{ik}\times\bar{b}_{kj}$; the *double precision matrix pseudo-product*: $\bar{A}\times\times\bar{B}=\bar{C}$ with $\bar{c}_{ij}=\sum^{*}\bar{a}_{ik}\bar{b}_{kj}$.

The only ordinary law of algebra which is not invalidated by the transition from true operations to pseudo-operations is, as in §2.4, the commutative law of multiplication. It holds for the true inner product, but not for the true matrix product, hence we obtain only these pseudo-relations:

$$(3.28.a) \qquad (\xi \bigcirc \bar{\eta}) = (\bar{\eta} \bigcirc \xi),$$

$$(3.28.b) \qquad (\xi \bigcirc\bigcirc \bar{\eta}) = (\bar{\eta} \bigcirc\bigcirc \xi).$$

The other laws are, as in §2.4, replaced by inequalities involving the round off error $\beta^{-s}/2$.

In order to obtain a first orientation concerning these, we begin by restating from (e) and (f) in (2.3):

$$(3.29.a) \qquad |(\xi, \bar{\eta}) - (\xi \bigcirc \bar{\eta})| \leqq n\beta^{-s}/2 \qquad \text{(strict)},$$

$$(3.29.b) \qquad (\xi, \bar{\eta}) - (\xi \bigcirc \bar{\eta}) \text{ has a Mean} = 0 \text{ and a Dispersion} \leqq .29n^{1/2}\beta^{-s} \qquad \text{(probabilistic)},$$

$$(3.29.c) \qquad |(\xi, \bar{\eta}) - (\xi \bigcirc\bigcirc \bar{\eta})| \leqq \beta^{-s}/2 \qquad \text{(strict)}.$$

We now pass to $\bar{A} \times \bar{B}$ and $\bar{A} \times\times \bar{B}$. The elements of these matrices and the corresponding ones in $\bar{A}\bar{B}$ are built exactly like the expressions $(\xi \bigcirc \bar{\eta})$, $(\xi \bigcirc\bigcirc \bar{\eta})$ and $(\xi, \bar{\eta})$. We have, therefore, in complete analogy with (3.29.a)–(3.29.c):

For $\bar{A}\bar{B} - \bar{A} \times \bar{B} = (\rho_{ij})$

$$(3.30.a) \qquad |\rho_{ij}| \leqq n\beta^{-s}/2 \qquad \text{(strict)},$$

$$(3.30.b) \qquad \rho_{ij} \text{ has a Mean} = 0 \text{ and a Dispersion} \leqq .29n^{1/2}\beta^{-s} \qquad \text{(probabilistic)},$$

for $\bar{A}\bar{B} - \bar{A} \times\times \bar{B} = (\sigma_{ij})$

$$(3.30.c) \qquad |\sigma_{ij}| \leqq \beta^{-s}/2 \qquad \text{(strict)}.$$

(3.17.b) permits us to infer from (3.30.a) and (3.30.c):

$$\left.\begin{array}{ll} (3.31.a) & |\bar{A}\bar{B} - \bar{A} \times \bar{B}| \leqq n^2\beta^{-s}/2 \\ (3.31.c) & |\bar{A}\bar{B} - \bar{A} \times\times \bar{B}| \leqq n\beta^{-s}/2 \end{array}\right\} \qquad \text{(strict)}.$$

Drawing a probabilistic inference from (3.30.b) is more difficult. Using some results of V. Bargmann[14] it is possible to show this:

(3.31.b) $|\bar{A}\bar{B} - \bar{A} \times \bar{B}| \leqq kn\beta^{-s}$ has a probability nearly 1 for moderately large values of k.

It seems worth noting that the estimates of (3.31.b) and (3.31.c) are of the same order of magnitude (that is, they involve the same power of n), which is not true for the estimates of (3.29.b) and (3.29.c), on which they are based. However, we do not propose to pursue the probabilistic estimates of the type (b) any further in this paper, although they are interesting and practically very relevant. We shall consider them at a later occasion. We shall instead continue here with the analysis of the strict estimates of the types (a) and (c).

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[14] These results are contained in a manuscript entitled *Statistical distribution of proper values*. This work was done under the auspices of the U. S. Navy, Bureau of Ordnance, under Contract NORD9596 (1946), and will be published elsewhere.

In this connection we wish to mention further work done on matrix inversion by the iteration method. It was done under the same contract and appeared in a report by V. Bargmann, D. Montgomery, and J. von Neumann, entitled *Solution of linear systems of high order*.

(3.31.a), (3.33.c) give

$$(3.32.\text{a})\qquad |\bar{A}\times(\bar{B}+\bar{C})-(\bar{A}\times\bar{B}+\bar{A}\times\bar{C})| \leqq 3n^2\beta^{-s}/2,$$

$$(3.32.\text{c})\qquad |\bar{A}\times\times(\bar{B}+\bar{C})-(\bar{A}\times\times\bar{B}+\bar{A}\times\times\bar{C})| \leqq 3n\beta^{-s}/2.$$

Further

$$\bar{A}\times(\bar{B}\times\bar{C})-\overline{ABC}=(\bar{A}\times(\bar{B}\times\bar{C})-\bar{A}(\bar{B}\times\bar{C}))+\bar{A}(\bar{B}\times\bar{C}-\overline{BC}),$$

and similarly

$$\begin{aligned}\bar{A}\times\times(\bar{B}\times\times\bar{C})-\overline{ABC}=(\bar{A}\times\times(\bar{B}\times\times\bar{C})&-\bar{A}(\bar{B}\times\times\bar{C}))\\ &+\bar{A}(\bar{B}\times\times\bar{C}-\overline{BC}),\end{aligned}$$

hence (3.31.a), (3.31.c) also give

$$(3.33.\text{a})\qquad |\bar{A}\times(\bar{B}\times\bar{C})-\overline{ABC}| \leqq (1+|\bar{A}|)n^2\beta^{-s}/2,$$

$$(3.33.\text{c})\qquad |\bar{A}\times\times(\bar{B}\times\times\bar{C})-\overline{ABC}| \leqq (1+|\bar{A}|)n\beta^{-s}/2.$$

Interchanging $\bar{A}$, $\bar{C}$ and adding gives

$$(3.34.\text{a})\qquad \begin{aligned}&|\bar{A}\times(\bar{B}\times\bar{C})-(\bar{A}\times\bar{B})\times\bar{C}|\\ &\qquad\leqq (2+|\bar{A}|+|\bar{C}|)n^2\beta^{-s}/2,\end{aligned}$$

$$(3.34.\text{c})\qquad \begin{aligned}&|\bar{A}\times\times(\bar{B}\times\times\bar{C})-(\bar{A}\times\times\bar{B})\times\times\bar{C}|\\ &\qquad\leqq (2+|\bar{A}|+|\bar{C}|)n\beta^{-s}/2.\end{aligned}$$

In comparing (3.33.a)–(3.34.c) with (2.12), (2.13), it should be remembered that we had there $|\bar{a}|\leqq 1$, $|\bar{c}|\leqq 1$, whereas now we have $|\bar{a}_{ij}|\leqq 1$, $|\bar{c}_{ij}|\leqq 1$, but from this we can infer (by (3.17.b)) only $|\bar{A}|\leqq n$, $|\bar{C}|\leqq n$.

More detailed evaluations will be derived when we get to our primary problems in Chapter VI.

CHAPTER IV. THE ELIMINATION METHOD

4.1. Statement of the conventional elimination method. In order to have a fixed point of reference, and also in order to introduce the notations that will be used in the subsequent sections of this paper, we described first the conventional elimination method—using true operations, and not yet pseudo-operations.

The elimination method is usually viewed as one for equation-solving and not for matrix-inverting, but this actually amounts to the same thing: Given a nonsingular matrix $A = (a_{ij})$ $(i, j = 1, \cdots, n)$ and the corresponding equation system

$$\sum_{j=1}^{n} a_{ij}x_j = y_i \qquad (i = 1 \cdots n), \tag{4.1}$$

that is,

$$A\xi = \eta, \tag{4.1'}$$

the solution

$$\sum_{j=1}^{n} t_{ij}y_j = x_i \qquad (i = 1, \cdots, n), \tag{4.2}$$

that is,

$$T\eta = \xi, \tag{4.2'}$$

is clearly furnishing the desired inverse:

$$T = A^{-1}. \tag{4.3}$$

Given the system of n equations (4.1) with the n unknowns $x_1, \cdots, x_n$, the solution by elimination proceeds in the following, familiar way:

Assume that the $k-1$ first unknowns $x_1, \cdots, x_{k-1}$ $(k = 1, \cdots, n-1)$ have already been eliminated, and that, for the remaining $n-k+1$ unknowns $x_k, \cdots, x_n$, $n-k+1$ equations have been derived:

$$\sum_{j=k}^{n} a_{ij}^{(k)} x_j = y_i^{(k)} \qquad (i = k, \cdots, n). \tag{4.4}$$

Then the elimination of the next unknown, x_k, is effected by subtracting the $a_{ik}^{(k)}/a_{kk}^{(k)}$-fold of equation number k from equation number i $(i = k+1, \cdots, n)$. This gives a new set of equations

$$\sum_{j=k+1}^{n} a_{ij}^{(k+1)} x_j = y_i^{(k+1)} \qquad (i = k + 1, \cdots, n), \tag{4.5}$$

where

$$(4.6)\qquad a_{ij}^{(k+1)} = a_{ij}^{(k)} - a_{ik}^{(k)} a_{kj}^{(k)} / a_{kk}^{(k)} \qquad (i, j = k + 1, \cdots, n),$$

$$(4.7)\qquad y_i^{(k+1)} = y_i^{(k)} - (a_{ik}^{(k)} / a_{kk}^{(k)}) y_k^{(k)} \qquad (i = k + 1, \cdots, n).$$

The transition from (4.4) to (4.5) is clearly an inductive step from k to $k+1$. This induction begins, of course, with the original equations (4.1), that is, we have

$$(4.8)\qquad a_{ij}^{(1)} = a_{ij} \qquad (i, j = 1, \cdots, n),$$

$$(4.9)\qquad y_i^{(1)} = y_i \qquad (i = 1, \cdots n).$$

The induction produces (4.4) successively for $k=1, \cdots, n$, that is, it produces

$$(4.10)\qquad a_{ij}^{(k)} \qquad (k = 1, \cdots, n;\ i, j = k, \cdots, n),$$

$$(4.11)\qquad y_i^{(k)} \qquad (k = 1, \cdots, n;\ i = k, \cdots, n).$$

After all n systems (4.4) have been derived, the first equation of each system is selected, and these are combined to a new system of n equations with the n original unknowns $x_1, \cdots, x_n$:

$$(4.12)\qquad \sum_{j=k}^{n} a_{kj}^{(k)} x_j = y_k^{(k)} \qquad (k = 1, \cdots, n).$$

These are now solved by a backward induction over $k=n, \cdots, 1$:

$$(4.13)\qquad x_k = \frac{1}{a_{kk}^{(k)}} y_k^{(k)} - \sum_{j=k+1}^{n} \frac{a_{kj}^{(k)}}{a_{kk}^{(k)}} x_j.$$

4.2. **Positioning for size in the intermediate matrices.** Before we undertake to analyze the procedure of §4.1, we note this:

The inductive step from k to $k+1$ (on (4.4)) involves a division by $a_{kk}^{(k)}$, and this division reappears in the k-step of (4.13). Hence it is important, from the abstract point of view, that $a_{kk}^{(k)} \neq 0$ and, from the actual computational point of view, that $a_{kk}^{(k)}$ be essentially as large as possible.

It is, however, perfectly conceivable, that an $a_{kk}^{(k)}$ turns out to be small, or even zero, although A is nonsingular: The simplest example is furnished by the possibility of $a_{11}^{(1)} = a_{11} = 0$ (that is, $k=1$), although A is nonsingular. In the actual, numerical uses of the elimination method this point is fully appreciated: It is customary to make arrangements to have $a_{kk}^{(k)}$ possess the largest absolute value among all $a_{ij}^{(k)}$ $(i, j=k, \cdots, n)$.[25] This is done by permuting the $i=k, \cdots, n$ and the $j=k, \cdots, n$ (separately) in such a way that $\mathrm{Max}_{i,j=k,\cdots,n} |a_{ij}^{(k)}|$ is assumed for $i=j=k$. This permutation is effected just before the operations that lead from (4.4) to (4.5) are undertaken, that is, just before the inductive step from k to $k+1$. This occurs $n-1$ times: For $k=1, \cdots, n-1$.

We call these permutations of i and j *positioning for size.*

Note that this positioning for size will produce an $a_{kk}^{(k)} \neq 0$ (in its k-step, $k=1, \cdots, n$), unless $\mathrm{Max}_{i,j=k,\cdots,n} |a_{ij}^{(k)}| = 0$, that is, unless $a_{ij}^{(k)} = 0$ for all $i, j=k, \cdots, n$.[26]

Now we prove:

(4.14) If A is nonsingular, then positioning for size will always produce an $a_{kk}^{(k)} \neq 0$ $(k=1, \cdots, n)$, that is, never $a_{ij}^{(k)}=0$ for all $i, j=k, \cdots, n$.

Assume the opposite: Let $k=k_1$ $(=1, \cdots, n)$ be the smallest k so that $a_{ij}^{(k)}=0$ for all $i, j=k, \cdots, n$. The system of equations (4.1) is clearly equivalent to the system of equations (4.4) with $k=k_1$, together with the system of equations (4.13) with $k=k_1-1, \cdots, 1$. Now our assumption amounts to stating that the left-hand sides in the system (4.4) vanish identically. Hence the system (4.4), (4.13), that is, the equivalent system (4.1), cannot have a unique solution $x_1, \cdots, x_n$. This however is in contradiction with the nonsingularity of the matrix $A=(a_{ij})$ of (4.1).

(4.14) is the rigorous justification for the operation of positioning for size. Throughout what follows, we shall keep pointing out whether the positioning for size is or is not assumed to have taken place in any particular part of the discussion.

4.3. Statement of the elimination method in terms of factoring A into semi-diagonal factors C, B'. We return now to the procedure of §4.1, without positioning for size, for the balance of this chapter.

Summing (4.7) over $k=1, \cdots, i-1$, and remembering (4.9), gives

$$y_i = y_i^{(i)} + \sum_{k=1}^{i-1} \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} y_k^{(k)}. \tag{4.15}$$

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[25] Or at least one which has the same order of magnitude as the maximum in question. We propose, however, to disregard this possible relaxation of the requirement. We shall postulate that $|a_{kk}^{(k)}|$ be strictly equal to $\mathrm{Max}_{i,j=k,\cdots,n} |a_{ij}^{(k)}|$.

[26] Positioning for size, as described above, occurs only for $k=1, \cdots, n-1$. For $k=n$, however, $a_{nn}^{(n)}$ is the only $a_{ij}^{(n)}$, hence the assertion is trivial.

We have $\xi = (x_i)$, $\eta = (y_i)$, let us introduce in addition $\zeta = (y_i^{(i)})$. Then (4.12) and (4.15) express two very simple matrix relations between ξ and ζ and between η and ζ. If we define

$$(4.16) \qquad B' = (b'_{ij}) \qquad (i, j = 1, \cdots, n)$$
$$\text{with} \quad b'_{ij} = \begin{cases} a_{ij}^{(i)} & \text{for } i \leq j \\ 0 & \text{for } i > j \end{cases}$$

$$(4.17) \qquad C = (c_{ij}) \qquad (i, j = 1, \cdots, n)$$
$$\text{with} \quad c_{ij} = \begin{cases} a_{ij}^{(j)}/a_{jj}^{(j)} & \text{for } i \geq j \\ \left[\begin{array}{ll} \text{hence} & \\ 1 & \text{for } i = j \end{array}\right] & \\ 0 & \text{for } i < j \end{cases}$$

then (4.12), (4.15) become

$$(4.18) \qquad B'\xi = \zeta,$$

$$(4.19) \qquad C\zeta] = \eta.$$

Since these are identities with respect to the original variables $x_1, \cdots, x_n$, that is, with respect to ξ, therefore comparison of (4.18), (4.19) with (4.1′) gives

$$(4.20) \qquad A = CB'.$$

From (4.20)

$$(4.21) \qquad A^{-1} = B'^{-1}C^{-1},$$

and (4.16), (4.17) show that B', C are semi-diagonal (upper and lower, that is, in $\mathcal{C}_+$ and $\mathcal{C}_-$ respectively). Furthermore, (4.7), (4.13), which represent the conventional way of expressing the elimination method, are clearly the inductive processes that invert (4.15), (4.12), that is, (4.19), (4.18), that is, they invert the matrices C, B'. C, B' are semi-diagonal, and renewed inspection of (4.7), (4.13) shows at once that these are indeed the inductive processes that are required to invert semi-diagonal matrices. (In this connection cf. the remark at the end of §3.4, and the explicit expressions (4.29), (4.30).)

We may therefore interpret the elimination method as one which bases the inverting of an arbitrary matrix A on the combination of two tricks: First, it decomposes A into a product of two semi-diagonal matrices C, B', according to (4.20), and consequently the inverse of

A obtains immediately from those of C, B', according to (4.21).[17] Second, using the semi-diagonality of C, B', it forms their inverses by a simple, explicit, inductive process.

4.4. **Replacement of C, B' by B, C, D.** The discussion of §4.3 is complete, but it suffers from a certain asymmetry: B', C play quite symmetric roles, being upper and lower semi-diagonal, and the right- and left-factors of the decomposition (4.20) of A; however, all diagonal elements of C are identically 1, whereas those of B' are not.

This is easily remedied: Put

$$\begin{aligned} &D = (d_i\delta_{ij}) && (i, j = 1, \cdots, n) \\ \text{with } &d_i = \overset{(i)}{a_{ii}} \end{aligned} \tag{4.22}$$

$$\begin{aligned} &B = (b_{ij}) && (i, j = 1, \cdots, n) \\ \text{with } &b_{ij} = \begin{cases} \overset{(i)}{a_{ij}}/\overset{(i)}{a_{ii}} & \text{for } i \leqq j, \\ \left[\begin{matrix} \text{hence} & \\ 1 & \text{for } i = j, \end{matrix}\right] \\ 0 & \text{for } i > j. \end{cases} \end{aligned} \tag{4.23}$$

Then clearly

$$B' = DB, \tag{4.24}$$

hence (4.20) becomes

$$A = CDB, \tag{4.25}$$

and (4.21) becomes

$$A^{-1} = B^{-1}D^{-1}C^{-1}. \tag{4.26}$$

To sum up:

(4.27) B, C, D fulfill (4.25). They belong to $\mathcal{C}_+$, $\mathcal{C}_-$, $\mathcal{C}_0$, respectively (cf. 3.4). All diagonal elements of B, C are identically 1.

Now (4.26) furnishes the desired A^{-1}, based on B^{-1}, C^{-1}, D^{-1}. D^{-1} is immediately given by

$$D^{-1} = (d_i^{-1}\delta_{ij}) \qquad (i, j = 1, \cdots, n), \tag{4.28}$$

and B^{-1}, C^{-1} obtain from simple, explicit, inductive algorithms which involve no divisions:

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[17] C, B' could not both belong to the same class $\mathcal{C}_\pm$, since each class $\mathcal{C}_\pm$ is reproduced by multiplication (cf. 3.26), and A is, of course, not assumed to belong to either (to be semi-diagonal). Indeed, C is in $\mathcal{C}_-$ and B' in $\mathcal{C}_+$.

$$B^{-1} = R = (r_{ij}) \qquad (i, j = 1, \cdots, n)$$

$$\text{(4.29)} \qquad \text{with } r_{ij} = \begin{cases} -\sum_{k=i+1}^{j} b_{ik} r_{kj} & \text{for } i < j, \\ 1 & \text{for } i = j, \\ 0 & \text{for } i > j, \end{cases}$$

$$C^{-1} = S = (s_{ij}) \qquad (i, j = 1, \cdots, n)$$

$$\text{(4.30)} \qquad \text{with } s_{ij} = \begin{cases} -\sum_{j+1}^{i} s_{ik} c_{kj} & \text{for } i > j, \\ 1 & \text{for } i = j, \\ 0 & \text{for } i < j. \end{cases}$$

Note that (4.29) obtains from $BR = I$, and gives for every fixed j $(=1, \cdots, n)$ an inductive definition over $i = j, \cdots, 1$; while (4.30) obtains from $SC = I$, and gives for every fixed i $(=1, \cdots, n)$ an inductive definition over $j = i, \cdots, 1$.

4.5. **Reconsideration of the decomposition theorem. The uniqueness theorem.** The decisive relation (4.24) or (4.25) can also be derived directly from (4.6).

Indeed, consider two fixed $i, j = 1, \cdots, n$. Put $i' = \text{Min}\ (i, j)$. Form (4.6) for $k = 1, \cdots, i' - 1$, and note that

$$\text{(4.6')} \qquad 0 = a_{ij}^{(k)} - a_{ik}^{(k)} a_{kj}^{(k)} / a_{kk}^{(k)}$$

for $k = i$ and for $k = j$, that is, for $k = i'$. Summing all these equations, and remembering (4.8), gives

$$\text{(4.31)} \qquad a_{ij} = \sum_{k=1}^{i'} \frac{a_{ik}^{(k)} a_{kj}^{(k)}}{a_{kk}^{(k)}}.$$

By (4.17), (4.22), (4.23) this may be written

$$\text{(4.32)} \qquad a_{ij} = \sum_{k=1}^{i'} c_{ik} d_k b_{kj} = \sum_{k=1}^{n} c_{ik} d_k b_{kj},$$

and this is precisely the statement of (4.25).

We give this alternative derivation of (4.25), because it is ex post more direct than the original one (in §§4.3, 4.4), and because our final discussion for pseudo-operations will have to follow this pattern (cf. §§5.2 and 6.1, especially (6.3)).

To conclude, we show:

(4.33) Given A, (4.25) and (4.27) determine B, C, D uniquely.

Let $A = CDB$ and $A = C_1D_1B_1$ be two decompositions that fulfill (4.27). A is nonsingular, hence the same is true for B, C, D, B_1, C_1, D_1. $CDB = C_1D_1B_1$, hence $C_1^{-1}C = D_1B_1B^{-1}D^{-1}$. Now C, C_1 belong to $\mathcal{C}_-$, hence $C_1^{-1}C$ belongs to $\mathcal{C}_-$. B, B_1 belong to $\mathcal{C}_+$, D, D_1 belong to $\mathcal{C}_0$, hence also to $\mathcal{C}_+$, so $D_1B_1B^{-1}D^{-1}$ belongs to $\mathcal{C}_+$. (For this, and what follows, cf. §3.4.) Thus $C_1^{-1}C$ belongs to $\mathcal{C}_-$ and to $\mathcal{C}_+$, hence it belongs to $\mathcal{C}_0$, that is, it is diagonal. C, C_1 are in $\mathcal{C}_-$ and have diagonal elements 1, therefore the same is true for $C_1^{-1}C$. Owing to the above this means that $C_1^{-1}C = I$, that is, $C = C_1$. Similarly (or by interchanging rows and columns) $B = B_1$. Now $CDB = C_1D_1B_1$ gives $D = D_1$. Hence B, C, D coincide with B_1, C_1, D_1, as desired.

Note that all these results were formulated and derived without the assumption of positioning for size.

CHAPTER V. SPECIALIZATION TO DEFINITE MATRICES

5.1. Reasons for limiting the discussion to definite matrices. We have not so far been able to obtain satisfactory error estimates for the pseudo-operational equivalent of the elimination method in its general form, that is, for the equivalent of §§4.3–4.5. The reason for this is that any such estimate would have to depend on the bounds of some or of all of the matrices B, B^{-1}, C, C^{-1}, D, D^{-1}, that is, on $|B|$, $|B|_l$, $|C|$, $|C|_l$, $|D|$, $|D|_l$ (cf. (3.5. d)). It would be necessary to correlate these quantities, or possibly other, allied ones, to $|A|$, $|A|_l$.[28] As stated above, we have not so far been able to derive such correlations to any adequate extent.[29]

We did, however, succeed in securing everything that is needed in the special case of a definite A. Furthermore, the inverting of an unrestricted (but, of course, nonsingular) A is easily derivable from the inverting of a definite one: Indeed, by (3.20), A^*A is always definite and, by the considerations that preceded (3.12.b), A^{-1} exists if and only if $(A^*A)^{-1}$ exists and then $A^{-1} = (A^*A)^{-1}A^*$.

For these reasons, which may not be absolutely and permanently valid ones, we shall restrict the direct application of the elimination method, or rather of its pseudo-operational equivalent, to definite matrices A.

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[28] Cf. the corresponding results in §5.4, where the efforts in this direction prove successful.

[29] Such correlations would probably also have to depend on the positioning for size, in the sense of §4.2. Cf. also the discussion following (5.7).

In addition various important categories of matrices are per se definite, for example, all correlation matrices.

The considerations of this chapter will still take place in terms of true, and not of pseudo-operations. They do, however, set the pattern for the subsequent pseudo-operatorial discussion of Chapter VI.

5.2. **Properties of our algorithm (that is, of the elimination method) for a symmetric matrix A. Need to consider positioning for size as well.** We shall show that if A is (nonsingular and) definite, then all matrices

$$A^{(k)} = (a_{ij}^{(k)}) \qquad (k = 1, \cdots, n;\ i, j = k, \cdots, n) \tag{5.1}$$

are also definite. Let us, however, consider first the connection between A and the $A^{(k)}$ with respect to the weaker property of symmetry.

We continue without positioning for size for a short while yet.

It is clear from (4.6) that $a_{ij}^{(k)} = a_{ji}^{(k)}$ (for all $i, j = k, \cdots, n$) implies $a_{ij}^{(k+1)} = a_{ji}^{(k+1)}$ (for all $i, j = k+1, \cdots, n$), that is, that the symmetry of $A^{(k)}$ implies that of $A^{(k+1)}$ $(k = 1, \cdots, n-1)$. If we begin with $A = A^{(1)}$, then we have:

(5.2) If A is symmetric, then all $A^{(k)}$ $(k = 1, \cdots, n)$ are.

From this we can infer:

(5.3) The symmetry of A is equivalent to having $C = B^*$ in (4.25), that is, to (4.25) assuming the form $A = B^*DB$.

Indeed: If A is symmetric, then by (5.2) always $a_{ij}^{(k)} = a_{ji}^{(k)}$, hence, by (4.23), (4.17), $c_{ij} = b_{ji}$, that is, $C = B^*$. Conversely, $C = B^*$, that is, $A = B^*DB$ implies $A^* = B^*D^*B^{**} = B^*DB = A$.

Let us now introduce positioning for size. This can disrupt the validity of (5.2), (5.3) above. Indeed: If for any k $(= 1, \cdots, n-1)$ $\operatorname{Max}_{i,j=k,\cdots,n} |a_{ij}^{(k)}|$ is assumed for no pair i, j with $i = j$, then the required permutations of $i = k, \cdots, n$ and of $j = k, \cdots, n$ are unavoidably different (cf. §4.2). Hence these permutations will disrupt the symmetry of $A^{(k)}$ inasmuch as it determines $A^{(k+1)}$ by (4.6), and therefore they will a fortiori disrupt the symmetry of $A^{(k+1)}$. Thus (5.2) fails, and consequently (5.3) fails, too.

The behavior of the $a_{ij}^{(k)}$, that is, of $A^{(k)}$, to which we refer, is perfectly possible. Clearly $A = A^{(1)}$ itself may be like this.

This discussion shows that it is unsafe to postpone the consideration of the problems of positioning for size any further. We shall therefore face them from now on, and we assume accordingly that positioning for size does take place in the cases which follow.

5.3. **Properties of our algorithm for a definite matrix A.** Consider now the property of definiteness, that is, the case of a (nonsingular and) definite A. It is best to derive a number of intermediate propositions in succession.

(5.4) Let $M=(m_{ij})$ be a definite matrix. Then we have:

(a) Always $m_{ii} \geqq 0$.

(b) Always $m_{ii}m_{jj} \geqq m_{ij}^2$.

(c) $m_{ii}=0$ implies $m_{ij}=0$ for all j.

(d) If $\mathrm{Max}\, m_{ii}$ is assumed for $i=h$, then $m_{hh} \geqq |m_{ij}|$ for all i, j.

Indeed: The diagonal minors of M are definite along with M, and hence their determinants are greater than or equal to 0. Applying this to the first and second order minors gives (a), (b), respectively. (c), (d) are immediate consequences of (b) with (a).

(5.5) If an $A^{(k)}=(a_{ij}^{(k)})$ is definite, and if $\mathrm{Max}_{i=k,\cdots,n}\, a_{ii}^{(k)}$ is assumed for $i=h$, then $\mathrm{Max}_{i,j=k,\cdots,n} |a_{ij}^{(k)}|$ is assumed for $i=j=h$.[30] We can therefore choose the same permutation for $i=k, \cdots, n$ and for $j=k, \cdots, n$ when we comply with the requirements of positioning for size. We propose to do this in all cases where it is possible. Hence if $A^{(k)}$ is definite, the positioning for size will not disrupt its symmetry.

The first assertion follows from (5.4.d), all other assertions are immediate consequences of the first one.

(5.6) If $A=A^{(1)}$ is (nonsingular and) definite, then the same is true for $A^{(2)}$, and

$$0 < |A|_l = |A^{(1)}|_l \leqq |A^{(2)}|_l \leqq |A^{(2)}| \leqq |A^{(1)}| = |A|.$$

Because of (3.5) the nonsingularity of A implies the assertion $0<|A|_l$ and conversely the assertion $0<|A^{(2)}|_l$ implies the equally asserted nonsingularity of $A^{(2)}$. Of the remaining relations (equalities and inequalities) only

(a) $$|A^{(2)}| \leqq |A^{(1)}|$$

and

(b) $$|A^{(2)}|_l \geqq |A^{(1)}|_l$$

---

[30] Note that we do not claim that $\mathrm{Max}_{i=k,\cdots,n}\, a_{ii}^{(k)}$ need be assumed for one $i=h$ only, nor that $\mathrm{Max}_{i,j=k,\cdots,n} |a_{ij}^{(k)}|$ may not also be assumed for pairs i, j with $i \neq j$.

require proof.

Use for the moment the definiteness of $A^{(1)}$, $A^{(2)}$. Then by (3.22.a) and (3.22.b) $|A^{(k)}| \leqq \lambda'$ or $|A^{(k)}|_l \geqq \mu'$ ($k=1, 2$) is equivalent to having all proper values of $A^{(k)} \leqq \lambda'$ or $\geqq \mu'$, respectively; that is, all proper values of $\lambda' \cdot I - A^{(k)}$ or $A^{(k)} - \mu' \cdot I$, respectively, $\geqq 0$; that is, by (3.21.a) to the definiteness of $\lambda' \cdot I - A^{(k)}$ or $A^{(k)} - \mu' \cdot I$, respectively. These, in turn, may be written $(A^{(k)}\xi, \xi) \leqq \lambda' |\xi|^2$ and $(A^{(k)}\xi, \xi) \geqq \mu' |\xi|^2$, respectively. So we see:

(a′) $|A^{(k)}| \leqq \lambda'$ is equivalent to $(A\xi, \xi) \leqq \lambda' |\xi|^2$ for all ξ,

(b′) $|A^{(k)}|_l \geqq \mu'$ is equivalent to $(A\xi, \xi) \geqq \mu' |\xi|^2$ for all ξ.

Put $\lambda' = |A^{(1)}|$, $\mu' = |A^{(1)}|_l$. Then (a′), (b′) with $k=1$ show that

(c) $\lambda' |\xi|^2 \geqq (A^{(1)}\xi, \xi) \geqq \mu' |\xi|^2$ for all ξ,

and (a′), (b′) with $k=2$ show that (a), (b) are equivalent to

(d) $\lambda' |\xi|^2 \geqq (A^{(2)}\xi, \xi) \geqq \mu' |\xi|^2$ for all ξ.

Note that the ξ of (c) are n-dimensional vectors: $\xi = (x_1, \cdots, x_n)$, while the ξ of (d) are $n-1$-dimensional vectors: $\xi = (x_2, \cdots, x_n)$.

Since $A^{(1)} = A$ is definite, it follows that we need to prove only two things: The definiteness of $A^{(2)}$ and (d).

$A^{(1)} = A$ is definite, hence symmetric, hence by (5.5) the positioning for size subjects i and j to the same permutation. Consequently the form of (4.6) is unaffected, and $A^{(1)}$ as well as $A^{(2)}$ remain symmetric. Therefore the definiteness of $A^{(2)}$ is secure if $(A^{(2)}\xi, \xi) \geqq 0$. This, however, follows from (d).

Thus we need to prove (d) only.

Put, with $\xi = (x_2, \cdots, x_n)$,

$$x_i' = \begin{cases} 0 \text{ for } i = 1, \\ x_i \text{ for } i = 2, \cdots, n, \end{cases} \qquad \xi' = (x_1', x_2', \cdots, x_n'),$$

$$x_i'' = \begin{cases} -\sum_{j=2}^{n} \dfrac{a_{1j}^{(1)}}{a_{11}^{(1)}} x_j \text{ for } i = 1, \\ x_i \text{ for } i = 2, \cdots, n, \end{cases} \qquad \xi'' = (x_1'', x_2'', \cdots, x_n'').$$

A simple calculation based on (4.6) gives

$$\sum_{i,j=2}^{n} a_{ij}^{(2)} x_i x_j = \begin{cases} \sum_{i,j=1}^{n} a_{ij}^{(1)} x_i' x_j' - a_{11}^{(1)} (x_1'')^2 \leqq \sum_{i,j=1}^{n} a_{ij}^{(1)} x_i' x_j', \\ \sum_{i,j=2}^{n} a_{ij}^{(1)} x_i'' x_j'' \end{cases}$$

that is,

$$(A^{(2)}\xi, \xi) \begin{cases} \leqq (A^{(1)}\xi', \xi'), \\ = (A^{(1)}\xi'', \xi''). \end{cases}$$

Using (c), the first relation gives

$$(A^{(2)}\xi, \xi) \leqq (A^{(1)}\xi', \xi') \leqq \lambda' |\xi'|^2 = \lambda' |\xi|^2,$$

and the second relation gives

$$(A^{(2)}\xi, \xi) = (A^{(1)}\xi'', \xi'') \geqq \mu' |\xi''|^2 \geqq \mu' |\xi|^2,$$

establishing together (d), as desired.

(5.6′) If A is (nonsingular and) definite then the same is true for all $A^{(k)} (k = 1, \cdots, n)$, and

$$\begin{aligned} 0 < |A|_l = |A^{(1)}|_l \leqq |A^{(2)}|_l \leqq \cdots \leqq |A^{(n)}|_l \leqq |A^{(n)}| \leqq \cdots \\ \leqq |A^{(2)}| \leqq |A^{(1)}| = |A|. \end{aligned}$$

This is immediate, by applying (5.6) to $A = A^{(1)}$ and to $A^{(2)}, \cdots, A^{(n-1)}$ in succession, in place of A.

We interrupt at this point our chain of deductions, in order to make a subsidiary observation.

(5.7) If A is (nonsingular and) definite, then all $a_{ii}^{(k)} > 0$.

Assume that some $a_{ii}^{(k)}$ is not greater than 0. $A^{(k)}$ is definite, hence by (5.4.a) this $a_{ii}^{(k)}$ is 0, and by (5.4.c) $a_{ij}^{(k)} = 0$ for this (k and) i and for all $j = k, \cdots, n$. Hence $A^{(k)}$ is singular, in contradiction with (5.6′).

(5.7) shows that for a (nonsingular and) definite A the elimination method could have been carried out without positioning for size in the sense of 4.2, since all $a_{ii}^{(k)} > 0$ automatically, that is, just in that case where positioning for size creates no difficulties (cf. the remarks at the end of 5.2), it seems to be superfluous.

This, however, is not the complete truth. A (nonsingular and) definite A could indeed be put through the algorithm of 4.1 in the rigorous sense, without positioning for size. However, if pseudo-operations are used, no satisfactory estimates seem to be obtainable, unless positioning for size is also effected. This will become apparent in several instances in Chapter VI, primarily inasmuch as the estimate (6.8), which is identical with (6.23.d′), depends directly on the positioning for size, and this (6.8) is the basis for the decisive estimates (6.12). (6.25). This is our true reason for insisting on positioning for size in the situation that we are going to discuss.

We return now to the main line of our deductions.

(5.8) If A is (nonsingular and) definite and all its elements lie in $-1, 1$, then all $A^{(k)}$ $(k=1, \cdots, n)$ are also definite and all their elements also lie in $-1, 1$.

The matter of definiteness was settled in (5.6′). By (5.4) all $a_{ii}^{(k)} \geqq 0$, by (4.6) $a_{ii}^{(k+1)} = a_{ii}^{(k)} - (a_{ik}^{(k)})^2 / a_{kk}^{(k)} \leqq a_{ii}^{(k)}$, hence $a_{ii}^{(k)} \leqq a_{ii}^{(k-1)} \leqq \cdots \leqq a_{ii}^{(1)} = a_{ii} \leqq 1$. Thus $0 \leqq a_{ii}^{(k)} \leqq 1$. Now (5.4.b) gives $|a_{ij}^{(k)}| \leqq 1$, that is, all $a_{ij}^{(k)}$ lie in $-1, 1$ as desired.

(5.9) Under the same assumptions as in (5.8) we have further:

(a) For all elements d_i of D, $0 < d_i \leqq 1$.

(b) All elements of B lie in $-1, 1$.

Proof of (a): By (4.22) $d_i = a_{ii}^{(i)}$, hence by (5.8) d_i lies in $-1, 1$, and by (4.14) it is not equal to 0. Furthermore, d_i is greater than or equal to 0 by (5.4.a). All these give together $0 < d_i \leqq 1$, as desired.

Proof of (b): Since the positioning for size has taken place, we have $|a_{kk}^{(k)}| \geqq |a_{ij}^{(k)}|$ (for all $i, j = k, \cdots, n$). Hence (4.23) guarantees $|b_{ij}| \leqq 1$ (for all $i, j = k, \cdots, n$), that is all b_{ij} lie in $-1, 1$, as desired.

5.4. **Detailed matrix bound estimates, based on the results of the preceding section.** For the balance of this chapter we assume A to be (nonsingular and) definite, and positioning for size to have taken place in the sense of (5.5). This implies (5.6′), (5.7), and hence (5.3), too. We may therefore restate (4.27) (together with (4.25)) as follows:

(5.10) B, D fulfill

$$A = B^* D B.$$

They belong to $\mathcal{C}_+$, $\mathcal{C}_0$, respectively. All diagonal elements of B are identically 1.

We now proceed to derive estimates for $|B|$, $|B|_l$, $|D|$, $|D|_l$, and some other, allied quantities, in terms of $|A|$, $|A|_l$, in the sense of §5.1.

Let $\lambda_1, \cdots, \lambda_n$ be the proper values of A, ordered in a monotone non-increasing sequence, cf. (3.21.b). We recall (3.22.a), (3.22.b) and define λ, μ:

$$(5.11.a) \qquad |A| = \lambda = \lambda_1,$$

$$(5.11.b) \qquad |A|_l = \mu = \lambda_n.$$

From (4.22)

$$D^{v} = (d_i^{v}\delta_{ij}) \qquad (i, j = 1, \cdots, n) \tag{5.12}$$

for any exponent v; we shall use this for $v = \pm 1, \pm 1/2$.

Now $(D^{1/2}B)^{*}\cdot D^{1/2}B = B^{*}D^{1/2}\cdot D^{1/2}B = B^{*}DB = A$. Hence (3.12.a), (3.12.b) permit us to infer from (5.11.a), (5.11.b)

$$|D^{1/2}B| = \lambda^{1/2}, \tag{5.13.a}$$

$$|D^{1/2}B|_l = \mu^{1/2}. \tag{5.13.b}$$

$D^{1/2}B$ is in $\mathcal{C}_+$ and its diagonal elements are those of $D^{1/2}$, the $d_i^{1/2}$. Consequently $(D^{1/2}B)^{-1}$ is also in $\mathcal{C}_+$ and its diagonal elements are the $d_i^{-1/2}$. (Cf. (3.26).) Hence by (3.17.b) (first half) $|d_i^{1/2}| \leqq |D^{1/2}B| = \lambda^{1/2}$, $|d_i^{-1/2}| \leqq |(D^{1/2}B)^{-1}| = |D^{1/2}B|_l^{-1} = \mu^{-1/2}$. By (5.9.a), $0 < d_i \leqq 1$. Hence

$$\mu \leqq d_i \leqq \lambda \text{ and } 1. \tag{5.14}$$

Now (5.12) and (3.27) give

$$|D^{v}| \left\{\begin{array}{lll} \leqq \lambda^{v} & \text{and } 1 & \text{for } v \geqq 0 \\ \geqq \mu^{v} & & \text{for } v \leqq 0 \end{array}\right| , \tag{5.15.a}$$

$$|D^{v}|_l \left\{\begin{array}{lll} \leqq \mu^{v} & & \text{for } v \geqq 0 \\ \geqq \lambda^{v} & \text{and } 1 & \text{for } v \leqq 0 \end{array}\right| . \tag{5.15.b}$$

Combining (5.13.a), (5.13.b) with (5.15.a), (5.15.b), $v = \pm 1/2$, gives

$$|DB| \leqq \lambda \text{ and } \lambda^{1/2}, \tag{5.16.a}$$

$$|DB|_l \geqq \mu^{1/2}, \tag{5.16.b}$$

$$|B| \leqq (\lambda/\mu)^{1/2}, \tag{5.17.a}$$

$$|B|_l \geqq (\mu/\lambda)^{1/2} \text{ and } \mu^{1/2}. \tag{5.17.b}$$

The estimates (5.13.a)–(5.13.b) justify this conclusion: The primary estimates, on which all others are based, are those concerning $D^{1/2}B$, that is, (5.13.a), (5.13.b). These are consequently the sharpest ones, as can also be inferred from the fact that they alone are equalities, all others being inequalities. Hence $D^{1/2}B$ is the truly fundamental quantity in preference to B, DB, and even to D.

Now the method of inversion discussed in §4.3 is based on $B' = DB$ and on C, which is now equal to B^{*}. In §4.4 (specifically: (4.25), (4.26)) we used B, C (which is now equal to B^{*}) and D. It follows from the above that, if we use these matrices, the methods of estimating should nevertheless emphasize $D^{1/2}B$. It will become apparent in several places throughout §6.6 and in parts of §6.8 how we endeavor to follow this principle.

CHAPTER VI. THE PSEUDO-OPERATIONAL PROCEDURE

6.1. Choice of the appropriate pseudo-procedures, by which the true elimination will be imitated. After the preparations in the foregoing chapters we can now attack our main problem: The pseudo-operational matrix inversion by means of the elimination method, the latter being reinterpreted, modified and specialized in the sense of chapters IV, V.

We consider accordingly a digital matrix $\bar{A}=(\bar{a}_{ij})$ $(i,j=1,\cdots,n)$ (cf. §3.5), of which we assume that it is nonsingular and definite.

We have to begin by performing the pseudo-operational equivalent of the manipulations of §4.1 on $\bar{A}$. This means that we define a sequence of digital matrices $\bar{A}^{(k)}=(\bar{a}_{ij}^{(k)})$ $(i,\ j=k,\cdots,n)$ for $k=1,\cdots,n$. The induction begins for $k=1$ with $\bar{A}^{(1)}=\bar{A}$, that is,

$$\bar{a}_{ij}^{(1)}=\bar{a}_{ij}\qquad (i,j=1,\cdots,n), \tag{6.1}$$

following (4.8). The inductive step from k to $k+1$ $(k=1,\cdots,n-1)$ has to follow (4.6). This creates a new problem: How are the true operations of the expression $a_{ik}^{(k)}a_{kj}^{(k)}/a_{kk}^{(k)}$ in (4.6) to be replaced by pseudo-operations?

There are obviously several ways of doing this. The simplest ones, which are most economical in the number of operations to be performed, are these:

$$(\bar{a}_{ik}^{(k)}\times\bar{a}_{kj}^{(k)})\div\bar{a}_{kk}^{(k)}, \tag{6.2.a}$$

$$\bar{a}_{ik}^{(k)}\times(\bar{a}_{kj}^{(k)}\div\bar{a}_{kk}^{(k)}), \tag{6.2.b}$$

$$(\bar{a}_{ik}^{(k)}\div\bar{a}_{kk}^{(k)})\times\bar{a}_{kj}^{(k)}. \tag{6.2.c}$$

The build of (6.2.b) and (6.2.c) is so similar that it suffices to discuss (6.2.a) and (6.2.b). Comparing these with $\bar{a}_{ik}^{(k)}\bar{a}_{kj}^{(k)}/\bar{a}_{kk}^{(k)}$ which they are supposed to approximate, we obtain

$$\begin{aligned}&\left|\,\bar{a}_{ik}^{(k)}\bar{a}_{kj}^{(k)}/\bar{a}_{kk}^{(k)}-(\bar{a}_{ik}^{(k)}\times\bar{a}_{jk}^{(k)})\div\bar{a}_{kk}^{(k)}\,\right|\\&\qquad\qquad\qquad\leqq(1+|\,\bar{a}_{kk}^{(k)}\,|^{-1})\beta^{-s}/2,\end{aligned} \tag{6.3.a}$$

$$\begin{aligned}&\left|\,\bar{a}_{ik}^{(k)}\bar{a}_{kj}^{(k)}/\bar{a}_{kk}^{(k)}-\bar{a}_{ik}^{(k)}\times(\bar{a}_{kj}^{(k)}\div\bar{a}_{kk}^{(k)})\,\right|\\&\qquad\qquad\qquad\leqq(1+|\,\bar{a}_{kk}^{(k)}\,|)\beta^{-s}/2.\end{aligned} \tag{6.3.b}$$

(Cf. these estimates with the very similar ones which were derived and discussed at the end of §2.4: (2.14), (2.15).) Clearly the estimate (6.3.a) is considerably less favorable than (6.3.b) (by a factor

$|\bar{a}_{kk}^{(k)}|^{-1}$) and altogether unsatisfactory per se: It involves the terms $|\bar{a}_{kk}^{(k)}|^{-1}$ which may be arbitrarily and unpredictably large. We reject, therefore, (6.2.a). (6.2.b), on the other hand, has this flaw: If, in anticipation of the symmetry of $\bar{a}_{ij}^{(k)}$, we write it in the form

$$\bar{a}_{ki}^{(k)} \times (\bar{a}_{kj}^{(k)} \div \bar{a}_{kk}^{(k)}), \tag{6.2.b$'$}$$

then the pseudo-character of its operations prevents it from being symmetric in i, j. We overcome this difficulty by using (6.2.b′) only when $i \leqq j$, and interchanging i, j when $i > j$. That is, we use this expression:

$$\begin{gathered} \bar{a}_{ki'}^{(k)} \times (\bar{a}_{kj'}^{(k)} \div \bar{a}_{kk}^{(k)}), \\ \text{where } i' = \mathrm{Min}\,(i, j),\ j' = \mathrm{Max}\,(i, j). \end{gathered} \tag{6.2.b$''$}$$

We formulate therefore the inductive step by replacing $\bar{a}_{ik}^{(k)}\bar{a}_{kj}^{(k)}/\bar{a}_{kk}^{(k)}$ in (4.6) by (6.2.b″). The inductive step consequently assumes this form:

$$\begin{gathered} \bar{a}_{ij}^{(k+1)} = \bar{a}_{ij}^{(k)} - \bar{a}_{ki'}^{(k)} \times (\bar{a}_{kj'}^{(k)} \div \bar{a}_{kk}^{(k)}), \\ \text{where } i' = \mathrm{Min}\,(i, j),\ j' = \mathrm{Max}\,(i, j) \quad (i, j = k+1, \cdots, n) \end{gathered} \tag{6.3}$$

for $k = 1, \cdots, n-1$.

We also assume that positioning for size takes place, but we anticipate the equivalent of (5.5) by permitting only one joint permutation for i and j. We define accordingly:

(6.4) Before (6.3) is effected, it must be made certain by an appropriate joint permutation of $i, j = k, \cdots, n$ that $\mathrm{Max}_{i=k,\cdots,n}\, \bar{a}_{ii}^{(k)}$ is assumed for $i = k$.

Thus (6.1) and (6.3), (6.4) define the $\bar{a}_{ij}^{(k)}$ $(k = 1, \cdots, n;\ i, j = k, \cdots, n)$. We also define the digital matrices

$$\bar{A}^{(k)} = (\bar{a}_{ij}^{(k)}) \qquad (k = 1, \cdots, n;\ i, j = k, \cdots, n). \tag{6.5}$$

It remains to show that all these definitions are indeed possible, that is, that all numbers produced by (6.1), (6.3), (6.4) (including the intermediate expressions $\bar{a}_{kj}^{(k)} \div \bar{a}_{kk}^{(k)}$ and $\bar{a}_{ki}^{(k)} \times (\bar{a}_{kj}^{(k)} \div \bar{a}_{kk}^{(k)})$ in (6.3)), and designated as digital numbers, are properly formed and, in particular, lie in $-1, 1$. We shall prove this in 6.2 below. The inductive proof which establishes this will also secure the equivalents of (5.5)–(5.7).

6.2. **Properties of the pseudo-algorithm.** Assume that for a $k = 1, \cdots, n$ this is true:

(6.6) (a) It is possible to form $\bar{A}^{(1)}, \cdots, \bar{A}^{(k)}$, obtaining properly formed, digital numbers throughout.

(b) $\bar{A}^{(k)}$ is (nonsingular and) definite.

(c) All $\bar{a}_{ii}^{(k)} \leqq 1$.

By (5.4.a) we have $\bar{a}_{ii}^{(k)} \geqq 0$. From $\bar{a}_{ii}^{(k)} = 0$ we could infer, as in the proof of (5.7), that $\bar{A}^{(k)}$ is singular. Hence

$$0 < \bar{a}_{ii}^{(k)} \leqq 1. \tag{6.7}$$

Next by (5.4.d):

$$\left| \bar{a}_{ij}^{(k)} \right| \leqq \bar{a}_{kk}^{(k)}. \tag{6.8}$$

(6.7), (6.8) give together:

$$\text{All } \bar{a}_{ij}^{(k)} \text{ lie in } -1, 1. \tag{6.9}$$

Now assume $k \neq n$, so that the validity of (6.6) for $k+1$ can be considered. (6.7)–(6.9) show that $\bar{A}^{(k+1)}$ can be formed, obtaining properly formed digital numbers throughout, with the one limitation, that it remains to be proved that the elements $\bar{a}_{ij}^{(k+1)}$ lie in $-1, 1$. Assume, furthermore, that

$$\left| \bar{A}^{(k)} \right|_l > (n-k)\beta^{-s}. \tag{6.10}$$

Since $\bar{A}^{(k)}$ is definite, it is also symmetric. (6.4) does not affect the form of (6.3), and $\bar{A}^{(k)}$ as well as $\bar{A}^{(k+1)}$ remain symmetric.

We may rewrite (6.3) as

$$\bar{a}_{ij}^{(k+1)} = \bar{a}_{ij}^{(k)} - \bar{a}_{ki'}^{(k)}\bar{a}_{kj'}^{(k)}/\bar{a}_{kk}^{(k)} + \eta_{ij}^{(k)}$$
$$(i, j = k+1, \cdots, n;\ i' = \operatorname{Min}(i, j),\ j' = \operatorname{Max}(i, j)),$$

that is, as

$$\bar{a}_{ij}^{(k+1)} = \bar{a}_{ij}^{(k)} - \bar{a}_{ik}^{(k)}\bar{a}_{kj}^{(k)}/\bar{a}_{kk}^{(k)} + \eta_{ij}^{(k)} \qquad (i, j = k+1, \cdots, n), \tag{6.11}$$

where by (6.8), (6.9)

$$\left| \eta_{ij}^{(k)} \right| \leqq \beta^{-s}. \tag{6.12}$$

Put

$$\bar{A}^{(k+1)} = \left(\bar{a}_{ij}^{(k)} - \bar{a}_{ik}^{(k)}\bar{a}_{kj}^{(k)}/\bar{a}_{kk}^{(k)}\right) \qquad (i, j = k+1, \cdots, n), \tag{6.11.a}$$

$$H^{(k+1)} = \left(\eta_{ij}^{(k)}\right) \qquad (i, j = k+1, \cdots n), \tag{6.11.b}$$

then (6.11) becomes

$$\bar{A}^{(k+1)} = \tilde{A}^{(k+1)} + H^{(k+1)}. \tag{6.11'}$$

Now we may replace $A = A^{(1)}$, $A^{(2)}$ in (5.6) by $\bar{A}^{(k)}$, $\tilde{A}^{(k+1)}$; this gives

$$|\tilde{A}^{(k+1)}|_l \geqq |\bar{A}^{(k)}|_l,$$

and hence, by (d) in the proof of (5.6),

$$(\tilde{A}^{(k+1)}\xi, \xi) \geqq \mu^* |\xi|^2 \qquad \text{where } \mu^* = |\bar{A}^{(k)}|_l. \tag{6.13}$$

Next by (3.17.b) and (6.12) (note that we have here $n-k$ in place of n)

$$|H^{(k+1)}| \leqq (n-k)\beta^{-s},$$

and so

$$|(H^{(k+1)}\xi, \xi)| \leqq (n-k)\beta^{-s}|\xi|^2. \tag{6.14}$$

(6.11′) together with (6.13) and (6.14) gives

$$(\bar{A}^{(k+1)}\xi, \xi) \geqq (|\bar{A}^{(k)}|_l - (n-k)\beta^{-s})|\xi|^2. \tag{6.15}$$

By (6.10) this implies that

$$\bar{A}^{(k+1)} \text{ is definite.} \tag{6.16}$$

We can now argue, exactly as in the derivation of (b′) in the proof of (5.6), that (6.15) is equivalent to

$$|\bar{A}^{(k+1)}|_l \geqq |\bar{A}^{(k)}|_l - (n-k)\beta^{-s}. \tag{6.17}$$

Now (6.10) secures $|\bar{A}^{(k+1)}|_l > 0$, that is, by (3.5)

$$\bar{A}^{(k+1)} \text{ is nonsingular.} \tag{6.18}$$

For $i=j$ we have $i'=j'=i=j$, hence the two factors of the second term of the right-hand side in (6.3) have the same sign. Hence that subtrahend is greater than or equal to 0. (If two digital numbers have the same sign, then no round off rule will impair the nonnegativity of their pseudo-product and pseudo-quotient. Cf. footnote 6.) Consequently

$$\bar{a}_{ii}^{(k+1)} \leqq \bar{a}_{ii}^{(k)}. \tag{6.19}$$

Now we have established all parts of (6.6) for $k+1$: (a) follows from the remark immediately preceding (6.10) with the one limitation that it remains to be proved that the elements $\bar{a}_{ij}^{(k+1)}$ lie in -1, 1; (b) is contained in (6.18), (6.16); (c) is contained in (6.7), (6.19). On the basis of these we can now infer (6.7)–(6.9) for $k+1$, too, and hence we have the last part of (a): All $\bar{a}_{ij}^{(k+1)}$ lie in -1, 1. So we see:

(6.20) (6.6) for k and (6.10) imply (6.6) for $k+1$ and (6.17).

Consider now the condition

$$(6.21) \qquad |\bar{A}|_l > (n(n-1)/2)\beta^{-s}.$$

Then applying (6.20), (6.17) for all $k=1, \cdots, n-1$ in succession gives:

(6.22) If (6.6) holds for $k=1$, that is, for $A=A^{(1)}$, and if (6.21) holds, then (6.6) holds for all $k=1, \cdots, n$.

We restate this more explicitly, together with the inferences (6.7)–(6.9) from (6.6):

(6.23) Assume the following:
(a) $\bar{A}$ is (nonsingular and) definite.
(b) All $\bar{a}_{ii} \leqq 1$.
(c) $|\bar{A}|_l > (n(n-1)/2)\beta^{-s}$.
Then we have:
(a′) It is possible to form all $\bar{A}^{(k)}$, obtaining only properly formed, digital numbers throughout.
(b′) All $\bar{A}^{(k)}$ are (nonsingular and) definite.
(c′) Always $0 < \bar{a}_{ii}^{(k)} \leqq 1$.
(d′) Always $|\bar{a}_{ij}^{(k)}| \leqq \bar{a}_{ii}^{(k)}$.
(e′) All $\bar{a}_{ij}^{(k)}$ lie in $-1, 1$.

Note that the assumptions (a), (b) above are reasonable in view of the nature of our problem. (c) will be eliminated (or rather absorbed by a stronger condition) after (6.67).

With (6.23) the question at the end of §6.1 is completely answered: The processes (6.1), (6.3), (6.4) (including those of the intermediate expressions $\bar{a}_{ij}^{(k)} \div \bar{a}_{kk}^{(k)}$ and $\bar{a}_{kk}^{(k)} \times (\bar{a}_{ij}^{(k)} \div \bar{a}_{kk}^{(k)})$ in (6.3)) produce indeed digital numbers, which are properly formed, and, in particular, lie in $-1, 1$. Furthermore, we have the certainty, as we had it in the corresponding situation in (5.5)–(5.6), that all the $\bar{A}^{(k)} (k=1, \cdots, n)$ that we produce are (nonsingular and) definite.

6.3. **The approximate decomposition of $\bar{A}$, based on the pseudo-algorithm.** We now proceed to derive the approximate equivalent of (5.10) (that is, of (4.25)).

Rewrite (6.3) as

$$\bar{a}_{ij}^{(k+1)} = \bar{a}_{ij}^{(k)} - (\bar{a}_{ki'}^{(k)} \div \bar{a}_{kk}^{(k)})\bar{a}_{kk}^{(k)}(\bar{a}_{kj'}^{(k)} \div \bar{a}_{kk}^{(k)}) + \theta_{ij}^{(k)}$$

$$(i, j = k+1, \cdots, n;\ i' = \operatorname{Min}(i, j),\ j' = \operatorname{Max}(i, j)),$$

that is, as

$$\bar{a}_{ij}^{(k+1)} = \bar{a}_{ij}^{(k)} - (\bar{a}_{ki}^{(k)} + \bar{a}_{kk}^{(k)})\bar{a}_{kk}^{(k)}(\bar{a}_{kj}^{(k)} + \bar{a}_{kk}^{(k)}) + \theta_{ij}^{(k)} \qquad (k = 1, \cdots, i' - 1;\ i' = \operatorname{Min}(i, j)), \tag{6.24}$$

where by (6.23.d′), (6,23.e′)

$$|\theta_{ij}^{(k)}| \leqq \beta^{-s}. \tag{6.25}$$

(It should be noted that (6.24), (6.25) are analogous to (6.11), (6.12), but not the same.)

We observe next that

$$0 = \bar{a}_{ij}^{(k)} - (\bar{a}_{ki}^{(k)} + \bar{a}_{kk}^{(k)})\bar{a}_{kk}^{(k)}(\bar{a}_{kj}^{(k)} + \bar{a}_{kk}^{(k)}) + \theta_{ij}^{(k)} \tag{6.26}$$

for $k=i$ and for $k=j$, that is, for $k=i'$, with

$$|\theta_{ij}^{(k)}| \leqq \beta^{-s}/2. \tag{6.27}$$

Summing all these equations (that is, (6.24) for $k=1, \cdots, i'-1$ and (6.26) for $k=i'$), and remembering (6.1), gives

$$\bar{a}_{ij} = \sum_{k=1}^{i'} (\bar{a}_{ki}^{(k)} \div \bar{a}_{kk}^{(k)})\bar{a}_{kk}^{(k)}(\bar{a}_{kj}^{(k)} + \bar{a}_{kk}^{(k)}) + \zeta_{ij} \tag{6.28}$$

where $\zeta_{ij} = \sum_{k=1}^{i'} \theta_{ij}^{(k)}$, hence by (6.25), (6.27)

$$|\zeta_{ij}| \leqq (i' - 1/2)\beta^{-s}. \tag{6.29}$$

The relation (6.28) is the analog of (4.31), therefore we now wish to perform the analog of the transition from (4.31) to (4.32). For this purpose we define first, in analogy with (4.22), (4.23)

$$\bar{D} = (\bar{d}_i\delta_{ij}) \quad (i, j = 1, \cdots, n) \quad \text{with} \quad \bar{d}_i = \bar{a}_{ii}^{(i)} \tag{6.30}$$

$$\bar{B} = (\bar{b}_{ij}) \quad (i, j = 1, \cdots, n) \quad \text{with} \quad \bar{b}_{ij} = \begin{cases} \bar{a}_{ij}^{(i)} + \bar{a}_{ii}^{(i)} & \text{for } i \leqq j, \\ \left[\begin{matrix}\text{hence} \\ 1\end{matrix} \qquad \text{for } i = j,\right] & \\ 0 & \text{for } i > j. \end{cases} \tag{6.31}$$

Now

$$\sum_{k=1}^{i'} (\bar a_{ki}^{(k)} \div \bar a_{kk}^{(k)}) \bar a_{kk}^{(k)} (\bar a_{kj}^{(k)} \div \bar a_{kk}^{(k)}) \tag{6.32}$$
$$= \sum_{k=1}^{i'} \bar b_{ki} \bar d_k \bar b_{kj} = \sum_{k=1}^{n} \bar b_{ki} \bar d_k \bar b_{kj}.$$

We may, therefore, write for (6.28)

$$\bar A = \bar B^* \bar D \bar B + \mathrm{Z} \qquad \text{where } \mathrm{Z} = (\zeta_{ij}). \tag{6.33}$$

Next by (6.29)

$$(N(\mathrm{Z}))^2 = \sum_{i,j=1}^{n} (\zeta_{ij})^2 \leqq \sum_{i,j=1}^{n} \left(i' - \frac{1}{2}\right)^2 \beta^{-2s}$$
$$= \sum_{i'=1}^{n} (2n - 2i' + 1)\left(i' - \frac{1}{2}\right)^2 \beta^{-2s} = \frac{n^2(2n^2 + 1)}{12} \beta^{-2s},$$

hence by (3.17.a)

$$|\mathrm{Z}| \leqq N(\mathrm{Z}) \leqq (n^2(2n^2 + 1)/12)^{1/2} \beta^{-s}.$$

For $n \to \infty$, $(n^2(2n^2+1)/12)^{1/2} \sim n^2/6^{1/2} = .4084n^2$, and already, for $n \geqq 3$, $(n^2(2n^2+1)/12)^{1/2} \leqq .42n^2$. Hence (we assume from now on that $n \geqq 3$)

$$|\bar A - \bar B^* \bar D \bar B| \leqq .42 n^2 \beta^{-s}. \tag{6.34}$$

6.4. **The inverting of $\bar B$ and the necessary scale factors.** (6.34) is the approximate analog of (5.10). It is, therefore, the point of departure for inverting $\bar A$ in the sense of (5.10), that is, of (4.27), that is, of the formulae (4.25), (4.26). We proceed, therefore, in this direction.

In other words: $\bar A$ is approximated by $\bar B^* \bar D \bar B$, therefore $\bar A^{-1}$ will be approximated by $\bar B^{-1} \bar D^{-1} (\bar B^{-1})^*$, or, rather, by some approximant of this expression. In any case we need $\bar B^{-1}$ and $\bar D^{-1}$, or approximants of these. Hence, we must analogise the formulae (4.29) and (4.28), which gave B^{-1} and D^{-1} respectively.

We begin by considering $\bar B^{-1}$, that is, by analogising B^{-1} and (4.29). The obvious way of analogising (4.29) in terms of pseudo-operations would seem to be to define

$$\bar X = (\bar x_{ij}) \qquad (i, j = 1, \cdots, n).$$

$$\text{with } \bar x_{ij} = \begin{cases} -\sum_{k=i+1}^{j} \bar b_{ik} \times \bar x_{kj} & \text{for } i < j, \\ 1 & \text{for } i = j, \\ 0 & \text{for } i > j. \end{cases} \tag{6.35'}$$

This, however, is unfeasible; the use of the notation $\bar{x}_{ij}$ for the quantities obtained from (6.35′), which implies that they are digital numbers, is improper: These quantities will not lie in general in $-1, 1$, and in order to obtain an algorithm in terms of digital numbers it is now necessary to introduce scale factors, in the sense of (b) in §2.1 and of §2.5. In order to effect this efficiently we note the following facts:

First, the $\bar{x}_{ij}$ with $i>j$ present no difficulty at all. They are all zero, and they do not enter into the definitions of the others. Second, as far as the other $\bar{x}_{ij}$, that is, those with $i\leqq j$, are concerned, the definition interrelates only $\bar{x}_{ij}$'s with the same j. Third, for these interrelated families of $\bar{x}_{ij}$'s, that is, the $\bar{x}_{ij}$, $i=1, \cdots, j$, with a fixed j $(=1, \cdots, n)$, the definition is inductive in i, but it proceeds in the direction of decreasing i: The $\bar{x}_{ij}$'s are obtained in this order: $i=j, j-1, \cdots, 1$.

The scale factors must, therefore, be introduced separately for every j, and must there be built up successively as i goes through the values $j, j-1, \cdots, 1$ (in this order).

Since the sequence $\bar{x}_{ij}$ (j fixed) begins with $\bar{x}_{jj}=1$ the scale factors must only be used to reduce the size of $\bar{x}_{ij}$. Hence the considerations of §2.5 apply: The scaling of $\bar{x}_{ij}$ will be effected by (pseudo-) division by an appropriate 2^p $(p=0, 1, 2, \cdots)$, say $2^{p_{ij}}$. Denote the quantity which corresponds to $\bar{x}_{ij}$ scaled down by the factor $2^{p_{ij}}$ by $\bar{y}_{ij}$. In the formula (6.35′), case $i<j$, the replacement of $\bar{x}_{ij}$ by $\bar{y}_{ij}$ and of the $\bar{x}_{kj}$ by the $\bar{y}_{kj}$ requires, of course, that all of them be affected with the same scale factor. Hence, the scale factor which applies to the left-hand side, $2^{p_{ij}}$, must be a multiple of those which apply to the terms of the right-hand side, the $2^{p_{kj}}$ $(k=i+1, \cdots, j)$. That is, $p_{ij}\geqq p_{kj}$ for $k=i+1, \cdots, j$. Then the kj-term of the right-hand side must be "adjusted to scale" by (pseudo-) division by $2^{p_{ij}-p_{kj}}$. Furthermore, p_{ij} must be chosen large enough to make the resulting $\bar{y}_{ij}$ lie in $-1, 1$. After all $\bar{y}_{ij}$, $i=j, j-1, \cdots, 1$, have been obtained, they must all be "adjusted to scale" with each other, by (pseudo-) division (of $\bar{y}_{ij}$) by $2^{p_{1j}-p_{ij}}$. We call this "readjusted" form of $\bar{y}_{ij}$, $\bar{z}_{ij}$. Thus $\bar{z}_{ij}$ corresponds to $\bar{x}_{ij}$ scaled down by $2^{p_{1j}}$.

The dependence of this scale factor on the column-index is worth noting, it appears to be essential for the obtaining of efficient estimates. (This is connected with the use of the vectors $U^{(j)}$ of (6.44).)

We can now reformulate (6.35′), and state it in its corrected and valid form:

$$(6.35)\quad \begin{aligned} &(\text{a})\ \begin{array}{l} \bar{y}_{jj} = 1, \\ p_{jj} = 0. \end{array} \\ &(\text{b}) \begin{cases} \bar{y}_{ij} = -\sum_{k=i+1}^{j} (\bar{b}_{ik} \times \bar{y}_{kj}) \div 2^{p_{ij}-p_{kj}}, \\ p_{ij} \text{ is the smallest integer } \geqq p_{i+1,j}\ (\geqq p_{i+2,j} \geqq \cdots \geqq p_{jj}) \\ \text{which makes } \bar{y}_{ij} \text{ lie in } -1,\ 1\ (i = j-1, \cdots, 1). \end{cases} \\ &(\text{c})\ \bar{z}_{ij} = \begin{cases} \bar{y}_{ij} \div 2^{p_{1j}-p_{ij}} & \text{for } i \leqq j, \\ 0 & \text{for } i > j. \end{cases} \end{aligned}$$

As a result of (6.35) all $\bar{y}_{ij}$ and all $\bar{z}_{ij}$ lie in $-1, 1$. By (6.23) all $\bar{b}_{ij}$ lie in $-1, 1$, too. Hence the processes (6.35) including those of the intermediate expressions $\bar{b}_{ik} \times \bar{y}_{kj}$ and $(\bar{b}_{ik} \times \bar{y}_{kj}) \div 2^{p_{ij}-p_{kj}}$ produce properly formed digital numbers, lying in $-1, 1$.

6.5. **Estimates connected with the inverse of $\bar{B}$.** Having defined the digital numbers $\bar{z}_{ij}$ by (6.35), we now form the (upper semi-diagonal) digital matrix

$$(6.36)\qquad \bar{Z} = (\bar{z}_{ij}) \qquad (i, j = 1, \cdots n)$$

and proceed to investigate the properties of $\bar{Z}$ and of the $\bar{z}_{ij}$.

We begin by proving:

$$(6.37)\qquad 1/2 \leqq \operatorname*{Max}_{i=1,\cdots,j} |\bar{z}_{ij}| \leqq 1 \qquad (j = 1, \cdots, n).$$

Choose the largest $i = 1, \cdots, j$ with $p_{ij} = p_{1j}$. If $i = j$, then $p_{1j} = p_{jj} = 0$, hence $\bar{y}_{jj} = \bar{z}_{jj} = 1$. If $i < j$, then $p_{i+1,j}$ exists, and is necessarily unequal to p_{1j}, that is, $p_{i+1,j} < p_{1j}$. Hence $p_{ij} > p_{i+1,j}$. The reason for choosing $p_{ij} > p_{i+1,j}$ could only have been that this was necessary to make $|\bar{y}_{ij}| \leqq 1$. Since p_{ij} must have been the smallest integer not less than $p_{i+1,j}$ which has this effect, therefore $p_{ij} - 1$ (which is also not less than $p_{i+1,j}$) cannot have had this property. This excludes $|\bar{y}_{ij}| < 1/2$. Hence $|\bar{y}_{ij}| \geqq 1/2$. Now $\bar{z}_{ij} = \bar{y}_{ij} \div 2^{p_{1j}-p_{ij}} = \bar{y}_{ij}$, hence $|\bar{z}_{ij}| \geqq 1/2$.

Thus we have in any event a $|\bar{z}_{ij}| \geqq 1/2$. We know that all $|\bar{z}_{ij}| \leqq 1$. These two facts together establish (6.37).

Let us now evaluate the elements of the matrix $\bar{B}\bar{Z}$ (true multiplication!).

The ij-element of this matrix is $\sum_{k=1}^{n} \bar{b}_{ik}\bar{z}_{kj}$. Since $\bar{B}$ and $\bar{Z}$ are both upper semi-diagonal, we can conclude: For $i > j$ all terms in this sum are zero, so that the sum itself is zero. For $i = j$ the sum has precisely one nonzero term: $\bar{b}_{ii}\bar{z}_{ii}$. Since $\bar{b}_{ii}=1$, $\bar{z}_{ii}=\bar{y}_{ii}+2^{p_{1i}-p_{ii}}=1+2^{p_{1i}}$, the sum is equal to $1+2^{p_{1i}}$. This deviates from $2^{-p_{1i}}$ by not more than $\beta^{-s}/2$. For $i<j$ the only nonzero terms in the sum are those with $i\leqq k\leqq j$. The $k=i$ term is $\bar{b}_{ii}\bar{z}_{ij}=\bar{z}_{ij}$. Hence in this case the sum is

$$\bar{z}_{ij}+\sum_{k=i+1}^{j}\bar{b}_{ik}\bar{z}_{kj}=-\left(\sum_{k=i+1}^{j}(\bar{b}_{ik}\times\bar{y}_{kj})+2^{p_{1j}-p_{ij}}\right)+2^{p_{1j}-p_{ij}}$$

$$+\sum_{k=i+1}^{j}\bar{b}_{ik}(\bar{y}_{kj}+2^{p_{1j}-p_{kj}}).$$

If we replace all pseudo-operations by true ones, then both terms on the right-hand side go over into the same expression: $\sum_{k=i+1}^{j}\bar{b}_{ik}\bar{y}_{kj}/2^{p_{1j}-p_{kj}}$. By (2.21), (2.22) the error caused by these transitions is not more than $(j-i)\beta^{-s}$ in either term. Hence the right-hand side, which is the difference of these two terms, deviates from zero by not more than $2(j-i)\beta^{-s}$.

To sum up: Put

$$\bar{B}\bar{Z}=\Delta+U,\qquad \Delta=(2^{-p_{1j}}\delta_{ij}),\qquad U=(u_{ij}),\tag{6.38}$$

then

$$|u_{ij}|\begin{cases}=0 & \text{for } i>j,\\ \leqq\beta^{-s}/2 & \text{for } i=j,\\ \leqq 2(j-i)\beta^{-s} & \text{for } i<j.\end{cases}\tag{6.39}$$

We are interested in the vectors $U^{(j)}=(u_{ij})$ and in the matrix U. We have

$$\begin{aligned}|U^{(j)}|^2=\sum_{i=1}^{n}u_{ij}^2&\leqq\left(\sum_{i=1}^{j-1}4(j-i)^2+\frac{1}{4}\right)\beta^{-2s}\\&=\left(\sum_{k=1}^{j-1}4k^2+\frac{1}{4}\right)\beta^{-2s}\\&=\left(4\frac{j(j-1)(2j-1)}{6}+\frac{1}{4}\right)\beta^{-2s},\end{aligned}$$

that is,

$$|U^{(j)}|^2\leqq(2j(j-1)(2j-1)/3+1/4)\beta^{-2s}.\tag{6.40}$$

The right-hand side is not greater than

$$(2n(n-1)(2n-1)/3+1/4)\beta^{-2s}$$
$$=(2(n-1)(2n-1)/3n^3+1/4n^4)\,n^4\beta^{-2s}.$$

For $n \geqq 10$ this is not greater than $.115\, n^4\beta^{-2s}$. Hence (we assume from now on that $n \geqq 10$)

$$|U^{(j)}| \leqq .34 n^2\beta^{-s}. \tag{6.40'}$$

Next by (6.40)

$$(N(U))^2 = \sum_{j=1}^{n} |U^{(j)}|^2 \leqq \left(\sum_{j=1}^{n} \frac{2}{3} j(j-1)(2j-1) + \frac{n}{4}\right)\beta^{-2s}$$
$$= \left(\frac{n^2(n^2-1)}{3} + \frac{n}{4}\right)\beta^{-2s} \leqq \frac{1}{3} n^4\beta^{-2s},$$

hence

$$|U| \leqq N(U) \leqq (1/3^{1/2}) n^2\beta^{-s},$$

that is,

$$|U| \leqq .58 n^2\beta^{-s}. \tag{6.40''}$$

To conclude this section, we define

$$\alpha = n^2\beta^{-s}/\mu. \tag{6.41}$$

Then (6.34), (6.40''), (6.40') become:

$$|\bar{A} - \bar{B}^*\bar{D}\bar{B}| \leqq .42\mu\alpha, \tag{6.42}$$

$$|U| \leqq .58\mu\alpha, \tag{6.43}$$

$$|U^{(j)}| \leqq .34\mu\alpha. \tag{6.44}$$

6.6. **Continuation.** (6.42)–(6.44) (together with (6.38)) are the decisive estimates on which we base all others. We proceed as follows. $|A| = \lambda$, $|A|_l = \mu$, hence (6.42) gives $|\bar{B}^*\bar{D}\bar{B}| \leqq \lambda + .42\mu\alpha \leqq \lambda(1+.42\alpha)$, $|\bar{B}^*\bar{D}\bar{B}|_l \geqq \mu - .42\mu\alpha = \mu(1-.42\alpha)$. Next, $(\bar{D}^{1/2}\bar{B})^*(\bar{D}^{1/2}\bar{B}) = \bar{B}^*\bar{D}\bar{B}$, hence $|\bar{D}^{1/2}\bar{B}|^2 = |\bar{B}^*\bar{D}\bar{B}|$, $|\bar{D}^{1/2}\bar{B}|_l^2 = |\bar{B}^*\bar{D}\bar{B}|_l$. Consequently

$$|\bar{D}^{1/2}\bar{B}| \leqq (\lambda(1+.42\alpha))^{1/2}, \tag{6.45.a}$$

$$|\bar{D}^{1/2}\bar{B}|_l \geqq (\mu(1-.42\alpha))^{1/2}. \tag{6.45.b}$$

We note two additional, minor facts:

(6.30), (6.23.c') imply

$$|D^v| \leqq 1 \qquad \text{for } v \geqq 0. \tag{6.46}$$

Since all $|a_{ij}| \leqq 1$, therefore $t(A) = \sum_{i=1}^{n} a_{ii} \leqq n$. On the other hand $t(A) = \sum_{i=1}^{n} \lambda_i \geqq n\mu$. Hence

$$\mu \leqq 1. \tag{6.47}$$

We now pass to considering the vectors $Z^{[j]} = (z_{ij})$ and $\Delta^{[j]} = (2^{-p_{1j}}\delta_{ij}) = 2^{-p_{1j}}I^{[j]}$. By (6.38)

$$\begin{aligned}\bar{D}^{1/2}\bar{B}(Z^{[j]}) = \bar{D}^{1/2}(\bar{B}Z)^{[j]} &= \bar{D}^{1/2}(\Delta^{[j]} + U^{[j]}) \\ &= \bar{d}_j^{1/2}2^{-p_{1j}}I^{[j]} + \bar{D}^{1/2}(U^{[j]}).\end{aligned}$$

By (6.45.b), (6.37)

$$|\bar{D}^{1/2}\bar{B}(Z^{[j]})| \geqq |\bar{D}^{1/2}\bar{B}|_l|Z^{[j]}| \geqq (1/2)(\mu(1 - .42\alpha))^{1/2};$$

obviously

$$|\bar{d}_j^{1/2}2^{-p_{1j}}I^{[j]}| = 2^{-p_{1j}}\bar{d}_j^{1/2}$$

and by (6.46), (6.44)

$$|\bar{D}^{1/2}(U^{[j]})| \leqq |\bar{D}^{1/2}||U^{[j]}| \leqq .34\mu\alpha.$$

From all these relations we can infer that

$$\begin{aligned}(\mu(1 - .42\alpha))^{1/2}/2 &\leqq 2^{-p_{1j}}\bar{d}_j^{1/2} + .34\mu\alpha, \\ 2^{-p_{1j}}\bar{d}_j^{1/2} &\geqq (\mu(1 - .42\alpha))^{1/2}/2 - .34\mu\alpha \\ &= ((\mu(1 - .42\alpha))^{1/2} - .68\mu\alpha)/2\end{aligned}$$

and, remembering (6.47),

$$\begin{aligned}2^{-p_{1j}}\bar{d}_j^{1/2} &\geqq (\mu(1 - .42\alpha) - 2\cdot(.68)\mu\alpha)^{1/2}/2 \\ &= (\mu(1 - 1.78\alpha))^{1/2}/2,\end{aligned}$$

that is,

$$2^{-p_{1j}}\bar{d}_j^{1/2} \geqq (\mu(1 - 1.78\alpha))^{1/2}/2. \tag{6.48}$$

Since $\Delta\bar{D}^{1/2} = (2^{-p_{1i}}\bar{d}_i^{1/2}\delta_{ij})$, (6.48) may also be written

$$|\Delta\bar{D}^{1/2}|_l \geqq (\mu(1 - 1.78\alpha))^{1/2}/2. \tag{6.48$'$}$$

Next by (6.38), (6.45.b), (6.46), (6.43), (6.48′)

$$\bar{D}^{1/2}\bar{B}Z\Delta^{-1}\bar{D}^{-1/2} = \bar{D}^{1/2}(\Delta + U)\Delta^{-1}\bar{D}^{-1/2} = I + \bar{D}^{1/2}U\Delta^{-1}\bar{D}^{-1/2},$$

$$\begin{aligned}|\bar{D}^{1/2}\bar{B}Z\Delta^{-1}\bar{D}^{-1/2}| &\geqq |\bar{D}^{1/2}\bar{B}|_l|Z\Delta^{-1}\bar{D}^{-1/2}| \\ &\geqq (\mu(1 - .42\alpha))^{1/2}|Z\Delta^{-1}\bar{D}^{-1/2}|, \\ |\bar{D}^{1/2}U\Delta^{-1}\bar{D}^{-1/2}| &\leqq |\bar{D}^{1/2}||U||\Delta\bar{D}^{1/2}|_l^{-1} \\ &\leqq .58\mu\alpha\cdot 2/(\mu(1 - 1.78\alpha))^{1/2} \\ &= 1.16\mu^{1/2}\alpha/(1 - 1.78\alpha)^{1/2}.\end{aligned}$$

Hence

$$(\mu(1 - .42\alpha))^{1/2} \left| \bar{Z}\Delta^{-1}\bar{D}^{-1/2} \right| \leqq 1 + 1.16\mu^{1/2}\alpha/(1 - 1.78\alpha)^{1/2}$$

and by (6.47)

$$\begin{aligned}(\mu(1 - .42\alpha))^{1/2} &\left| \bar{Z}\Delta^{-1}\bar{D}^{-1/2} \right| \\ &\leqq 1 + 1.16 \frac{\alpha}{(1 - 1.78\alpha)^{1/2}} = \frac{(1 - 1.78\alpha)^{1/2} + 1.16\alpha}{(1 - 1.78\alpha)^{1/2}} \\ &\leqq \frac{(1 - 1.78\alpha/2) + 1.16\alpha}{(1 - 1.78\alpha)^{1/2}} = \frac{1 + .27\alpha}{(1 - 1.78\alpha)^{1/2}}.\end{aligned}$$

Consequently

$$\left| \bar{Z}\Delta^{-1}\bar{D}^{-1/2} \right| \leqq \frac{1 + .27\alpha}{(1 - .42\alpha)^{1/2}(1 - 1.78\alpha)^{1/2}} \frac{1}{\mu^{1/2}}. \tag{6.49}$$

Since $\Delta^{-1}\bar{D}^{-1/2}\bar{Z}^* = (\bar{Z}\Delta^{-1}\bar{D}^{-1/2})^*$ and $\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^* = (\Delta^{-1}\bar{D}^{1/2}\bar{Z}^*)^* \cdot (\Delta^{-1}\bar{D}^{1/2}\bar{Z}^*)$ (remember that Δ and $\bar{D}$ commute, because they are diagonal matrices), therefore

$$\left| \Delta^{-1}\bar{D}^{-1/2}\bar{Z}^* \right| \leqq \frac{1 + .27\alpha}{(1 - .42\alpha)^{1/2}(1 - 1.78\alpha)^{1/2}} \frac{1}{\mu^{1/2}}, \tag{6.49'}$$

$$\left| \bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^* \right| \leqq \frac{(1 + .27\alpha)^2}{(1 - .42\alpha)(1 - 1.78\alpha)} \frac{1}{\mu}. \tag{6.49''}$$

6.7. **Continuation.** Choose q ($=0, 1, 2, \cdots$) minimal with

$$2^q > \frac{4}{1 - 1.78\alpha} \frac{1}{\mu}, \tag{6.50.a}$$

that is, having in addition the property

$$2^q \leqq \frac{8}{1 - 1.78\alpha} \frac{1}{\mu}. \tag{6.50.b}$$

By (6.23.c′), (6.30), $0 < \bar{d}_j \leqq 1$. Hence we can form the maximal $r = 0, 1, 2, \cdots$ with $2^r d_j \leqq 1$, say r_j. We have

$$1/2 < 2^{r_j}\bar{d}_j \leqq 1. \tag{6.51}$$

By (6.48), (6.50.a), $2^q\bar{d}_j > {}^{2p_{1j}}$, $2^{q-2p_{1j}}\bar{d}_j > 1$. Hence $q - 2p_{1j} > r_j$, that is,

$$q - 2p_{1j} \geqq r_j + 1. \tag{6.52}$$

From (6.51), (6.52) we can infer:

(6.53) $\bar{e}_j^{(q)} = (1/2 + 2^{r_j}\bar{d}_j) + 2^{q-2p_{jj}-r_j-1}$ is, together with all its intermediate expressions, a digital number and well formed.[21]

Next by (6.50.a), (6.49″)

$$(6.54) \qquad \left| 2^{-q} Z \Delta^{-1} \bar{D}^{-1} Z^* \right| \leq \frac{1}{4} \frac{(1 + .27\alpha)^2}{1 - .42\alpha}.$$

Furthermore form

$$(6.55) \qquad \bar{W}^{(q)} = (\bar{w}_{ij}^{(q)}), \qquad \bar{w}_{ij}^{(q)} = \sum_{k=1}^{n} (z_{i'k} \times \bar{e}_k^{(q)}) \times z_{j'k}$$

$$(i' = \text{Min}\,(i, j),\ j' = \text{Max}\,(i, j)).$$

(We must still establish the digital character of $\bar{W}^{(q)}$ and of the $\bar{w}_{ij}^{(q)}$, which in this case amounts to showing that all $\bar{w}_{ij}^{(q)}$ lie in $-1, 1$. For this cf. (6.58), (6.59).) Now

$$\bar{W}^{(q)} - 2^{-q} Z \Delta^{-1} \bar{D}^{-1} Z^*$$

$$= \left(\sum_{k=1}^{n} (z_{i'k} \times \bar{e}_k^{(q)}) \times z_{j'k} - \sum_{k=1}^{n} z_{i'k} 2^{-q+2p_{kk}} \bar{d}_k^{-1} z_{j'k} \right).$$

(Note that we replaced in the second term on the right-hand side i, j by i', j'. This is permissible, since it is symmetric in i, j.) The ij-element of the right-hand side can be written

$$\sum_{k=1}^{n} ((\bar{z}_{i'k} \times \bar{e}_k^{(q)}) \times z_{j'k} - z_{i'k} 2^{-q+2p_{kk}} \bar{d}_k^{-1} z_{j'k}),$$

or, since $Z = (z_{ij})$ is upper semi-diagonal,

$$\sum_{k=j'}^{n} ((z_{i'k} \times \bar{e}_k^{(q)}) \times z_{j'k} - z_{i'k} 2^{-q+2p_{kk}} \bar{d}_k^{-1} z_{j'k}), \qquad j' = \text{Max}\,(i, j).$$

For this we can further write

$$\sum_{k=j'}^{n} \left\{ (\bar{z}_{i'k} \times \bar{e}_k^{(q)}) \times z_{j'k} - z_{i'k} \bar{e}_k^{(q)} z_{j'k}) + z_{i'k} (\bar{e}_k^{(q)} - 2^{-q+2p_{kk}} \bar{d}_k^{-1}) z_{j'k} \right\}.$$

By (6.51),

$$\left| \frac{1}{2} + 2^{r_j} \bar{d}_j - 2^{-r_j-1} \bar{d}_j^{-1} \right| \leq \beta^{-s}/2,$$

---

[21] We are assuming here that 1/2 is a digital number. This is only true when the base β is even. This limitation could be removed with little trouble, but it does not seem worthwhile, since $\beta = 2$ and $\beta = 10$ are both even.

hence, by (2.22) and (6.53), $|\bar{t}_j^{(q)} - 2^{-q+2p_0}\bar{d}_j^{-1}| \leqq \beta^{-s}$. Therefore the second term in the $\{\cdots\}$ above has an absolute value not greater than β^{-s}. The first term in this $\{\cdots\}$ has clearly an absolute value not greater than $\beta^{-s}/2+\beta^{-s}/2=\beta^{-s}$. Hence this $\{\cdots\}$ has an absolute value not greater than $2\beta^{-s}$, and so the entire expression in question is not greater than $2(n-j'+1)\beta^{-s}$. Consequently

$$(N(\bar{W}^{(q)} - 2^{-q}\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^*))^2$$

$$\leqq \sum_{i,j=1}^{n} 4(n-j'+1)^2\beta^{-2s} = \sum_{j'=1}^{n} 4(2j'-1)(n-j'+1)^2\beta^{-2s}$$

$$= \sum_{k=1}^{n} 4(2n-2k+1)k^2\beta^{-2s} = \frac{2n(n+1)(n^2+n+1)}{3}\beta^{-2s}$$

and ($n \geqq 10$, cf. the remark preceding (6.40′)) is not greater than

$$.82n^4\beta^{-2s},$$

hence

$$|\bar{W}^{(q)} - 2^{-q}\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^*| \leqq N(\bar{W}^{(q)} - 2^{-q}\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^*)$$
$$\leqq .91n^2\beta^{-s} = .91\mu\alpha,$$

that is,

$$(6.56) \qquad |\bar{W}^{(q)} - 2^{-q}\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^*| \leqq .91\mu\alpha.$$

From (6.54), (6.56), remembering (6.47), we get

$$(6.57) \qquad |\bar{W}^{(q)}| \leqq \frac{1}{4}\cdot\frac{(1+.27\alpha)^2}{1-.42\alpha} + .91\alpha.$$

We shall see later (cf. (6.67)) that it is reasonable to assume that $\alpha \leqq .1$. This implies that the right-hand side of (6.57) is not greater than .37. Consequently

$$(6.57') \qquad |\bar{W}^{(q)}| \leqq .37.$$

From this, (3.17.b) (first half) permits us to infer

$$(6.57'.a) \qquad |\bar{w}_{ij}^{(q)}| \leqq 1 \qquad (i, j = 1, \cdots, n).$$

Next, since $\bar{W}^{(q)}(I^{[j]}) = (\bar{W}^{(q)})^{[j]}$ and $|I^{[j]}| = 1$, (6.57′) implies $|(\bar{W}^{(q)})^{[j]}| \leqq .37$, that is,

$$\sum_{i=1}^{n} (\bar{w}_{ij}^{(q)})^2 \leqq (.37)^2 \leqq .14.$$

If we replace in this sum $(\bar{w}_{ij}^{(q)})^2$ by $\bar{w}_{ij}^{(q)} \times \bar{w}_{ij}^{(q)}$, then the total change is not greater than $n \cdot \beta^{-s}/2 = n^2\beta^{-s}/2n = \mu\alpha/2n$. Since $\alpha \leqq .1$, $n \geqq 10$ (cf. above) and $\mu \leqq 1$ (by (6.47)), therefore this is not greater than .005. Consequently

$$\sum_{i=1}^{n} \bar{w}_{ij}^{(q)} \times \bar{w}_{ij}^{(q)} \leqq .99 \qquad (j = 1, \cdots, n). \tag{6.57'.b}$$

We sum up:

(6.58) For the minimal q $(=0, 1, 2, \cdots)$ of (6.50.a) the conditions (6.53), (6.57′) are fulfilled. The latter implies (6.57′.a) and (6.57′.b).

In this section we use (6.57′.a); the need for (6.57′.b) will arise later. (Cf. (6.92).)

We define:

(6.59) Let q_0 be the minimal q $(=0, 1, 2, \cdots)$ for which the conditions (6.53), (6.57′.a) are fulfilled.

Note that these are simple, explicit conditions, which permit forming the q_0 in question directly.[23]

(6.58) shows that the minimal q of (6.50.a) is not less than q_0. Hence q_0, too, fulfills (6.50.b), that is,

$$2^{q_0} \leqq \frac{.8}{1 - 1.78\alpha} \frac{1}{\mu} \cdot \tag{6.60}$$

We now put

$$\bar{W}_0 = \bar{W}^{(q_0)} = (\bar{w}_{ij}^{(q_0)}). \tag{6.61}$$

The derivation of (6.56) was based on (6.53) alone, hence (6.56) holds for $\bar{W}_0 = \bar{W}^{(q_0)}$, too:

$$\left| \bar{W}_0 - 2^{-q_0}\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^* \right| \leqq .91\mu\alpha. \tag{6.62}$$

6.8. **Continuation. The estimates connected with the inverse of $\bar{A}$.** We are now able to effect our final estimates. The relevant auxiliary estimates are (6.42), (6.43), (6.45.a), (6.45.b), (6.46), (6.48′), (6.49′), (6.49′′), (6.60), (6.62). The procedure is as follows:

Put

$$A' = \bar{B}^*\bar{D}\bar{B}, \tag{6.63.a}$$

$$W' = 2^{-q_0}\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^*. \tag{6.63.b}$$

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[23] This would remain true if we replaced (6.57′.a) by (6.57′.b) (cf. (6.92)), but not if we reverted to (6.57′).

Owing to

$$2^{s_0}\bar{A}\bar{W}_0 - 2^{s_0}A'W' = (\bar{A} - A')2^{s_0}W' + 2^{s_0}\bar{A}(\bar{W}_0 - W'),$$

we have

$$|2^{s_0}\bar{A}\bar{W}_0 - 2^{s_0}A'W'| \leqq |\bar{A} - A'|\,|2^{s_0}W'| + 2^{s_0}|\bar{A}|\,|\bar{W}_0 - W'|.$$

By (6.42), (6.49′′), (6.60), (6.62) this is less than or equal to

$$.42\mu\alpha\cdot\frac{(1+.27\alpha)^2}{(1-.42\alpha)(1-1.78\alpha)}\,\frac{1}{\mu} + \frac{8}{1-1.78\alpha}\,\frac{1}{\mu}\,\lambda\cdot .91\mu\alpha$$

$$= \left[.42\frac{(1+.27\alpha)^2}{(1-.42\alpha)(1-1.78\alpha)} + 7.28\frac{1}{1-1.78\alpha}\lambda\right]\alpha.$$

Summing up:

$$\begin{aligned} &|2^{s_0}\bar{A}\bar{W}_0 - 2^{s_0}A'W'| \\ (6.63)\qquad &\leqq \left[.42\frac{(1+.27\alpha)^2}{(1-.42\alpha)(1-1.78\alpha)} + 7.28\frac{1}{1-1.78\alpha}\lambda\right]\alpha. \end{aligned}$$

Next

$$\begin{aligned} 2^{s_0}A'W' - I &= \bar{B}^*\bar{D}\bar{B}\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^* - I \\ &= \bar{B}^*\bar{D}(\Delta + U)\Delta^{-2}\bar{D}^{-1}\bar{Z}^* - I \\ &= (\bar{B}^*\Delta^{-1}\bar{Z}^* - I) + \bar{B}^*\bar{D}U\Delta^{-2}\bar{D}^{-1}\bar{Z}^* \end{aligned}$$

(remember that Δ and $\bar{D}$ commute, because they are diagonal matrices). Now

$$\begin{aligned} \bar{B}^*\Delta^{-1}\bar{Z}^* - I &= (\bar{Z}\Delta^{-1}\bar{B} - I)^* = (\bar{B}^{-1}(\bar{B}\bar{Z}\Delta^{-1} - I)\bar{B})^* \\ &= (\bar{B}^{-1}((\Delta + U)\Delta^{-1} - I)\bar{B})^* = (\bar{B}^{-1}U\Delta^{-1}\bar{B})^* \\ &= ((\bar{D}^{1/2}\bar{B})^{-1}\bar{D}^{1/2}U(\Delta^{-1}\bar{D}^{-1/2})(\bar{D}^{1/2}\bar{B}))^*, \end{aligned}$$

hence, by (6.45.b), (6.46), (6.43), (6.48′), (6.45.a),

$$\begin{aligned} &|\bar{B}^*\Delta^{-1}\bar{Z}^* - I| \\ &\qquad \leqq |\bar{D}^{1/2}\bar{B}|_l^{-1}\,|\bar{D}^{1/2}|\,|U|\,|\Delta\bar{D}^{1/2}|_l^{-1}\,|\bar{D}^{1/2}\bar{B}| \\ &\qquad \leqq \frac{1}{(\mu(1-.42\alpha))^{1/2}}\,.58\mu\alpha\,\frac{2}{(\mu(1-1.78\alpha))^{1/2}}\,(\lambda(1+.42\alpha))^{1/2} \\ &\qquad = 1.16\frac{(1+.42\alpha)^{1/2}}{(1-.42\alpha)^{1/2}(1-1.78\alpha)^{1/2}}\,\lambda^{1/2}\alpha. \end{aligned}$$

Furthermore

$$\bar{B}^*\bar{D}U\Delta^{-1}\bar{D}^{-1}\bar{Z}^* = (\bar{D}^{1/2}\bar{B})^*\bar{D}^{1/2}U(\Delta^{-1}\bar{D}^{-1/2})(\Delta^{-1}\bar{D}^{-1/2}\bar{Z}^*),$$

hence by (6.45.a), (6.46), (6.43), (6.48′), (6.49′)

$$\begin{aligned} |\bar{B}^*\bar{D}U\Delta^{-1}\bar{D}^{-1}\bar{Z}^*| &\leq |\bar{D}^{1/2}\bar{B}|\,|\bar{D}^{1/2}|\,|U|\,|\Delta\bar{D}^{1/2}|_l^{-1}\,|\Delta^{-1}\bar{D}^{-1/2}\bar{Z}^*| \\ &\leq (\lambda(1+.42\alpha))^{1/2}.58\mu\alpha\frac{2}{(\mu(1-1.78\alpha))^{1/2}} \\ &\quad\cdot\frac{1+.27\alpha}{(1-.42\alpha)^{1/2}(1-1.78\alpha)^{1/2}}\,\frac{1}{\mu^{1/2}} \\ &= 1.16\frac{(1+.42\alpha)^{1/2}(1+.27\alpha)}{(1-.42\alpha)^{1/2}(1-1.78\alpha)}\lambda^{1/2}\alpha. \end{aligned}$$

Summing up:

$$\begin{aligned} &|2^s A'W' - I| \\ &\leq 1.16\frac{(1+.42\alpha)^{1/2}}{(1-.42\alpha)^{1/2}}\left(\frac{1}{(1-1.78\alpha)^{1/2}} + \frac{1+.27\alpha}{1-1.78\alpha}\right)\lambda^{1/2}\alpha. \end{aligned} \tag{6.64}$$

Combining (6.63), (6.64) we obtain

$$\begin{aligned} &|2^s\bar{A}W_0 - I| \\ &\leq \Bigg[.42\frac{(1+.27\alpha)^2}{(1-.42\alpha)(1-1.78\alpha)} \\ &\quad + 1.16\frac{(1+.42\alpha)^{1/2}}{(1-.42\alpha)^{1/2}}\left(\frac{1}{(1-1.78\alpha)^{1/2}} + \frac{1+.27\alpha}{1-1.78\alpha}\right)\lambda^{1/2} \\ &\quad + 7.28\frac{1}{1-1.78\alpha}\lambda\Bigg]\alpha. \end{aligned} \tag{6.65}$$

If we now assume $\alpha \leq .1$, then the right-hand side of (6.65) is less than or equal to the expression

$$(.56 + 2.83\lambda^{1/2} + 8.86\lambda)\alpha. \tag{6.66}$$

It is indicated to scale $\bar{A} = (\bar{a}_{ij})$ so that $\mathrm{Max}_{i,j=1,\dots,n}(\bar{a}_{ij})$ is near 1, hence (by (3.17.b), first half) λ is near or more than 1. Hence the above coefficient of α is presumably greater than or equal to 10. This implies that for $\alpha = .1$ the right-hand side of (6.65) is presumably greater than or equal to 1. This, however, means that W_0 was not worth constructing since even 0 in place of W_0 would have given a right-hand side equal to 1. In other words: If $\alpha \leq .1$ does not hold, then the result (6.65) is without interest. If $\alpha \leq .1$ holds, then the right-hand side of (6.65) is less than or equal to the expression (6.66). It seems therefore logical to assume

$$(6.67)\qquad \alpha \leqq .1.$$

Note that this renders our earlier assumption (6.23.c) superfluous: $\alpha \leqq .1$ means that $n^2\beta^{-s}/\mu \leqq 1/10$, $10n^2\beta^{-s} \leqq \mu$, which implies the relation in question.

We can now incorporate (6.66) into (6.65). In this way we obtain:

$$(6.65')\qquad \left| 2^{q}\overline{A}\overline{W}_{q} - I \right| \leqq (.56 + 2.83\lambda^{1/2} + 8.86\lambda)\alpha.$$

Since $\lambda^{1/2} \leqq (1+\lambda)/2$ we can simplify (6.65′) to

$$(6.65'')\qquad \left| 2^{q}\overline{A}\overline{W}_{q} - I \right| \leqq (1.98 + 10.28\lambda)\alpha.$$

6.9. **The general $\overline{A}_I$. Various estimates.** (6.65″) (together with (6.41) and (6.67)) is our final result for the $\overline{A}$ fulfilling the conditions of (6.23) (that is, (a), (b) there, (c) was eliminated, cf. above after (6.67)). That is, this completes our work dealing with the definite $\overline{A}$. This result still requires some discussion and interpretation, but we postpone these until Chapter VII. At this point we turn our attention to the general (that is, nonsingular and not necessarily definite) $\overline{A}$. In order to emphasize the difference, we shall denote the general matrix in question by $\overline{A}_I$ instead of $\overline{A}$.

Let, then

$$(6.68)\qquad \overline{A}_I = (\bar{a}_{I,ij}) \qquad (i, j = 1, \cdots, n)$$

be a general (nonsingular), digital matrix.

Since we have solved the problem of inverting a matrix in the case when it is definite (cf. above), we wish to replace the problem of inverting A_I by that one of inverting an appropriate definite matrix. This should be done in analogy with the procedure suggested in §5.1. More specifically: The inverting of $\overline{A}_I$ should be based on that of $\overline{A}_I^*\overline{A}_I$. Since we are dealing with digital matrices, however, we have to consider $\overline{A}_I^* \times \overline{A}_I$ instead of $\overline{A}_I^*\overline{A}_I$. Furthermore, it will prove technically more convenient to use $\overline{A}_I \times \overline{A}_I^*$ rather than $\overline{A}_I^* \times \overline{A}_I$ (cf. the algebraical manipulations leading up to (6.100). We introduce accordingly

$$(6.69)\qquad \overline{A} = (\bar{a}_{ij}) = \overline{A}_I \times \overline{A}_I^*.$$

(6.69) indicates that $\overline{A}$ is a digital matrix. This, however, is not immediate: If we assume of $\overline{A}_I$ merely that all its elements $\bar{a}_{I,ij}$ lie in $-1, 1$, then the elements

$$\bar{a}_{ij} = \sum_{k=1}^{n} \bar{a}_{I,ik} \times \bar{a}_{I,jk} \tag{6.70}$$

of $\bar{A}$ need not lie in $-1, 1$. We shall rectify this shortcoming before long, but we prefer to disregard it for a moment, and discuss a few other matters first.

Put

$$|\bar{A}_I| = \lambda_I, \tag{6.71.a}$$

$$|\bar{A}_I|_l = \mu_I, \tag{6.71.b}$$

$$|\bar{A}| = \lambda, \tag{6.72.a}$$

$$|\bar{A}|_l = \mu. \tag{6.72.b}$$

(Cf. the analogous (5.11.a), (5.11.b).) Furthermore put

$$\alpha_I = n^2\beta^{-s}/\mu_I^2, \tag{6.73}$$

$$\alpha = n^2\beta^{-s}/\mu. \tag{6.74}$$

(Cf. the analogous (6.41). Note, however, that the denominator of the right-hand side of (6.73) is μ_I^2 and not μ_I.)

By (3.31.a)

$$|\bar{A} - \bar{A}_I\bar{A}_I^*| \leqq n^2\beta^{-s}/2 = \mu_I^2\alpha_I/2. \tag{6.75}$$

Hence

$$\begin{aligned}\lambda &= |\bar{A}| \leqq |\bar{A}_I\bar{A}_I^*| + \mu_I^2\alpha_I/2 = |\bar{A}_I|^2 + \mu_I^2\alpha_I/2 \\ &= \lambda_I^2 + \mu_I^2\alpha_I/2 \leqq \lambda_I^2(1 + \alpha_I/2), \\ \mu &= |\bar{A}|_l \geqq |\bar{A}_I\bar{A}_I^*|_l - \mu_I^2\alpha_I/2 = |\bar{A}_I|_l^2 - \mu_I^2\alpha_I/2 \\ &= \mu_I^2 - \mu_I^2\alpha_I/2 = \mu_I^2(1 - \alpha_I/2),\end{aligned}$$

that is,

$$\lambda \leqq \lambda_I^2(1 + \alpha_I/2), \tag{6.76.a}$$

$$\mu \geqq \mu_I^2(1 - \alpha_I/2). \tag{6.76.b}$$

From (6.76.b) and (6.73), (6.74)

$$\alpha \leqq \frac{\alpha_I}{1 - \alpha_I/2}. \tag{6.77}$$

Hence the condition (6.67), which we restate

$$\alpha \leqq .1, \tag{6.78}$$

is fulfilled, if

$$\alpha_I \leqq .095. \tag{6.78'}$$

Accordingly, we postulate (6.78′).

(6.78) implies $\mu_I \neq 0$, that is, the nonsingularity of $\bar{A}$. (6.75) implies

$$|(\bar{A}\xi, \xi) - (\bar{A}_I\bar{A}_I^*\xi, \xi)| \leqq \mu_I^2\alpha_I|\xi|^2/2.$$

Next

$$(\bar{A}_I\bar{A}_I^*\xi, \xi) = (\bar{A}_I^*\xi, \bar{A}_I^*\xi) = |\bar{A}_I^*\xi|^2 \geqq |\bar{A}_I^*|_l^2|\xi|^2.$$

Since $|\bar{A}_I^*|_l = |\bar{A}_I|_l = \mu_I$, this is greater than or equal to

$$\mu_I^2|\xi|^2.$$

Hence

$$(\bar{A}\xi, \xi) \geqq \mu_I^2|\xi|^2 - \mu_I^2\alpha_I|\xi|^2/2,$$

and so, by (6.78′), $(\bar{A}\xi, \xi) \geqq 0$. Therefore $\bar{A}$ is definite. We sum up:

(6.79) $\bar{A}$ is (nonsingular and) definite.

We conclude this section by securing the digital character of $\bar{A}$, that is, of all $\bar{a}_{ij}$. What is required is that all $\bar{a}_{ij}$ lie in $-1, 1$. Now by (6.70)

$$\left|\bar{a}_{ij} - \sum_{k=1}^{n} \bar{a}_{I,ik}\bar{a}_{I,jk}\right| \leqq n\beta^{-s}/2 = n^2\beta^{-s}/2n = \mu_I^2\alpha_I/2n,$$

and, since $n \geqq 10$, $\alpha_I < .1$, is less than or equal to

$$\frac{1}{200}\mu_I^2.$$

Hence

$$|\bar{a}_{ij}| \leqq \sum_{k=1}^{n} |\bar{a}_{I,ik}||\bar{a}_{I,jk}| + \mu_I^2/200. \tag{6.80}$$

Now

$$\begin{aligned} \sum_{k=1}^{n} |\bar{a}_{I,ik}||\bar{a}_{I,jk}| &\leqq \left(\sum_{k=1}^{n} (\bar{a}_{I,ik})^2\right)^{1/2}\left(\sum_{k=1}^{n} (\bar{a}_{I,jk})^2\right)^{1/2} \\ &\leqq \underset{i=1,\cdots,n}{\operatorname{Max}} \sum_{k=1}^{n} (\bar{a}_{I,ik})^2, \end{aligned} \tag{6.81}$$

and, since $A_I(I^{(h)}) = A_I^{(h)}$ and $|I^{(h)}| = 1$,

$$\mu_I^2 \leqq |A_I^{(h)}|^2 = \sum_{k=1}^{n} (\bar{a}_{I,hk})^2,$$

that is,

$$\mu_I^2 \leqq \underset{h=1,\cdots,n}{\mathrm{Min}} \sum_{k=1}^{n} (\bar{a}_{I,hk})^2. \tag{6.82}$$

By (6.81), (6.82) we obtain from (6.80)

$$|\bar{a}_{ij}| \leqq 1.005 \underset{h=1,\cdots,n}{\mathrm{Max}} \sum_{k=1}^{n} (\bar{a}_{I,hk})^2. \tag{6.83}$$

Again

$$\left| \sum_{k=1}^{n} \bar{a}_{I,hk} \times \bar{a}_{I,hk} - \sum_{k=1}^{n} (\bar{a}_{I,hk})^2 \right| \leqq \frac{n\beta^{-s}}{2}$$

and, as above, is less than or equal to

$$\frac{\mu_I^2}{200}.$$

Hence

$$\sum_{k=1}^{n} \bar{a}_{I,hk} \times \bar{a}_{I,hk} \geqq \sum_{k=1}^{n} (\bar{a}_{I,hk})^2 - \frac{\mu_I^2}{200}$$

and by (6.82) is greater than or equal to

$$.995 \sum_{k=1}^{n} (\bar{a}_{I,hk})^2.$$

Consequently

$$\sum_{k=1}^{n} (\bar{a}_{I,hk})^2 \leqq \frac{1}{.995} \sum_{k=1}^{n} \bar{a}_{I,hk} \times \bar{a}_{I,hk}. \tag{6.84}$$

Thus

$$\sum_{j=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij} \leqq .99 \qquad (i = 1, \cdots, n) \tag{6.85}$$

implies by (6.84) and (6.82), (6.83)

$$\mu_I \leqq 1, \tag{6.82'}$$

$$|\bar{a}_{ij}| \leqq 1. \tag{6.83'}$$

We assume the validity of (6.85). Then the digital character of $\bar{A}$ and of the $\bar{a}_{ij}$ is secured.

6.10. **Continuation.** By (6.78), (6.79), and the remark after (6.83′), $\bar{A}$ fulfills the conditions (6.23a), (6.23b), (6.67). Hence our results on inverting a definite matrix apply to it, and we can form the q_0 and $\bar{W}_0$ of §§6.7, 6.8 for this $\bar{A}$.

(6.65″) shows that $2^{q_0}\bar{W}_0$ is an approximate inverse of $\bar{A}$, and (6.75) shows that $\bar{A}$ is approximately $\bar{A}_I\bar{A}_I^*$. It is therefore reasonable to expect that $2^{q_0}\bar{A}_I^*\bar{W}_0$ will be usable as an approximate inverse of $\bar{A}_I$. Since we want a digital matrix, we should consider $2^{q_0}\bar{A}_I^*\times\bar{W}_0$ instead of $2^{q_0}\bar{A}_I^*\bar{W}_0$.

The digital character is, of course, desired for $\bar{A}_I^*\times\bar{W}_0$, and not for $2^{q_0}\bar{A}_I^*\times\bar{W}_0$. (Cf. with respect to this the last remark in §7.6.) However, the digital character of $\bar{A}_I^*\times\bar{W}_0$ is open to the same doubts, which we discussed immediately after (6.69) in connection with $\bar{A}_I\times\bar{A}_I^*$: We know that the elements $\bar{a}_{I,ji}$ of $\bar{A}_I^*$ and the elements $\bar{w}_{ij}^{(q_0)}$ of $\bar{W}_0$ lie in $-1, 1$, but this does not guarantee that the elements $\bar{\mathfrak{z}}_{ij}=\sum_{k=1}^{n}\bar{a}_{I,ki}\times\bar{w}_{kj}^{(q_0)}$ of $\bar{A}_I^*\times\bar{W}_0$ lie also in $-1, 1$. It is therefore necessary that we make certain that the $\bar{\mathfrak{z}}_{ij}$ do lie in $-1, 1$.

Write q for q_0, and put

$$\bar{\mathfrak{z}}_{ij}^{(q)} = \sum_{k=1}^{n} \bar{a}_{I,ki}\times\bar{w}_{kj}^{(q)}. \tag{6.86}$$

We argue as in the corresponding part of 6.9: By (6.86)

$$\left|\bar{\mathfrak{z}}_{ij}^{(q)} - \sum_{k=1}^{n} \bar{a}_{I,ki}\bar{w}_{kj}^{(q)}\right| \leqq n\beta^{-s}/2 = n^2\beta^{-s}/2n = \mu_I^2\alpha_I/2n,$$

and, since $n\geqq 10$, $\alpha_I\leqq .1$, and by (6.82′), is less than or equal to

$$\frac{1}{200}.$$

Hence

$$\left|\bar{\mathfrak{z}}_{ij}^{(q)}\right| \leqq \sum_{k=1}^{n}\left|\bar{a}_{I,ki}\right|\left|\bar{w}_{kj}^{(q)}\right| + \frac{1}{200}. \tag{6.87}$$

Now

$$\sum_{k=1}^{n}\left|\bar{a}_{I,ki}\right|\left|\bar{w}_{kj}^{(q)}\right| \leqq \left(\sum_{k=1}^{n}(\bar{a}_{I,ki})^2\right)^{1/2}\left(\sum_{k=1}^{n}(\bar{w}_{kj}^{(q)})^2\right)^{1/2}. \tag{6.88}$$

Furthermore

$$\left|\sum_{k=1}^{n} \bar{a}_{I,ki} \times \bar{a}_{I,ki} - \sum_{k=1}^{n} (\bar{a}_{I,ki})^2\right| \leqq \frac{n\beta^{-s}}{2} \leqq \frac{1}{200}, \tag{6.89.a}$$

$$\left|\sum_{k=1}^{n} \bar{\mathfrak{w}}_{kj}^{(q)} \times \bar{\mathfrak{w}}_{kj}^{(q)} - \sum_{k=1}^{n} (\bar{\mathfrak{w}}_{kj}^{(q)})^2\right| \leqq \frac{n\beta^{-s}}{2} \leqq \frac{1}{200}\cdot \tag{6.89.b}$$

Therefore

$$\sum_{k=1}^{n} \bar{a}_{I,ki} \times \bar{a}_{I,ki} \leqq .99, \tag{6.90.a}$$

$$\sum_{k=1}^{n} \bar{\mathfrak{w}}_{kj}^{(q)} \times \bar{\mathfrak{w}}_{kj}^{(q)} \leqq .99 \tag{6.90.b}$$

imply by (6.89.a), (6.89.b) with (6.88) and (6.87), that

$$\left| \mathfrak{z}_{ij}^{(q)} \right| \leqq 1. \tag{6.87$'$}$$

That is, in this case $\mathfrak{z}_{ij}^{(q)}$ lies in -1, 1, as desired.

We propose to treat (6.90.a), that is,

$$\sum_{i=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij} \leqq .99 \qquad (j = 1, \cdots, n), \tag{6.91}$$

like the analogous (6.85), as a postulate concerning $\bar{A}_I$. On the other hand (6.90.b), which coincides with (6.57′.b), will be secured by an appropriate choice of q. In other words:

The q_0, which was the value given to q throughout §6.8, was defined by (6.59) in §6.7. It was the minimal q $(=0, 1, 2, \cdots)$ fulfilling (6.53), (6.57′.a). We have seen that we should now replace (6.57′.a) by (6.57′.b), and thereby give q a new value q_1, instead of q_0.

We define accordingly:

(6.92) Let q_1 be the minimal q $(=0, 1, 2, \cdots)$ for which the conditions (6.53), (6.57′.b) are fulfilled. (Cf. the remark after (6.58) and footnote 32.)

We can now repeat a good deal of the argument following upon (6.58) with practically no change:

(6.58) shows that the minimal q of (6.50.a) is not more than q_1. Hence q_1, too, fulfills (6.50.b), that is,

$$2^{q} \leqq \frac{8}{1 - 1.78\alpha}\,\frac{1}{\mu}\cdot \tag{6.93}$$

We put

$$(6.94) \qquad \bar{W}_1 = \bar{W}^{(q_1)} = (\bar{w}_{ij}^{(q_1)}).$$

The derivation of (6.56) was based on (6.53) alone, hence (6.56) holds for $\bar{W}_1 = \bar{W}^{(q_1)}$, too:

$$(6.95) \qquad |\bar{W}_1 - 2^{-q}\bar{Z}\Delta^{-2}\bar{D}^{-1}\bar{Z}^*| \leqq .91\mu\alpha.$$

(6.93)–(6.95) are the precise equivalents of (6.60)–(6.62). Therefore the entire argument of §6.8 can be repeated unchanged, and we obtain the equivalent of (6.65''):

$$(6.96) \qquad |2^q\bar{A}\bar{W}_1 - I| \leqq (1.98 + 10.28\lambda)\alpha.$$

In addition to this we know now that $\bar{A}_I^* \times \bar{W}_1$ is a properly formed digital matrix.

6.11. Continuation. The estimates connected with the inverse of $\bar{A}_I$. We are now able to effect the final estimates of the general case. The relevant auxiliary estimates are (6.75), (6.76.a), (6.76.b), (6.77), (6.93), (6.96), as well as the two following ones: By (3.31.a)

$$(6.97) \qquad |\bar{A}_I^* \times \bar{W}_1 - \bar{A}_I\bar{W}_1| \leqq n^2\beta^{-s}/2 = \mu_I^2\alpha_I/2.$$

By (6.96)

$$|\bar{A}|_l |2^q\bar{W}_1| \leqq 1 + (1.98 + 10.28\lambda)\alpha,$$

$$|\bar{A}|_l = \mu$$

hence

$$|2^q\bar{W}_1| \leqq \frac{1 + (1.98 + 10.28\lambda)\alpha}{\mu},$$

and by (6.76.a), (6.76.b), (6.77) this is less than or equal to

$$\frac{1 + (1.98 + 10.28(1 + \alpha_I/2)\lambda_I^2)\alpha_I/(1 - \alpha_I/2)}{\mu_I^2(1 - \alpha_I/2)}.$$

Since $\alpha_I \leqq .1$, this is less than or equal to

$$\frac{1.20 + 1.20\lambda_I^2}{\mu_I^2},$$

that is,

$$(6.98) \qquad |2^{q_1}\bar{W}_1| \leqq \frac{1.20 + 1.20\lambda_I^2}{\mu_I^2}.$$

These being understood, the procedure is as follows:

Put

$$\bar{S} = \bar{A}_I^* \times \bar{W}_1. \tag{6.99}$$

($\bar{S}$ is digital, cf. the end of §6.10.)

Owing to

$$\begin{aligned} 2^q\bar{A}_I\bar{S} - I &= 2^q\bar{A}_I(\bar{A}_I^* \times \bar{W}_1) - I \\ &= 2^q\bar{A}_I(\bar{A}_I^* \times \bar{W}_1 - \bar{A}_I^*\bar{W}_1) - (\bar{A} - \bar{A}_I\bar{A}_I^*)\cdot 2^q\bar{W}_1 \\ &\quad + (2^q\bar{A}\bar{W}_1 - I), \end{aligned}$$

we have

$$\begin{aligned} |2^q\bar{A}_I\bar{S} - I| \leqq 2^q|\bar{A}_I|\,|\bar{A}_I^* \times \bar{W}_1 - \bar{A}_I^*\bar{W}_1| \\ + |\bar{A} - \bar{A}_I\bar{A}_I^*|\,|2^q\bar{W}_1| \\ + |2^q\bar{A}\bar{W}_1 - I|. \end{aligned} \tag{6.100}$$

By (6.93), (6.97) the first term of the right-hand side is less than or equal to

$$\frac{8}{1 - 1.78\alpha}\,\frac{1}{\mu}\cdot\lambda_I\cdot\frac{1}{2}\,\mu_I^2\alpha_I,$$

and by (6.77), (6.76.b) this is less than or equal to

$$4\,\frac{1}{1 - 1.78\alpha_I/(1 - \alpha_I/2)}\,\frac{1}{1 - \alpha_I/2}\,\lambda_I\alpha_I,$$

and, since $\alpha_I \leqq .1$, is less than or equal to

$$5.20\lambda_I\alpha_I.$$

By (6.75), (6.98) the second term is less than or equal to

$$\frac{1}{2}\,\mu_I^2\alpha_I\cdot\frac{1.20 + 1.20\lambda_I^2}{\mu_I^2} = (.60 + .60\lambda_I^2)\alpha_I.$$

By (6.96) the third term is less than or equal to

$$(1.98 + 10.28\lambda)\alpha,$$

and by (6.76.a), (6.77) this is less than or equal to

$$\left(1.98 + 10.28\left(1 + \frac{\alpha_I}{2}\right)\lambda_I^2\right)\frac{\alpha_I}{1 - \alpha_I/2}\cdot$$

Since $\alpha_I \leqq .1$, this is less than or equal to

$$(2.09 + 11.35\lambda_I^2)\alpha_I.$$

Summing up, and using (6.100), we obtain:

$$\left| 2^{q_1}\overline{A}_I\overline{S} - I \right| \leqq (2.69 + 5.20\lambda_I + 11.95\lambda_I^2)\alpha_I. \tag{6.101}$$

Since $\lambda_I \leqq (1+\lambda_I^2)/2$, we can simplify (6.101) to

$$\left| 2^{q_1}\overline{A}_I\overline{S} - I \right| \leqq (5.29 + 14.55\lambda_I^2)\alpha_I. \tag{6.102}$$

CHAPTER VII. EVALUATION OF THE RESULTS

7.1. Need for a concluding analysis and evaluation. Our final result is stated in (6.65″) for (nonsingular and) definite matrices $\overline{A}$ and in (6.102) for (nonsingular) general matrices $\overline{A}_I$. These statements and the considerations which led up to them form a logically complete whole. Nevertheless a concluding analysis and evaluation of these results, including a connected restatement of their underlying assumptions and of their constituent computations and discriminations, is definitely called for. Indeed, both the assumptions and the computations are dispersed over the length of Chapter VI and are not easy to visualise in their entirety; furthermore the assumptions were repeatedly modified, merged and rearranged. In addition, and this is more important, there entered into the procedure various quantities and properties which cannot be supposed to be known when the problem of inverting a matrix $\overline{A}$ or $\overline{A}_I$ comes up: For some of these the determination involves problems which are at least as difficult as that of inverting $\overline{A}$ or $\overline{A}_I$, and may even be in themselves closely connected or nearly equivalent to that inverting. Examples of such quantities or properties are: $|\overline{A}| = \lambda$, $|\overline{A}|_l = \mu$, $|\overline{A}_I| = \lambda_I$, $|\overline{A}_I|_l = \mu_I$, the nonsingularity of $\overline{A}$ or of $\overline{A}_I$, the definiteness of $\overline{A}$. Indeed, the basic quantities $\alpha = n^2\beta^{-s}/\mu$ and $\alpha_I = n^2\beta^{-s}/\mu_I^2$ belong to this category, and with them the final estimates (6.65″) and (6.102) and their preliminary conditions (6.67) (or (6.78)) and (6.78′).

We shall clarify these matters, and show that our procedure is actually self-consistent and leads directly to those types of results that one can reasonably desire for a problem like ours.

In connection with this we shall also estimate the amount of computation work that our procedure involves, and say something about the standards by which this amount may be judged.

7.2. Restatement of the conditions affecting $\overline{A}$ and $\overline{A}_I$: $(\mathcal{A})-(\mathcal{D})$. We assume, as we did throughout Chapter VI, that $n \geqq 10$. Indeed, for smaller values of n the problem of inverting a matrix hardly justifies this thorough analysis.

Let us now consider the hypotheses concerning $\bar{A}$ and $\bar{A}_I$.

First, both are introduced as digital matrices. This secures automatically that all their elements lie in $-1, 1$. This implies, of course, (6.23.b) for $\bar{A}$. In the case of $\bar{A}_I$ we need more: (6.85), (6.91), that is,

$$(7.1_I.\text{a}) \qquad \sum_{j=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij} \leqq .99 \qquad (i = 1, \cdots, n),$$

$$(7.1_I.\text{b}) \qquad \sum_{i=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij} \leqq .99 \qquad (j = 1, \cdots, n).$$

Second, nonsingularity is required for both $\bar{A}$ and $\bar{A}_I$, but this means $\mu \neq 0$ and $\mu_I \neq 0$, which is obviously subsumed in (6.67) (or (6.78)) and (6.78′). We restate, however, these latter conditions:

$$(7.2) \qquad \alpha \leqq .1, \quad \text{that is,} \quad \mu \geqq 10n^2\beta^{-s},$$

$$(7.2_I) \qquad \alpha_I \leqq .095, \quad \text{that is,} \quad \mu_I \geqq 10.5n^2\beta^{-s}.$$

Thus (6.23.c) (cf. the remark after (6.67)) and part of (6.23.a) for $\bar{A}$ are taken care of.

Third, $\bar{A}$ has to be definite, which covers the residual part of (6.23.a).

This is a complete list of our requirements. We restate it.

$(\mathcal{A})$ $\bar{A}$ and $\bar{A}_I$ are digital.

$(\mathcal{B})$ $\bar{A}_I$ fulfills (7.1$_I$.a), (7.1$_I$.b).

$(\mathcal{C})$ $\bar{A}$ and $\bar{A}_I$ fulfill (7.2) and (7.2$_I$), respectively.

$(\mathcal{D})$ $\bar{A}$ is definite.

$(\mathcal{A})$, $(\mathcal{B})$ are explicit conditions, of which $(\mathcal{A})$ is automatic and $(\mathcal{B})$ immediately verifiable by digital computation. $(\mathcal{C})$, $(\mathcal{D})$, on the other hand, represent the difficult type to which we referred in 7.1.

It is desirable to say a few things in connection with $(\mathcal{A})$, $(\mathcal{B})$ before we begin the analysis of $(\mathcal{C})$, $(\mathcal{D})$.

7.3. **Discussion of $(\mathcal{A})$, $(\mathcal{B})$: Scaling of $\bar{A}$ and $\bar{A}_I$.** We noted already that $(\mathcal{A})$ is automatically fulfilled. $(\mathcal{B})$ can be satisfied by an appropriate "scaling down" of $\bar{A}_I$, for example, by applying the operation $\div 2^p$ with a suitable p $(=0, 1, 2, \cdots)$ to all its elements.

On the other hand, we may if necessary "scale up" $\bar{A}$ or $\bar{A}_I$, for example, by multiplying it by $2^{p'}$ with a suitable $p'(=0, 1, 2, \cdots)$. In the case of $\bar{A}$, by choosing p' maximal without violating $(\mathcal{A})$, we can make $\text{Max}_{i,j=1,\cdots,n} |\bar{a}_{ij}|$ greater than or equal to one-half its permissible maximum, that is,

$$(7.3) \qquad \frac{1}{2} \leqq \operatorname*{Max}_{i,j=1,\cdots,n} |\bar{a}_{ij}| \leqq 1.$$

In the case of $\bar{A}_I$: if no p was needed (that is, if $p=0$), we can choose p' maximal without violating ($\mathcal{A}$), ($\mathcal{B}$); or, if p was needed (that is, if $p=0$ conflicts with ($\mathcal{B}$)), we can choose p minimal without violating ($\mathcal{B}$) and omit p' (that is, put $p'=0$). This makes $\operatorname{Max}_{j=1,\cdots,n \text{ or } i=1,\cdots,n}(\sum_{i=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij}, \sum_{j=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij})$ greater than or equal to one-quarter its permissible maximum, that is,

$$(7.3_I) \qquad \frac{.99}{4} \leqq \operatorname*{Max}_{j=1,\cdots,n \text{ or } i=1,\cdots,n} \left(\sum_{i=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij}, \sum_{j=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij} \right) \leqq .99.$$

We assume that these scaling operations have been effected, so that we have (7.3) and (7.3_I) for $\bar{A}$ and $\bar{A}_I$, respectively.

(7.3) implies by (3.17.b) (first half)

$$(7.4) \qquad \lambda \geqq 1/2.$$

From (7.3_I), on the other hand, we can infer this. $\sum_{i=1}^{n} \bar{a}_{I,ij} \times \bar{a}_{I,ij}$ and $\sum_{i=1}^{n} \bar{a}_{I,ij}^2$ differ by not more than $n\beta^{-s}/2 = \mu_I^2 \alpha_I / 2n$. Since we shall assume $\alpha_I \leqq .1$, we can argue, as we did at the end of 6.9, that this quantity is not greater than 1/200. Hence

$$\sum_{i=1}^{n} \bar{a}_{I,ij}^2 \geqq .24.$$

Now since $A_I(I^{(j)}) = A_I^{(j)}$ and $|I^{(j)}| = 1$, so

$$\lambda_I^2 \geqq |A_I^{(j)}|^2 = \sum_{i=1}^{n} \bar{a}_{I,ij}^2 \geqq .24,$$

that is,

$$(7.4_I) \qquad \lambda_I \geqq .49.$$

We sum up:

($\mathcal{A}$), ($\mathcal{B}$) can and will be satisfied, indeed strengthened to (7.3) and (7.3_I), by scaling $\bar{A}$ and $\bar{A}_I$ by appropriate powers of 2.

7.4. **Discussion of ($\mathcal{C}$): Approximate inverse, approximate singularity.** Let us now consider ($\mathcal{C}$).

Whether ($\mathcal{C}$) is fulfilled or not cannot be decided in advance by any direct method, or to be more precise, it constitutes a problem that is rather more difficult than the inverting of $\bar{A}$ or of $\bar{A}_I$. Accordingly, we do not propose to decide this in general. What we do propose to do instead is this:

Given $\bar{A}$ or $\bar{A}_I$, we wish to obtain an approximate inverse of $\bar{A}$ (or $\bar{A}_I$) by computational methods. Now it is clear that the solution of this problem cannot consist of furnishing such an (approximate) inverse, since $\bar{A}$ (or $\bar{A}_I$) may not have any inverse, that is, since it may be singular. Whether $\bar{A}$ (or $\bar{A}_I$) is singular or not is in general (that is, disregarding certain special situations) a problem of about the same character and depth as the finding of an (approximate) inverse (if one exists). Consequently the proper formulation of our problem is not this: "Find an (approximate) inverse of $\bar{A}$ (or $\bar{A}_I$)," but rather this: "Either find an (approximate) inverse of $\bar{A}$ (or $\bar{A}_I$), or guarantee that none exists."

Let us consider each one of these two alternatives more closely.

An approximate inverse of a matrix $\bar{P}$ might be defined as one which lies close to the exact inverse $\bar{P}^{-1}$. From the point of view of numerical procedure it seems more appropriate, however, to interpret it as the inverse P'^{-1} of a matrix P' that lies close to $\bar{P}$ that is, to permit an uncertainty of, say, ϵ in every element of $\bar{P}$. Thus we mean by an approximate inverse of $\bar{P}$ a matrix $Q = P'^{-1}$, where all elements of $\bar{P} - P'$ have absolute values not greater than ϵ.[33]

The nonexistence of an approximate inverse of a matrix should now be interpreted in the same spirit. From the point of view of numerical procedure it seems appropriate to interpret it as meaning that, with the uncertainty ϵ which affects every element of $\bar{P}$, $\bar{P}$ is not distinguishable from a singular matrix. That is, that there exists a singular matrix P'' such that all elements of $\bar{P} - P''$ have absolute values not greater than ϵ. (Cf. again footnote 33 above.)

We can now correlate our results concerning $\bar{A}$ and $\bar{A}_I$ with ($\mathcal{C}$): We had, for $\bar{A}$, (6.65″) based on (7.2) (that is, (6.67) or (6.78)), and, for $\bar{A}_I$, (6.102) based on (7.2$_I$) (that is, (6.78′)). The conditions (7.2) and (7.2$_I$) correspond, of course, to ($\mathcal{C}$).

We restate (6.65″) and (6.102):

$$(7.5) \qquad \left| 2^{p}\bar{A}\bar{W}_0 - I \right| \leqq (1.98 + 10.28\lambda)n^2\beta^{-s}/\mu,$$

$$(7.5_I) \qquad \left| 2^{q}\bar{A}_I\bar{S} - I \right| \leqq (5.29 + 14.55\lambda_I^2)n^2\beta^{-s}/\mu_I^2,$$

respectively. By (7.4), (7.4$_I$) these imply

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[33] This corresponds, of course, to the source of errors (B) in §1.1. As we pointed out before, the effects of (B) are not the subject of this paper. It seems nevertheless reasonable to take (B) into consideration at this stage, when we analyse what concept and what degree of approximation is to be viewed as significant.

$$(7.5') \qquad |2^{q}\bar{A}\bar{W}_0 - I| \leqq 14.24(\lambda/\mu)n^2\beta^{-s},$$

$$(7.5'_I) \qquad |2^{q_1}\bar{A}_I\bar{S} - I| \leqq 36.58(\lambda_I^2/\mu_I^2)n^2\beta^{-s},$$

respectively. (7.2) and (7.2_I) are (sufficient) conditions for the validity of these relations. We restate instead their negations, which are alternatives to the relations in question. They are:

$$(7.6) \qquad \mu < 10n^2\beta^{-s},$$

$$(7.6_I) \qquad \mu_I^2 < 10.5n^2\beta^{-s},$$

respectively.

Thus we have either (7.5′) or (7.6) for $\bar{A}$, and either $(7.5'_I)$ or (7.6_I) for $\bar{A}_I$. Now (7.5′), $(7.5'_I)$ on one hand and (7.6), (7.6_I) on the other correspond just to the two alternatives mentioned above:

(7.5′) and $(7.5'_I)$ express that $2^{q}\bar{W}_0$ and $2^{q_1}\bar{S}$ are approximate inverses of $\bar{A}$ and $\bar{A}_I$, respectively. (7.6) and (7.6_I) express that $\bar{A}$ and $\bar{A}_1$, respectively, are approximately singular. We leave the working out of the details, which can be prosecuted in several different ways, to the reader.

7.5. **Discussion of ($\mathfrak{D}$): Approximate definiteness.** We come finally to ($\mathfrak{D}$).

Whether ($\mathfrak{D}$) is fulfilled or not, that is, whether $\bar{A}$ is definite or not, is again difficult to ascertain. In this case, however, the situation is somewhat different from what it was in the preceding ones.

($\mathfrak{D}$) arises for $\bar{A}$ only. Indeed, it is the extra condition by which the inverting of $\bar{A}$ is distinguished from the inverting of $\bar{A}_I$, that is, which justifies the use of the more favorable estimates (7.5′), (7.6) that apply to the former, instead of the less favorable estimates $(7.5'_I)$, (7.6_I) that apply to the latter. It states that $\bar{A}$ is definite.

One will therefore let the need for ($\mathfrak{D}$) arise, that is, want to use the $\bar{A}$-method, only when it is known a priori that ($\mathfrak{D}$) is fulfilled, that is, that $\bar{A}$ is definite; that is, only when $\bar{A}$ originates in procedures which are known to produce definite matrices only.[34]

This might seem to dispose of ($\mathfrak{D}$), but there is one minor observation that might be made profitably:

$\bar{A}$ will have been obtained by numerical procedures, which are affected by (round off) errors. In spite of this we can assume that $\bar{A}$ is symmetric, but it need not be definite, only approximately definite. That is, there will be an estimate, by virtue of which this can be asserted: For a suitable definite matrix A' all elements of $\bar{A}-A'$ have absolute values, say not greater than ϵ. (This may be interpreted as a violation of the principle stated at the end of (d) in §2.2.)

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[34] For example, when $\bar{A}$ is a correlation matrix.

This does not, of course, guarantee that $\bar{A}$ is definite. It does, however, imply this:

(7.7) If $\bar{A}$ is not definite, then $\mu \leqq n\epsilon$.

Indeed: Assume that $\bar{A}$ is not definite. It is assumed to be symmetric, hence by (3.21.a) it has a proper value $\lambda_i < 0$. Since λ_i is a proper value, there exists a $\xi \neq 0$ with

$$\bar{A}\xi = \lambda_i \xi. \tag{7.8}$$

Hence $(\bar{A}\xi, \xi) = \lambda_i |\xi|^2 < 0$, that is,

$$(\bar{A}\xi, \xi) < 0. \tag{7.9}$$

Since A' is definite, therefore

$$(A'\xi, \xi) \geqq 0. \tag{7.10}$$

By (3.17.b) (second half) $|\bar{A} - A'| \leqq n\epsilon$ hence by (3.10)

$$|((\bar{A} - A')\xi, \xi)| \leqq n\epsilon |\xi|^2. \tag{7.11}$$

(7.9), (7.10), (7.11) imply together

$$-n\epsilon |\xi|^2 \leqq (\bar{A}\xi, \xi) \leqq 0. \tag{7.12}$$

Since $(\bar{A}\xi, \xi) = \lambda_i |\xi|^2$, (7.12) implies $-n\epsilon |\xi|^2 \leqq \lambda_i |\xi|^2 \leqq 0$, hence

$$|\lambda_i| \leqq n\epsilon. \tag{7.13}$$

Now (7.8), (7.13) gives $|\bar{A}\xi| \leqq n\epsilon |\xi|$, and therefore $\mu = |\bar{A}|_l \leqq n\epsilon$, as desired.

The significance of (7.7) is that it produces for $(\mathfrak{D})$ the same type of alternative which we obtained, and found satisfactory, in the last part of §7.4 for $(\mathfrak{C})$. Indeed, (7.7) guarantees that $\bar{A}$ is either definite, that is, $(\mathfrak{D})$ is fulfilled, or that

$$\mu \leqq n\epsilon \tag{7.7'}$$

and (7.7′) is exactly of the same type as the alternative conditions (7.6) and (7.6_I) in the part of §7.4 referred to.

7.6. **Restatement of the computational prescriptions. Digital character of all numbers that have to be formed.** We can sum up our conclusions reached so far as follows:

$\bar{A}$ and $\bar{A}_I$ must be scaled by an appropriate power of 2 as indicated in §7.3, that is, according to (7.3) and (7.3_I), respectively. If $\bar{A}$ is to be used, we assume in addition that it is symmetric and approximately definite (in the sense of §7.5, within a termwise error of, say, *e*). Our computational prescriptions then furnished the matrices $2^{q_0}\bar{W}_0$ and $2^{q_1}\bar{W}_1$, such that either (7.5) or (7.6) or (7.7′) holds in the case of $\bar{A}$, and either (7.5$_I$) or (7.6$_I$) holds in the case of $\bar{A}_I$. (7.5) and (7.5$_I$) mean that we found an approximate inverse; (7.6) or (7.7′), and (7.6$_I$) mean that the matrix was not inverted, because we found it to be approximately singular.

To this the following further remarks should be added:

The computational prescriptions to which we referred above are:

In the case of $\bar{A}$: Form the $\bar{a}_{ij}^{(k)}$ $(k=1, \cdots, n;\ i, j=k, \cdots, n)$ according to (6.3), (6.4). From these obtain the $\bar{d}_i$ $(i=1, \cdots, n)$ by (6.30) and the $\bar{b}_{ij}$ $(i, j=1, \cdots, n)$ by (6.31). From these obtain the $\bar{p}_{ij}, \bar{q}_{ij}, \bar{z}_{ij}$ $(i, j=1, \cdots, n)$ by (6.35). Then form the r_j $(j=1, \cdots, n)$ by (6.51). From all these form the $\bar{z}_j^{(q)}$ $(j=1, \cdots, n)$ by (6.53) and the $\bar{w}_{ij}^{(q)}$ $(i, j=1, \cdots, n)$ by (6.55), obtaining $q=q_0$ from (6.59) (that is, with the help of (6.53), (6.57′.a)). Finally put $\bar{W}_0=(\bar{w}_{ij}^{(n)})$.

In the case of $\bar{A}_I$: Form $\bar{A}$ by (6.69). Then proceed exactly as in the case of $\bar{A}$ above, with this exception: Instead of $q=q_0$ obtain $q=q_1$ from (6.92) (that is, with the help of (6.53), (6.57′.b)). Then form $\bar{S}$ by (6.99).

All these constructions were carried out and discussed in Chapter VI. We also showed in the course of those discussions that all the numbers to which we referred above, as well as all the intermediate ones which occur in their constructions, are properly formed digital numbers, and, in particular, lie in $-1, 1$ (except q_0, 2^{q_0} and q_1, 2^{q_1}, cf. below). This depended however on our assumptions concerning $\bar{A}$ and $\bar{A}_I$, which were summarized in $(\mathcal{A})-(\mathcal{D})$ in §7.2. Now $(\mathcal{A})-(\mathcal{D})$ are either contained in the assumptions made at the beginning of the present section, or expressed by the alternative possibilities (7.5), (7.6), (7.7′) and (7.5$_I$), (7.6$_I$) enumerated there. We can therefore assert this:

Either all the constructions that we enumerated above produce only digital numbers (including all the intermediate ones which occur in these constructions), which are properly formed, and, in particular, lie in $-1, 1$; or one of the alternative conditions must hold: (7.6) or (7.7′) for $\bar{A}$, (7.6$_I$) for $\bar{A}_I$.

We conclude this section by noting: The approximate inverses of $\bar{A}$ and $\bar{A}_I$ are, by (7.5) and (7.5$_I$), $2^{q_0}\bar{W}_0$ and $2^{q_1}\bar{S}$, respectively. $\bar{W}_0$ and $\bar{S}$ are digital matrices, but $2^{q_0}\bar{W}_0$ and $2^{q_1}\bar{S}$ need not be: Their elements need, of course, not lie in $-1, 1$. This is clearly unavoidable for an approximate inverse. Since we want to use only digital numbers, $2^{q_0}\bar{W}_0$, $2^{q_1}\bar{S}$ should be formed and recorded by keeping 2^{q_0}, 2^{q_1} and $\bar{W}_0$, $\bar{S}$ separately. $\bar{W}_0$, $\bar{S}$ are digital matrices, so they offer no difficulty. 2^{q_0}, 2^{q_1} are not digital, but we may form and record the digital $2^{e_0}\beta^{-s}$, $2^{e_1}\beta^{-s}$ or the equally digital $q_0\beta^{-s}$, $q_1\beta^{-s}$ instead. All the computations to which we referred at the beginning of this section deal, however, with digital numbers only, as we pointed out further above.

7.7. **Number of arithmetical operations involved.** It may be of interest to count the arithmetical operations that are involved in our computational prescriptions, as stated at the beginning of §7.6. Since most computations are dominated by the multiplications and divisions they contain, we shall only count these.

Referring back to the enumeration at the beginning of §7.6, we find:

In the case of $\bar{A}$:

	Multiplications	*Divisions*
(6.3)	$n(n+1)(n+2)/6$[35]	$n(n+1)/2$[35]
(6.4)	—	—
(6.30)	—	—
(6.31)	—	known from (6.3)
(6.35)	$(n-1)n(n+1)/6$	—[35]
(6.51)	—[35]	—
(6.53)	—[35]	n[35]
(6.55)	$n(n+1)(n+2)/6$	—
(6.59)	—	—
Total	$(n^3+2n^2+n)/2$	$(n^2+3n)/2$

Additional in the case of $\bar{A}_I$:

From $\bar{A}$	$(n^3+2n^2+n)/2$	$(n^2+3n)/2$
(6.69)	$n^2(n+1)/2$	—
(6.99)	n^3	—
	$(4n^3+3n^2+n)/2$	$(n^2+3n)/2$

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[35] We do not omit trivial multiplications like $a \times 1$ and trivial divisions like $a \div a$, since their numbers are irrelevant compared to the whole. We do, however, omit scaling operations $2^s a$ and $a \div 2^s$, since these are likely to be effected in simpler ways than by full-sized multiplications and divisions. Besides their numbers, too, are irrelevant.

Since we are interested in large values of n (at least $n \geqq 10$), we can use the asymptotic forms: For $\bar{A}$: $n^3/2$ multiplications, $n^2/2$ divisions. For $\bar{A}_I$: $2n^3$ multiplications, $n^2/2$ divisions. The divisions are presumably irrelevant in comparison with the multiplications. Hence our final result is: For $\bar{A}$: $n^3/2$ multiplications, for $\bar{A}_I$: $2n^3$ multiplications.

Note that an ordinary matrix multiplication consists of n^3 (number) multiplications. Hence the $\bar{A}$-method of inversion is comparable to half a matrix multiplication, while the $\bar{A}_I$-method of inversion is comparable to two matrix multiplications. It is a priori plausible that an inversion should be a more complicated operation than a multiplication. Thus we have in the above a quantitative measure of the high efficiency of inverting a matrix by elimination.

7.8. **Numerical estimates of precision.** In conclusion, it seems desirable to make some effective numerical evaluations of our estimates: Of (7.5′), (7.6) for $\bar{A}$ (definite case) and of ($7.5_I'$), (7.6_I) for $\bar{A}_I$ (general case).

Since the intervening quantities λ, μ and λ_I, μ_I are not known in general (or even, in most special cases, in advance), this cannot be done without some additional hypotheses.

We shall introduce such a hypothesis in the form of the statistical results of V. Bargmann, referred to in footnote 24. According to these, we can assert for a "random" matrix $\bar{A}_I$ (which we may assume to have been scaled in the sense of §7.3, that is, according to (7.3_I)) that λ_I, μ_I have with a probability ~ 1 the following sizes:

$$\lambda_I \sim n^{1/2}, \tag{7.14$_I$.a}$$

$$\mu_I \sim 1/n^{1/2}, \tag{7.14$_I$.b}$$

and hence

$$\lambda_I/\mu_I \sim n. \tag{7.14$_I$.c}$$

In order to reduce the probabilistic uncertainties to reasonably safe levels, we allow a factor 10 in excess of each estimate (7.14_I.a)–(7.14_I.c):

$$\lambda_I \leqq 10n^{1/2}, \tag{7.14$_I'$.a}$$

$$\mu_I \geqq 1/10n^{1/2}, \tag{7.14$_I'$.b}$$

$$\lambda_I/\mu_I \leqq 10n. \tag{7.14$_I'$.c}$$

For the definite (or approximately definite) matrices it seems very unreasonable to introduce any direct "randomness," since they are usually secondary, originating in other, general matrices. It does not seem unreasonable to estimate their λ, μ about as the squares of the above λ_I, μ_I; but we shall not attempt to analyze this hypothesis here any further. If it is accepted, we obtain from $(7.14_I'.\text{a})$–$(7.14_I'.\text{c})$:

$$\lambda \leqq 100n, \tag{7.14'.a}$$

$$\mu \geqq 1/100n, \tag{7.14'.b}$$

$$\lambda/\mu \leqq 100n^2. \tag{7.14'.c}$$

Now both estimates (7.6), (7.6_I) imply (approximately):

$$n \geqq .1\beta^{s/3}. \tag{7.15}$$

That is, an approximate inverse will usually be found if n does not fulfill (7.15), that is, if

$$n < .1\beta^{s/3}. \tag{7.15'}$$

Furthermore, the right-hand sides of both estimates (7.5′), $(7.5_I')$ become (approximately):

$$\leqq 2{,}000n^4\beta^{-s}. \tag{7.16}$$

(The factor 2,000 should actually have been ~1,500 in the case of (7.5′), and ~3,500 in the case of $(7.5_I')$. We replaced these by the common (and, for the second alternative, low) value 2,000 in order to simplify matters. This change is irrelevant, because in passing to (7.16′) a fourth root is being extracted. Besides, the estimate by which we passed from (7.5_I) to $(7.5_I')$ was very generous, because λ_I is likely to be essentially larger than indicated by (7.4_I).)

Hence this is less than 1 if

$$n < .15\beta^{s/4}, \tag{7.16'}$$

that is, an approximate inverse will usually be found if n fulfills (7.16′). Its (relative) precision is measured by the fourth power of the factor by which n is below the limit of (7.16′), or by the first power of the factor by which β^s is above it.

(7.16′) is more stringent than (7.15′) if $.15\beta^{s/4} \leqq .1\beta^{s/3}$, which is equivalent to $\beta^{s/12} \geqq 1.5$, $\beta^s \geqq 1.5^{12} \approx 130$. This is the case for all precisions at which a calculation of the type that we consider is likely to be carried out. (It is hardly conceivable that there should not be $\beta^s \geqq 10^8$.) We may say therefore:

(7.16′) is the critical condition, regarding the (relative) precision of the approximate inverse, cf. the remark after (7.16′).

Let us now consider some plausible precisions:

(7.17.a) $$\beta^s \sim 10^8 \sim 2^{27},$$

(7.17.b) $$\beta^s \sim 10^{10} \sim 2^{33},$$

(7.17.c) $$\beta^s \sim 10^{12} \sim 2^{40}.$$

(7.16′) becomes:

(7.18.a) $$n < 15,$$

(7.18.b) $$n < 50,$$

(7.18.c) $$n < 150,$$

respectively. As we saw in §7.7, these n correspond to maximally $\sim n^3 \sim 3{,}500$; 120,000; 3,500,000 multiplications. This might provide a basis for estimating what degrees of precision are called for in this problem by various possible types of procedures and equipment.

INSTITUTE FOR ADVANCED STUDY

THE SUMMER MEETING IN NEW HAVEN

The fifty-third Summer Meeting of the Society was held at Yale University, New Haven, Connecticut, Tuesday through Friday, September 2–5, 1947. The Mathematical Association of America met on September 1–2 and The Institute of Mathematical Statistics on September 2–4. Over seven hundred people attended these meetings among whom were the following four hundred forty-three members of the Society:

J. C. Abbott, C. R. Adams, R. B. Adams, R. P. Agnew, E. J. Akutowicz, E. S. Allen, C. B. Allendoerfer, Warren Ambrose, R. C. Archibald, R. F. Arens, H. A. Arnold, H. E. Arnold, L. A. Aroian, M. G. Arsove, R. N. Ascher, H. T. R. Aude, Silvio Aurora, M. C. Ayer, Frank Ayres, W. L. Ayres, H. M. Bacon, F. E. Baker, D. H. Ballou, Joshua Barlaz, J. L. Barnes, H. P. Beard, F. S. Beckman, E. G. Begle, P. O. Bell, A. A. Bennett, Stefan Bergman, Felix Bernstein, A. H. Black, Garrett Birkhoff, D. H. Blackwell, R. P. Boas, H. W. Bode, J. W. Bower, J. G. Bowker, H. R. Brahana, A. T. Brauer, Richard Brauer, J. V. Breakwell, Robert Breusch, J. R. Britton, Foster Brooks, Bailey Brown, R. H. Brown, E. F. Buck, R. C. Buck, L. H. Bunyan, M. L. Burke, F. J. H. Burkett, W. K. Burroughs, Hobart Bushey, J. H. Bushey, W. H. Bussey, Albert Cahn, S. S. Cairns, B. H. Camp, G. C. Campbell, R. E. Carr, W. B. Carver, G. F. Carrier, C. R. Cassity, W. B. Caton, Omar Catunda, K. Chandrasekharan, Harold Chatland, W. F. Cheney, W. L. Chow, D. E. Christie, K. L. Chung, C. E. Clark, H. E. Clarkson, A. B. Coble, I. S. Cohen, L. W. Cohen, Nancy Cole, R. H. Cole, L. A. Colquitt, Esther Comegys, Erben Cook, T. F. Cope, L. P. Copeland, W. H. H. Cowles, V. F. Cowling, M. J. Cox, H. S. M. Coxeter, C. C. Craig, Jane Cronin, L. L. Cronvich, C. W. Crouse, E. L. Crow, H. B. Curry, G. B. Dantzig, M. D. Darkow, F. H. Davidson, D. R. Davis, M. D. Davis, F. F. Decker, C. E. Dimick, Bernard Dimsdale, L. L. Dines, M. P. Dolciani, M. D. Donsker, H. L. Dorwart, Y. N. Dowker, Arnold Dresden, Nelson Dunford, W. L. Duren, William H. Durfee, Ben Dushnik, Beno Eckmann, Samuel Eilenberg, Churchill Eisenhart, F. M. Ellis, Murray Ellis, Bernard Epstein, Paul Erdös, M. W. Eudey, H. P. Evans, B. G. Farley, Herbert Federer, J. M. Feld, William Feller, A. D. Fialkow, F. A. Ficken, N. J. Fine, H. A. Fisher, W. B. Fite, W. W. Flexner, M. M. Flood, L. R. Ford, R. M. Foster, F. H. Fowler, A. H. Fox, R. H. Fox, J. S. Frame, Orrin Frink, R. E. Fullerton, H. L. Garabedian, L. J. Gårding, C. S. Gardner, R. S. Gardner, H. M. Gehman, Hilda Geiringer, B. H. Gere, Irving Gerst, B. P. Gill, R. E. Gilman, Wallace Givens, A. M. Gleason, H. E. Goheen, Michael Goldberg, R. O. Goodman, M. J. Gottlieb, M. C. Graustein, H. J. Greenberg, Leonard Greenstone, Edison Greer, V. G. Grove, Theodore Hailperin, D. W. Hall, Marshall Hall, P. R. Halmos, O. H. Hamilton, R. W. Hamming, G. H. Handelman, B. I. Hart, K. E. Hazard, G. A. Hedlund, A. E. Heins, M. H. Heins, R. G. Helsel, Aaron Herschfeld, Fritz Herzog, Edwin Hewitt, T. H. Hildebrandt, Einar Hille, I. I. Hirschman, G. P. Hochschild, Banesh Hoffmann, T. R. Hollcroft, R. H. Hoskins, Harold Hotelling, A. S. Householder, C. C. Hsiung, H. T. Hsu, E. M. Hull, Witold Hurewicz, W. A. Hurwitz, R. E. Huston, L. C. Hutchinson, S. B. Jackson, Nathan Jacobson, J. J. Janko, Herbert Jehle, E. D. Jenkins, S. A. Jennings, R. N. Johanson, R. B. Johnson, R. F. Johnson, C. W. Jordan, Aida Kalish, Irving Kaplansky, William Karush, M. E. Kellar, L. M. Kelly, A. J. Kempner, D. E. Kibbey, Fred Kiokemeister,

W. J. Klimczak, E. R. Kolchin, H. S. Konijn, C. F. Kossak, M. Z. Krzywoblocki, Jack Laderman, W. D. Lambert, H. G. Landau, R. E. Langer, E. H. Larguier, J. P. LaSalle, H. L. Lee, J. R. Lee, E. L. Lehmann, Marguerite Lehr, R. A. Leibler, Walter Leighton, F. C. Leone, Howard Levene, W. J. LeVeque, Norman Levinson, D. C. Lewis, F. A. Lewis, M. A. Lipschutz, B. J. Lockhart, Z. L. Loflin, A. J. Lohwater, W. R. Longley, Lee Lorch, C. I. Lubin, V. O. McBrien, E. D. McCarthy, R. B. McClenon, Dorothy McCoy, N. H. McCoy, E. A. McDougle, W. H. McEwen, L. A. MacColl, G. W. Mackey, G. R. MacLane, H. F. MacNeish, H. B. Mann, Morris Marden, M. H. Martin, W. T. Martin, P. R. Masani, F. J. Massey, F. I. Mautner, A. E. Meder, G. M. Merriman, E. L. Mickelson, E. J. Miles, A. N. Milgram, Knox Millsaps, S. Minakshisundaram, H. J. Miser, J. M. Mitchell, E. B. Mode, Deane Montgomery, J. C. Montgomery, F. V. E. Morfoot, C. N. Moore, T. W. Moore, E. M. Morenus, R. K. Morley, D. S. Morse, Frederick Mosteller, H. T. Muhly, C. G. Mumford, W. R. Murray, J. R. Musselman, A. L. Nelson, P. F. Neményi, E. D. Nering, O. E. Neugebauer, John von Neumann, C. V. Newsom, E. N. Nilson, G. E. Noether, P. B. Norman, E. P. Northrop, H. W. Norton, Lawrence Norwood, R. E. O'Connor, P. S. Olmstead, Paul Olum, T. G. Ostrom, E. R. Ott, M. H. Payne, D. K. Pease, W. H. Pell, P. M. Pepper. R. I. Pepper, Sam Perlis, H. R. Phalen, George Piranian, J. G. Pocock, Harry Pollard, C. R. Polley, George Pólya, William Prager, G. B. Price, A. L. Putnam, G. Y. Rainich, J. F. Randolph, Anatol Rapoport, R. B. Rasmusen, C. H. Rawlins, G. E. Raynor, L. J. Reed, C. J. Rees, Mina Rees, W. T. Reid, Irving Reiner, B. P. Reinsch, Helene Reschovsky, C. N. Reynolds, R. G. D. Richardson, C. E. Rickart, P. R. Rider, F. D. Rigby, John Riordan, J. F. Ritt, E. K. Ritter, H. E. Robbins, H. M. Roberts, M. S. Robertson, L. B. Robinson, L. V. Robinson, Saul Rosen, J. B. Rosser, M. F. Rosskopf, E. H. Rothe, Herman Rubin, Charles Saltzer, H. E. Salzer, Hans Samelson, Arthur Sard, Max Sasuly, F. E. Satterthwaite, A. T. Schafer, R. D. Schafer, M. M. Schiffer, A. E. Schild, O. F. G. Schilling, R. W. Schmied, I. J. Schoenberg, K. C. Schraut, J. E. Schubert, Abraham Schwartz, G. E. Schweigert, W. T. Scott, C. E. Sealander, C. H. W. Sedgewick, I. E. Segal, A. S. Shapiro, H. N. Shapiro, H. S. Sharp, L. W. Sheridan, Seymour Sherman, Max Shiffman, Marlow Sholander, D. T. Sigley, Milton da Silva Rodrigues, D. N. Silver, M. M. Slotnick, D. M. Smiley, M. F. Smiley, C. V. L. Smith, F. C. Smith, P. A. Smith, R. E. Smith, W. M. Smith, Ernst Snapper, W. S. Snyder, Andrew Sobczyk, Herbert Solomon, E. H. Spanier, A. H. Sprague. C. E. Springer, D. E. Spencer, V. E. Spencer, M. E. Stark, E. P. Starke, Rothwell Stephens, F. M. Stewart, R. W. Stokes, R. R. Stoll, R. C. Strodt, Walter Strodt, M. C. Suffa, A. C. Sugar, M. M. Sullivan, J. L. Synge, Gabor Szegö, A. H. Taub, J. S. Taylor, Feodor Theilheimer, J. E. Thompson, D. L. Thomsen, R. M. Thrall, W. J. Thron, H. S. Thurston, Gerhard Tintner, Leonard Tornheim, M. M. Torrey, J. I. Tracey, W. J. Trjitzinsky, C. A. Truesdell, Y. W. Tschen, Bryant Tuckerman, J. W. Tukey, H. S. Uhler, S. M. Ulam, E. P. Vance, H. E. Vansant, Philon Vassiliou, D. F. Votaw, Abraham Wald, H. M. Walker, R. J. Walker, Henry Wallman, J. L. Walsh, R. M. Walter, J. B. Walton, C. W. Watkeys, G. C. Webber, Alexander Weinstein, B. A. Welch, D. W. Western, G. W. Whitehead, A. L. Whiteman, P. M. Whitman, Hassler Whitney, D. V. Widder, L. R. Wilcox, S. S. Wilks, M. F. Willerding, John Williamson, G. M. Wing, R. M. Winger, H. A. Wood, Jacob Wolfowitz, F. M. Wright, Bertram Yood, J. W. T. Youngs, J. A. Zilber, Oscar Zariski.

On Tuesday afternoon, Wednesday, Thursday and Friday mornings, Professor Oscar Zariski of Harvard University gave the Colloquium Lectures on *Abstract algebraic geometry*. President Einar Hille, Professor A. B. Coble, Professor T. H. Hildebrandt, and Vice President L. R. Ford presided in turn at these lectures.

On Thursday afternoon Professor S. S. Wilks of Princeton University gave an address on *Sampling theory of order statistics*. Professor Harold Hotelling presided.

Presiding officers for the sessions of contributed papers were: Applied Mathematics, Tuesday afternoon, Professor William Prager; Geometry, Tuesday afternoon, Reverend R. E. O'Connor; Analysis, Wednesday morning, Professor W. T. Reid; Algebra, Wednesday morning, Professor P. M. Whitman; Applied Mathematics, Thursday morning, Professor W. T. Martin; Topology, Thursday morning, Professor G. E. Schweigert; Statistics and Probability, Thursday afternoon, Professor J. W. Tukey; Analysis, Thursday afternoon, Professor Gabor Szegö; Analysis, Friday morning, Professor G. B. Price; Algebra, Friday morning, Professor M. F. Smiley.

There was a business meeting Wednesday morning at which President Einar Hille presided.

The general sessions were all held in the auditorium of Sprague Hall and the sections were held in this room and in 201 Harkness Hall.

Registration headquarters were in Branford College. Rooms in Branford, Jonathan Edwards, and Saybrook Colleges were available to those attending the meetings. Meals were served in the Cafeteria of the University Dining Hall. On Wednesday and Thursday mornings three tours to different parts of the campus were conducted simultaneously. One included the Sterling Memorial Library and the Payne Whitney Gymnasium; another the Gallery of Fine Arts, and the third the Peabody Museum. The picnic and swimming at Hammonasset State Park on Wednesday afternoon were enjoyed by over four hundred people. Wednesday noon there was a trip to visit the insurance companies in Hartford, Connecticut. The mathematicians were the guests of the Connecticut Mutual Life Insurance Company at a luncheon served in the Company's cafeteria. On Wednesday just before the picnic there was a tour to the Connecticut Agricultural Experiment Station. Throughout the entire week there was an exhibition of early books outstanding in the history of mathematics.

On Tuesday afternoon, the ladies of the mathematics faculty served tea in the President's Room, Woolsey Hall.

On Tuesday evening, an organ recital was given by Professor H. Frank Bozyan of Yale University.

A group photograph of those attending the meeting was taken on Wednesday at 1:30 P.M.

The dinner for the three organizations was served in the University Dining Hall. Professor W. R. Longley was toastmaster. Director E. W. Sinnott of the Sheffield Scientific School welcomed the mathematicians and their guests. President Einar Hille of the Society, President L. R. Ford of the Association and President William Feller of the Institute talked about the rapid growth of the three organizations in membership and activities. Professor P. R. Rider proposed a resolution, unanimously approved, expressing the appreciation of the members of the three organizations to the administration of Yale University, the local committee and all who contributed to the success of the meetings.

The Board of Trustees of the Society met Wednesday at 2:00 P.M.

The Council met in the lounge of Jonathan Edwards College at 8:00 P.M. on Tuesday.

The Secretary announced the election of the following eighty-eight persons to ordinary membership in the Society:

Professor José Abdelhay, University of Brazil;
Mr. William Attix Allen, Naval Research Laboratory, Washington, D. C.;
Mr. Alfred Titus Anderson, Sewanhaka High School, Floral Park, N.Y., and Cooper Union, New York, N. Y.;
Mr. John Jacob Andrews, St. Louis University;
Dr. Franz L. Alt, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.;
Mr. Frank Arena, University of Michigan;
Mr. Eugene August Behmer, Minnesota State Highway Department, St. Paul, Minn.;
Mr. Paul Walter Berg, New York, N. Y.;
Dr. Felix Adalbert Behrend, University of Melbourne;
Professor Enrique Loizelier Blanco, University of Madrid;
Mr. Henry David Block, Iowa State College of Agriculture and Mechanical Arts;
Mr. Darrell Hastings Browne, Buffalo, N. Y.;
Mr. Elmer Jesse Bryson, Shell Development Company, Emeryville, Calif.;
Mr. Carlos Alberto Aragão de Carvalho, Federal District, Brazil;
Mr. Charles William Cassel, Jr., University of Dayton;
Professor Benedito Castrucci, University of São Paulo;
Mr. Leo George Chélius, Carbide and Carbon Chemicals Corporation, Oak Ridge, Tenn.;
Mr. Peter Joseph Cocuzza, Brooklyn, N. Y.;
Mr. Robert Reginald Coveyou, Monsanto Chemical Company, Knoxville, Tenn.;
Mr Charles William Crouse, Manufacturers Casualty Insurance Company, Philadelphia, Pa.;
Miss Nancy Lee Davidson, U. S. Naval Ordnance Test Station, Inyokern, Calif.;
Mr. David Bliss Dekker, University of California;
Mr. Lars Niclas Enequist, Bendix Aviation Corporation, Teterboro, N. J.;
Mr. Joseph Epstein, Philosophy Department, Columbia University;
Mr. Robert M. Exner, Syracuse University;
Mr. Edison Farah, University of São Paulo;
Mr. Thorleif Moland Fostvedt, University of Alberta;
Mr. Peter Alden Franken, New York, N. Y.;

Professor Fernando Furquim de Almeida, University of São Paulo;
Mr. Landis Gephart, University of Dayton;
Mr. Alvercio Moreira Gomes, University of Brazil;
Miss Elza Furtado Gomide, University of São Paulo;
Mr. Harry Herbert Goode, Office of Naval Research, Sands Point, N. Y.;
Mr. Earl Vincent Greer, Bethany-Peniel College, Bethany, Okla.;
Mr. Ralph Hafner, University of Dayton;
Mr. Norman Shannon Hall, Jr., University of California;
Mr. Warren Maurice Hirsch, New York University;
Mr. Temple Rice Hollcroft, Jr., N. C. A., Washington, D. C.;
Mr. Morton Allan Hyman, Naval Ordnance Laboratory, Washington, D. C.;
Mr. Eri T. Jabotinsky, Jerusalem, Palestine;
Mr. Reginald Charles Jacka, University of Alberta;
Professor Jaroslav Joseph Janko, University of Technical Sciences of Prague;
Mr. Robert Kahal, Polytechnic Institute of Brooklyn;
Mr. Herman Kahn, Los Angeles, Calif.;
Professor Joseph Kampé de Fériet, University of Lille;
Mr. Edward Lynn Kaplan, Naval Ordnance Laboratory, Washington, D. C.;
Professor Hendrik Douwe Kloosterman, University of Leiden;
Mr. Simon Harold Lachenbruch, National Bureau of Standards, Washington, D. C.;
Mr. Mario Laserna, New York, N. Y.;
Dean Ta Chung-Heng Li, National Fuh Tan University, Shanghai, China;
Professor Candido Lima a Silva Dias, University of São Paulo;
Miss Miriam Amalie Lipschutz, Massachusetts Institute of Technology;
Mr. Sherman Cabot Lowell, New York University;
Mr. Harold Edwin Malde, Portland, Ore.;
Dr. J. Carson Mark, Los Alamos Scientific Laboratories, Los Alamos, N. Mex.;
Mr. David G. Mead, West Orange, N. J.;
Mr. Sidney Melmore, Acomb, York, England;
Mr. Hasan Hasan Mostafa, Syracuse University;
Professor Antonio Aniceto Monteiro, University of Brazil;
Mr. Luiz Henrique Jacy Monteiro, University of São Paulo;
Mr. Wolfe Mostow, Bureau of Ships, Navy Department, Washington, D. C.;
Professor Leopoldo Nachbin, University of Brazil;
Mr. K. R. Narayana Row, Carnegie Institute of Technology;
Professor Ernesto Luis Oliveira, Jr., University of Brazil;
Professor Francisco Mendes de Oliveira Castro, School of Engineering, University of Brazil;
Professor Theodore Gleason Ostrom, Montana State University;
Dr. Mary Hewlett Payne (Mrs. W. T.), University of Detroit;
Professor Marillia Chaves Peixoto, School of Engineering, University of Brazil;
Professor Maurico Matos Peixoto, School of Engineering, University of Brazil;
Dr. Charles Dillon Perrine, Córdoba, Argentina;
Mr. Robert Richard Putz, Extension Division, University of California;
Professor Cortez Benjamin Rader, University of Houston;
Mr. Otho Mills Rasmussen, University of Kansas;
Professor C. Carl Robusto, St. John's University;
Miss Sally Irene Rothstein, University of Wisconsin;
Dr. John Paul Scholz, Bartol Research Foundation, Franklin Institute, Swarthmore, Pa.
Mr. Benjamin L. Schwartz, Carnegie Institute of Technology;

Professor Raymond Coffey Staley, University of North Dakota;
Miss Irene Anne Stegun, Mathematical Tables Project, National Bureau of Standards, New York, N. Y.;
Professor Haldum Koca Taskin, Milwaukee School of Engineering, Milwaukee, Wis.;
Mr. Gino Turrin, Buenos Aires, Argentina;
Professor Paul Vincensini, University of Besançon;
Mr. John Howell White, U. S. Naval Academy;
Dr. Margaret Frances Willerding, Washington University;
Professor Martin Wright, University of Houston;
Mr. Norman Joseph Zabb, Brooklyn, N. Y.;
Mr. Lotfi Asker Zadeh, Electrical Engineering Department, Columbia University;
Mr. Julian Joel Zeig, College of the City of New York.

It was reported that the following had been elected to membership on nomination of institutional members as indicated:

Harvard University: Dean Gordon Maskew Fair;
Institute for Advanced Study: Professor Wei Liang Chow and Mr. Sven Herbert Hilding.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker Vereinigung: Professor Paul Funk, Technische Hochschule, Vienna, Austria; Dr. Maximilian Joseph Pinl, University of Cologne; Professor Hans Zassenhaus, University of Hamburg. London Mathematical Society: Professor Harold Davenport, University of London; Dr. Arthur Erdélyi, Edinburgh University; Dr. Hans Arnold Heilbronn, University of Bristol; Dr. Archibald James Macintyre, University of Aberdeen; Dr. Dan Pedoe, Westfield College, London; Professor Harry Raymond Pitt, Queen's University, Belfast, Northern Ireland; Professor Harold Stanley Ruse, University of Leeds; Professor Arthur Geoffrey Walker, University of Sheffield. Swiss Mathematical Society: Professor Johannes Gualtherus van der Corput, University of Amsterdam; Professor Walter Saxer, Eidgenössische Technische Hochschule, Zurich; Professor Albert Pfluger, Eidgenössische Technische Hochschule, Zurich; Professor Eugenio G. Togliatti, University of Genoa; Professor Georges Valiron, University of Paris. Unione Matematica Italiana: Professor Carlo Miranda, University of Naples.

The Council had adopted by mail vote the report of the Policy Committee for Mathematics, favoring the meeting in connection with the Mexico City meeting of UNESCO for a preliminary discussion of the re-establishment of an International Mathematical Union. Subsequent to this report of the Policy Committee and in view of the unwillingness of the International Council of Scientific Unions to sponsor such a meeting in Mexico City, it was agreed by the Policy

Committee to abandon the meeting that had been contemplated for Mexico City.

The Council had also approved by mail vote the publication of the Proceedings of the Symposium on *Non-linear problems in mechanics of continua*, held at Brown University on August 2–4, 1947.

The following appointments by President Einar Hille were reported: Professor Nelson Dunford as representative of the Society at the inauguration of Rosemary Park as President of Connecticut College on May 17, 1947; Professor Morris Marden as representative of the Society at the inauguration of Nelson Vance Russell as President of Carroll College on May 17, 1947; Dr. G. E. Forsythe as representative of the Society at the inauguration of Raymond Bernard Allen as President of the University of Washington on May 22–24, 1947; Professor H. S. MacDougal as representative of the Society at the inauguration of Fred H. Leinbach as President of South Dakota State College on June 2, 1947; Professor J. W. Tukey as representative of the Society on the Joint Committee for the Development of Statistical Applications in Engineering and Manufacturing; Professors P. R. Rider (Chairman), R. H. Bruck, J. E. Case, T. L. Downs, Walter Leighton, Messrs. C. W. Mathews, Marlow Sholander, Professor W. H. Roever as a Committee on Arrangements for the November, 1947, meeting at Washington University; Professors T. H. Hildebrandt (Chairman), Arnold Dresden, J. R. Kline, Marston Morse, P. A. Smith, Dr. Warren Weaver as a committee to study the problems of the library and housing (joint committee of Council and Trustees); Professors Eric Reissner (three years), J. J. Stoker (two years), William Prager (one year) as an Editorial Committee for Applied Mathematics Symposium Proceedings, with Professor Reissner appointed Chairman of the Committee as now constituted.

The Naval Ordnance Laboratory has appointed Dr. C. A. Truesdell as a member of the Laboratory's Committee of Professional Society Representatives for the specific purpose of liaison with the American Mathematical Society.

It was reported that Professor Hermann Weyl had accepted the invitation to deliver the Josiah Willard Gibbs Lecture at the 1948 Annual Meeting.

The following dates of meetings in 1948 were approved: February 28 in New York City; February 28 in Chicago, Illinois; April 16–17 in New York City; April 16–17 in Ann Arbor, Michigan; April 17 in Berkeley, California. The Council also voted in favor of holding a meeting in Vancouver, British Columbia in June, 1948, the details to be left to the Associate Secretary for the far west.

On recommendation of the Colloquium Editorial Committee, the Council voted to accept for publication in the Colloquium Series the manuscript of Professor R. L. Wilder, entitled *Topology of manifolds*. It was further voted to invite Professor G. A. Hedlund of the University of Virginia to deliver a series of Colloquium Lectures at the 1949 Summer Meeting.

On recommendation of the Mathematical Surveys Editorial Committee, the Council voted to accept for publication in that series a manuscript entitled *Valuation theory* by Professor O. F. G. Schilling and one entitled *Geometry of the zeros of a polynomial* by Professor Morris Marden.

The Finance Committee for the Birkhoff Memorial Project reported that, as of August 18, 1947, a total of $6,616.20 had been received in subscriptions and contributions as a result of the preliminary solicitation among friends of Professor Birkhoff and members of the Society. The Council and Trustees had recommended at an earlier date that the project proceed as soon as the sum of $6,000 had been received. Hence the Editorial Committee for the project has now been requested to begin the task of preparing the material for publication. The Council and Trustees approved the tentative price of $18.00 per set and the special pre-publication discount of 30 percent to members of the Society. It was further agreed by these two groups that the final price of the volumes should not be set before the contract for their printing has been let and that no subscriptions at the pre-publication rate should be accepted after December 31, 1947.

It was agreed to change the Society's Committee on Publicity to a joint Committee on Publicity of the Society and the Association. This action had been concurred in by the Board of Governors of the Association. Representatives of other organizations which may meet from time to time with the Society and the Association will be added to the committee as the occasion arises.

On recommendation of the Committee on Applied Mathematics, President Hille was authorized to appoint a Committee on Known Results. This committee is to make a survey of available compendia, including tables of infinite series, solutions of differential equations, et cetera, but not numerical tables; it is also hoped that this committee will propose possible action by the Society to remedy deficiencies revealed by such a survey.

Professor Szolem Mandelbrojt has been invited to deliver an address at the November, 1947, meeting at Washington University.

Professors R. P. Agnew and W. M. Whyburn were appointed representatives of the Society on the Council of the American Association for the Advancement of Science for the year 1948.

Professor T. H. Hildebrandt and Dean M. H. Ingraham were nominated as representatives of the Society in the Division of Physical Sciences of the National Research Council for the three-year period beginning July 1, 1948.

It was announced that the Secretary had submitted a report to the Rockefeller Foundation in July, 1947, regarding the activities of the Policy Committee for Mathematics. The Council and Trustees voted to appropriate a sum of $500 per year for the work of the Policy Committee which has been financed in the past by generous grants from the Rockefeller Foundation.

The Committee on Aid to Devastated Libraries reported that the first shipment of books and journals was on its way to twelve libraries in ten different countries. In response to the appeal published in the spring, a number of cash contributions, amounting in total to somewhat over $300, had been received. Among these contributions there are several from student bodies in different institutions. A gift of $250 was made by the Mathematical Association of America at the New Haven meeting. Further gifts of books and of periodicals were received from a number of individuals as well as from a number of mathematical journals.

The Council recommended to the Trustees that the Editors of Mathematical Reviews be authorized to publish in alternate years a complete list of the journals from which abstracts are made.

The President was authorized to appoint a committee to study the present organization of the Society and to make recommendations at an early date for such changes as they deem to be necessary to adjust the organization of the Society to present conditions.

In the Council Meeting and later at the Business Meeting of the Society it was announced that Mrs. Genevra Barrett Hutchinson, who died recently, had made a bequest of one thousand dollars to the Society in memory of her late husband, John Irwin Hutchinson, and in accordance with his wishes. John Irwin Hutchinson (1867–1935) was successively instructor, associate professor, and professor of mathematics at Cornell University from 1894 to 1935. He was one of the first cooperating editors of the Transactions of the Society, serving from 1902 to 1915. In 1910 he was vice president of the Society. He made frequent contributions of research papers, chiefly in analysis, but also in algebra and geometry. Two of his achievements deserve particular mention for their permanent value and the attention they attracted here and abroad: the introduction of the isometric circle in connection with automorphic functions, and the discovery of the infinite group of birational transformations of the general Kummer surface.

Abstracts of most of the papers read by title were published in the September issue of this Bulletin. Abstracts of the rest of the papers read by title and all of the papers read in person follow below. Abstracts whose numbers are followed by the letter "*t*" represent papers presented by title. Mr. Graves was introduced by Professor R. H. Cameron, Mr. Hahn by Professor J. H. Roberts, Mr. Reynolds by Professor A. T. Brauer, and Mr. Wall by Dr. H. W. E. Schwerdtfeger. The following papers were read by: 366, Professor Perlis; 367, Professor Hall; 378, Dr. Ayer; 381, Dr. Piranian; 386, Professor Helsel; 403, Dr. Ulam; 410, Professor Synge; 411, Dr. Neményi; 412, Mr. Prim; 414, Dr. Mary H. Payne; 420, Mr. Schwartz; 430, Dr. Aroian; 434, Dr. Lehmann; 438, Professor Fox; 439, Professor Samelson.

ALGEBRA AND THEORY OF NUMBERS

362. A. T. Brauer: *On the irreducibility of polynomials with large third coefficient.*

Let $f(x)=x^n+a^{n-1}+\cdots+a_n$ be a polynomial with integral rational coefficients. It was proved by Perron (J. Reine Angew. Math. vol. 132 (1907) pp. 288–307) that $f(x)$ is irreducible in the field of rational numbers if $a_n\neq 0$ and $a_2>16^{n-1}(|a_1|+|a_3|+\cdots+|a_n|)^2$. Lipka (Math. Ann. vol. 118 (1941–1943) pp. 235–245) proved that 16^{n-1} can be replaced by 9/2. In this paper this result is improved further. (Received July 28, 1947.)

363. Richard Brauer: *On the Schur index of the representations of groups of finite order.* Preliminary report.

It is shown that the Schur index of a representation F of a group G of finite order with regard to a field K of characteristic 0 can be expressed as a product of Schur indices of representations of suitable subgroups H of G. Each group H contains a cyclic normal subgroup A such that H/A is a group of prime power order p^r; the order of A is not divisible by p. The Schur index of the representation of H is the exact contribution of p to the Schur index of F. This shows in particular that the whole problem can be reduced to one dealing with soluble groups. (Received July 30, 1947.)

364. Mary P. Dolciani: *On the representation of integers by quadratic forms.*

Let S be a quadratic form $\sum_{i,j=1}^{n} a_{ij}x_ix_j$ in the variables $x_1, \cdots, x_n$ in which the a_{ii} and $2a_{ij}$ are rational integers; and let m, the rank of S, be greater than or equal to four. For a given integer c the problem is to determine whether or not there exist integral solutions of the equation $S=c$ and to give a quantitative measure of the number of such solutions of $S=c$. Associating with S a particular diagonal positive form P, the author considers the sum $A(s)=\sum \exp(-\pi sP)$ extended over all integral solutions of $S=c$. By use of a generalization of a method employed by Siegel (Amer. J. Math. vol. 63 (1941) pp. 658–680), it is proved that if S is an indefinite form which represents c in the ring of p-adic integers for all primes p, then, as ϵ tends to zero through positive values, the expression $\epsilon^{(m/2)-1}A(\epsilon)$ approaches a positive limit. Analogous results are also obtained for S a definite form, this case having been already treated by Kloosterman (Acta Math. vol. 49 (1926) pp. 407–464) and by Ross and Pall (Amer. J. Math. vol. 68 (1946) pp. 47–65). (Received July 26, 1947.)

365. Ben Dushnik: *A property of a set of permutations.*

Let S be a set of m elements. For any positive integer $k \leq m$, let T be a set of permutations of the elements of S which possesses the following property: (α) For any k elements of S, say a_1 $a_2, \cdots, a_k$, and for $j \leq k$ there exists a permutation in T in which a_j follows all the other elements a_i with $i \neq j$. The smallest number of permutations which a set T with property (α) must possess is denoted by $D(m, k)$. For a given m, it is difficult to obtain the value of $D(m, k)$ for small k (>2); however, it is possible to obtain an explicit formula for $D(m, k)$ if k is sufficiently large. (Received July 28, 1947.)

366. John Dyer-Bennet and Sam Perlis: *On the center of a nilpotent algebra.*

Let N be a nilpotent algebra with center C. Then C is an ideal of N if N has index two or three, but an example is given in which C is not an ideal, N having order seven and index four, N^2 having order four, N^3 having order one. It is shown that C always is an ideal if N has order $n<7$, and, more generally, if N has index four or five and the orders of certain difference algebras N^i-N^{i+1} are suitably restricted. If N has order n and index $n+1$ (the maximum index for the given order), then N is commutative, being generated by a single quantity. (Received July 25, 1947.)

367. Marshall Hall and Tibor Rado: *On Schreier systems in free groups.*

A system of elements in a free group is called a Schreier system if it contains the beginning sections of every element in the system. Proceeding in a way inverse to that followed by Schreier, it is shown that every Schreier system can be the coset representatives of a subgroup, and in fact a parametric representation is given for all subgroups with a fixed Schreier system of coset representatives. The word problem is solvable for subgroups given by this parametric representation. For a shortest Schreier system belonging to a subgroup, it is shown that the subgroup generators derived from the system automatically satisfy the Nielsen conditions for reduced generators. (Received May 19, 1947.)

368. S. A. Jennings: *The u-algebra of a nilpotent restricted Lie algebra.* Preliminary report.

Jacobson has shown that the Birkhoff-Witt algebra u belonging to a nilpotent restricted Lie algebra L over a field of characteristic p is itself a nilpotent algebra (Trans. Amer. Math. Soc. vol. 50 (1941) pp. 15–25). In this note bases for the ideals u^w, $w=1, 2, \cdots$, are determined in terms of the elements of a series of ideals of L analogous to the dimensional subgroups mod p of a p-group. These results are used to investigate the structure of certain Lie algebras associated with p-groups. Similar methods may be used to study the Birkhoff-Witt algebra of any nilpotent Lie algebra over an arbitrary field. (Received July 31, 1947.)

369. Irving Kaplansky: *Rings with a polynomial identity.*

M. Hall has observed that any division ring satisfying $(xy-yx)^2z=z(xy-yx)^2$ is a field or quaternion algebra. This is generalized as follows: if D is a division ring for which there exists a non-commutative polynomial $f(x_i)$ such that $f=0$ for x_i in D, then D is finite-dimensional over its center C. If the degree of f is less than 6, $[D:C]$ is 1 or 4. An essential step in the proof is the fact that the free non-commutative algebra generated by two indeterminates has a complete set of finite-dimensional representations. It is conversely true that any algebra of finite order satisfies a polynomial identity. (Received July 25, 1947.)

370. W. J. LeVeque: *The order of magnitude of number-theoretic functions.*

Let $f(p)$ be a strongly additive arithmetic function, with $|f(p)|\leq 1$, let $A_n=\sum_{p\leq n}f(p)/p$, $B_n=(\sum_{p\leq n}f^2(p)/p)^{1/2}$, where p is a prime number, and let ω_1, ω_2, ω be real numbers. Then if $B_n\rightarrow\infty$ with n, the number of integers m, $0<m\leq n$, for which simultaneously $f(m)<A_n+\omega_1B_n$, $f(m+1)<A_n+\omega_2B_n$ is $nD(\omega_1)D(\omega_2)+o(n)$, where $D(\omega)$ is the Gaussian integral. In particular, if $\nu(m)$ denotes the number of distinct prime divisors of m, the number of positive integers $m\leq n$ for which $\nu(m)<\log\log n+\omega_1(\log\log n)^{1/2}$ and $\nu(m+1)<\log\log n+\omega_2(\log\log n)^{1/2}$ is $nD(\omega_1)D(\omega_2)+o(n)$. It can be shown from this that the number of $m\leq n$ for which $\nu(m)<\nu(m+1)+\omega(2\log\log n)^{1/2}$ is $nD(\omega)+o(n)$, and hence that the number of $m\leq n$ for which $d(m)<d(m+1)\cdot 2^{\omega}(2\log\log n)^{1/2}$ is $nD(\omega)+o(n)$, if $d(m)$ is the number of divisors of m. A simpler proof is given for the special case of a theorem of Erdös and Kac (Amer. J. Math. vol. 62 (1940) pp. 738–742) where $f(m)=\nu(m)$. The proofs rely on Brun's method and the central limit theorem from the calculus of probability. The first theorem mentioned above was stated without proof by Erdös (Ann. of Math. vol. 47 (1946) p. 17). (Received July 23, 1947.)

371*t*. D. C. Murdoch: *A theorem on nilpotent groups.*

Corresponding to a fixed element a of a group G a series of subgroups $G_a^{(0)}\supseteq G_a^{(1)}\supseteq G_a^{(2)}\supseteq\cdots$ may be defined where $G_a^{(0)}=G$ and $G_a^{(i)}$ is the subgroup generated by all commutators $g^{-1}a^{-1}ga$ where g is an element of $G_a^{(i-1)}$. Each $G_a^{(i)}$ is a normal subgroup of $G_a^{(i-1)}$. If this series terminates in the unit element the element a will be called a nilpotent element of G. If G is finite it is shown that the nilpotence of every element of G implies the nilpotence of G in the usual sense. (Received July 30, 1947.)

372. Irving Reiner: *A generalisation of Meyer's theorem.*

The following theorem of A. Meyer (Journal für Mathematik vol. 103 (1888) p. 98) is generalized: Every properly primitive binary quadratic form represents infinitely many primes in any preassigned arithmetic progression $Mx+N$ consistent with the generic character of the form, where M and N are relatively prime. The generalization consists of replacing the set of classes of properly primitive binary quadratic forms of given determinant D, which forms a group under composition, by an arbitrary abelian group H whose elements are denoted by θ_i. A correspondence is assumed between H and the set G of numbers $m>0$ which are prime to M and of which D is a quadratic residue. By considering the Dirichlet series $\sum a_m(\theta_i)\chi(\theta_i)\tau(m)m^{-s}$ summed over all θ_i in H and all m in G, where $a_m(\theta_i)$ is an integer determined by the correspondence, $\chi(\theta_i)$ is a group character for H, and $\tau(m)$ is a character modulo m, a result is obtained which includes Meyer's theorem as a special case, but is more general. (Received July 16, 1947.)

373. M. F. Smiley: *The radical of an alternative ring*. Preliminary report.

N. Jacobson's definition of the radical of an associative ring (Amer. J. Math. vol. 67 (1945) pp. 300–320) is applied to alternative rings. It is shown that the resulting set is an ideal and certain elementary properties of the radical are established. This generalizes a result of M. Zorn (Ann. of Math. (2) vol. 42 (1941) pp. 676–686) concerning hypercomplex alternative rings, as well as a result of Dubisch and Perlis (Bull. Amer. Math. Soc. vol. 50 (1944) p. 53) concerning alternative algebras of finite order. A typical lemma encountered in the proof is that *if* $a+b+ab=0$ *in an alternative ring* A, *then* $a(bc)=(ab)c$ *for every* $c\in A$. (Received June 16, 1947.)

374t. R. R. Stoll: *Coset systems in semigroups*. Preliminary report.

Let S denote a semigroup ($=$a system closed with repect to an associative binary operation). A partition $P=P(S_\alpha)$ of S is called a *left* (*right*) *coset system* for S if xS_α ($S_\alpha x$) is contained in a P-block S_β for every x. P is *normal* if it is both a left and right coset system and moreover is permutable with every coset system in S. The last condition means: for every coset system Q the cover of a Q-block by P-blocks is a sum of Q-blocks. Analogues of the group theoretic results on composition and normal series are obtained. A theory of monomial representations is developed which generalizes those obtained by Rees (Proc. Cambridge Philos. Soc. vol. 36 (1940) pp. 387–400) for completely simple semigroups. (Received July 30, 1947.)

375. L. R. Wilcox: *Closure operations*. Preliminary report.

Let M be a set and $U(M)$ the set of all subsets of M. A *semi-closure* operation of $U(M)$ is an order homomorphism between $U(M)$ and itself such that $f(S)\supset S$ for $S\in U(M)$. If $f(f(S))=f(S)$, then f is a *closure operation*. The present paper obtains formulas for the join and intersection of any set of closure operations and studies the composition $S\rightarrow f(g(S))$ of two closure operations. The composition, usually only a semi-closure operation, determines uniquely a closure operation by means of a transfinite sequence of extensions. Application is made to the case $M=A\times A$ (A a set); the elements $R\in U(M)$ are then binary relations on $A\times A$. Closure operations carrying any R into its reflexive, symmetric and transitive extensions, respectively, are defined and found to possess many interrelations. Any relation R gives rise to a unique closure operation in $U(A)$. It is proved that for such an operation the closed (that is, invariant) sets constitute a complete ring of sets; conversely, any complete ring arises in this manner from a unique reflexive and transitive R. (Received July 28, 1947.)

376. Margaret F. Willerding: *Determination of all classes of positive quaternary quadratic forms which represent all* (*positive*) *integers*.

It was apparently known to Diophantus and first proved by Lagrange (*Oeuvres*, vol. III, 1869, pp. 189–201) that the form $x^2+y^2+z^2+w^2$ represents all positive integers. Ramanujan (Proc. Cambridge Philos. Soc. vol. 19 (1916) pp. 11–12) proved that there are only 54 sets of positive integers a, b, c, d, such that $F=ax^2+by^2+cz^2+du^2$ represents all positive integers. Dickson (Amer. J. Math. vol. 49 (1927) pp. 39–56) called all such forms universal. Since these are reduced forms the result may be stated: the number of classes of forms which are equivalent to a form without cross products is equal to 54. It is of interest to ask what is the number of universal classes of forms if cross product terms are allowed. If the cross product terms have odd coefficients the number is quite large. If only the so-called "classic" forms, that is, forms whose cross product terms have necessarily even coefficients, are considered the number is not large. Employing an extension of the method of Ramanujan the theorem is proved: There are 178 classes of universal classic positive quaternary quadratic forms. (Received July 16, 1947.)

ANALYSIS

377. R. P. Agnew: *Divergent series evaluable by matrix methods.*

Let $y_n=\sum_{k=1}^{\infty}a_{nk}x_k$ be a multiplicative matrix transformation with multiplier ρ; a sequence x_n is evaluable A to L if $y_n\rightarrow L$ as $n\rightarrow\infty$. It is shown that if $\rho=0$, then A evaluates each sequence x_n (bounded or unbounded) which (1) oscillates sufficiently slowly and (2) is dominated by a sequence of positive numbers which becomes infinite sufficiently slowly. If $\rho\neq 0$ and $x_1, x_2, \cdots$ is a *bounded* divergent sequence evaluable A to L, then the sequence $(x_n-L/\rho)\xi_n$ is evaluable A whenever the sequence ξ_n satisfies the conditions (1) and (2). (Received July 24, 1947.)

378. Miriam C. Ayer and Tibor Rado: *A note on convergence in length.*

Let $\mathfrak{x}_n(t)$, $t\in I$, $n=0, 1\ 2, \cdots$, represent a sequence of vector functions with components $x_n(t)$, $y_n(t)$, $z_n(t)$ which are defined, continuous, and of bounded variation on the interval I: $a\leqq t\leqq b$, and which converge uniformly on I to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively. Let $L(\mathfrak{x}_n)$ denote the length of the curve determined by the equations $x=x_n(t)$, $y=y_n(t)$, $z=z_n(t)$, $a\leqq t\leqq b$, $n=0, 1, 2, \cdots$. Under these hypotheses the sequence $L(\mathfrak{x}_n)$ may or may not converge—the general situation being that $\liminf L(\mathfrak{x}_n)\geqq L(\mathfrak{x}_0)$. It is also well known that if $L(\mathfrak{x}_n)\rightarrow L(\mathfrak{x}_0)$ then the total variation of $c_1x_n(t)+c_2y_n(t)+c_3z_n(t)$ on I converges to the total variation of $c_1x_0(t)+c_2y_0(t)+c_3z_0(t)$ on I for every choice of the (real) constants c_1, c_2, c_3. It is the purpose of this note to prove that conversely the convergence of the total variation of $c_1x_n(t)+c_2y_n(t)+c_3z_n(t)$ on I to the total variation of $c_1x_0(t)+c_2y_0(t)+c_3z_0(t)$ on I for every choice of the constants c_1, c_2, c_3 implies convergence of $L(\mathfrak{x}_n)$ to $L(\mathfrak{x}_0)$. This is a generalization of a result of A. P. Morse. (Received July 29, 1947.)

379*t*. R. P. Boas: *Simple sets of polynomials.* Preliminary report.

A simple set of polynomials $p_n(z)$ is one in which $p_n(z)$ is of degree n; we may write $p_n(z)=z^n+\sum_{k=0}^{n-1}p_{nk}z^k$. The idea of the present investigation is that $p_n(z)$ is "nearly" z^n if $|z|$ is large enough. The comparison theorem of Paley and Wiener (*Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 19, New York, 1934, p. 100) then leads to the following general result. If there is a sequence of constants β_n such that $|p_{nk}|\leqq\beta_{n-k}$, the basic series (in the terminology of J. M. Whittaker) in terms of $p_n(z)$ represents in $|z|<r$ every function analytic in $|z|<r$ provided that $\sum_{k=1}^{\infty}\beta_k r^{-k}<1$; moreover, the expansion is unique.

Several known theorems are special cases. The following corollaries appear to be new. If $\lim\sup \beta_k^{1/k} < \infty$, the basic series represents every function analytic in a sufficiently large circle. If $p_n(s)$ is an Appell set $(p_n'(s) = np_{n-1}(s))$, the basic series represents all entire functions of order 1, type less than r, if $\sum_{n=1}^{\infty} |p_{n0} r^{-n}/n| < 1$. (Received July 22, 1947.)

380*t*. H. B. Curry and I. J. Schoenberg: *On spline distributions and their limits: the Pólya distribution functions.*

Define the truncated power function x_+^{k-1} as x^{k-1} if $x \geq 0$, and as 0 if $x<0$ $(k=1, 2, \cdots)$. Let $M_k(x; s) = k(s-x)_+^{k-1}$, $\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_k)$, $x_0 < x_1 < \cdots < x_k$. The divided difference of order k of $M_k(x; s)$, with respect to the variable s for the values $s = x_0, x_1, \cdots, x_k$, that is (in Steffensen's notation) the function $M_k(x) = M_k(x; x_0, x_1, \cdots, x_k) = \sum_{\nu=0}^{k} k(x_\nu - x)_+^{k-1}/\omega'(x_\nu)$, is called a spline frequency function of order k. Its integral $\int_{-\infty}^{x} M_k(x)dx$ is called a spline distribution function. These functions have a number of interesting elementary properties of which the most noteworthy is the following identity concerning the divided difference of order k of a function $f(x)$: $f(x_0, x_1, \cdots, x_k) = (1/k!)\int_{-\infty}^{\infty} M_k(x; x_0, \cdots, x_k) f^{(k)}(x)dx$. The main result is as follows: Let $M_k(x)$ be a spline frequency function and assume that for a sequence of values of $k \to \infty$, $\lim \int_{-\infty}^{x} M_k(x)dx = L(x)$ as $k \to \infty$, in all continuity points of the distribution function $L(x)$. Then $L(x)$ is a Pólya distribution function (see Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 11–17), that is, its moment generating function $F(s) = \int_{-\infty}^{\infty} \exp(xs)dL(x)$ is the reciprocal of an entire function of the type $1/F(s) = e^{-\gamma s^2 + \delta s}\prod_{\nu=1}^{\infty}(1+\delta_\nu s)e^{-\delta_\nu s}$ $(\gamma \geq 0;\ \delta,\ \delta_\nu$ real). Conversely, every Pólya d.f. is the limit of a sequence of spline distribution functions. (Received July 28, 1947.)

381. W. F. Eberlein and George Piranian: *Ergodicity under semi-groups of Toeplitz transformations.*

The authors study the spectral analysis of Abelian semi-groups of regular Toeplitz transformations. The transformations are considered as mappings of the Banach space m/c_0 of equivalence classes of bounded sequences into itself (two sequences $x = x_n$ and $y = y_n$ are defined to be equivalent provided the sequence $x - y = x_n - y_n$ converges to zero). Special results are obtained for the semi-group generated by the Hölder transformation and for other semi-groups of Hausdorff transformations. For example, the set of points that are invariant under a given semi-group G (that is, the set of fixed points with respect to G) is examined, and it is shown that the equivalence class of every sequence that oscillates sufficiently slowly is invariant under all regular Hausdorff transformations; in particular, the e.c. of the sequence $x_n = \sin \log \log n$ is invariant under the semi-group generated by the Hölder transformation, and the e.c. of the sequence $x_n = \sin \log \log \log n$ is invariant under the semi-group of regular Hausdorff transformations. A study is also made of the sets of points X in m/c_0 that are ergodic with respect to a given Abelian semi-group G, that is, for which some sequence of transforms $B_iX = \sum_{j=0}^{i} a_{ij}A_{ij}X$ $(i = 0, 1, \cdots;\ \sum_{j=0}^{i} a_{ij} = 1,\ a_{ij} \geq 0,\ A_{ij}$ in $G)$ converges to a fixed point Y. (Received July 28, 1947.)

382*t*. Marianne Freundlich: *On normed rings.*

A completely continuous element of a normed ring R is a ring-element which is completely continuous considered as a linear operator on the underlying Banach space. It is shown that if the ring is irreducible then completely continuous elements are in the radical. Weakly complete continuity is defined similarly and the set of weakly completely continuous elements is shown to be a closed ideal. If the radical of R is the zero-ideal, then ring multiplication is continuous in both factors simultaneously and in each factor separately with no restrictions on R. Weakly complete continuity of every element is shown to be equivalent with reflexivity of R if R has a unit element, but the theorem may fail to hold in the absence of a unit. (Received July 28, 1947.)

383. R. E. Fullerton: *Representations of linear operators.*

The operators considered are bounded linear operators from a Banach space X with range in a space of functions summable over a σ-field. The expression for representing an operator from a general Banach space to an L space is given together with an expression for its norm. From this theorem representation theorems for operators from various function spaces to an L space are derived. It is shown in particular that an operator from the space L^p to the space L has the form $Tx = d/ds\int_a^b K(s, t)x(t)dt$ and its norm satisfies the inequality $\sup_s [\int_a^b |K(s, t)|^{p'}dt]^{1/p'} \leq \|T\| \leq 2 \sup_s [\int_a^b |K(s, t)|^{p'}dt]^{1/p'}$ where $1/p'+1/p=1$. If T is a non-negative operator the equality holds between the first two members of the above expression. Various other representation theorems are given including new proofs of some already known. (Received July 28, 1947.)

384. R. E. Graves: *An alternative definitition of an integral over the Wiener space.* Preliminary report.

Let C be the space of all continuous functions $x(t)$ on $[0, 1]$ which vanish at $t=0$, and let $\{\alpha_k(t)\}$ $(k=1, 2, \cdots)$ be a complete orthonormal set on $[0, 1]$, each of whose elements is of bounded variation. An n-dimensional α-quasi-interval Q is defined to be the set of all $x(\cdot)\in C$ such that $\xi_i<\int_0^1\alpha_i(t)dx(t)\leq\eta_i$ $(i=1, 2, \cdots, n)$, and the area of Q is defined as $\pi^{-n/2}\int\cdots\int\exp\left[-\sum_1^n u_j^2\right]du_1\cdots du_n$, where the range of integration is $\xi_i\leq u_i\leq\eta_i$ $(i=1, 2, \cdots, n)$. The author shows that it is possible to construct a measure theory from this definition of the area of an α-quasi-interval. An integral over C, termed the α-integral, is defined on the basis of this measure theory. The author then proves that every α-measurable functional is Wiener-measurable and that its α-integral is equal to its Wiener integral. The paper concludes with a proof that, with certain restrictions on the set $\{\alpha_k(t)\}$, every Wiener-measurable functional is α-measurable, and hence that the α and Wiener integrals are completely equivalent. (Received July 28, 1947.)

385. P. R. Halmos: *The range of a vector measure.*

This paper contains a simple proof of the fact that the range of a countably additive finite measure with values in a finite-dimensional real vector space is closed and, in the non-atomic case, convex. (Received July 9, 1947.)

386. R. G. Helsel and E. J. Mickle: *The Kolmogoroff principle for the Lebesgue area.*

Let $T:\mathfrak{x}=\mathfrak{x}(u, v)$ and $T^*:\mathfrak{x}=\mathfrak{x}^*(u, v)$, where $\mathfrak{x}$ is the vector with components (x, y, z), be any two continuous transformations from the unit disk $D: u^2+v^2\leq 1$ into xyz-space. If the inequality $|\mathfrak{x}(u_1, v_1)-\mathfrak{x}(u_2, v_2| \geq |\mathfrak{x}^*(u_1, v_1)-\mathfrak{x}^*(u_2, v_2)|$ is satisfied for every pair of points (u_1, v_1), (u_2, v_2) in D, then the Lebesgue areas of the Fréchet surfaces determined by T and T^* satisfy the inequality $A(T)\geq A(T^*)$. This result may be interpreted as showing that the Lebesgue area of a Fréchet surface satisfies the Kolmogoroff principle. (Received May 1, 1947.)

387. Fritz Herzog: *On upper and lower relative frequencies.*

Let $\{a_k\}$ be an infinite sequence of positive integers. Let $P(m; n)$ denote the relative frequency with which the positive integer m occurs among the first n of the a_k. Then the numbers $H(m) = \lim \sup_{n\to\infty} P(m; n)$ and $h(m) = \lim \inf_{n\to\infty} P(m; n)$ are, respectively, the *upper* and *lower relative frequencies* of m in the sequence. If the a_k are allowed to assume all positive integers (*infinite* case) then the $H(m)$ and $h(m)$ satisfy the relations $0 \leq h(m) \leq H(m) \leq 1$ and $H(m) - h(m) + \sum_{r=1}^{\infty} h(r) \leq 1$ for all positive integral m. It is shown moreover that these relations are the only restrictions placed upon the $H(m)$ and $h(m)$. That means that to any given set of values for $H(m)$ and $h(m)$, satisfying the above conditions, there exists a sequence $\{a_k\}$, having the prescribed values of $H(m)$ and $h(m)$ as its upper and lower frequencies. If the a_k are restricted to the integers $1, 2, \cdots, M$ (*finite* case) then the most general relations (in the same sense as above) between the $H(m)$ and $h(m)$ are: $0 \leq h(m) \leq H(m) \leq 1$, $H(m) - h(m) + \sum_{r=1}^{M} h(r) \leq 1$ and $h(m) - H(m) + \sum_{r=1}^{M} H(r) \geq 1$ for $m = 1, 2, \cdots, M$. (Received July 25, 1947.)

388. W. H. McEwen: *Derived series associated with an integro-differential system.*

Consider the integro-differential system $M(u) + \lambda u = 0$, $U_j(u) = 0$ $(j = 1, 2, \cdots, n)$ in which $M(u) = u^{(n)} + p_1 u^{(n+1)} + \cdots + p_n u - \int_a^b k(x, \xi) u(\xi) d\xi + \sum_1^n \phi_j(x) l_j(u)$, $U_j(u) = W_j(u) - \int_a^b \psi_j(\xi) u(\xi) d\xi$, and $l_j(u)$, $W_j(u)$ are linearly independent two-point boundary forms. Let $S_N(x)$ and $\sigma_N(x)$ denote the partial sums of the series expansions of the functions $f(x)$ and $M^q(f)$ relative to this system, q being a given positive integer. If $f(x)$ satisfies $U_j(f) = 0$, $U_j(M(f)) = 0$, $\cdots$, $U_j(M^{q-1}(f)) = 0$ $(j = 1, 2, \cdots, n)$ it is shown that $|f^{(k)} - S_N^{(k)}| \leq K \int_a^b |M^q(f) - \sigma_N| dx$ for $k = 0, 1, \cdots, qn - 1$; if in addition $M^q(f)$ is of bounded variation on (a, b) these quantities converge uniformly to zero on (a, b). (Received June 30, 1947.)

389. G. R. MacLane: *Approximation by finite Blaschke products.*

Let R be the set of all non-constant functions $g(z)$ holomorphic in $|z| \leq 1$ such that $|g(t)| = 1$ for $|t| = 1$. R consists of all products of a finite number of linear transformations leaving the interior of the unit circle invariant. Let E be the set of all functions $f(z)$ holomorphic in $|z| < 1$ with $|f(z)| \leq 1$. *Theorem*: R is dense in E; that is, given $f(z) \in E$, there exists $g_n(z) \in R$ $(n = 1, 2, \cdots)$ such that $g_n(z) \to f(z)$ uniformly on any closed subset of $|z| < 1$. Three different proofs are given: (1) starting with the Riemann image of $|z| < 1$ by $w = f(z)$, a sequence of Riemann configurations corresponding to functions g_n of R is constructed so that $g_n \to f$. (2) Factoring out a Blaschke product and taking logarithms the desired approximation follows from the definition of an integral since the kernel of the Poisson integral is readily obtained as a special limit. (3) A third proof follows from the fact that there exists a function of R of degree n which assumes the same values as $f(z)$ at any n given points. (Received July 14, 1947.)

390. Morris Marden: *A refinement of Pellet's theorem.*

The principal theorem proved, applied and generalized in this paper is the following. Let $G(r_0, r; p, \alpha)$ be the boundary curve of the gear shaped region obtained on deleting from the circle $|z| \leqq r$ the points common to the annular ring $r_0 < |z| < r$ and the p angles ($k = 0, 1, 2, \cdots, p-1$) $(2\alpha_0 - \pi + 4\pi k)/2p \leqq \arg z \leqq (2\alpha_0 + \pi + 4\pi k)/2p$. Let $f(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots + a_n z^n$, $a_p \neq 0$, be a polynomial such that the real polynomial $F_p(z) = |a_0| + |a_1| z + \cdots + |a_{p-1}| z^{p-1} - |a_p| z^p + |a_{p+1}| z^{p+1} + \cdots + |a_n| z^n$ has two positive zeros r and R with $r < R$. Then the real polynomial $\Phi_p(z) = [F_p(z) - a_0]/z$ has two positive zeros r_0 and R_0 with $r_0 < r < R < R_0$ and the polynomial $f(z)$ has p zeros in or on the curve $G(r_0, r; p, \alpha_0)$ where $\alpha_0 = \arg a_0/a_p$ and has no zeros between the curves $G(r_0, r; p, \alpha_0)$ and $G(R, R_0; p, \alpha_0)$. When $p = n$, this theorem coincides with the result due to Lipka in Monatshefte für Mathematik und Physik vol. 50 (1944) pp. 209–221. (Received July 16, 1947.)

391. H. J. Miser: *Generalized conformal representations of Fréchet surfaces.*

The author defines in terms of elementary properties an essentially nondegenerate transformation $T(P) = P^*$, where T is continuous, P is a 2-cell, and P^* is a subset of Euclidean 3-space. If one of two Fréchet equivalent transformations is essentially nondegenerate, the other is also. Let K be the class of Fréchet surfaces whose representations are essentially nondegenerate. Then saddle-surfaces and Lebesgue monotone surfaces (cf. McShane, Trans. Amer. Math. Soc. vol. 35 (1933) pp. 716–733) and nondegenerate surfaces (cf. Morrey, Amer. J. Math. vol. 57 (1935) pp. 692–702) are members of the class K. If a Fréchet surface S is of class K and has finite Lebesgue area, then S admits of a generalized conformal representation; this theorem includes similar theorems of McShane (loc. cit.) and Morrey (loc. cit.). No "middle-space topology" is used in establishing the above results, while Morrey's proof that his theorem includes McShane's uses middle-space topology. Finally, using middle-space topology the author shows that the class K is the class of nondegenerate surfaces, so that the new theorem stated here is equivalent to Morrey's theorem. (Received July 28, 1947.)

392. C. N. Moore: *Generalized limits in general analysis.* IV.

The present paper constitutes an extension to multiple limits of a theorem in general analysis dealing with simple limits, presented at a previous meeting (Bull. Amer. Math. Soc. Abstract 53-5-220). It deals with necessary and sufficient conditions that a certain functional operation J, which includes as special cases sequence to function transformations and function to function transformations by means of infinite integrals, shall be a limit-preserving transformation. (Received July 28, 1947.)

393. G. B. Price: *Researches in the theory of functions of several real variables.* I. *Functions, limits, continuity.*

The functions considered are general transformations $F^{(m)}$: $y_j = f_j(x_1, \cdots, x_n)$, $j = 1, \cdots, m$. The $k+1$ points $X_0^{(n)}, X_1^{(n)}, \cdots, X_k^{(n)}$, $k \geqq 1$, determine an increment $\Delta[X^{(n)}; X_0^{(n)} X_1^{(n)} \cdots X_k^{(n)}]$ of the independent variable $X^{(n)}$: $(x_1, \cdots, x_n)$. The images $F^{(m)}(X_i^{(n)})$, $i = 0, 1, \cdots, k$, of these points determine the corresponding increment $\Delta[F^{(m)}; X_0^{(n)} X_1^{(n)} \cdots X_k^{(n)}]$ of the function $F^{(m)}$. The values of $\Delta[X^{(n)}; X_0^{(n)} X_1^{(n)} \cdots X_k^{(n)}]$ and $\Delta[F^{(m)}; X_0^{(n)} X_1^{(n)} \cdots X_k^{(n)}]$ are the k-measures of the k-cells determined by the points $X_i^{(n)}$ and $F^{(m)}(X_i^{(n)})$, $i = 0, 1, \cdots, k$, respectively. The limits considered are limits of expressions such as $\Delta[F^{(m)}; X_0^{(n)} X_1^{(n)} \cdots X_k^{(n)}]$ as $X_1^{(n)}, \cdots, X_k^{(n)}$ tend to $X_0^{(n)}$. If the increment of $F^{(m)}$ tends to zero as $X_1^{(n)}, \cdots, X_k^{(n)}$ tend to $X_0^{(n)}$ subject to certain specified conditions, then $F^{(m)}$ has the corresponding type of k-dimensional continuity. The results depend in an essential manner on the restrictions imposed on $X_1^{(n)}, \cdots, X_k^{(n)}$ as they tend to $X_0^{(n)}$. The case $k=1$ yields the ordinary theory of continuity. This paper contains an investigation of the various types of continuity and of their interrelations. The theory of determinants provides an essential tool. (Received July 24, 1947.)

394. W. T. Reid: *Expansion methods for the isoperimetric problem of Bolza in non-parametric form.*

In this paper sufficient conditions for a strong relative minimum in the isoperimetric problem of Bolza in non-parametric form are established by suitable modifications of the expansion method of proof previously used by the author (Ann. of Math. vol. 38 (1937) pp. 662–678; Trans. Amer. Math. Soc. vol. 42 (1937) pp. 183–190) for ordinary problems of Bolza in the calculus of variations. The present method of proof avoids the use of a Lindeberg theorem, in contrast to an earlier proof by the author (Duke Math. J. vol. 5 (1939) pp. 675–691) of a sufficiency theorem that is equivalent to the one herein given. (Received July 29, 1947.)

395. Arthur Sard: *Integral representations of remainders.*

Suppose that X, Y, Z are Banach spaces; that U is a linear operation on X to all of Y; and that V is a linear operation on X to Z. If $Vx=0$ whenever $Ux=0$, $x\in X$, there is a linear operation T on Y to Z such that $Vx=TUx$, $x\in X$. This theorem and known representations of linear functionals on particular function spaces lead to explicit direct procedures for obtaining integral representations of the remainder in many processes of computation. Particular examples are cited. Among the theorems proved is the following. Let C_m be the space of functions $x=x(s)$ with continuous mth derivative on the finite interval $a\leq s\leq b$, norm, addition, and scalar multiplication being defined in the customary way. If the functional Vx is linear on C_{n-p}, $0\leq p\leq n$, and vanishes whenever x is a polynomial of degree n, then $Vx=\int_a^b x^{(n+1)}(s)\beta(s)\,ds$ for any function x with absolutely continous nth derivative, where β is a p-fold integral of a function of bounded variation and $\beta(s')=-V\phi_{s'}$, $\phi_{s'}=\phi_{s'}(s)=(s-s')^n/n!$ if $s\leq s'$, $\phi_{s'}(s)=0$ if $s>s'$. The present paper is related to work of E. J. Rémès (1939, 1940). (Received July 28, 1947.)

396. W. T. Scott: *The corresponding continued fraction of a J-fraction.*

The author shows that there is a one-to-one correspondence between J-fractions, $K_1^\infty[c_p/(z+d_p)]$, and corresponding type continued fractions (C-fractions) of the form $-K_1^\infty(-1/b_pz^{\beta_p})$, where $\beta_1=1$, β_p is 0 or 1, $\beta_p+\beta_{p+1}>0$, $b_p\neq 0$. The methods used for J-fractions are applicable to these C-fractions, and a theory parallel to that for J-fractions is obtained. This includes a theorem of invariability and a positive definite case related to those of Hellinger and Wall (Ann. of Math. vol. 44 (1943) pp. 103–127), and a treatment of the indeterminate case similar to that of Wall (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 671–679). (Received July 17, 1947.)

397. F. C. Smith: *An alternative form for some theorems of Ford and Newsom.*

In this paper, the author obtains a new form for the asymptotic expansion of a class of functions previously studied by W. B. Ford in his *Asymptotic developments of* *functions defined by Maclaurin series* (University of Michigan Press, 1936), chap. 4. The results can be generalized to give a theorem parallel to that proved by C. V. Newsom (Amer. J. Math. vol. 40 (1938) pp. 561–572). As in Ford's and Newsom's work, the proofs are based upon the use of contour integrals and the calculus of residues. (Received July 24, 1947.)

398. Walter Strodt: *Note on linear difference equations*. Preliminary report.

Given (1) $\sum_{j=0}^{n} A_j y(x+\omega_j)=\phi(x)$, with $\phi(x)$ *of type* (M, γ). (That is, $\phi(x)$ is analytic at $x=0$ and in the sector $\mathfrak{S}: |\arg x| < \gamma$, and for every positive pair ϵ, δ there is a positive $C(\epsilon, \delta)$ such that $|\phi(x)| < C(\epsilon, \delta) \exp((M+\epsilon)|x|)$ whenever $|\arg x| < \gamma-\delta$.) Given that $\omega_0=0$ and that $\omega_1, \cdots, \omega_n$ are in $\mathfrak{S}$. Let $H(s)=1/[\sum_{j=0}^{n} A_j \exp(\omega_j s)]$. Let B be any point such that $H(B)$ is finite. Corresponding to M, but independently of $\phi(x)$, a constant g and finitely many functions $I_m(t)$ $(m=0, 1, \cdots, s)$ are, by contour integration, constructed from the principal parts of $H(s)$, half-lines $\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_s$ lying in $\mathfrak{S}$ are determined, and finitely many poles $p_1, \cdots, p_r$ of $H(s)$ (orders $j_1, \cdots, j_r$) are selected, such that the most general solution of (1) of type (M, γ) is $g\phi(x)+\int_0^\infty \phi(t)I_0(x-t)dt+\sum_{m=1}^{s}\int[B\phi(x+t)-\phi'(x+t)]I_m(t)dt+D(x)$, where the second integral is taken over $\mathcal{L}_m$ and $D(x)$ is a linear combination, with arbitrary constant coefficients, of $x^j \exp(p_k x)(k=1, \cdots, r;\ j=0, 1, \cdots, j_k-1)$. The basic method is a generalization of the author's method of *approximating q-difference equations* (Ann. of Math. (1943); Amer. J. Math. (1947)). (Received July 29, 1947.)

399. Gerhard Tintner: *Restricted maxima and minima which yield solutions homogeneous in some parameters.*

There are $m+1$ functions f, $g^{(j)}$ $(j=1, \cdots, m)$ of n variables $x_1 \cdots x_n$ and of $(N-n)$ parameters $x_{n+1} \cdots x_N$. By introducing m Lagrange multipliers λ_j, form the function $F=f+\sum_j \lambda_j g^{(j)}$. Derivatives are denoted by subscripts. Necessary conditions for an extremum of f with $g^{(j)}=0$ are: $F_r=0$ $(r=1, \cdots, n)$. Sufficient conditions involve a set of restricted Hessians. They have the sign (-1^m) for a minimum, are alternatively positive and negative for a maximum. A criterion for the solution x_r to be homogeneous of zero degree in the parameters is the determinant $E_r=0$. E_r is formed by substituting into the rth column of the *last* restricted Hessian the column: $-\{\sum_k x_k F_{1k} \cdots \sum_k x_k F_{nk} \sum_k x_k g_k^{(1)} \cdots \sum_k x_k g_k^{(m)}\}(k=n+1, \cdots, N)$. Important applications of homogeneous solutions occur in the modern theory of general equilibrium and employment in mathematical economics. (Received May 15, 1947.)

400. Leonard Tornheim: *On n-parameter families of functions and associated convex functions.*

An n-parameter family is a set of single-valued, real, continuous functions $f(x)$ on the interval $a \leqq x \leqq b$ such that for every set of points (x_i, y_i) $(i=1, \cdots, n)$ and $a \leqq x_1 < x_2 < \cdots < x_n \leqq b$ there is exactly one $f(x)$ with $f(x_i)=y_i$. An associated convex function is a function that intersects no member of the family more than n times. Beckenbach and Bing have discussed 2-parameter families and T. Popoviciu has investigated the case of linear systems. Among the results obtained is the following: if $2n$ sequences (x_{i1}), (y_{i1}); $\cdots$; (x_{in}), (y_{in}) $(i=1, 2, \cdots)$ converge to $x_1, y_1; \cdots; x_n, y_n$ respectively and if $x_{ij} \neq x_{ik}$ and $x_{ik} \neq x_{ij}$ $(j \neq k)$ then the sequence of functions

$f_t(x)$ of the n-parameter family determined by (x_{tj}, y_{tj}) $(j=1, \cdots, n)$ converges uniformly to the function determined by $(x_1, y_1), \cdots, (x_n, y_n)$. (Received July 29, 1947.)

401. W. J. Trjitzinsky: *Singular elliptic-parabolic partial differential equations.*

This is a continuation of the author's extensive investigation of singular elliptic and hyperbolic partial differential equations, currently being published in Rec. Math. (Mat. Sbornik). In the present paper he is concerned chiefly with singular equations of elliptic-parabolic type; these are transformed (throughout the domain of definition) into integral equations in appearance of the first and second kind. (Received May 12, 1947.)

402t. S. M. Ulam: *On quasi-fixed points for transformations in function spaces.*

A transformation T of a topological space E into itself has a quasi-fixed point—with respect to a system of k real-valued functions $f_1(p), \cdots, f_k(p)$ defined on E—if there exists a point p_0 such that $f_1(p_0)=f_1(T(p_0))$; $f_2(p_0)=f_2(T(p_0))$; $\cdots$; $f_k(p_0)=f_k(T(p_0))$. This notion (of coincidence in the sense of Lefschetz between the transformation L given by $f_1(p) \cdots f_k(p)$ and $L(T)$ is of interest in the case where E is a function space—T a (fixed point free) operator and $f_1, \cdots, f_k$ k given "properties" of the function which is an element of E, for example the first k coefficients in a series development, the first k moments of the function, and so on. Results are established on the existence of quasi-fixed points in the above sense for various special transformations—and systems of functionals occurring in analysis. The geometric properties of the manifold of all the quasi-fixed points in E are investigated in the case where T is an algebraic operator. (Received July 28, 1947.)

403. S. M. Ulam and John von Neumann: *On combination of stochastic and deterministic processes.* Preliminary report.

A computational procedure for the study of various differential equations—ordinary or partial—is investigated. It consists of a statistical model of the corresponding physical problem and involves a process which is a combination of deterministic and stochastic processes (see Bull. Amer. Math. Soc. Abstract 51-9-165). This procedure is analogous to the playing of a series of "solitaire" card games and is performed on a computing machine. It requires, among others, the use of "random" numbers with a given distribution. Various distributions of such numbers can, however, be obtained by deterministic processes. For example, starting with almost every x_1 (in the sense of Lebesgue measure) and *iterating* the function $f(x)=4x\cdot(1-x)$ one obtains a sequence of numbers on $(0, 1)$ with a computable algebraic distribution. By playing suitable *games* with numbers "drawn" in this fashion, one can obtain various other distributions, either given explicitly or satisfying given differential or integral equations. (Received September 3, 1947.)

APPLIED MATHEMATICS

404t. Stefan Bergman: *A representation for the generating function of an operator in the theory of a compressible fluid.*

The generating function E^* of the integral operator of the second kind $\mathrm{P}_2(f)=\int_{t=-1}^{1} H\mathrm{E}^* f[\mathrm{Z}(1-t^2)/2]dt/(1-t^2)^{1/2}$ (see formulas [55] and [69]; here and in the following, numbers in brackets refer to NACA, TN 972), which operator generates streamfunctions $\psi(\lambda, \theta) = \mathrm{Im}[\mathrm{P}_2(f)]$ of compressible fluid flows, can be represented in the form $\mathrm{E}^* = \sum_{\kappa=1}^{2} A_\kappa \sum_{n=0}^{\infty} q^{(n,\kappa)}/(-t^2 Z)^{n-1/2+(2/3)\kappa}$, where A_κ are constants, $Z = \lambda + i\theta$; $\lambda = \lambda(M)$, and θ are cartesian coordinates of the "pseudo-logarithmic" plane (see [48]). The functions $q^{(n,\kappa)} = \sum_{\nu=0} C_\nu^{(n,\kappa)}(-\lambda)^{n-1/2+(2/3)(\kappa+\nu)}$ are connected by the equations $2(n-1/2+2\kappa/3)q_\lambda^{(n,\kappa)} + q_{\lambda\lambda}^{(n+1,\kappa)} + 4Fq^{(n+1,\kappa)} = 0$, $n = 1, 2, \cdots$; $\kappa = 1, 2$; $q_\lambda^{(n)} = dq^{(n)}/d\lambda, \cdots$, where $4F = \lambda^{-2}S$, $S = (5/36) + \sum_{\nu=1}^{\infty} a_\nu(-\lambda)^{\nu/3}$, is given by [71]. E^* satisfies [88] (with F_m replaced by F), and $|C_\nu^{(n,\kappa)}| \leq C2^n/s_0^\nu$, where s_0 is the radius of convergence of $S(s)$, $s = (-\lambda)^{2/3}$, C being a suitably chosen constant. Continuing the solutions $\psi(\lambda, \theta)$ to complex values of the first argument, assuming $\Lambda = 0$, in $\psi(\lambda + i\Lambda, \theta)$ in the subsonic case, and $\lambda = 0$ in the supersonic case, the author obtains "mixed" flow patterns. (Received September 1, 1947.)

405*t*. Stefan Bergman: *Two-dimensional mixed flow patterns.*

In this paper the coefficient $l(\mathrm{H})$ in the equation $S(\psi) = l(\mathrm{H})\psi_{\theta\theta} + \psi_{\mathrm{HH}} = 0$ for the streamfunction of a compressible fluid is replaced by $-c_1\mathrm{H}$, c_1 being a constant (for notation see NACA, TN No. 972, p. 25). The new equation $S\dagger(\psi) = 0$ admits a comparatively simple mathematical treatment. In particular the generating function $\mathrm{E}\dagger$ of the integral operator $\mathrm{P}\dagger(f) = \int_{t=-1}^{1} \mathrm{E}\dagger(\lambda, \theta, t) f[2^{-1}(\lambda\dagger + i\theta)(1-t^2)]dt/(1-t^2)^{1/2}$ of the second kind which generates solutions of $S\dagger(\psi) = 0$ (see the above TN, p. 28) can be represented in the form $\mathrm{E}\dagger = c_2(-2\lambda\dagger)^{-1/6}F(1/6, 5/6, 1/2, \chi) + c_3(-2\lambda\dagger)^{-2/3} \cdot [-t^2(\lambda\dagger + i\theta)]^{1/2}F(2/3, 4/3, 3/2, \chi) = c_4[-t^2(\lambda\dagger + i\theta)]^{-1/6}F(1/6, 2/3, 1/3, 1/\chi) + c_5(-2\lambda\dagger)^{2/3}[-t^2(\lambda\dagger + i\theta)]^{5/6}F(5/6, 4/3, 5/6, 1/\chi)$ where $\chi = [t^2(\lambda\dagger - i\theta)/2\lambda\dagger]$, $-\lambda\dagger = c_6(-\mathrm{H})^{2/3}$, F the hypergeometric series, c_κ are constants. Analogous formulas hold for the supersonic region, in which case $\lambda\dagger$ is replaced by $\Lambda = i\lambda\dagger$, $\Lambda = c_7\mathrm{H}^{2/3}$. Sufficient conditions for f are given in order that the streamfunction $\psi = \mathrm{Im}[\mathrm{P}\dagger(\psi)]$ defined in the subsonic may be continued to the supersonic region. It appears that the results can be extended to the case of the exact equation $S(\psi) = 0$. (Received August 18, 1947.)

406. H. E. Goheen: *A non-rigorous suggestion for numerical solution of certain problems in heat generation.*

The partial differential equation (1) $\partial T/\partial t = \partial^2 T/\partial x^2 + (2/x)(\partial T/\partial x) + e^{-1/T}$ with boundary conditions (a) at $t = 0$, $T = \theta_0$, $0 < x < a$, and $T = \theta_1$, $x > a$; (b) at $x = \infty$, $T = w$ in which w is the solution of the ordinary differential equation $w' = e^{-1/w}$ with $w(0) = \theta_1$, and the partial differential equation (2) $\partial u/\partial t = \partial^2 u/\partial x^2 + e^{-1/u}$ with boundary conditions (a) at $t = 0$, $T = \theta_0$ if $|x| < a$ and $u = \theta_1$ if $|x| > a$; (b) at $|x| = \infty$, $u = w$, as above, are discussed. Assuming $T(x, t)$ in (1) and $u(x, t)$ in (2) to possess derivatives of all orders with respect to x at $x = 0$, a technique is developed which reduces the determination of each of these functions and their derivatives w.r.t. x at $x = 0$ to an infinite set of ordinary differential equations. Making a second assumption that the solution of each of these sets is approximated sufficiently well by a finite number (two or three) of them, an integration technique is determined. Making a third assumption that $T(x, t)$ and $u(x, t)$ are analytic functions of x at $x = 0$ with infinite radius of convergence, these values of derivatives and functions may be used to obtain the values of $T(x, t)$ and $u(x, t)$ for any x and t. (Received June 13, 1947.)

407*t*. F. G. Gravalos: *A consequence of Karman's law of similarity.*

A complete logical exposition of Karman's law of similarity of turbulent flow is lacking. Based on Karman's own remarks, an attempt is made to fill that gap. As a consequence of such analysis, the phenomenology of turbulent boundary layers may be given a new formulation; namely that, "The boundary layer's Reynolds number is a functional of the local velocity profile." (Received July 24, 1947.)

408. A. E. Heins: *Water waves over a channel of infinite depth with a dock.*

The author is concerned with the solution of $(*)\Delta\phi(x, y) - k^2\phi(x, y) = 0$ in the half-space $-\infty < x < \infty$, $y \geqq 0$ subject to the boundary conditions (a) $\partial\phi/\partial y = 0$ at $y = 0$, $x < 0$ and $\partial\phi/\partial y = \beta\phi$ at $y = 0$, $x > 0$ $(\beta > 0)$. The velocity potential $\phi(x, y)$ is assumed to possess a logarithmic singularity at $x = 0$ in order that $(*)$ have a traveling wave term for $x \gg 0$. It is further assumed that $\phi(x, y) = O(e^{\gamma r})$, $r = (x^2 + y^2)^{1/2}$ for $r \to \infty$ and $\gamma < k$. The surface $y = 0$, $x < 0$ is the dock-like surface, while $y = 0$, $x > 0$ is the free surface of the liquid. This boundary value problem may be formulated as an integral equation of the Wiener-Hopf type in terms of a Green's function kernel which may be exhibited explicitly. The Fourier transform (in the complex domain) is used to solve this integral equation. It is found that there exists a traveling surface wave on the surface $x > 0$, $y = 0$ provided $\beta^2 > k^2$. Fourier methods are not applicable to the case $k = 0$. It is possible to provide the asymptotic form of $\phi(x, y)$ for $x, y \gg 0$ as well as $x, y \to 0$. The technique of solution of the Wiener-Hopf integral equation has been recently applied by H. Feshbach and the present writer to a related boundary value problem in acoustics. (Received July 16, 1947.)

409. M. Z. Krzywoblocki: *On the two-dimensional steady turbulent flow of a compressible fluid far behind a solid symmetrical body (momentum transfer theory).*

Schlichting solved the equations of a steady turbulent flow of an incompressible fluid in a wake far behind a solid symmetrical body by application of the method of successive approximations. In the present paper this method is extended to apply to the solution of the equations of a steady turbulent flow of a compressible fluid in a wake far behind a solid symmetrical body. As is known, the complete tensor form for the eddy stress is due to Prandtl. Two other complete tensor forms for the stresses due to density fluctuations are proposed by the author. Throughout all the calculations in the paper the monentum transfer theory was used. This was done purposely. In the theory of incompressible turbulent flow it is the opinion that the velocity distributions by the momentum transfer theory or the vorticity transfer theory are the same. The question arises whether it is true for a compressible turbulent flow. To answer this question, the author intends to solve the entire system of equations based on both theories. In the present paper he solves the first part. The second part will be treated in a separate paper. (Received July 11, 1947.)

410. C. G. Maple and J. L. Synge: *Aerodynamic symmetry of projectiles.*

According to the usual hypothesis, the aerodynamic force system (force plus couple) on a projectile is a function of its instantaneous motion (velocity plus angular velocity). If the projectile possesses symmetry, the dependence of the force system on the instantaneous motion is conditioned by this symmetry. In this work, a projectile with n-gonal symmetry is considered, with or without reflectional symmetry relative to a plane containing the axis of n-gonal symmetry, and it is assumed that the elements of the force system can be expanded in power series in the components of cross-velocity and cross-spin, the coefficients depending on the axial components of velocity and spin. The effect of symmetry is to make some of these coefficients disappear, and to establish odd or even dependence on axial spin in the case of surviving coefficients. If axial and cross-spin vanish, expansion for the axial torque starts with the nth power of the cross-velocity. (Received July 28, 1947.)

411. P. F. Neményi and R. C. Prim: *On the equations of plane rotational flow of a perfect gas in natural coordinates and on plane rotational gas flow with constant velocity magnitude or constant vorticity along the streamlines.*

The partial differential equations governing steady plane rotational flow of a perfect gas are referred to a natural coordinate system consisting of the streamlines and an orthogonal family of curves. These equations are used to investigate the possibility of flows in which the velocity is constant along each streamline. It is proved that these two types of flow are possible only in streamline patterns consisting of concentric circles or of parallel lines. (Received July 30, 1947.)

412. P. F. Neményi and R. C. Prim: *Some patterns of vorticose flow of a perfect gas.*

Complete analytic solutions of definite boundary value problems of compressible, rotational flows of a perfect gas are extremely difficult to obtain. Therefore, in the present stage of study of rotational gas flow the inverse, or semi-inverse, approach promises to give most instructive solutions. In the present paper broad families of flow patterns satisfying the differential equations of steady, rotational gas flow and having throughout the field certain prescribed geometric or kinematic properties are studied. Beltrami and Massotti have investigated classes of incompressible flow patterns characterized by certain specific relations between the velocity and vorticity vector fields. In the present paper the implications of the same kinematic conditions are investigated for the case of compressible flow. In addition, an exhaustive survey is made of all steady flows of a perfect gas for which, in a cylindrical, isometric coordinate system, the three velocity components depend only on one coordinate. It is proved that the only generating coordinate net for such solutions consists of logarithmic spirals and their limiting cases. (Received July 30, 1947.)

413*t*. John von Neumann and H. H. Goldstine: *Some estimates on the numerical stability of the elimination method for inverting matrices of high order.*

Since computing machines give necessarily only approximately correct products and quotients, it is important, especially for high speed devices, to determine the loss of precision caused by given numerical procedures in their application to specific problems. The well known elimination method for inverting high order matrices is particularly important since it involves highly iterative as well as implicit algorithms. In the present paper it is shown that if the matrix at hand is definite, a slightly modified version of this method yields an inverse in error below $20n^2\epsilon/\mu^2$ where n is the order of the matrix, ϵ the roundoff error, and where μ is the smallest proper value of the matrix, if the latter is so adjusted that all its elements are below 1/2. It is observed that the method depends on the decomposition of the matrix in question into a product C^*C, where C is semidiagonal. The proposed algorithm produces a matrix C_1 for which $C_1^*C_1$ lies very close to the correct C^*C. This estimate proves to be the really relevant one, and not that one of the deviation of C_1 from C. Extension to indefinite matrices is also discussed. (Received August 21, 1947.)

414. Mary H. Payne and W. T. Payne: *Geometrical spinor theory.*

The bipolar coordinates of a line element on the light cone are defined and their transformation properties stated. The transformation properties are the same as for a first rank spinor, so that the bipolar coordinates of a line element on the light cone may be regarded as the components of a first rank spinor. The scalar and spinor products of two first rank spinors are defined and interpreted geometrically. The direct product is shown to lead to a second rank spinor, and certain geometrical properties of second rank spinors are discussed. (Received July 23, 1947.)

415. George Pólya: *Torsional rigidity increases by symmetrization.*

For example, a semicircle with radius a is changed by symmetrization with respect to its bounding diameter into an ellipse with semiaxes a and $a/2$. The torsional rigidity for the former cross-section is $.296\, a^4G$, for the latter $.314\, a^4G$. The general theorem is closely analogous to the following: *The pitch of the gravest tone of a membrane decreases by symmetrization.* In fact, both theorems can be proved by the same method, given for the membrane by G. Pólya and G. Szegö, Amer. J. Math. vol. 67 (1945) p. 14. Incomparably simpler is the proof for the following theorem: *The polar moment of inertia of a plane area with respect to its centroid decreases by symmetrization.* (Received July 25, 1947.)

416. L. V. Robinson: *The solution of linear partial differential equations by differential operator methods.* Preliminary report.

By making use of generalized differential operators, the formulas which give the solution of an ordinary linear differential equation can be extended to solve partial differential equations of the type $\partial u/\partial x+P(x, y, z)\partial u/\partial y+Q(x, y, z)\partial u/\partial z+R(x, y, z)u=S(x, y, z)$. These operators are of the type $\exp\left[G_1(x, y, z)\partial/\partial y+G_2(x, y, z)\partial/\partial z\right]$; and a table is given for the various forms of the functions, G. The work involved in solving such equations is thereby considerably reduced. (Received July 28, 1947.)

417. A. E. Schild: *Discrete space-time and integral Lorentz transformations.*

A cubic lattice of points in Minkowski space-time is considered, the four coordinates of the lattice points being integers. The "integral" Lorentz transformations, which leave the cubic lattice invariant as a whole, form an infinite, discrete six-parameter group. In a certain sense the cubic lattice is homogeneous and spatially isotropic. It may therefore be suitable as a background for physical theory. (Received July 3, 1947.)

418. A. C. Sugar: *The coding of boundary value problems for an electronic computer.* Preliminary report.

First, using the Liebmann form of the finite difference method for finding an approximate solution of the Dirichlet problem (which is actually a specialization of the Gauss-Seidel iterative process), a code is prepared for carrying out this process on an electronic digital computer. A generalization (by abridgment) of the UNIVAC instruction Code C-5 was used. First the rectangular boundary was considered and then the non-rectangular boundary. It is planned to consider the biharmonic equation for rectangular and non-rectangular boundaries and finally to code the more general Gauss-Seidel process and by comparison determine the extent of the advantages derived from the specialized codes. (Received September 3, 1947.)

419. C. A. Truesdell: *On the reliability of the membrane theory of shells of revolution.*

The author investigates the correctness of the stress and moment resultants predicted by the membrane theory of shells of revolution when the load and support are not rotationally symmetric and when the boundary conditions are consistent with a state of membrane stress. These distributions are consistent with the results of ordinary three-dimensional elasticity when the apex of the shell is open or when it is closed but flat, but he shows that they may be unrealistically sensitive to slight changes in the meridian curve when the apex is pointed. The method of proof throughout depends upon the stress functions of Némenyi and the author's prior investigation of the membrane theory (Trans. Amer. Math. Soc. vol. 58 (1945) pp. 96–166). The author suggests a possible substitute theory based on stress and moment averages rather than stress and moment resultants, and not requiring that the shell radius be much smaller than its thickness. This theory if properly constructed agrees with the present membrane theory when that theory is correct, but deviates from it in problems involving pointed shells. The equilibrium equations of this theory are easy to establish, but it is not clear what the proper approximate relations giving the stress and moment averages in terms of the displacements should be. (Received June 30, 1947.)

420. C. A. Truesdell and R. Schwartz: *On the Newtonian mechanics of continua.*

The author states the principles governing the motion of all Newtonian continua and gives formulations of the theories of perfectly elastic bodies and of viscous, compressible, heat conducting fluids. The analysis is valid for inhomogeneous, nonisotropic media. To the familiar principles of conservation of mass and momentum must be added the principle of conservation of energy stated sufficiently generally to be valid in the motion of inhomogeneous media. To this end a local phenomenological theory of the thermodynamics of inhomogeneous media is constructed. Murnaghan's theory of finite deformations of an elastic solid is generalized so as to apply to nonhomogeneous bodies non-uniformly heated. A viscous fluid is defined as a continuum in which the rate of dissipation of energy per unit volume is a function only of the thermodynamic state, the mean stress, and the deformation rate tensor. This definition leads to a nonlinear theory of viscous fluids, to which the Navier-Stokes theory is a first order approximation. (Received July 30, 1947.)

421. Henry Wallman: *Electronic general-transform computer.*

A proposed method is described for rapid electronic evaluation of general integral transforms of the form $F(x)=\int_0^1 f(t)K(x,t)dt$, where $f(t)$ is an arbitrary real function and $K(x, t)$ an arbitrary real kernel. The complete computation consumes about .01 second and is repetitively presented on the face of a cathode-ray tube. The instrument consists of (a) a high-speed one-variable function table for $f(t)$, scanned in 100μ sec, (b) a high-speed two-variable function table for $K(x, t)$, scanned in two dimensions, as in television, in $10{,}000\mu$ sec, (c) an electronic multiplier, (d) an electronic averager, (e) a synchronized cathode-ray tube system. Easily treated special cases include the sine and cosine transform and the convolution integral. Because of its speed and the ease of varying the one-variable function table, the instrument promises to be capable of solving integral equations of various types, and, as a special case, of inverting matrices. (Received August 3, 1947.)

422. Alexander Weinstein: *On the torsion of shafts.*

The author investigates the torsion of a family of shafts with axial semi-infinite holes and with flat bottoms. The corresponding profiles are obtained by the method of sources and sinks in a five-dimensional space, the sources being distributed on a four-dimensional disk of variable radius and of a constant total strength. The investigation is based on previous results by the author on many-valued stream functions in generalized potential theory (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 59, 494). (Received July 15, 1947.)

GEOMETRY

423. P. O. Bell: *Projective curvature and projective torsion of a curve on an analytic surface.*

The author presents new geometric characterizations for Bompiani's projective curvature and the author's projective torsion of a curve C_λ on an analytic surface S at a point P_x. Let t_λ be the tangent to C_λ at P_x; l' a line through P_x intersecting the quadric of Lie in two points P_x and Q; and K the cone, with vertex at P_x, which contains the conic of intersection of the tangent plane to the quadric of Lie at Q with the quadric of Wilczynski at P_x. To characterize projective curvature let l' be the projective normal to S at P_x, and let π_n, π, π_0, $\pi_\pm$ be respectively the plane determined by t_λ and l', the tangent plane to S at P_x, the osculating plane of C_λ at P_x and the planes through t_λ which are tangent to the cone K, respectively. The projective curvature is given by the cross-ratio equations $k = \pm(\pi_n, \pi, \pi_0, \pm\pi)$. To characterize the projective torsion τ of C_λ at P_x replace in these cross-ratio equations k by 2τ, the plane π_n by π_0, and the planes π_0, $\pi_\pm$ by the planes p_3, $p_\pm$, respectively, defined as follows: p_3 is the plane determined by t_λ and the point d^3x/ds^3 where ds is the Fubini element of projective arc length, and $p_\pm$ are the planes through t_λ tangent to the cone K defined for the line l' which joins the points P_x and d^2x/ds^2. The covariant points d^2x/ds^2, d^3x/ds^3 have been geometrically characterized by the author (Trans. Amer. Math. Soc. vol. 50 (1941) p. 539). (Received July 25, 1947.)

424. H. S. M. Coxeter: *Coordinates in the projective plane.*

To define the homogeneous coordinates of a point P for a given triangle of reference XYZ and unit point U: on the conic $PUXYZ$, take an arbitrary point I; set up a parameter (or "abscissa") on the conic, taking the values 0, 1, ∞ at the respective points U, I, P; let x, y, z be the values of the parameter at X, Y, Z; then x, y, are the desired coordinates for P. They are homogeneous, since a different choice of I would have the effect of multiplying them all by the same constant. Initially, x, y, z

must be all distinct and different from zero, but this restriction can be removed later. Coordinates in n dimensions can be set up in an analogous manner, using a rational normal n-ic curve instead of a conic. (Received June 23, 1947.)

425. J. M. Field: *A kinematic characterisation of series of lineal elements in the plane and of their differential invariants under the group of whirl-similitudes and some of its subgroups.*

In a previous paper (Bull. Amer. Math. Soc. vol. 48 (1942)) the author investigated the geometry of lineal elements under the seven-parameter group of whirl-similitudes G_7. Now he determines a set of fundamental differential invariants of series of lineal elements under G_7 and under two of its subgroups: G_6 (group of whirl-motions) and G_3 (group of whirls). It is shown that series and their differential invariants under these groups can be given simple kinematic characterizations. (Received March 31, 1947.)

426. Michael Goldberg: *Circular-arc rotors in regular polygons.*

A rotor is defined here as a convex closed curve which remains tangent to all the sides of a fixed polygon during a complete rotation of the curve. The curves of constant width (which may rotate within a pair of parallel fixed lines) are a special class of rotors. The conception of rotors in regular polygons, the proof of their existence, and a method of constructing them have been attributed to Kakeya by Fujiwara (Tôhoku Science Reports vol. 4 (1915) pp. 303–312). Among these rotors are curves composed entirely of arcs of circles, but such curves have been described previously only for the equilateral triangle, the square and the regular pentagon. The author gives a general construction for obtaining a circular-arc rotor for each regular polygon. It has been shown by Blaschke (Math. Ann. vol. 76 (1914) pp. 504–513) and Fujiwara (Proc. Imp. Acad. Tokyo vol. 3 (1927) pp. 307–309 and vol. 7 (1931) pp. 300–302) that the rotors of least area for the triangle and the square are circular-arc rotors. The other circular-arc rotors, described in this paper, may conceivably share this minimal property. (Received April 2, 1947.)

427. V. G. Grove; *Generalized canonical lines.*

A generalization of the class of canonical lines associated with a point on an analytic surface is presented in this paper. To this class of generalized canonical (g.c.) lines belong the classical canonical lines. A geometric method of generating these g.c. lines is made to depend on an invariant family of curves, and an invariant point on the tangent to that curve of the family through the generic point of the surface. In particular the invariant family of curves used in the generation of the classical canonical lines is any one of the family of Segre curves. Generalizations of the integral invariants used by Wilkins to generate the classical lines may be used to generate the g.c. lines. Generalizations of the scroll directrices of Sullivan follow from these integral invariants. (Received June 23, 1947.)

428. S. B. Jackson: *Geodesic vertices on surfaces of constant curvature.*

In a former paper by the writer (Bull. Amer. Math. Soc. vol. 50 (1944) pp. 564–578) there were obtained five structural properties of plane curves with only two vertices. The present paper investigates the extent to which these results can be extended to curves having only two extrema of the geodesic curvature and lying in a simply-connected region of a surface of constant curvature. Four of the properties are still valid while the fifth has to be modified. A new property of such curves is established, namely that such curves have only a finite number of double points and arcs. The methods parallel closely those of the earlier paper, and use is made of the former results by mapping into the plane. Two theorems are established, analogous to well known results in the plane, relating the number of geodesic vertices on a curve to its number of intersections with a geodesic circle. It is not necessary in any of the results to specify the curvature of the surface or even its sign. (Received July 15, 1947.)

429*t*. W. H. Roever: *A geometric construction for choosing in their proper relation the projective scales on the axonometric axes of perspective axonometry.*

During the recent war much use was made of production illustration in the design of airplanes. The construction given by the author is applicable not only to the production of such illustration but also to the solution in the picture itself of the geometric problems of space from the data contained in the picture. (Received August 4, 1947.)

STATISTICS AND PROBABILITY

430. L. A. Aroian and Marguerite D. Darkow: *The fourth degree exponential function.*

It is shown that the fourth degree exponential function is supported by the Bernoulli probability function and the hypergeometric probability function as well as being the function for which the method of moments is the "best" method according to the criterion of maximum likelihood. In the general situation six moments, at most, are needed. The function is classified into two general groups depending on symmetry or asymmetry and each case is divided again into unimodal and bimodal distributions. Examples show that the function is very successful in graduating the main Pearson types and the Gram-Charlier Type A frequency function. Various generalizations of the exponential function are indicated. In addition to its wide generality, the greatest practical advantage of the new system is the simplicity of the numerical calculations. (Received July 23, 1947.)

431. K. L. Chung: *On the maximum partial sums of sequences of independent random variables.*

The asymptotic behavior of the maximum partial sums of a sequence of independent random variables is studied in this paper. Two groups of new limit theorems are established under general conditions. The first group deals with theorems of the "weak" type. The limiting distribution of the maximum partial sums is obtained with an estimate of the remainder, thus improving a recent result of Erdös and Kac. Another estimate is obtained for a different domain of variation, which plays an essential role in the sequel. These results correspond to the sharper forms of the central limit theorem. In the second group theorems of the "strong" type are obtained, giving precise lower bounds (in the sense of probability) for the maximum partial sums. These results form the exact counterpart to the general form of the law of the iterated logarithm, due to Feller, which give the precise upper bounds. A summary of the main results and methods has appeared in Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947), pp. 132–136. (Received July 21, 1947.)

432. Harold Hotelling: *Effects of non-normality at high significance levels.*

The effects of non-normality in the underlying population on the probabilities of "significance" by customary statistical tests are not well understood, in spite of numerous attacks, both mathematical and experimental, on the problem. Chung's proof that the distribution of the Student ratio t has in samples from an arbitrary population a distribution approaching normality for large samples tends to confirm the idea that non-normality makes little difference if only the sample is fairly large. But this holds only for a fixed range of values of t while the sample number N increases. If p is the probability that $|t| > t_0$ in samples of N from a normal population and p' is the corresponding probability for another population, it is shown that $\lim_{N\to\infty}\{\lim_{t_0\to\infty}(p'/p)\}$ may be zero or infinite or may take any finite value. Conditions that this limit be unity are concerned only with the "shoulders" of the population histogram, and have nothing to do with its moments or behavior at infinity. This suggests caution in applying familiar tests with high significnce levels and that the use of moments cannot lead to the most appropriate criterion of non-normality for this purpose. (Received July 16, 1947.)

433*t*. P. L. Hsu: *A general weak limit theorem for independent distributions.*

For every positive integer n let there be n distribution functions $F_{n1}(x)$, $F_{n2}(x)$, $\cdots$, $F_{nn}(x)$. Assume that $\lim_{n\to\infty} \max_{1\leq j\leq n}\{1-F_{nj}(x)+F_{nj}(-x)\}=0$. Let $F(x)$ be the convolution $F_{n1}(x)*F_{n2}(x)*\cdots*F_{nn}(x)$. Let $\psi(t)=mit+\int_{-\infty}^{+\infty}(e^{itx}-1-itx/(1+x^2))\cdot((1+x^2)/x^2)dG(x)$, with $G(x)\uparrow$ and $G(\infty)-G(-\infty)<\infty$. Let $F(x)$ be the (infinitely divisible) distribution law having $\exp\psi(t)$ as its characteristic function. In order to have $\lim_{n\to\infty}F_n(x)=F(x)$ at every continuity point of $F(x)$, it is necessary and sufficient that the following relations hold at every x greater than 0 such that $\pm x$ are continuity points of $G(y)$: (I) $\lim_{n\to\infty}\sum_{j=1}^{n}\int_{|y|\geq x}dF_{nj}(y)=\int_{|y|\geq x}((1+y^2)/y^2)dG(y)$, (II) $\lim_{n\to\infty}\sum_{j=1}^{n}\{\int_{|y|<x}y^2dF_{nj}(y)-(\int_{|y|<x}ydF_{nj}(y))^2\}=\int_{|y|<x}(1+y^2)dG(y)$, (III) $\lim_{n\to\infty}\sum_{j=1}^{n}\int_{|y|<x}ydF_{nj}(y)=m+\int_{|y|<x}ydG(y)-\int_{|y|<x}(1/y)dG(y)$. (Received June 1, 1947.)

434. E. L. Lehmann and Henry Scheffé: *On the problem of similar regions.*

If $X=(X_1,\cdots,X_n)$ is a set of random variables with a joint probability density depending on a set of parameters $\theta=(\theta_1,\cdots,\theta_m)$, and if $T=(T_1,\cdots,T_m)$ is a set of sufficient statistics for θ, then Neyman (Philos. Trans. Roy. Soc. London Ser. A vol. 236 (1937) pp. 333–380) has proved that a region w in the space of X is similar with respect to θ (that is, has constant probability for all θ) if it has the following structure: The intersections $w(t)$ of w with the surfaces $T=t$ have the property that the conditional probability of the sample point X falling into w given that $T=t$ does not depend on t. In the present paper a necessary and sufficient condition is found for the regions with the above structure to be the only similar regions. This condition is shown to be satisfied for a certain class K of probability densities which contains as special cases all densities for which the totality of similar regions has been previously determined. In particular the partial differential equations which Neyman (Ann. Math. Statist.

vol. 12 (1941) pp. 46–76) assumed were satisfied in his solution of the problem of similar regions are solved and it is shown that any density satisfying these equations belongs to the above class K. (Received July 10, 1947.)

435. Herman Rubin: *Some results on the distribution of quadratic forms from Gaussian stochastic processes.* Preliminary report.

If one considers the estimation of the parameters of a Gaussian stochastic process whose elements are continuous functions from the functional values over a finite interval, one often finds that certain parameters can be estimated exactly, and certain parameters can not. This result often depends on the distribution of quadratic functionals whose arguments are elements of the stochastic process under consideration. In this paper, it is shown that the elements of a certain class of quadratic functionals have distributions concentrated at a point, and that the elements of a different class do not; in this latter case, the characteristic function is computed. (Received July 26, 1947.)

436. D. F. Votaw: *Testing compound symmetry in a normal multivariate distribution.*

Let X be a t-order vector variate ($t \geqq 3$) having d.f. $F(X)$. Let the components of X be divided into mutually exclusive subsets. $F(X)$ is said to be *compound symmetric*, for the given division of its variates into subsets, if it is invariant over all permutations of its variates within these subsets. $F(X)$ is *completely symmetric* if the invariance holds over all permutations of its variates. In a normal, compound symmetric $F(X)$ there is equality of means, of variances, and of covariances within each subset of variates and equality of covariances between any two subsets of variates. In this paper likelihood ratio criteria are developed for various hypotheses involving compound symmetry in a normal distribution and in k normal distributions ($k \geqq 2$). Given that the corresponding null hypothesis is true, the moments of each criterion are obtained explicitly and the distribution of each criterion is identified as the product of independent beta variates. For the one-sample case, the distributions are given explicitly for $t=3$, 4, 5 for certain divisions of the variates into subsets. In a previous paper Wilks gave a very thorough treatment of the case of complete symmetry. (Received July 21, 1947.)

437. J. E. Walsh: *Some significance tests for the mean using the sample range and midrange.*

Consider a sample of size n ($2 \leqq n \leqq 10$) drawn from a normal population with mean μ. Let x_n be the largest value and x_1 the smallest value of the sample. Significance tests are developed to compare μ with a given hypothetical value μ_0 by use of the sample. These tests are based on the quantity $D = [(x_1+x_n)/2-\mu_0]/(x_n-x_1)$ $= [(\text{sample midrange})-(\text{hypothetical mean})]/(\text{sample range})$. One-sided and symmetrical tests are considered. Values of D_α such that $\Pr(D>D_\alpha | \mu=\mu_0)=\alpha$ are computed for $\alpha=.05$, .025, .01, .005. These values of D_α can be used to obtain one-sided tests at the .05, .025, .01, .005 significance levels and symmetrical tests at the .10, .05, .02, .01 significance levels. Efficiencies are computed for one-sided tests at the .05 and .01 significance levels. The efficiency is at least 90 percent for $n \leqq 6$ at the .05 significance level and for $n \leqq 8$ at the .01 level. The range-midrange test can be applied without computation through the use of an easily constructed graph. The application of a test requires only the plotting of the sample point (x_1, x_n) on this graph. (Received July 14, 1947.)

TOPOLOGY

438. Emil Artin and R. H. Fox: *On the complementary domains of spheres and simple closed curves in 3-space.*

In spherical 3-space examples are constructed of (a) a simply connected closed surface whose interior, though simply connected, is not homeomorphic to the interior of the unit sphere; (b) a simply connected closed surface each of whose complementary domains is homeomorphic to the interior of the unit sphere although no homeomorphism of space upon itself can transform the surface into the unit sphere; (c) a simple closed curve which bounds a 2-cell and whose complementary domain has an infinite cyclic fundamental group although no homeomorphism of space upon itself can transform the curve into the unit circle. (Received July 25, 1947.)

439. Beno Eckmann, Hans Samelson and G. W. Whitehead: *A note on fiber bundles.*

It is proved that it is impossible to fiber the n-sphere S_n (the Euclidean n-space E_n) with the torus T_m as fiber if $m>1$, for odd n, and if $m \geqq 1$, for even n. The proof makes use of the Eckmann-Hurewicz-Steenrod relations for homotopy groups in fiber spaces, and the Eilenberg-MacLane relations between homotopy and homology groups. (Received July 29, 1947.)

440. Samuel Eilenberg: *Homology groups of cartesian products.*

The cartesian product $K \times L$ is defined for abstract complexes K, L. If in each dimension the integral groups of chains of K and L are free abelian groups, then the integral homology group $H_r(K \times L)$ is isomorphic with the direct sum $H_p(K, H_q(L))$, $p+q=r$, of the homology groups of K with coefficients in the integral homology groups of L. Using universal coefficient theorems various other forms of the relation are obtained. (Received July 28, 1947.)

441. O. H. Hamilton: *Fixed points for interior transformations.*

It is shown that, if T is an interior continuous transformation of a bounded locally connected plane continuum M which does not separate the plane onto a topological 2-cell which contains M, then T leaves a point of M invariant. It is shown further that if T is an interior continuous transformation of a topological 2-cell I onto a continuum M which contains I, then T leaves a point of I invariant. (Received August 6, 1947.)

442. E. H. Larguier: *Homology bases with applications to local connectedness.*

The author studies sequences of direct-sum decompositions of the (augmented) homology group $H^r(S, \mathfrak{F})$ of a compact Hausdorff space S with coefficients over a field $\mathfrak{F}$. These give rise to sequences of subspaces of $H^r(S, \mathfrak{F})$, called *decomposing sequences*, and also to associated sequences of r-cycles of S, which serve to determine bases for $H^r(S, \mathfrak{F})$, suitably topologized. The converse procedure from sequences of r-cycles to decomposing sequences and finally to direct-sum decompositions is also possible. Conditions are given under which $H^r(S, \mathfrak{F})$ becomes a metric space; and further restrictions are indicated which assure the completeness of $H^r(S, \mathfrak{F})$. The homology group of a compact G_δ subset of a locally compact lc^r space verifies this abstract situation. It is further shown that if there exists a sequence of 0-cycles which can be associated with a decomposing sequence of subspaces, then there also exists a sequence of nontrivial 0-cycles carried by *pairs of points* associated with the same decomposing sequence. The application of the preceding concepts leads to necessary and sufficient conditions for imbedding a class of compact spaces in "minimal" 0-*lc* spaces, and incidentally to an extension of certain results of Wilder (Fund. Math. vol. 19, pp. 45–64) to non-metric compact spaces. (Received July 9, 1947.)

443. Paul Olum: *Homology with operators and mapping theory.*

Let K be a finite connected simplicial n-complex and T an arcwise connected and locally arcwise connected topological space. The author is principally concerned with the problem of determining the homotopy classes of mappings of K into T. Let $\pi_r(T)$ denote the rth homotopy group of T. Then the classification is known when $\pi_r(T)=0$ for $r>1$, and when $\pi_r(T)=0$ for $r<n$, as well as for some special cases. The author gives a simple and effective classification in the case $\pi_r(T)=0$ for $1<r<n$. The classification is given in terms of the following topological quantities associated with K and T: the fundamental group of K, $\pi_1(K)$; the groups $\pi_1(T)$ and $\pi_n(T)$; the operation of $\pi_1(T)$ on $\pi_n(T)$; a function k_{n+1} which assigns to $n+1$ ordered elements of $\pi_1(T)$ an element of $\pi_n(T)$; and an n-dimensional cohomology group with operators in K. This last is substantially the same as Steenrod's cohomology with local coefficients. There is also proved an extension theorem for the mappings of an $(n+1)$-complex into such a space T. This gives a necessary and sufficient condition that a mapping of the n-skeleton into T can be extended over the whole complex. (Received July 25, 1947.)

444. E. H. Spanier: *Cohomology theory for general spaces.*

Using a definition due to A. D. Wallace, a new cohomology theory for general spaces is defined in terms of functions from sets of points in the space to the coefficient group. This definition is quite similar to some definitions previously given by J. W. Alexander (Proc. Nat. Acad. Sci. U.S.A. vol. 21 (1935) pp. 509–512 and Ann. of Math. vol. 37 (1936) pp. 698–708). It is shown that this theory satisfies the cohomology axioms of Eilenberg and Steenrod (Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 117–120). The new groups coincide with the Čech cohomology groups on compact Hausdorff spaces and agree with the singular cohomology groups on locally finite polyhedra. Hence, the new theory is distinct from any of the theories now in use. (Received May 15, 1947.)

445t. A. D. Wallace: *Group-invariant continua.*

Let X be a compact connected Hausdorff space, Z a topological space which is also a group and f a map of $Z\times X$ into X such that $f(e, x)=x$, $f(s, (w, x))=f(sw, x)$ where e is the identity in Z. Define $s(x)=f(s, x)$. The author is concerned with the proposition: There exists a subcontinuum E of X with no cutpoint such that $s(E)=E$ for all s in Z. In this note the proposition is shown to be valid provided Z is abelian or connected or finite. When X is assumed metric and Z cyclic the result was first proved by W. L. Ayres (X locally connected) and extended by J. L. Kelley. (Received June 4, 1947.)

446. J. W. T. Youngs: *Extensions of a homeomorphism defined on the boundary of a 2-manifold.*

Given a 2-manifold, M, with boundary, B, and a homeomorphism, $k: B \to B$, this paper concerns the extension of k to a homeomorphism $k^*: M \to M$. If M is orientable an extension need not exist; there is a necessary and sufficient condition for the extendability of k. If M is non-orientable, then k can always be extended. (Received July 24, 1947.)

T. R. HOLLCROFT,
Associate Secretary

NOTES

Within the National Bureau of Standards a new division has been established, the National Applied Mathematics Laboratories. This division has been organized to conduct research and provide services in the field of applied mathematics. Dr. J. H. Curtiss has been named chief of the National Applied Mathematics Laboratories. There will be four separate laboratories: the Institute of Numerical Analysis, of which Mr. Albert S. Cahn has been appointed executive officer; the Computation Laboratory, of which Dr. A. N. Lowan has been appointed chief; the Statistical Engineering Laboratory, of which Dr. Churchill Eisenhart has been appointed head; and the Machine Development Laboratory, of which Dr. E. W. Cannon has been appointed chief.

Beginning with the October issue, the quarterly journal Mathematical Tables and other Aids to Computation will publish a new feature section, Automatic Computing Machinery, designed to disseminate information and news on research and development in the field of high-speed automatic calculating machinery.

Assistant Professor Florence E. Allen of the University of Wisconsin has been made assistant professor emeritus.

Professor H. E. Buchanan of Tulane University of Louisiana has retired with the title emeritus.

Professor H. B. Phillips of the Massachusetts Institute of Technology has been made professor emeritus.

Professor A. A. Albert of the University of Chicago is giving a course on the modern theory of linear algebras at the University of Buenos Aires.

Dr. G. E. Albert of the Naval Ordnance Plant, Indianapolis, Indiana, has been appointed to an associate professorship at the University of Tennessee.

Dr. R. F. Arens of the Institute for Advanced Study has been appointed to an assistant professorship at the University of California at Los Angeles.

Dr. P. G. Bergmann of the Navy Department has been appointed to an associate professorship of physics at Syracuse University.

Associate Professor A. W. Boldyreff of Wittenberg College, Springfield, Ohio, has been appointed to an associate professorship at the University of New Mexico.

Assistant Professor H. K. Brown of Lehigh University has been appointed to an assistant professorship of mechanical engineering at Northeastern University.

Associate Professor A. V. Bushkovitch of the College of Charleston has been appointed to an associate professorship of physics at St. Louis University.

Professor C. E. Carey of State Teachers College, Clarion, Pennsylvania, has been appointed to an associate professorship at Greenville College, Greenville, Illinois.

Assistant Professor F. M. Carpenter of the Missouri School of Mines has been appointed to an assistant professorship at the Colorado School of Mines.

Dr. P. W. Carruth of the University of Illinois has been appointed to an assistant professorship at Swarthmore College.

Mr. W. C. Carter of Harvard University has accepted a position as mathematician at the Ballistic Research Laboratories, Aberdeen, Proving Ground, Maryland.

Assistant Professor Mary Dean Clement of Wells College has been appointed to an assistant professorship at the University of Miami.

Professor L. W. Cohen of the University of Kentucky has been appointed to an assistant professorship at Queens College of the City of New York.

Assistant Professor Nancy Cole of Connecticut College has been appointed to an assistant professorship at Syracuse University.

Dr. J. B. Coleman has been appointed to a professorship at Presbyterian College, Clinton, South Carolina.

Dr. Ky Fan of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Notre Dame.

Adjunct Professor A. D. Fialkow of the Polytechnic Institute of Brooklyn has been appointed to an associate professorship.

Dr. N. J. Fine of the Navy Department has been appointed to an assistant professorship at the University of Pennsylvania.

Assistant Professor Caspar Goffman of the University of Kentucky has been appointed to an associate professorship at the University of Oklahoma.

Professor M. O. González of the University of Havana has been appointed to a visiting professorship at the University of Alabama.

Professor L. M. Graves of the University of Chicago has been appointed to a visiting professorship at Indiana University for the current academic year.

Mr. William Gustin of the University of California at Los Angeles has been appointed to an assistant professorship at Indiana University.

Dr. N. A. Hall of United Aircraft Corporation has been appointed to a professorship of thermodynamics at the University of Minnesota.

Assistant Professor R. C. Hildner of the College of Wooster, Wooster, Ohio, has been appointed to an associate professorship at the University of New Mexico.

Dean C. C. Hurd of Allegheny College, Meadville, Pennsylvania, has accepted the position of theoretical research engineer at Carbide and Carbon Chemicals Corporation, Oak Ridge, Tennessee.

Associate Professor Nathan Jacobson of Johns Hopkins University has been appointed to an associate professorship at Yale University.

Professor B. W. Jones of Cornell University will be on leave for the current academic year and will be at the California Institute of Technology.

Dr. H. A. Jordan has been appointed to an associate professorship at Colby College, Waterville, Maine.

Professor H. D. Larsen of the University of New Mexico has been appointed to a professorship at Albion College, Albion, Michigan.

Dr. W. S. Loud of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of Minnesota.

Dr. A. V. Martin of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of Nevada.

Mr. F. J. Massey of the University of California has been appointed to an assistant professorship at the University of Maryland.

Dr. A. K. Mitchell of Pratt and Whitney Aircraft Division of United Aircraft Corporation has been appointed to an associate professorship at the University of Maryland.

Associate Professor Josephine M. Mitchell of Texas State College for Women has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

Professor F. S. Nowlan of the University of British Columbia has been appointed to a visiting professorship at the Chicago Undergraduate Division of the University of Illinois for this academic year.

Assistant Professor O. G. Owens of the University of Nevada has accepted a position as research mathematician at New York University.

Assistant Professor S. T. Parker of the University of Louisville has been appointed to an assistant professorship at Kansas State College of Agriculture and Applied Science.

Dr. M. H. Protter of Brown University has been appointed to an assistant professorship at Syracuse University.

Professor F. M. Pulliam of Lambuth College, Jackson, Tennessee, has been appointed to an assistant professorship at the University of Kentucky.

Professor C. E. Rhodes of Washington College, Chestertown, Maryland, has been appointed to a professorship at Alfred University, Alfred, New York.

Assistant Professor L. A. Ringenberg of the University of Maryland has been appointed to a professorship at Eastern Illinois State College, Charleston, Illinois.

Associate Professor L. F. S. Ritcey of United College, Winnipeg, Manitoba, has been appointed to a professorship at the University of Manitoba.

Assistant Professor A. E. Schild of Carnegie Institute of Technology is on leave of absence and will be at the Institute for Advanced Study for the current academic year.

Mr. Henry Schutzberger of the University of Illinois has accepted a position in the Research and Development Division of the New Mexico School of Mines.

Assistant Professor C. E. Sealander of Iowa State College of Agriculture and Mechanical Arts has been appointed to an assistant professorship at Ohio State University.

Dr. Hyman Serbin of Fairchild Aircraft Company has been appointed to a professorship of aeronautical engineering at Purdue University.

Dr. E. B. Shanks of the University of Illinois has been appointed to an assistant professorship at Vanderbilt University.

Assistant Professor Herbert Solomon of Stanford University has been appointed lecturer at the Newark College of Engineering.

Professor F. F. Stephan of Cornell University has been appointed to a professorship of social statistics at Princeton University.

Professor C. F. Stephens of Prairie View University, Prairie View, Texas, has been appointed to a professorship at Morgan State College, Baltimore, Maryland.

Assistant Professor George Whaples of the University of Wisconsin has been appointed to an associate professorship at Indiana University.

Dr. Y. C. Wong of the University of Pennsylvania has been appointed to a professorship at the National Sun Yat-Sen University.

The following promotions are announced:

D. H. Blackwell, Howard University, to a professorship.

W. H. Bradford, John McNeese Junior College of Louisiana State University, Lake Charles, Louisiana, to an associate professorship.

Ethel B. Callahan, Hartwick College, Oneonta, New York, to a professorship.

Randolph Church, United States Naval Postgraduate School, to a professorship.

H. A. Davis, West Virginia University, to a professorship.

D. W. Hall, University of Maryland, to a professorship.

G. H. Handelman, Brown University, to an assistant professorship of engineering.

Fritz Herzog, Michigan State College, to an associate professorship.

W. J. Jaffe, Newark College of Engineering, to an assistant professorship.

A. R. Kirby, Fordham University, to an assistant professorship in the school of education.

E. L. Lehmann, University of California, to an assistant professorship.

D. H. Lehmer, University of California, to a professorship.

J. C. R. Li, Oregon State College, to an assistant professorship.

K. H. Murphy, West Virginia University, to an assistant professorship.

I. D. Peters, West Virginia University to an assistant professorship.

Edmund Pinney, University of California, to an assistant professorship.

George Piranian, University of Michigan, to an assistant professorship.

R. A. Rosenbaum, Reed College, to an associate professorship.

Abraham Seidenberg, University of California, to an assistant professorship.

Otto Szász, University of Cincinnati, to a professorship.

H. P. Thielman, Iowa State College of Agriculture and Mechanic Arts, to a professorship.

L. V. Toralballa, Fordham University, to an assistant professorship.

C. H. Vehse, West Virginia University, to a professorship.

František Wolf, University of California, to an associate professorship.

The following appointments to instructorships are announced: Berea College, Berea, Kentucky: Miss Ruth E. Porter; Brown University: Dr. A. L. Stewart; University of California: Dr. H. L. Alder, Mr. S. P. Diliberto, Dr. R. C. James, Mr. Frank Harary (Extension Division); Barnard College, Columbia University: Miss Grace L. Bolton; Johns Hopkins University: Mr. D. N. Silver; Lehigh University: Mr. J. W. Mettler; Los Angeles City College: Mr. R. E.

Horton; MacMurray College for Women, Jacksonville, Illinois: Mr. J. A. Schumaker; University of Michigan: Mr. P. S. Jones, Mr. E. E. Moise; Modesto Junior College, Modesto, California: Mr. H. J. Osner; University of Oklahoma: Miss Elaine V. Nantkes; University of Oregon: Mrs. Pearl Van Natta; University of Pennsylvania: Mr. Bernard Epstein; Queens College of the City of New York: Dr. J. E. Eaton; University of Rochester: Mr. W. J. Klimczak; Walter Hervey Junior College, New York, New York: Dr. Carl Hammer; Washington University: Dr. Franklin Haimo; West Virginia University: Mr. Thomas Bauserman, Miss Marcia Saile.

The School of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1948–1949. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of mathematics, Institute for Advanced Study, Princeton, New Jersey, and are returnable by February 1, 1948.

Mr. Bernard Mason of Hofstra College died August 25, 1947, at the age of thirty-four years.

Mr. F. M. Pederson, formerly associate professor at the College of the City of New York, died August 20, 1947, at the age of seventy-eight years. Mr. Pederson became a member of the Society in 1906 and was a member for many years.

NEW PUBLICATIONS

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Cartan, E. Leçons sur la géométrie des espaces de Riemann. 2d ed., revised. Paris, Gauthier-Villars, 1946. 378 pp.

Davaux, E. See Liapounoff, A.

Denjoy, A. L'énumération transfinie. Vol. 1: La notion de rang. Paris, Gauthier-Villars, 1946. 27+206 pp. 550 fr.

Fan, K. See Fréchet, M.

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Fréchet, M., and Fan, K. Introduction à la topologie combinatoire. Paris, Vuibert, 1946. 88 pp.

Humbert, P. Cet effrayant génie. · · · L'oeuvre scientifique de Blaise Pascal. Paris, Michel, 1947. 263 pp. 200 fr.

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Tables of spherical Bessel functions. Vol. 2. Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1947. 20+232 pp. $7.50.

Weaver, W., Edited by. The scientists speak. New York, Boni and Gaer, 1947. 13+369 pp. $3.75.

ON GREEN'S AND NEUMANN'S FUNCTIONS IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

S. BERGMAN AND M. SCHIFFER[1]

Introduction. The purpose of this paper is to describe an effective construction of Green's and Neumann's functions for a general class of linear, second-order partial differential equations of elliptic type in terms of a set of continuously differentiable functions, complete and orthonormal with respect to the domain considered. This result completes a previous paper[2] by us in which an analogous construction was given for the difference between the Green and Neumann functions. In order to obtain representations of the Green and Neumann functions separately, one had to assume in the former paper the knowledge of a fundamental solution of the differential equation considered. Since it is by no means an easy task to construct for a given linear partial differential equation of second order a fundamental solution, and since the method of this paper permits a direct construction of the Green and Neumann functions without knowledge of such a fundamental solution, our result seems to constitute a certain advance in the application of orthonormal sets to the theory of linear partial differential equations.

1. Generalities and notations. We choose a finite domain B in the plane (x, y) which is bounded by a set of M closed smooth curves C_ν $(\nu=1, 2, \cdots, M)$. We shall denote the boundary of B by $C=\sum_{\nu=1}^{M} C_\nu$. Let $P(x, y)$ be a continuous positive function in the closed region $B+C$. Consider the linear partial differential equation of elliptic type

$$(1) \qquad \frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=P(x, y)\phi, \qquad P(x, y)>0.$$

We associate with this equation the following Dirichlet integral

$$(2) \qquad D\{\phi, \psi\}=\iint_B\left(\frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y}+P\phi\psi\right) dxdy,$$

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[1] Research paper done under Navy Contract NOrd 8555-Task F at Harvard University. The ideas expressed in this paper represent the personal views of the authors and are not necessarily those of the Bureau of Ordnance.

[2] S. Bergman and M. Schiffer, *A representation of Green's and Neumann's functions* . . . , Duke Math. J. vol. 14 (1947).

which defines a scalar product for each pair of functions ϕ and ψ which are continuously differentiable in B and for which the integrals $D\{\phi,\phi\}$ and $D\{\psi,\psi\}$ taken over B in the Lebesgue sense converge. All functions of this type form a linear space Ω and the scalar product (2) defines a metric in this space.

For the sake of a concise notation, we shall throughout this paper denote a point (x, y) of the domain B by Z and the corresponding complex number $x+iy$ by z. Another point (u, v) of B will be denoted by W and analogously $u+iv=w$; in the same way, $(r, s)=T$, $r+is=t$.

A function $S(Z, W)$ is called a fundamental solution of (1) in B if it is a solution of (1) as a function of $Z\in B$ for fixed $W\in B$, which is finite everywhere in B with the exception of the point W, where it has a logarithmic singularity. It may be written in the form

$$S(Z, W) = A(Z, W) \log \frac{1}{|z-w|} + \alpha(Z, W) \tag{3}$$

where $A(Z, W)$ is a solution of (1) of class Ω which is symmetric in Z and W throughout the whole domain B. We may even require additionally

$$A(W, W) = 1. \tag{3'}$$

$\alpha(Z, W)$ is finite and continuously differentiable in Z at every point of B, but it is in general not a solution of (1). There exists an infinity of fundamental solutions of (1). We distinguish two particular fundamental solutions which are of special interest for our theory:

Green's function $G(Z, W)$ of the domain B with respect to the differential equation (1) is that fundamental solution which vanishes, if, for $W\in B$ fixed, Z converges to the boundary C of B.

Neumann's function $N(Z, W)$ of the domain B with respect to the differential equation (1) is that fundamental solution whose normal derivative vanishes on the boundary C of B for fixed $W\in B$, that is,

$$\frac{\partial N(Z, W)}{\partial n_z} = 0, \qquad Z \in C, \tag{4}$$

where n_z denotes the interior normal to C at the point $Z\in C$.

2. **Orthonormal sets of solutions of** (1). Let $\{\phi_\nu(Z)\}$ be a set of solutions of (1) each of which belongs to the class Ω. We call this set orthonormalized if it satisfies the conditions

$$D\{\phi_\nu, \phi_\mu\} = \delta_{\nu\mu}, \qquad \delta_{\nu\mu} = \begin{cases} 1 & \text{if } \nu = \mu, \\ 0 & \text{if } \nu \neq \mu. \end{cases} \tag{5}$$

Let Γ be the subclass of Ω consisting of all solutions $\phi(Z)$ of (1). We call the system $\{\phi_\nu(Z)\}$ complete with respect to the class Γ if every element of Γ may be approximated arbitrarily by a linear aggregate of functions $\phi_\nu(Z)$ uniformly in every closed subdomain of B. If this set is complete and orthonormalized, every solution $\phi(Z)$ of class Γ may be developed into a series[3]

$$\phi(Z) = \sum_{\nu=1}^{\infty} a_\nu \phi_\nu(Z), \qquad a_\nu = D\{\phi_\nu, \phi\}, \tag{6}$$

which converges uniformly in every closed subset of B.

The series

$$K(Z, W) = \sum_{\nu=1}^{\infty} \phi_\nu(Z)\phi_\nu(W) \tag{7}$$

extended over a complete orthonormalized system $\{\phi_\nu(Z)\}$ converges uniformly in every closed subdomain of B and represents there a symmetric function of Z and W which is in each variable a solution of (1). K is called the kernel function of the orthonormal set $\{\phi_\nu(Z)\}$.

By means of (5), (6) and (7), one proves easily that for every element $\phi(Z)$ of class Γ

$$\phi(Z) = D_W\{K(Z, W), \phi(W)\}, \tag{8}$$

where the subscript W of D denotes the variable with respect to which differentiation and integration are to be carried out. The characteristic property (8) of reproducing every function of class Γ after scalar multiplication determines the kernel function in a unique way, as may easily be shown. Therefore, the kernel function $K(Z, W)$ is independent of the particular orthonormal set $\{\phi_\nu(Z)\}$ from which it was constructed. This fact proves of great importance in the theory of orthonormal functions and permits numerous applications.

The significance of the kernel function was discussed in our previous paper and we proved the identity

$$K(Z, W) = \frac{1}{2\pi}[N(Z, W) - G(Z, W)]. \tag{9}$$

From (9) it follows that $K(Z, W)$ is a symmetric regular function of Z and W and a solution of (1) which has on C the same boundary values as $(1/2\pi)N(Z, W)$ and the same normal derivatives as $-(1/2\pi)G(Z, W)$.

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[3] S. Bergman, *On functions satisfying partial differential equations of elliptic type and their representation*, Duke Math. J. vol. 14(1947) pp. 349–366.

We proved further: If $f(Z)$ is a continuously differentiable function of Z except for isolated points where it may have logarithmic infinities and if $D\{f, f\} < \infty$ the function

$$\psi(Z) = D_W\{K(Z, W), f(W)\} \tag{10}$$

is a regular solution of (1) which has on C the same boundary values as $f(Z)$. This result leads at once to the construction of Green's function if an arbitrary fundamental solution is known. For if $S(Z, W)$ is this fundamental solution,

$$\sigma(Z, W) = D_T\{K(Z, T), S(T, W)\} \tag{11}$$

is a solution of (1) of the class Ω having for $Z \in C$ the same values as $S(Z, W)$. Hence

$$G(Z, W) = S(Z, W) - \sigma(Z, W) \tag{11'}$$

is just the required Green's function.

In this way we effect the construction of Green's function by means of an arbitrary fundamental solution and the kernel function. In view of (9) this leads to an explicit expression of Neumann's function, too. It is, however, often very difficult to obtain any fundamental solution $S(Z, W)$ and we shall therefore give in the following sections a construction of the Neumann function which does not presuppose the knowledge of any fundamental solution.

3. **An identity for the Neumann function.** Let $f(Z)$ be continuously differentiable in B, with the possible exception of a finite number of points where it may have logarithmic infinities. Let further $D\{f, f\} < \infty$. We suppose at first that $f(Z)$ has on the boundary C of B continuous boundary values. By virtue of Green's identity we have then

$$\begin{aligned} D_W\{N(Z, W), f(W)\} = & -\oint_C f(W) \frac{\partial}{\partial n_W} N(Z, W) ds_W \\ & - \sum_{\mu=1}^{r} \oint_{k_\mu(\rho)} f(W) \frac{\partial}{\partial n_W} N(Z, W) ds_W + \epsilon(\rho) \end{aligned} \tag{12}$$

where the $k_\mu(\rho)$ denote small circles of radius ρ around the point $W = Z$ and the infinities of $f(Z)$, and where all integrations are taken in the positive sense with respect to the domain B from which the circles $k_\mu(\rho)$ have been eliminated. The term $\epsilon(\rho)$ converges to zero if ρ does so. Because of the symmetry of the Neumann function in Z and W and in view of the vanishing of its normal derivative on C, the first integral on the right-hand side is zero. If we let $\rho \to 0$, all the integrals around the infinities of $f(z)$ converge to zero, while the integral around the point $Z=W$ tends to the limit $-2\pi f(Z)$. Hence we get in the limit $\rho=0$ finally

$$D_W\left\{\frac{1}{2\pi}N(Z, W), f(W)\right\} = f(Z) \tag{13}$$

for every function $f(Z)$ satisfying the aforementioned quite general conditions.

We may even drop the assumption on the boundary values of $f(Z)$ on C. For though it played a certain part during the proof, the boundary C has vanished in the final identity (13). We may approximate more general functions $f(Z)$ in B by functions which have continuous boundary values and by the usual considerations extend the results to the class Λ of all functions $f(Z)$ which have continuous derivatives in B except for a finite number of logarithmic infinities and for which the Lebesgue integral $D\{f, f\}$ converges.

We see from (13) that the Neumann function $N(Z, W)$ has with respect to the class Λ of continuously differentiable functions with a finite number of logarithmic infinities in B just the same characteristic property as has the kernel function (cf. (8)) with respect to the class Γ of solutions of (1). One might therefore expect that the Neumann function appears as a kernel function of a certain orthonormal set of functions of class Λ. A certain difficulty arises, however, from the possible logarithmic infinities of the functions considered and the logarithmic pole of $N(Z, W)$ itself. This fact leads to a certain complication in all our considerations.

4. The construction of the Neumann function by means of an orthonormal set. Consider the class Ω of all functions $g(Z)$ which have continuous derivatives in B and for which $D\{g, g\}<\infty$. Obviously $\Omega\subset\Lambda$. Let $\{g_\nu(Z)\}$ be a complete orthonormal set of the class Ω. In the case of a finite smoothly-bounded domain B, the existence of such a system is ensured by Weierstrass' approximation theorem, and it may even be chosen as a sequence of polynomials in x and y.

Since the Neumann function becomes infinite for $Z=W$, we have no possibility of developing it into a series of the $g_\nu(Z)$ which converges in every closed subdomain of B. But if we denote Green's function of B with respect to the ordinary Laplace equation by $\gamma(Z, W)$, we find immediately from (3) and (3′) that

$$l(Z, W) = N(Z, W) - \gamma(Z, W) \tag{14}$$

belongs to the class Ω. In fact, $N(Z, W)$ being a fundamental solution of (1), we may write

$$(14') \qquad l(Z, W) = [A(Z, W) - 1] \log \frac{1}{|z - w|} + \beta(Z, W)$$

where $\beta(Z, W)$ is of class Ω. The term $A(Z, W) - 1$ vanishes for $Z = W$ at least to the second order; for, upon introducing $N(Z, W)$ into the equation (1) and using the fact that $A(Z, W)$ is itself a solution of (1), one finds

$$(14'') \qquad \frac{\partial A(Z, W)}{\partial x} \frac{x - u}{|z - w|^2} + \frac{\partial A(Z, W)}{\partial y} \frac{y - v}{|z - w|^2} = \chi(Z, W)$$

where $\chi(Z, W)$ remains finite as $Z \to W$. Hence, one concludes easily that

$$(14''') \qquad \frac{\partial A(Z, W)}{\partial x} = O(|z - w|), \qquad \frac{\partial A(Z, W)}{\partial y} = O(|z - w|).$$

From (3') and (14''') we recognize at once that $l(Z, W)$ is of class Ω.

Let us compute now the Fourier coefficients of $l(Z, W)$ with respect to the set $\{g_\nu(W)\}$. In view of (13) and (14) we obtain

$$(15) \qquad \begin{aligned} c_\nu(Z) &= D_W\{l(Z, W), g_\nu(W)\} \\ &= 2\pi g_\nu(Z) - D_W\{\gamma(Z, W), g_\nu(W)\}. \end{aligned}$$

We associate with every function $g(Z)$ of class Ω a transformed function

$$(16) \qquad h(Z) = -\frac{1}{2\pi} D_W\{\gamma(Z, W), g(W)\}.$$

Then we may write instead of (15)

$$(15') \qquad c_\nu(Z) = D_W\{l(Z, W), g_\nu(W)\} = 2\pi[g_\nu(Z) + h_\nu(Z)]$$

where the $h_\nu(Z)$ are associated with $g_\nu(Z)$ by (16).

One proves in the usual way the Parseval equality

$$(17) \qquad \sum_{\nu=1}^{\infty} c_\nu(Z)^2 = 4\pi^2 \sum_{\nu=1}^{\infty} [g_\nu(Z) + h_\nu(Z)]^2 = D_W\{l(Z, W), l(Z, W)\}$$

and obtains in particular the convergence of the left-hand sum.

Consider next the function $l(Z, W) + l(T, W)$ which is, as a function of W, of class Ω. Its Fourier coefficient with respect to $g_\nu(W)$ is obviously $c_\nu(Z) + c_\nu(T)$. Applying now the Parseval equality with respect to this function, we find

$$\sum_{\nu=1}^{\infty} [c_\nu(Z) + c_\nu(T)]^2 = D_W\{l(Z, W) + l(T, W), l(Z, W) + l(T, W)\}. \tag{18}$$

By a simple transformation and using the identity (17), we obtain

$$\sum_{\nu=1}^{\infty} c_\nu(Z)c_\nu(T) = D_W\{l(Z, W), l(T, W)\}. \tag{19}$$

The left-hand sum converges in view of the convergence of the sum (17) and of the Schwarz inequality.

This result may be simplified again by using the definition (14) of $l(Z, W)$ and the identity (13). We find easily

$$\begin{aligned}\sum_{\nu=1}^{\infty} c_\nu(Z)c_\nu(T) = {} & 2\pi N(Z, T) - 4\pi\gamma(Z, T) \\ & + D_W\{\gamma(Z, W), \gamma(T, W)\}.\end{aligned} \tag{20}$$

We transform further the last term on the right-hand side. By definition (2) and Green's identity one obtains almost immediately (with $\Delta_W = \partial^2/\partial u^2 + \partial^2/\partial v^2$)

$$\begin{aligned} D_W\{\gamma(Z, W), \gamma(T, W)\} = {} & 2\pi\gamma(Z, T) - \oint_C \gamma(Z, W)\frac{\partial\gamma(T, W)}{\partial n_W}ds_W \\ & - \iint_B \gamma(Z, W)[\Delta_W\gamma(T, W) - P\gamma(T, W)]du\,dv \end{aligned} \tag{21}$$

whence, by the definition of Green's function $\gamma(Z, W)$ of the Laplace equation,

$$\begin{aligned} & D_W\{\gamma(Z, W), \gamma(T, W)\} \\ & \qquad = 2\pi\gamma(Z, T) + \iint_B P(W)\gamma(Z, W)\gamma(T, W)du\,dv. \end{aligned} \tag{21'}$$

From (15'), (20) and (21') we obtain finally the following representation for the Neumann function

$$\begin{aligned} N(Z, T) = {} & \gamma(Z, T) - \frac{1}{2\pi}\iint_B P(W)\gamma(Z, W)\gamma(T, W)du\,dv \\ & + 2\pi\sum_{\nu=1}^{\infty}[g_\nu(Z) + h_\nu(Z)][g_\nu(T) + h_\nu(T)]. \end{aligned} \tag{22}$$

On the right-hand side of this equation occur terms which are not directly connected with the differential equation (1). One has only Green's function for the Laplace equation, a complete orthonormal set for the class Ω, and certain other expressions which are obtained from the foregoing by simple processes of differentiation and integration. (22) represents the main result of this paper. It permits an effective calculation of Neumann's function with respect to the partial differential equation (1) by means of a complete set of functions of class Ω orthonormalized by the Dirichlet integral (2) connected with that differential equation.

5. The construction of the Green function by means of an orthonormal set. Since Green's function is related by (9) with the kernel function and the Neumann function, we solved in the preceding paragraph also the problem of constructing Green's function by orthonormal functions. We may, however, find a procedure which permits us to find Green's function directly in terms of a particular orthonormal set.

For this purpose consider the expression

$$D_W\{G(Z, W), f(W)\} = F(Z) \tag{23}$$

where $f(W)$ is an arbitrary function of the class Λ defined in §3. This expression may be transformed by means of (9), (10) and (13); one obtains easily

$$F(Z) = 2\pi[f(Z) - \psi(Z)] \tag{24}$$

where $\psi(Z)$ is that function of class Γ which has on C the same boundary values as $f(Z)$.

We define the class Ω^* of functions $f^*(Z) \in \Omega$ which have on C the boundary value 0. From (23) and (24) we conclude

$$D_W\left\{\frac{1}{2\pi} G(Z, W), f^*(W)\right\} = f^*(Z), \qquad \text{for } f^*(Z) \in \Omega^*, \tag{23'}$$

since the only function of class Γ which vanishes on C is the constant zero.

Green's function has with respect to the class Ω^* the characteristic property of reproducing each function of this class by scalar multiplication. We may, therefore, construct Green's function by means of a complete orthonormal system with respect to this class in exactly the same way as we did for Neumann's function by means of a complete orthonormal system with respect to Ω. In fact, let $\{g_\nu^*(Z)\}$ be a complete orthonormal system with respect to Ω^* and $\{k_\nu^*(Z)\}$ the set of functions associated with it by means of (16).

Then, the conclusions of §4 lead now to the formula

$$
\begin{aligned}
G(Z, T) = \gamma(Z, T) &- \frac{1}{2\pi}\iint_B P(W)\gamma(Z, W)\gamma(T, W)\,du\,dv \\
&+ 2\pi\sum_{\nu=1}^{\infty}\left[g_\nu^\bullet(Z) + h_\nu^\bullet(Z)\right]\left[g_\nu^\bullet(T) + h_\nu^\bullet(T)\right]
\end{aligned}
\tag{25}
$$

since $l^\bullet(Z, W) = G(Z, W) - \gamma(Z, W)$ obviously belongs to the class $\Omega^\bullet$.

Thus we may summarize our results in the following way:

I. Consider the sub-class Γ of Ω consisting of the solutions of (1). The kernel function of any complete orthonormal system with respect to Γ is $K(Z, W)$.

II. Consider the class $\Omega^\bullet$ of all functions $f^\bullet \in \Omega$ which vanish on the boundary C. Constructing the kernel function of any complete orthonormal system with respect to $\Omega^\bullet$ and taking the necessary steps in order to guarantee convergence leads to Green's function.

III. Carrying out the same processes as in II for a complete orthonormal system with respect to the general class Ω leads to Neumann's function.

The relation (9) between the kernel function and the Green and Neumann functions appears now as a simple consequence of the fact that the linear space Ω is the sum of the two complementary orthogonal spaces $\Omega^\bullet$ and Γ. Thus, we may construct a complete orthonormal system in Ω by combining any complete orthonormal system in $\Omega^\bullet$ with another in Γ. The kernel function in Ω appears, therefore, as the sum of the two kernel functions in $\Omega^\bullet$ and Γ.

A particular method for obtaining a complete orthonormal system for the class $\Omega^\bullet$ is given by the following procedure. Consider the differential equation

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - Pu + \lambda u = 0. \tag{1'}
$$

Determine the eigen values λ_ν for which (1′) has eigen functions $u_\nu(Z)$ which vanish on the boundary C of B. We suppose for sake of simplicity that all eigen values λ_ν are nondegenerate. By Green's identity one has for each pair of eigen functions

$$
D\{u_\nu, u_\mu\} = -\iint_B u_\nu[\Delta u_\mu - Pu_\mu]\,dx\,dy = \lambda_\mu \iint_B u_\nu u_\mu\,dx\,dy. \tag{26}
$$

In view of the symmetry of $D\{u_\nu, u_\mu\}$ and of the fact that $\lambda_\nu \neq \lambda_\mu$ for

$\nu \neq \mu$, (26) implies the orthogonality of the eigen functions in our metric. Assuming the eigen functions normalized in the usual way,

$$\iint_B u_\nu^2 dx dy = 1, \tag{27}$$

we have in

$$v_\nu(Z) = \frac{1}{\lambda_\nu^{1/2}} u_\nu(Z) \tag{28}$$

an orthonormal system in our metric.

By a classical result of the theory of integral equations, all $v_\nu(Z)$ form a complete system in $\Omega^{\bullet}$. Hence this orthonormal system may be used for the construction of the the Green function. This particular result leads to a well known theorem in the theory of eigen functions.

6. **Discussion of the result.** In order to understand the significance of our result, we have to investigate the meaning of the terms $g_\nu(Z)+h_\nu(Z)$. By definition (16), $h_\nu(Z)$ is of class Ω and has the boundary values 0. One easily shows further:

$$\Delta h_\nu(Z) = P(Z)g_\nu(Z) - \Delta g_\nu(Z), \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\cdot \tag{29}$$

Hence $g_\nu(Z)+h_\nu(Z)$ is a function of class Ω with the same boundary values on C as $g_\nu(Z)$ and satisfying in B the differential equation

$$\Delta[g_\nu(Z) + h_\nu(Z)] = P(Z)g_\nu(Z). \tag{29'}$$

This fact clarifies the meaning of (22). For $N(Z, T)$ satisfies as function of each argument the differential equation (1). If we apply in (22) the Laplace operator with respect to Z and T on both sides, we obtain formally

$$P(Z)P(T)N(Z, T) = P(Z)P(T)\cdot 2\pi \sum_{\nu=1}^{\infty} g_\nu(Z)g_\nu(T). \tag{22'}$$

If the right-hand side sum were convergent, this would give a representation of the Neumann function as the kernel function with respect to the class Ω. In order to ensure convergence, however, we had to introduce the $g_\nu(Z)+h_\nu(Z)$, that is, to carry out a certain integration of the formal identity (22'). Analogous considerations are valid in the case of formula (25) and the theory of Green's function.

One might inquire further concerning the importance of the Green's function $\gamma(Z, W)$ in our considerations. We used essentially the two following properties of $\gamma(Z, W)$:

(a) $\gamma(Z, W)$ has at $Z=W$ a logarithmic pole;

(b) $\gamma(Z, W)$ is a symmetric harmonic function;

while the third and most restrictive property of $\gamma(Z, W)$, that is, that it vanishes on the boundary C of B, was only used in the transformation of (21) into (21′). It might be more convenient for practical computations to use instead of $\gamma(Z, W)$ simply the singularity function $-\log|z-w|$ which satisfies both conditions (a) and (b). The representation of the Neumann function is in this case a little more complicated, but on the other hand one is saved the effort of computing the Green's function of the domain with respect to Laplace's equation.

Finally, we wish to point out that we dealt with a particular type of the partial differential equation (1) only for the sake of a simple representation of the method. We indicated in our previous paper the possible generalizations of the type of equation, and for all those partial differential equations obtained one may construct the Green and Neumann functions in an analogous way.

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THE ISOPERIMETRIC PROBLEM OF BOLZA WITH FINITE SIDE CONDITIONS

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1. Introduction. In view of recent improvements in the theory of the problem of Bolza in the calculus of variations, both with respect to the necessary [2, 6][1] and the sufficient conditions [3, 4, 5, 7], it seems worthwhile to investigate certain problems which, although not immediate special cases of the problem of Bolza, can be transformed by simple means into such problems. A comparatively simple problem of this nature is that of finding necessary conditions and sufficient conditions in order that an arc

$$C: \qquad a_k, \; y_i(x) \qquad (k = 1, \cdots, r; \; i = 1, \cdots, n; \; x_1 \leq x \leq x_2)$$

will minimize an expression of the form

$$(1.1) \qquad I(C) = g(a) + \int_{x_1}^{x_2} f(a, x, y, y')dx$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

in the class of arcs C which satisfy

$$x_s = x_s(a), \qquad y_i(x_s) = y_{is}(a) \qquad (s = 1, 2), \tag{1.2}$$

$$\phi_\sigma(a, x, y, y') = 0 \qquad (\sigma = q + 1, \cdots, q + m < n), \tag{1.3}$$

$$g_\rho(a) + \int_{x_1}^{x_2} f_\rho(a, x, y, y')dx = 0 \qquad (\rho = q + 1, \cdots, q + p), \tag{1.4}$$

and in addition finite side conditions of the form

$$\psi_\mu(a, x, y) = 0 \qquad (\mu = 1, \cdots, q). \tag{1.5}$$

It is assumed that the functions appearing in equations (1.1) to (1.5) are continuous and have continuous partial derivatives of the first three orders in a region R of points (a, x, y, y'). We suppose further that $x_1(a) < x_2(a)$ and that the matrix

$$\left\| \begin{matrix} \phi_{\sigma y'_i} \\ \psi_{\mu y_i} \end{matrix} \right\| \tag{1.6}$$

has rank $m+q$ on R.

For the case in which the parameters a_k are fixed and the conditions (1.3) and (1.4) are absent, Bower [1] has given a treatment of this problem.[2] Her treatment, however, is complicated by the fact that the associated isoperimetric problem of Bolza to be introduced in the next section is abnormal. Since normality is no longer needed as a hypothesis in either the necessary or the sufficient conditions, we can give a much simpler treatment of her problem. Introducing the parameters a_k and the conditions (1.3) and (1.4) provides no essential difficulty.

2. Reduction to an equivalent isoperimetric problem of Bolza. Suppose that C is an arc which satisfies (1.2) and (1.5). Then C also satisfies

$$\phi_\mu(a, x, y, y') \equiv \psi_{\mu x}(a, x, y) + \psi_{\mu y_i}(a, x, y)y'_i = 0, \tag{2.1}$$

$$g_{\mu s}(a) \equiv \psi_\mu[a, x_s(a), y_s(a)] = 0 \qquad \text{when } s = 1. \tag{2.2}$$

Conversely, if C satisfies (1.2), (2.1) and (2.2), then it satisfies (1.5). Therefore, C furnishes a weak or strong relative minimum for $I(C)$ in the class of arcs satisfying (1.2), (1.3), (1.4) and (1.5) if and only if C furnishes a weak or strong relative minimum for $I(C)$ in the class of arcs satisfying (1.2), (1.3), (1.4), (2.1) and (2.2). Since conditions (2.1) are ordinary differential equations, and since conditions (2.2) may be regarded as additional isoperimetric conditions, this second problem is an ordinary isoperimetric problem of Bolza.

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[2] This paper also contains references to the earlier work of Mayer, Bolza, Hadamard and Bliss.

3. **The necessary conditions.** Suppose that C is an arc of class C'' which minimizes $I(C)$ in the class of arcs satisfying (1.2), (1.3), (1.4) and (1.5). Then we can prove that for each variation

$$\gamma: \qquad \alpha_h, \quad \eta_i(x)$$

which satisfies

$$\eta_i(x_s) = [y_{ish} - y_i'(x_s)x_{sh}]\alpha_h, \tag{3.1}$$

$$\phi_{\sigma a_h}\alpha_h + \phi_{\sigma y_i}\eta_i + \phi_{\sigma y_i'}\eta_i' = 0, \tag{3.2}$$

$$\psi_{\mu a_h}\alpha_h + \psi_{\mu y_i}\eta_i = 0, \tag{3.3}$$

$$c_{\rho h}\alpha_h + \int_{x_1}^{x_2}(f_{\rho a_h}\alpha_h + f_{\rho y_i}\eta_i + f_{\rho y_i'}\eta_i')dx = 0, \tag{3.4}$$

in which $c_{\rho h}$ is defined as

$$c_{\rho h} = g_{\rho a_h} + f_\rho x_{sh}\Big|_{s=1}^{s=2},$$

and $x_{sh}=x_{sa_h}$, $y_{ish}=y_{isa_h}$, there exist constants $l_0\geqq 0$, l_ρ, $l_{\mu s}$, continuously differentiable functions $m_\sigma(x)$, and continuous functions $\lambda_\mu(x)$ which do not all vanish simultaneously at any point x, such that if

$$F(a, x, y, y', l, m, \lambda) \equiv l_0 f + l_\rho f_\rho + m_\sigma(x)\phi_\sigma + \lambda_\mu(x)\psi_\mu, \tag{3.5}$$

$$G(a, l) \equiv l_0 g + l_\rho g_\rho + l_{\mu s}g_{\mu s}, \tag{3.6}$$

then the following conditions, in addition to (1.2), (1.3), (1.4) and (1.5), hold along C:

$$\frac{d}{dx}F_{y_i'} = F_{y_i}; \tag{3.7}$$

$$G_{a_h} + \Big[(F - y_i'F_{y_i'})x_{sh} + F_{y_i'}y_{ish}\Big]_{s=1}^{s=2} + \int_{x_1}^{x_2}F_{a_h}dx = 0; \tag{3.8}$$

$$\begin{aligned} &E(a, x, y, y', Y', l, m, \lambda) \\ &\quad\equiv F(a, x, y, Y', l, m, \lambda) - F(a, x, y, y', l, m, \lambda) \\ &\qquad - (Y_i' - y_i')F_{y_i'}(a, x, y, y', l, m, \lambda) \geqq 0 \end{aligned} \tag{3.9}$$

whenever $(a, x, y, Y')\neq(a, x, y, y')$ and satisfies (1.3) and (2.1);

$$F_{y_i'y_j'}(a, x, y, y', l, m, \lambda)\pi_i\pi_j \geqq 0 \tag{3.10}$$

whenever $\pi \neq 0$ and is such that

$$\phi_{\sigma y_i'}(a, x, y, y')\pi_i = 0, \qquad \psi_{\mu y_i}(a, x, y)\pi_i = 0; \tag{3.11}$$

$$J_2(\gamma) = b_{hk}\alpha_h\alpha_k + \int_{x_1}^{x_2} 2\omega(\alpha, x, \eta, \eta')dx \geqq 0 \tag{3.12}$$

where

$$b_{hk} = G_{a_h a_k} + \Big[(F - y_i'F_{y_i'})x_{shk} + F_{y_i'}y_{ishk} + F_{a_k}x_{sh} + F_{a_h}x_{sk}$$

$$+ (F_x - y_i'F_{y_i})x_{sh}x_{sk} + F_{y_i}(x_{sh}y_{isk} + x_{sk}y_{ish}) \Big]_{s=1}^{s=2},$$

$$2\omega = F_{y_i y_j}\eta_i\eta_j + 2F_{y_i y_j'}\eta_i\eta_j' + F_{y_i'y_j'}\eta_i'\eta_j' + 2F_{y_i a_h}\eta_i\alpha_h$$

$$+ 2F_{y_i' a_h}\eta_i'\alpha_h + F_{a_h a_k}\alpha_h\alpha_k.$$

In fact, it is known [2] that if C is of class D' and if γ satisfies (3.1), (3.2), (3.4) and

$$\phi_{\mu a_h}\alpha_h + \phi_{\mu y_i}\eta_i + \phi_{\mu y_i'}\eta_i' = 0, \tag{3.13}$$

$$g_{\mu 1 a_h}\alpha_h = 0, \tag{3.14}$$

then[3] there exist continuous multipliers $l_0 \geqq 0$, l_ρ, l_μ, $m_\sigma(x)$, $m_\mu(x)$, which do not all vanish simultaneously at any point x, such that if

$$F_0 = l_0 f + l_\rho f_\rho + m_\sigma\phi_\sigma - m_\mu\phi_\mu, \qquad G_0 = l_0 g + l_\rho g_\rho + l_\mu g_{\mu 1}, \tag{3.15}$$

then conditions (3.8), (3.9), (3.10) and (3.12) hold when F and G are replaced by F_0 and G_0. Moreover, we also have

$$F_{0y_i'} = \int_{x_1}^{x} F_{0y_i}dx + c_i \tag{3.16}$$

for some constants c_i. An examination of the proof of this multiplier rule shows that if C is known to be of class C'', then the multipliers $m_\sigma(x)$, $m_\mu(x)$ are of class C'. Furthermore, (3.16) can be written in the form (3.7) with F replaced by F_0. If we now define $\lambda_\mu(x)$ and $l_{\mu s}$ by the equations

$$\lambda_\mu(x) = m_\mu'(x), \quad l_{\mu 1} = l_\mu + m_\mu(x_1), \quad l_{\mu 2} = - m_\mu(x_2), \tag{3.17}$$

then the multipliers l_0, l_ρ, $l_{\mu s}$, $m_\sigma(x)$, $\lambda_\mu(x)$ have the differentiability and continuity properties which they were asserted to have, and they vanish simultaneously at some point x if and only if the multipliers l_0, l_β, l_μ, $m_\sigma(x)$, $m_\mu(x)$ vanish simultaneously at some point x. If F and G are defined by (3.5) and (3.6) with these values of the multipliers, it is easy to see that conditions (3.7), (3.8), (3.9), (3.10) and (3.12) hold as stated when they hold for F_0 and G_0. Finally we observe that when C satisfies (1.2) and (1.5) and γ satisfies (3.1), then γ satisfies (3.13) and (3.14) if and only if γ satisfies (3.3).

---

[3] Cox derives these conditions for the nonisoperimetric problem. But the derivation of the necessary conditions for the isoperimetric problem from those for the nonisoperimetric problem is well known.

4. The sufficiency theorems.

These are now immediate consequences of the sufficiency theorems for the isoperimetric problem of Bolza [5, 7]. If for every nonidentically vanishing variation which satisfies (3.1), (3.2), (3.3) and (3.4) there exist multipliers (l, m, λ) satisfying the differentiability and continuity properties mentioned in §3 such that equations (3.7) and (3.8) hold, and for which

$$F_{y'_i y'_j}\pi_i\pi_j > 0, \qquad J_2(\gamma) > 0 \tag{4.1}$$

whenever $\pi \neq 0$ and is such that equations (3.11) hold, then [7] C gives $I(C)$ a proper weak relative minimum in the class of arcs satisfying (1.2), (1.3), (1.4) and (1.5). If in addition to (4.1) we require that (3.9) hold whenever the point $(a, x, y, y', l, m, \lambda)$ is in a neighborhood of those belonging to C (l_0 being held fast) and (a, x, y, Y') satisfies (1.3) and (2.1), then [5] C gives $I(C)$ a proper strong relative minimum in the class of arcs satisfying (1.2), (1.3), (1.4) and (1.5).

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A NOTE ON THE DERIVATIVES OF INTEGRAL FUNCTIONS

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1. **Introduction.** Let $f(z)=\sum_0^\infty a_n z^n$ be an integral function of order ρ and lower order λ, and $M(r)=\max_{|z|=r}|f(z)|$; $M'(r)=\max_{|z|=r}|f'(z)|$. In a recent paper [1][1] I have proved the following two theorems.

THEOREM A. *If $f(z)$ be any integral function of order ρ then*[2]

$$\limsup_{r\to\infty}\frac{\log\{rM'(r)/M(r)\}}{\log r}=\rho. \tag{1.1}$$

THEOREM B. *If $f(z)=\sum a_n z^n$ be an integral function of lower order λ and $a_n\geqq 0$ then*

$$\liminf_{r\to\infty}\frac{\log\{rM'(r)/M(r)\}}{\log r}=\lambda.$$

The condition that the coefficients a_n be real and non-negative is unnecessary. The purpose of this note is to prove the following two theorems and to deduce a number of interesting results.

THEOREM 1. *If $f(z)$ be an integral function of lower order λ $(0\leqq\lambda\leqq\infty)$ then*

$$\liminf_{r\to\infty}\frac{\log\{rM'(r)/M(r)\}}{\log r}=\lambda. \tag{1.2}$$

THEOREM 2. *For any integral function $f(z)$ we have*

$$\begin{aligned}\liminf_{r\to\infty} M'(r)/M(r) &\leqq \liminf_{r\to\infty}\nu(r)/r \leqq \limsup_{r\to\infty}\nu(r)/r\\ &\leqq \limsup_{r\to\infty} M'(r)/M(r),\end{aligned} \tag{1.3}$$

$$\begin{aligned}\liminf_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) &\leqq \liminf_{r\to\infty}\nu(r)/r \leqq \limsup_{r\to\infty}\nu(r)/r\\ &\leqq \limsup_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) \qquad (s=1,2,3,\cdots),\end{aligned} \tag{1.4}$$

where $f^{(s)}(z)$ is the sth derivative of $f(z)$, $M^{(s)}(r)=\max_{|z|=r}|f^{(s)}(z)|$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A glance at the proof [1, pp. 1–2] shows that the result (1.1) holds when ρ is infinite. An alternative proof of Theorem A is to employ Lemma 4 and relation (8) of my paper [1, p. 1].

and $\nu(r)$ *is the rank of the maximum term of* $f(z)$ *for* $|z| = r$.

2. Lemmas.

LEMMA 1. *Let*

(2.1) $F(x)$ *be a positive nondecreasing function for* $x > x_0$,

$$\liminf_{x\to\infty} F(x)/x = a \qquad (0 \leqq a < \infty); \tag{2.2}$$

then, corresponding to each pair of positive numbers b, c *satisfying the inequalities*

$$a < b; \qquad a/b < c < 1$$

there is a sequence $x_1, x_2, \cdots$, *tending to infinity such that*

$$F(x) < bx \qquad (cx_n \leqq x \leqq x_n).^3$$

PROOF. For there is a sequence $x_1, x_2 \cdots$ such that

$$F(x_n) < bcx_n.$$

Hence if $cx_n \leqq x \leqq x_n$

$$F(x) \leqq F(x_n) < bcx_n \leqq bx.$$

LEMMA 2. *Let* (2.1) *hold*,

$$\limsup_{x\to\infty} F(x)/x = a_1 \qquad (0 < a_1 \leqq \infty); \tag{2.3}$$

then, corresponding to each pair of positive numbers b_1, c_1, *satisfying the inequalities*

$$b_1 < a_1; \qquad 1 < c_1 < a_1/b_1$$

there is a sequence $X_1, X_2, \cdots$ *tending to infinity such that*

$$F(x) > b_1x \qquad (X_n \leqq x \leqq c_1X_n).$$

For there is a sequence $\{X_n\}$ tending to infinity such that

$$F(X_n) > b_1c_1X_n.$$

If $X_n \leqq x \leqq c_1X_n$

$$F(x) \geqq F(X_n) > b_1c_1X_n \geqq b_1x.$$

LEMMA 3 [2, p. 22]. *Let* (2.1) *hold*,

---

[3] It is always possible to choose the sequence $\{x_n\}$ such that two consecutive intervals have no common point. Similarly for the intervals of Lemmas 2, 3, and 4.

$$\liminf_{x\to\infty} \log F(x)/\log x = \alpha \qquad (0 \leqq \alpha < \infty); \tag{2.4}$$

then, corresponding to each pair of positive numbers β, γ *satisfying the inequalities*

$$\alpha < \beta; \qquad \alpha/\beta < \gamma < 1$$

there is a sequence $y_1, y_2, \cdots$ *tending to infinity such that*

$$F(x) < x^{\beta} \qquad (y_n^{\gamma} \leqq x \leqq y_n).$$

LEMMA 4. *Let* (2.1) *hold,*

$$\limsup_{x\to\infty} \log F(x)/\log x = \alpha_1 \qquad (0 < \alpha_1 \leqq \infty); \tag{2.5}$$

then, corresponding to each pair of positive numbers β_1, γ_1, *satisfying the inequalities*

$$\beta_1 < \alpha_1; \qquad 1 < \gamma_1 < \alpha_1/\beta_1$$

there is a sequence $Y_1, Y_2, \cdots$ *tending to infinity such that*[4]

$$F(x) > x^{\beta_1} \qquad (Y_n \leqq x \leqq Y_n^{\gamma_1}).$$

For there is a sequence $\{Y_n\}$ such that

$$\log F(Y_n) > \beta_1\gamma_1 \log Y_n.$$

If $Y_n \leqq x \leqq Y_n^{\gamma_1}$

$$\log F(x) \geqq \log F(Y_n) > \beta_1\gamma_1 \log Y_n = \beta_1 \log Y_n^{\gamma_1} \geqq \beta_1 \log x.$$

3. **Proof of Theorem 1.** Let

$$\theta(r) = \frac{\log \{rM'(r)/M(r)\}}{\log r}.$$

If λ be infinite, we have from the inequality [3]

$$M'(r) > (M(r) \log M(r))/(r \log r); \qquad r > r_0 = r_0(f) \tag{3.1}$$

that[5]

$$\liminf_{r\to\infty} \theta(r) = \infty.$$

We therefore suppose that $0 \leqq \lambda < \infty$. From the inequality (3.1) it follows that

$$\liminf_{r\to\infty} \theta(r) \geqq \lambda. \tag{3.2}$$

---

[4] We can deduce Lemmas 1 and 2 from Lemmas 3 and 4.

[5] r_0 and n_0 are not necessarily the same at each occurrence.

If $\nu(r)$ denotes the rank of the maximum term of $f(z)$, then we have [2, p. 21]

$$\liminf_{r\to\infty} \log \nu(r)/\log r = \lambda.$$

Hence by Lemma 3, corresponding to each pair of positive numbers β, γ satisfying the inequalities $\lambda<\beta$; $\lambda/\beta<\gamma<1$, there is a sequence $y_1, y_2, \cdots$ tending to infinity such that

$$\nu(r) < r^{\beta} \qquad (y_n^{\gamma} \leqq r \leqq y_n).$$

Let E_n denote the set of points r $(y_n^{\gamma}\leqq r\leqq y_n)$ and $E=E_1+E_2+\cdots$. Let F denote the set of points r which lie [4, p. 105] outside a set of exceptional segments in which, for $r>R$, the variation of $\log r$ is less than $K\nu(R/k)^{-1/12}$. Since the variation of $\log r$ in E_n is

$$\log y_n - \gamma \log y_n = (1-\gamma) \log y_n,$$

which tends to infinity with n, there are points in E_n which do not belong to the set of exceptional segments. The set EF therefore contains a sequence $e_1, e_2, \cdots$ tending to infinity. At these points [4, p. 105] e_n

$$rM'(r) \sim M(r)\nu(r), \qquad r = e_n, \tag{3.3}$$

$$rM^{(s+1)}(r) \sim M^{(s)}(r)\nu(r). \tag{3.4}$$

Hence[5] for $n>n_0$

$$e_nM'(e_n)/M(e_n) < 2\nu(e_n) < 2e_n^{\beta}$$

and so $\liminf_{r\to\infty} \theta(r)\leqq\beta$ and since $\beta-\lambda$ can be chosen arbitrarily small we have

$$\liminf_{r\to\infty} \theta(r) \leqq \lambda$$

and so $\liminf_{r\to\infty} \theta(r)=\lambda$.

4. **Proof of Theorem 2.** Let $\liminf_{r\to\infty}\nu(r)/r=a$ and suppose first that $a<\infty$. Then if

$$a < b, \qquad a/b < c < 1,$$

we have, by Lemma 1, $\nu(r)<br$ $(cx_n\leqq r\leqq x_n)$. Let E_n denote the set of points r $(cx_n\leqq r\leqq x_n)$ and $E=E_1+E_2+\cdots$. The variation of $\log r$ in E_n is $\log x_n-\log cx_n=\log 1/c$ which is not less than $K\nu(R/k)^{-1/12}$ if R be large enough. The total variation of $\log r$ in the intervals $\sum_{\nu=1}^{n}E_\nu$ tends to infinity with n. Hence the set EF contains [4, p. 105]

a sequence $e_1', e_2', \cdots$ tending to infinity. For $r=e_n'$ $(n>n_0)$

$$M'(r)/M(r) \sim \nu(r)/r < b, \tag{4.1}$$

$$M^{(s+1)}(r)/M^{(s)}(r) \sim \nu(r)/r < b \qquad (s = 1, 2, 3, \cdots). \tag{4.2}$$

Hence

$$\liminf_{r\to\infty} M'(r)/M(r) \leqq b.$$

Since $b-a$ can be chosen arbitrarily small

$$\liminf_{r\to\infty} M'(r)/M(r) \leqq a$$

which certainly holds if $a=\infty$. Also

$$\liminf_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) \leqq a \qquad (s = 1, 2, \cdots).$$

Let $\limsup_{r\to\infty} \nu(r)/r = a_1$ and suppose $a_1>0$. Let $b_1<a_1$, $1<c_1<a_1/b_1$. If G denotes the set of points formed by the intervals of Lemma 2 the set GF contains a sequence $g_1, g_2, \cdots, g_n, \cdots$ tending to infinity. For $r=g_n$ $(n>n_0)$

$$M'(r)/M(r) \sim \nu(r)/r > b_1, \tag{4.3}$$

$$M^{(s+1)}(r)/M^{(s)}(r) \sim \nu(r)/r > b_1 \qquad (s = 1, 2, \cdots). \tag{4.4}$$

Hence $\limsup_{r\to\infty} M'(r)/M(r) \geqq a_1$,

$$\limsup_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) \geqq a_1$$

which hold if $a_1=0$. Hence the theorem follows.

5. **Applications.** We have from (1.1) and (1.2)

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \rho &= 1 + \limsup_{r\to\infty} \frac{\log\{M'(r)/M(r)\}}{\log r} \\ &= 1 + \limsup_{r\to\infty} \frac{\log\{M^{(s+1)}(r)/M^{(s)}(r)\}}{\log r}, \end{aligned} \tag{5.1}$$

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log \nu(r)}{\log r} = \lambda &= 1 + \liminf_{r\to\infty} \frac{\log\{M'(r)/M(r)\}}{\log r} \\ &= 1 + \liminf_{r\to\infty} \frac{\log\{M^{(s+1)}(r)/M^{(s)}(r)\}}{\log r} \end{aligned} \tag{5.2}$$

$$(s = 1, 2, \cdots).$$

Let s denote any fixed positive integer and C, C_1 two positive constants.

(5.3) *If* $M'(r) \geqq CM(r)$ *for all* $r > r_0$ *then either* $\lambda > 1$ *or* $\lambda = 1$ *and* $\liminf_{r\to\infty} \nu(r)/r \geqq C$.

From (1.3) we have $\liminf_{r\to\infty} \nu(r)/r \geqq C$. Hence $\lambda \geqq 1$.

REMARK (i). This is a best possible result in the sense that there are functions for which $M'(r) \geqq CM(r)$ and $\nu(r) \sim Cr$. We may take for instance $f(z) = \exp\ (Cz)$.

(ii) The converse—if $\liminf_{r\to\infty} \nu(r)/r \geqq C$ then $M'(r) \geqq CM(r)$ for all $r > r_0$—is false. For consider $f(z) = \cosh Cz$. Here $\lambda = 1 = \rho$, $\nu(r) \sim Cr$.

$$\frac{M'(r)}{M(r)} = C\,\frac{e^{Cr} - e^{-Cr}}{e^{Cr} + e^{-Cr}} < C \qquad \text{for all } r > 0,$$

$$\frac{M''(r)}{M'(r)} = C\,\frac{e^{Cr} + e^{-Cr}}{e^{Cr} - e^{-Cr}} > C \qquad \text{for all } r > 0,$$

and so on.

(5.4) *If* $M^{(s+1)}(r) \geqq CM^{(s)}(r)$, $r > r_0$, *then* $\lambda \geqq 1$ *and* $\liminf_{r\to\infty} \nu(r)/r \geqq C$.

(5.5) *If* $M'(r) \leqq C_1 M(r)$, *or if* $M^{(s+1)}(r) \leqq C_1 M^{(s)}(r)$, *for all* $r > r_0$, *then either* $\rho < 1$ *or* $\rho = 1$ *and* $\limsup_{r\to\infty} \nu(r)/r \leqq C_1$. This follows from Theorem 2. If $f(z)$ is of order 1 then [5, p. 81] it follows that $\limsup_{r\to\infty} \log M(r)/r \leqq C_1$.

Let $\phi(r)$ be any function, nondecreasing and positive for $r > r_0$, and such that $\log \phi(r)/\log r$ tends to zero as r tends to infinity.

(5.6) *If* $M'(r) \geqq (1/\phi(r))M(r)$, *or if* $M^{(s+1)}(r) \geqq (1/\phi(r))M^{(s)}(r)$ *for a sequence of values of* r *tending to infinity, then* $\rho \geqq 1$.

This follows from (5.1). If this hypothesis holds for all $r > r_0$ then from (5.2) we get $\lambda \geqq 1$.

(5.7) *If* $M'(r) \leqq \phi(r)M(r)$, *or if* $M^{(s+1)}(r) \leqq \phi(r)M^{(s)}(r)$, *for a sequence of values of* r *tending to infinity, then* $\lambda \leqq 1$.

This follows from (5.2). If this hypothesis holds for all $r > r_0$ then from (5.1) we get $\rho \leqq 1$.

(5.8) *If* $1/\phi(r) \leqq M'(r)/M(r) \leqq \phi(r)$ *or if* $1/\phi(r) \leqq M^{(s+1)}(r)/M^{(s)}(r) \leqq \phi(r)$ *for all* $r > r_0$ *then* $\lambda = \rho = 1$.

This follows from (5.6) and (5.7).

(5.9) *If* $\rho < 1$ *then*

$$M(r) > \phi(r)M'(r) > \phi^2(r)M''(r) > \cdots > \{\phi(r)\}^s M^{(s)}(r)$$

for all $r > r_0$.

This follows[6] from (5.1).

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[6] Cf. (5.6) above.

(5.10) *If* $\lambda>1$ *then*

$$M(r) < \frac{1}{\phi(r)} M'(r) < \frac{1}{\phi^2(r)} M''(r) < \cdots < \frac{1}{\{\phi(r)\}^s} M^{(s)}(r)$$

for all $r>r_0$

This follows[7] from (5.2).

(5.11) *If* $\lambda=1$ *and* $\liminf_{r\to\infty} \nu(r)/(r \log r)>1$ *then*[8] $M(r)<M'(r)<M''(r)< \cdots <M^{(s)}(r)$ *for all* $r>r_0$. If $\liminf_{r\to\infty} \nu(r)/(r \log r)=l>1$ then since

$$\log M(r) > \int_{r_0}^{r} \{\nu(t)/t\} dt,$$

$$\liminf_{r\to\infty} \log M(r)/(r \log r) > 1.$$

Hence for all $r>r_0$

$$M'(r)/M(r) > \log M(r)/(r \log r) > 1,$$

$$M''(r)/M'(r) > \log M'(r)/(r \log r) > \log M(r)/(r \log r) > 1,$$

and so on.

(5.12) *If* $\lambda=1$ *and* $\liminf_{r\to\infty} \nu(r)/r<1$ *there is a sequence of numbers* r *tending to infinity for which*

$$M(r) > M'(r) > \cdots > M^{(s)}(r).$$

Let $\liminf_{r\to\infty} \nu(r)/r=a$ and $a<b<1$. The result follows from (4.1) and (4.2). This result does not hold if $\liminf_{r\to\infty} \nu(r)/r\geqq 1$. In fact for the function $f(z)=\cosh z$, $\nu(r)\sim r$ and the sequence $\{M^{(n)}(r)\}$ $(n=0, 1, 2, \cdots)$ is not monotonic for any $r>0$.

(5.13) *If* $\lambda<1$, *there is a sequence of numbers* r *tending to infinity for which*

$$M(r) > \phi(r)M'(r) > \phi^2(r)M''(r) > \cdots > \{\phi(r)\}^s M^{(s)}(r).$$

This follows from (3.3) and (3.4).

(5.14) *If* $\rho=1$ *and* $\limsup_{r\to\infty} \nu(r)/r>1$, *there is a sequence of numbers* r *tending to infinity for which*

$$M(r) < M'(r) < M''(r) < \cdots < M^{(s)}(r).$$

This follows from (4.3) and (4.4). It does not hold if $\limsup_{r\to\infty} \nu(r)/r\leqq 1$.

(5.15) *If* $\rho>1$, *there is a sequence of numbers* r *tending to infinity*

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[7] Cf. (5.7) above.

[8] S. K. Bose has proved (5.11) with the hypothesis $\lambda>2$.

for which

$$M(r) < \frac{1}{\phi(r)} M'(r) < \frac{1}{\phi^2(r)} M''(r) < \cdots < \frac{1}{\{\phi(r)\}^s} M^{(s)}(r).$$

Let $1<\beta_1<\rho$; $1<\gamma_1<\rho/\beta_1$; H the set of intervals of Lemma 4. The set FH contains a sequence of numbers r tending to infinity for which the above inequality holds.

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GENERALIZATION OF CERTAIN THEOREMS OF G. SZEGÖ ON THE LOCATION OF ZEROS OF POLYNOMIALS

GYULA DE SZ. NAGY

Using the theorem of Grace, the following result was obtained by G. Szegő:[1]

Let the polynomial

$$f(z) = z^n + A_1 z^{n-1} + \cdots + A_n$$

have no zeros in the circular region $|z| \leqq R$. *Then the "section"*

$$h(z) = f(z) - z^n = A_1 z^{n-1} + A_2 z^{n-2} + \cdots + A_n$$

has no zeros in the circular region $|z| \leqq R/2$.

In case of an even n the example $f(z) = (z-R)^n$ shows that the circle $|z| \leqq R/2$ can not be replaced by a larger concentric circle. But in case n is odd, according to Szegő the polynomial $h(z)$ is different from zero even in the circle $|z| \leqq (R/2) \sec (\pi/2n)$.[2]

This theorem can be generalized as follows:

I. *Let the polynomial*

$$f(z) = z^n + A_1 z^{n-1} + \cdots + A_n$$

have no zeros in the circular region $|z-\alpha| \leqq R$. *Then no polynomial*

$$h(z) = f(z) - \epsilon(z - \alpha)^n, \qquad |\epsilon| \leqq 1,$$

can have any zeros in the circle $|z-\alpha| \leqq R/2$.

The example $f(z) = (z-R)^n$, $\alpha = 0$, $\epsilon = 1$, shows that this theorem can not be refined even in the case of an odd n.

Theorem I is a consequence of the following more general theorem:

II. *Let the polynomial*

$$f(z) = (z - a_1)(z - a_2) \cdots (z - a_n) \tag{1}$$

have no zeros in the circle $|z-\alpha| \leqq R$; *and let the polynomial*

$$g(z) = (z - b_1)(z - b_2) \cdots (z - b_n) \tag{2}$$

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[1] G. Szegő, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Zeit. vol. 13 (1922) pp. 28–55.

[2] Loc. cit. p. 46.

have all its zeros in the circle $|z-\alpha| \leqq \rho$, $\rho < R$. *Then the polynomial*

$$h(z) = f(z) - \lambda g(z), \qquad |\lambda| \leqq t^n,\ 0 \leqq t < R/\rho, \tag{3}$$

can have no zero in the circle

$$|z - \alpha| \leqq r = \frac{R - \rho t}{1 + t}. \tag{4}$$

In order to prove this we observe first that for any zero ξ of the polynomial $h(z)$

$$\frac{f(\xi)}{g(\xi)} = \prod_{k=1}^{n} \frac{\xi - a_k}{\xi - b_k} = \lambda.$$

Hence $h(z_0) \neq 0$ in every point z_0 where

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n} \left|\frac{z_0 - a_k}{z_0 - b_k}\right| \neq |\lambda|.$$

Now at every point z_0 of the circular region (4)

$$\begin{aligned} |z_0 - a_k| &\geqq |a_k - \alpha| - |z_0 - \alpha| \geqq |a_k - \alpha| - r > R - r \\ &= (r + \rho)t, \\ |z_0 - b_k| &\leqq |z_0 - \alpha| + |b_k - \alpha| \leqq r + \rho, \end{aligned}$$

so that

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n} \left|\frac{z_0 - a_k}{z_0 - b_k}\right| > \left(\frac{R - r}{r + \rho}\right)^n = t^n \geqq |\lambda|.$$

This concludes the proof of II. In the case $b_1 = b_2 = \cdots = b_n = \alpha$, $|\lambda| = |\epsilon| = 1$ $(\rho = 0,\ t = 1)$, we obtain I.

Substituting

$$g(z) = z^n + A_k z^{n-k} = z^{n-k}(z^k + A_k), \quad \alpha = 0, \quad \rho = |A_k|^{1/k}, \quad \lambda = 1$$

in Theorem II we obtain:

III. *The polynomial*

$$f(z) = z^n + A_k z^{n-k} + A_{k+1} z^{n-k-1} + \cdots + A_n$$

has at least one zero in the circle $|z| \leqq 2r + |A_k|^{1/k}$ *provided the section*

$$h(z) = f(z) - z^n - A_k z^{n-k} \equiv A_{k+1} z^{n-k-1} + A_{k+2} z^{n-k-2} + \cdots + A_n$$

has at least one zero on the circle $|z| \leqq r$.

The proof of the following theorem is similar to that of II:

IV. *Let* $a_1, a_2, \cdots, a_n$ *be given points of the complex plane which are all different from* α. *Let* S_k *be the half-plane containing* α *and having as boundary the perpendicular bisector of* $\alpha_1\ a_k$. *Finally let* S^* *be the common part of the half-planes:* $S_1, S_2, \cdots, S_n$. *Then no polynomial*

$$h(z) = (z - a_1)(z - a_2) \cdots (z - a_n) - \epsilon(z - \alpha)^n, \qquad |\epsilon| \leqq 1,$$

can have a zero in the interior of the convex domain S^*.

Indeed in any point z_0 in the interior of S^* we have $|z_0 - a_k| > |z_0 - \alpha|$ so that

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n}\left|\frac{z_0 - a_k}{z_0 - \alpha_k}\right| > 1 \geqq |\epsilon|.$$

A theorem similar to II holds also if the zeros of the polynomials $f(z)$ and $g(z)$ are in arbitrary circular domains without common points. One of these circular domains is the interior of a circle, the other the exterior or interior of a circle or a half-plane. Corresponding to these cases three theorems can be obtained generalizing also certain theorems of G. Szegö.[3]

V. *Let the zeros of the polynomials* $f(z) = (z - a_1) \cdots (z - a_n)$ *and* $g(z) = (z - b_1)(z - b_2) \cdots (z - b_n)$ *be located in the circular regions*

$$|z - \alpha| \geqq \rho_1 \quad \text{and} \quad |z - \beta| \leqq \rho_2, \tag{5}$$

respectively. We assume that these regions have no points in common, that is,

$$\rho_1 - \rho_2 > 0, \qquad |\beta - \alpha| < \rho_1 - \rho_2. \tag{6}$$

Then no polynomial

$$h(z) = f(z) - \epsilon g(z), \qquad |\epsilon| \leqq 1, \tag{7}$$

can have a zero in the interior of the ellipse E *with foci at* α *and* β *and with the major axis* $\rho_1 - \rho_2$.

VI. *Let the zeros of the polynomials* $f(z)$ *and* $g(z)$ *be located in the circular regions*

$$|z - \alpha| \leqq \rho_1 \quad \text{and} \quad |z - \beta| \leqq \rho_2, \tag{8}$$

respectively, such that these regions have no points in common, that is,

$$|\beta - \alpha| > \rho_1 + \rho_2. \tag{9}$$

Then no polynomial

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[3] Loc. cit. pp. 47–48, Theorems 13–15.

$$h(z) = f(z) - \epsilon g(z), \qquad |\epsilon| = 1,$$

can have a zero in the interior of the hyperbola H with foci at α *and* β *and with the real axis* $\rho_1+\rho_2$.

VII. *Let the zeros of the polynomials* $f(z)$ *and* $g(z)$ *be located in the circular region* $|z-\alpha| \leq \rho$ *and in the half-plane S, respectively, such that these regions have no points in common. Let K be a conic section with* α *as focus and the boundary line L of the half-plane S as the directrix corresponding to* α.[4] *Then no polynomial*

$$f(z) - \lambda g(z) \tag{10}$$

with

$$|\lambda| \geq t^n = \left(e + \rho\frac{e+1}{\delta}\right)^n \tag{11}$$

can have a zero in the interior of the conic section K where e is the numerical eccentricity of K and δ *is the distance of* α *from the line L.*

By the interior of a conic section we mean the set of points from which no tangent can be drawn to the given conic section.

In order to prove V and VI, we denote by z_0 an arbitrary point in the interior of the conic sections E and H, respectively; let

$$|z_0 - \alpha| = r_1, \qquad |z_0 - \beta| = r_2.$$

As to the ellipse E we have

$$r_1 + r_2 < \rho_1 - \rho_2 \quad \text{or} \quad \rho_2 + r_2 < \rho_1 - r_1,$$
$$|z_0 - a_k| \geq |a_k - \alpha| - |z_0 - \alpha| = |a_k - \alpha| - r_1 > \rho_1 - r_1,$$
$$|z_0 - b_k| \leq |b_k - \beta| + |z_0 - \beta| \leq \rho_2 + r_2.$$

From this Theorem V follows since

$$\left|\frac{z_0 - a_k}{z_0 - b_k}\right| > \frac{\rho_1 - r_1}{\rho_2 + r_2} > 1 \quad \text{so that} \quad \left|\frac{f(z_0)}{g(z_0)}\right| > 1 \geq |\epsilon|.$$

As to the hyperbola H, we distinguish two cases regarding the position of z_0, according as z_0 is nearer to β than to α, or conversely. In the first case

$$r_1 > r_2, \quad r_1 - r_2 > \rho_1 + \rho_2, \quad \text{hence} \quad r_1 - \rho_1 > r_2 + \rho_2,$$
$$|z_0 - a_k| \geq |z_0 - \alpha| - |a_k - \alpha| \geq r_1 - \rho_1,$$
$$|z_0 - b_k| \leq |z_0 - \beta| + |b_k - \beta| \leq r_2 + \rho_2.$$

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[4] That is, the polar of α.

In the second case

$$r_1 < r_2, \quad r_2 - r_1 > \rho_1 + \rho_2, \quad \text{hence} \quad r_2 - \rho_2 > r_1 + \rho_1,$$
$$| z_0 - a_k | \leqq | z_0 - \alpha | + | a_k - \alpha | \leqq r_1 + \rho_1,$$
$$| z_0 - b_k | \geqq | z_0 - \beta | - | b_k - \beta | \geqq r_2 + \rho_2.$$

Thus we have in the first and second case,

$$\left| \frac{z_0 - a_k}{z_0 - b_k} \right| \geqq \frac{r_1 - \rho_1}{r_2 + \rho_2} > 1, \quad \text{and} \quad \left| \frac{z_0 - a_k}{z_0 - b_k} \right| \leqq \frac{r_1 + \rho_1}{r_2 - \rho_2} < 1,$$

respectively. This furnishes Theorem VI since we have

$$\left| \frac{f(z_0)}{g(z_0)} \right| > 1 = | \epsilon |, \quad \text{and} \quad \left| \frac{f(z_0)}{g(z_0)} \right| < 1 = | \epsilon |,$$

respectively.

This proof furnishes the following corollary:

VI′. *Let H be the hyperbola defined in Theorem* VI. *No polynomial* $f(z) - \lambda g(z)$ *with* $|\lambda| \geqq 1$ $(|\lambda| \leqq 1)$ *can have any zero inside the branch of* H *containing the focus* α (β).

Let z_0 be a point in the interior of the conic section K defined in Theorem VII; let r and d be the distance of z_0 from the focus α and the directrix L, respectively. Then

$$\frac{r}{d} < e \quad \text{and} \quad \frac{1}{d} < \frac{1}{d^*} = \frac{e+1}{\delta};$$

indeed if z^* is the point of K nearest to L and d^* is the distance of z^* from L, we have

$$\frac{| z^* - \alpha |}{d^*} = \frac{\delta - d^*}{d^*} = e.$$

But $|z_0 - a_k| \leqq |z_0 - \alpha| + |a_k - \alpha| \leqq r + \rho$ and $|z_0 - b_k| \geqq d$, so that

$$\left| \frac{z_0 - a_k}{z_0 - b_k} \right| \leqq \frac{r + \rho}{d} < e + \frac{\rho}{d^*} = e + \rho \frac{e+1}{\delta} = t,$$
$$\left| \frac{f(z_0)}{g(z_0)} \right| \leqq \left(\frac{r + \rho}{d} \right)^n < t^n = | \lambda |.$$

This establishes the proof of Theorem VII.

The special cases

$$b_1 = b_2 = \cdots = b_n = 0, \beta = 0, \rho_2 = 0, \epsilon = 1 \qquad \text{of V and VI,}$$

and

$$a_1 = a_2 = \cdots = a_n = 0, \alpha = 0, \rho = 0, \epsilon = 1, \lambda = 1 \qquad \text{of VII,}$$

are due to G. Szegö.

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SOME REMARKS ON POLYNOMIALS

P. ERDÖS

This note contains some disconnected remarks on polynomials.

Let $f_n(x) = \prod_{i=1}^{n}(x - x_i)$, $-1 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n \leqq 1$. Denote by $-1 \leqq y_1 \leqq \cdots \leqq y_{n-1} \leqq 1$ the roots of $f_n'(x)$. We prove the following theorem.

THEOREM 1. *For all* n

$$(1) \qquad |f_n(-1)| + |f_n(+1)| + \sum_{i=1}^{n-1} |f_n(y_i)| \leqq 2^n.$$

For $n \geqq 3$

$$(2) \qquad |f_n(-1)|^{1/2} + |f_n(+1)|^{1/2} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/2} \leqq 2^{n/2}.$$

For $n \geqq n_0(k)$

$$(3) \qquad |f_n(-1)|^{1/k} + |f_n(+1)|^{1/k} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/k} \leqq 2^{n/k}.$$

REMARK. If $y_i = y_{i+1}$ or $-1 = y_1$, $+1 = y_{n-1}$ the corresponding summands clearly vanish.

Clearly

$$|f_n(-1)| \leqq (1 - x_1)2^{n-1}, \qquad |f_n(y_i)| \leqq |y_i - x_{i+1}| 2^{n-1},$$
$$|f_n(+1)| \leqq (1 - x_n)2^{n-1}.$$

Thus

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$$|f_n(-1)| + |f_n(+1)| + \sum_{i=1}^{n-1} |f_n(y_i)|$$
$$\leqq \left((1 - x_1) + \sum_{i=1}^{n-1} (y_i - x_{i+1}) + (1 - x_n)\right) 2^{n-1} \leqq 2^n,$$

which proves (1)

We have by the inequality of the geometric and arithmetic mean

$$|f_n(-1)|^{1/2} \leqq \frac{(1 - x_1) + (1 - x_2)}{2} 2^{n/2-1},$$

$$|f_n(y_i)|^{1/2} \leqq \frac{(x_{i+1} - x_i)}{2} 2^{n/2-1},$$

$$|f_n(+1)|^{1/2} \leqq \frac{(1 - x_n) + (1 - x_{n-1})}{2} 2^{n/2-1}.$$

Thus we evidently have for $n \geqq 3$

$$|f_n(-1)|^{1/2} + |f_n(+1)|^{1/2} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/2} \leqq 2^{n/2},$$

which proves (2).

$f_1(x) = x$ and $f_2(x) = x^2/2 - 1$ shows that (2) is false for $n < 3$. Clearly equality occurs in (1) and (2) only for $\pm(1 \pm X)^n$.

The proof of (3) is more complicated and since the proof does not present any particular interest we are just going to sketch it. Let $f_n(x)$ be the polynomial which maximizes the sum (3). If (3) is not true we must have

$$|f_n(x_0)| = \max_{-1 \leqq x \leqq 1} |f_n(x)| > \frac{2^n}{n^k}\cdot$$

But then it is easy to see that x_0 does not lie in $(1-\epsilon, -1+\epsilon)$; without loss of generality we can assume that $1-\epsilon < x_0 \leqq 1$, and $n+o(n)$ of the x_i are in $(-1+\delta, -1)$. But then a simple computation shows that $f_n(x)$ has no roots in $(1-\epsilon, 1)$ and thus $x_0 = 1$. (This is clear since if we move any possible root of $f_n(x)$ in $(1-\epsilon, 1)$ to -1, we clearly increase the sum (3).) By the same argument we obtain by a simple calculation that all the roots of $f_n(x)$ have to be in -1, which proves (3) and completes the proof of Theorem 1.

At present I can not determine the exact value of $n_0(k)$.

Let $g_n(z) = \prod_{i=1}^{n} (z - z_i)$, $|z_i| = 1$. Denote by $u_1, u_2, \cdots$ the local extremal points of $g_n(z)$, that is, the points where the vector $g_n'(z)$

points either towards the origin or away from it. I conjectured that

$$\sum_i |g_n(u_i)| \leq 2^n.$$

Professor Breusch[1] proved this conjecture for sufficiently large n. The proof is complicated. For small values of n he showed by examples that the result is false.

Let $-1=x_0<x_1\leq \cdots \leq x_n=x_{n+1}=1$. Put $\omega(x)=\prod_{i=1}^{n} (x-x_i)$, $l_k(x)=\omega(x)/\omega(x_k)(x-x_k)$, the fundamental functions of Lagrange interpolation. The problem of determining the set for which

$$\max_{-1\leq x\leq 1} \sum_{k=1}^{n} | l_k(x) |$$

is minimal is still unsolved. It has been conjectured, but never proved, that the minimum is attained for the points for which all the $n+1$ sums

$$\max_{x_i\leq x\leq x_{i+1}} \sum_{k=1}^{n} | l_k(x) |, \qquad i = 0, 1, \cdots, n, \tag{4}$$

are equal. If the x_i are the roots of $T_n(x)$ (the nth Tchebychef polynomial), then a simple computation shows that the sums (4) all equal

$$\frac{2}{\pi} \log n + O(1).$$

S. Bernstein[2] proved that for any $-1\leq x_1\leq x_2\leq \cdots \leq x_n\leq 1$

$$\max_{-1\leq x\leq 1} \sum_{k=1}^{n} | l_k(x) | > (1+o(1)) \frac{2}{\pi} \log n,$$

and I proved[3] that

$$\max_{-1\leq x\leq 1} \sum_{k=1}^{n} | l_k(x) | > \frac{2}{\pi} \log n - c \qquad (c \text{ absolute constant}).$$

We consider a slightly different problem. We prove the following theorem.

THEOREM 2. *Let* $-1=x_0\leq x_1 \cdots \leq x_n\leq x_{n+1}=1$. *Then for some* i

$$\max_{x_i<x<x_{i+1}} \sum_{k=1}^{n} | l_k(x) | < n^{1/2}. \tag{5}$$

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[1] Oral communication.

[2] Bull. Acad. Sci. URSS. Sér. Math. (1931) pp. 1025–1050.

[3] Unpublished.

REMARK. $n^{1/2}$ in (5) can very likely be improved to $c\log n$. In fact it is likely that

$$\min_{0\leqq i\leqq n}\ \max_{x_i\leqq x\leqq x_{i+1}}\ \sum_{k=1}^{n}|l_k(x)|$$

assumes its maximum when all the sums (4) are equal.

If, for some i, $x_i=x_{i+1}$ then (5) is obvious. Assume that $x_i\neq x_{i+1}$, $0\leqq i\leqq n+1$. Consider the equation $\sum_{k=1}^{n} l_k^2(x)=1$. The number of solutions is not greater than $2n-2$ and $x_1, x_2, \cdots, x_n$ are solutions. Thus a simple argument shows that for some i, $1\leqq i\leqq n-1$,

$$\sum_{k=1}^{n} l_k^2(x) < 1 \qquad \text{for } x_i < x < x_{i+1}.$$

But then clearly (from Schwartz's inequality)

$$\sum_{k=1}^{n}|l_k(x)| < n^{1/2} \qquad \text{for } x_i \leqq x \leqq x_{i+1}$$

which proves Theorem 2.

In one of his interesting papers Schur[4] proves among others the following result: Let $a_0x^n+\cdots+a_n$ be a polynomial with integer coefficients, all whose roots are in $(-1, +1)$ and are different. Then for sufficiently large n, $|a_0|>(2^{1/2}-\epsilon)^n$. We prove the stronger theorem:

THEOREM 3. *Let $f_n(x)=a_0x^n+\cdots+a_n$ be a polynomial with integer coefficients and $f_n(-1)\neq 0$, $f_n(0)\neq 0$, $f_n(+1)\neq 0$. Then,* $|a_0|\geqq 2^{n/2}$.

We have ($x_1, x_2, \cdots, x_n$ are the roots of $f_n(x)$)

$$|f_n(-1)f_n^2(0)f_n(+1)| = \left|a_0^4\prod_{i=1}^{n}(1-x_i^2)x_i^2\right| \geqq 1.$$

But $|(1-x_i^2)x_i^2|\leqq 1/4$. Thus $|a_0|\geqq 2^{n/2}$ which completes the proof. $2^n(x-1/2)^n$ shows that $|a_0|\geqq 2^{n/2}$ is best possible.

Schur in his proof makes use of the fact that the discriminant of $f_n(x)$ has to be an integer. If we make use of this fact we easily obtain that, for large n, $|a_0|>(2^{1/2}+c)^n$. On p. 390 of his paper Schur constructs a polynomial of degree $2n$ with

$$a_0^{(2n)}=\frac{1}{2(2)^{1/2}}\left((1+2^{1/2})^{n+1}-(1-2^{1/2})^{n+1}\right).$$

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[4] Math. Zeit. vol. 1 (1918) pp. 377–402, see p. 389–391.

This seems the greatest possible value of $|a_0^{(2n)}|$.

In the same paper (Theorem XIII, pp. 397–398) Schur proves the following theorem: Let a_0 be a given integer, and let $f_n(z) = a_0z^n + \cdots + a_n$ be a polynomial with integer coefficients the roots of which either all have absolute value 1 and are different or all are in the interior of the unit circle (in which case multiple roots are permitted). Denote these roots by $z_1, z_2 \cdots, z_n$. Then

$$\limsup \frac{z_1 + z_2 + \cdots + z_n}{n} \leqq 1 - \frac{e^{1/2}}{2}. \tag{6}$$

Schur conjectures that the limit (6) is 0, and remarks that if $a_0 = 1$ this follows from a theorem of Kronecker, which asserts that in this case all the z_i are roots of unity. We now prove Schur's conjecture.

THEOREM 4. *Let the z_i be defined as above. Then*

$$\lim \frac{z_1 + z_2 + \cdots + z_n}{n} = 0.$$

First we can assume that n tends to infinity (that is, for every n there are only a finite number of equations satisfying the conditions of the theorem). Also if $f(z)$ has all its root in the interior of the unit circle then $z^k f(z) + z^n f(z^{-1}) = g(z)$ has all its roots on the unit circle and all are different. Also the sum of the roots of $f(z)$ and $g(z)$ are identical (p. 397). Thus it will suffice to consider polynomials having all their roots on the unit circle and distinct.

Therefore the discriminant of D satisfies the inequality

$$1 \leqq D = a_0^{2n-2} \prod_{i<j\leqq n} (z_i - z_j)^2. \tag{7}$$

It follows from a result of Pólya (p. 395) that

$$|f(z_1, z_2, \cdots, z_n)| = \prod_{i<j\leqq n} (z_i - z_j) \leqq n^n. \tag{8}$$

Thus for at least one z

$$\prod_{j\neq i} |(z_i - z_j)| \leqq n. \tag{9}$$

To prove Theorem 4 it clearly suffices to show that the z_i are uniformly distributed on the unit circle. Suppose this is not true. Then it follows from a result of Fekete[5] that there exists a z_0, $|z_0| = 1$, such that

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[5] Ann. of Math. vol. 41 (1940) pp. 162–173, see pp. 165–166.

$$(10) \qquad \prod_{j \neq i} |(z_0 - z_j)| > (1 + c_1)^n.$$

But then from (9) and (10)

$$|f(z_0, z_1, \cdots, z_{i-1}, z_{i+1} \cdots z_n) > \frac{(1 + c_1)^n}{n} f(z_1, z_2, \cdots, z_n).$$

If $z_1, z_2, \cdots, z_n$ are not uniformly distributed we can continue this process $c_2 n$ times, and thus obtain $y_1, y_2, \cdots, y_n$, $|y_i| = 1$, so that

$$(11) \qquad |f(y_1, y_2, \cdots, y_n)| > \frac{(1 + c_1)^{c_2 n^2}}{n^{c_2 n}} |f(z_1, z_2, \cdots, z_n)|.$$

But from (7) and (11) we obtain

$$|f(y_1, y_2, \cdots, y_n)| > (1 + c_2)^{n^2} \frac{1}{a_0^{2n-2}} > n^n$$

which contradicts (8) and completes the proof of Theorem 4.

Szegö[6] proved the following theorem: Let M be any closed set in the plane. Denote by $\omega_n(M, z_0)$ the maximum of $|f_n'(z_0)|$ for all polynomials $f_n(z)$ of degree n which satisfy $|f_n(z)| \leq 1$ for all z in M. Assume that the transfinite diameter of M is positive. Then

$$\lim \omega_n(M, z_0)^{1/n} < \infty.$$

Fekete[7] proved that if z_0 is not in M, $\lim \omega_n(M, z_0)^{1/n}$ exists and is finite if the transfinite diameter of M is positive, and is infinite if the transfinite diameter of M is 0.

Assume now that z_0 belongs to M. The following questions remain open: (1) Does $\lim \omega_n(M, z_0)^{1/n}$ exist? (2) Let the transfinite diameter of M be 0. Is $\lim \omega_n(M, z_0)^{1/n} = \infty$?[7]

We are going to answer both questions in the negative. In fact we prove the following theorem.

THEOREM 5. *Let M be the set consisting of* 0 *and* $1/2^k$, $k = 0, 1, 2, \cdots$. *Then*

$$\omega_n(M, 0) < c^n.$$

Clearly M is closed and countable, thus its transfinite diameter is 0.

LEMMA. *Let a, b, d be three real numbers, $d - b = b - a$. Then if $|f_n(z)| < 1$ for $a < z < b$,*

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[6] Math. Zeit. vol. 23 (1925) pp. 45–61.

[7] Math. Zeit. vol. 26 (1927) pp. 324–344.

$$f_n'(d) < c_1^n/(b-a).$$

If $a=0$, $b=1$ the lemma follows from Szegö's[4] result. The general case follows by a linear transformation. As a matter of fact it is well known that $\omega_n(M, d)$ is given in this case by the Tchebychef polynomial belonging to (a, b).

Now we prove Theorem 5. The equation $f_n^2(z)=1$ can have at most $2n$ real roots. Thus since $|f(1/2)^k)|<1$, $k=0, 1, \cdots$, we obtain that, for some $k>n+1$, $|f_n(z)|<1$ for all $1/2^k<z<1/2^{k+1}$. Thus by the lemma

$$|f_n'(0)| < 2^{n+1}c_1^n < c^n,$$

q.e.d.

THEOREM 6. *Let the set M be defined as follows: $n_1<n_2<\cdots$ tend to infinity sufficiently fast. M consists of the points* 0 *and* $1/2^u$ *where* $n_i\leqq u\leqq 2n_i+1$. *Then* lim $\omega_n(M, 0)^{1/n}$ *does not exist. In fact* lim sup $\omega_n(M, 0)^{1/n}=\infty$, lim inf $\omega_n(M, z_0)^{1/n}<\infty$.

As in Theorem 5 it follows that if $|f(1/2^u)|<1$ for $n_i\leqq u\leqq 2n_i+1$, $f(x)$ a polynomial of degree n_i, then $\omega_{n_i}(M, 0)<c^{n_i}$. Consider $f(x)=2^{n_i+1}\prod(x-1/2^k)$, $k\leqq 1, 2, \cdots, 2n_i+1$. The degree of $f(x)$ equals $2n_i+2$. Also $|f(x)|<1$ for all x in M, and if n_{i+1} tends to infinity sufficiently fast

$$(f'(0))^{1/2n_i+2} > (2^{n_i+1}/2^{(2n_i+1)^2})^{1/2n_i+2} \to \infty$$

q.e.d.

THEOREM 7. *Let $f_n(z)$ be a polynomial of degree n with real coefficients. $|f_n(z)|<1$ for $-1\leqq z\leqq 1$. Then if $|z_0|\geqq 1$*

$$|f_n(z_0)| \leqq |T_n(z_0)|.$$

Equality holds only for $f_n(z)=\pm T(z)$.

In case z_0 is real this result is well known.

We are going to prove the following more general result: Let $|f_n(x_i)|\leqq 1$ where $x_1=-1$, x_i, $i=1, 2, \cdots, n-1$, are the roots of $T_n'(x)$ and $x_n=1$. Then for $|z_0|\geqq 1$

$$|f_n(z_0)| \leqq |T_n(z_0)|. \tag{12}$$

We have $(l_i(x)=\omega(x)/\omega'(x_i)(x-x_i)$, $\omega(x)=(1-x^2)T_n'(x))$

$$f_n(z_0) = \sum_{i=0}^{n} y_i l_i(x_0), \qquad |y_i| \leqq 1, \ y_i \text{ real.}$$

We evidently have for complex numbers A and B, $\max(|A+B|, |A-B|>A$. Thus $|f_n(z)|$ will be maximal if $y_i=\pm 1$. A simple geometric argument shows that the angle between any two of the vectors $(-1)^i l_i(z_0)$ is less than $\pi/2$ (since the interval $(-1, +1)$ subtends from z_0 at an angle not greater than $\pi/2$). But then clearly $|f_n(z_0)|$ is maximal if

$$f_n(z_0) = \pm \sum_{i=0}^{n} (-1)^i l_i(z_0) = \pm T_n(z_0).$$

Equality clearly occurs only if $f(z) = \pm T(z)$.

COROLLARY. *Let* $|f_n(z)| \leqq 1$ *for* $-1 \leqq z \leqq 1$, *also let* $f_n(z)$ *have real coefficients. Then for* $|z| \leqq 1$, $|f_n(z)| < |T_n(i)|$.

If we do not assume that the coefficients of $f_n(z)$ are real it is easy to give examples which show that $|f_n(z)|$ does not have to be less than $|T_n(i)|$. Trivially $|f_n(z)| \leqq \sum_{k=0}^{n} |l_k(i)|$. But in general $\max |f_n(z)| < \sum_{k=0}^{n} |l_k(i)|$. I can not at present determine $\max |f_n(z)|$ for $|z| \leqq 1$.

In the same way we can prove that if $f(z) = a_0 z^n + \cdots + a_n$ has real coefficients and $|f(z)| \leqq 1$ for $-1 \leqq z \leqq 1$ then $\sum_{k=0}^{n} |a_k|$ is maximal for $f(z) = \pm T_n(z)$. Szegö[8] proved the following stronger result: $|a_{2k}| + |a_{2k+1}|$ is maximal for $f(z) = \pm T_n(z)$.

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[8] Oral communication.

PAIRS OF INVERSE MODULES IN A SKEWFIELD

F. W. LEVI

Let Σ be a skewfield. If J and J' are submodules of Σ such that the nonzero elements of J are the inverse elements of those of J', then J and J' form a "pair of inverse modules." A module admitting an inverse module will be called a J-module and a selfinverse module containing 1 will be called an S-module. In an earlier paper[1] the author has shown that if Σ is a (commutative) field of characteristic not equal to 2, then every S-module is a subfield of Σ. Only in fields of characteristic 2, nontrivial S-modules can be found. A corresponding distinction of that characteristic does not hold for skewfields. Even the skewfield of the quaternions contains nontrivial S-modules, for examples the module generated by 1, j, k. In the present paper some properties of S-modules and J-modules will be discussed. For example it will be proved that when an S-module contains the elements a, b and ab, it contains all the elements of the skewfield which is generated by a and b. By a similar method it will be shown that finite S-modules are necessarily Galois-fields.

1. Necessary and sufficient conditions for J-modules.

THEOREM 1. *A submodule J of Σ is a J-module if and only if $a \in J$ and $b \neq 0 \in J$ imply $ab^{-1}a \in J$.*

PROOF. Let J be a J-module. Without loss of generality suppose that $a \neq 0$, $b-a=c\neq 0$. Then $k=a^{-1}+c^{-1}\in J'$ since J' is closed under addition and subtraction. As $k=a^{-1}(c+a)c^{-1}$, $k^{-1}=cb^{-1}a$; hence $a-k^{-1}=ab^{-1}a$ is contained in J. Let now J be a module satisfying the condition mentioned above. To prove that J is a J-module, we shall show that when a and c are nonzero elements in J, but otherwise arbitrary, then $a^{-1}+c^{-1}$ is either 0 or the inverse of an element of J. The first alternative holds when $b=a+c=0$; if however $b\neq 0$, then $a^{-1}+c^{-1}=(a-ab^{-1}a)^{-1}$ is the inverse of an element of J. Hence the theorem.

COROLLARY 1. *The meet of any (finite or infinite) set of J-modules in Σ is a J-module in Σ.*

This corollary shows that the J-modules in Σ form a lattice with the set-inclusion as the defining order-relation. $J_1 \wedge J_2$ denotes the ordi-

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[1] *Pairs of inverse moduls*, J. Indian Math. Soc. N.S. vol. 3 (1936) pp. 295–306.

nary meet, whereas $J_1 \vee J_2$ is the meet of all the J-modules in Σ which contain J_1 and J_2. This lattice is in general not a sublattice of the lattice of all the submodules of Σ.

COROLLARY 2. *If a and b are elements of Σ and J is a J-module in Σ, then aJb is also a J-module in Σ.*

COROLLARY 3. *If the J-module J contains* 1, *then J is an S-module.*

PROOF. From $1 \cdot b^{-1} \cdot 1 \in J$, follows $J' \subseteq J$. As $1^{-1} = 1 \in J'$, the inverse inequality holds. Hence $J = J'$ is selfinverse and contains 1.

COROLLARY 4. *If $a' \in J'$, then $a'J = S$ is an S-module, or the zero-module.*

Therefore every J-module which contains nonzero elements can be denoted by $J = aS$, where $a \neq 0$ is an otherwise arbitrary element of J. The S-module S depends on the selection of a. For the following proofs, it is important to remember that when a and b belong to an S-module S, then

$$(1) \qquad a + b, \qquad a - b, \qquad aba, \quad \text{and,} \quad \text{for} \quad a \neq 0, \quad a^{-1}$$

also belong to S.

COROLLARY 5. *If $a \neq 0$ and $a \in J$, then $a J'a = J$.*

PROOF. From Theorem 1 it follows that $aJ'a \subseteq J$ and $a^{-1}Ja^{-1} \subseteq J'$. The second formula furnishes $J \subseteq aJ'a$; hence the corollary.

In S-modules (and other selfinverse modules) every element $a \neq 0$ of S generates a module-automorphism $S \rightarrow aSa$.

2. **Skewfields in S-modules.** Obviously the primefield of Σ is contained in every S-module of Σ. We shall investigate now the conditions for S to contain the skewfield

$$(2) \qquad F(a, b)$$

which is generated in Σ by the elements a and b (that is, the meet of all the sub-skewfields containing a and b). That S may contain a and b but not $F(a, b)$ appears from the example mentioned above, where Σ is the skewfield of the quaternions and S is the module generated by $1, j, k$.

LEMMA 1. *If an S-module S contains $a \neq 0$, it contains a^m (for $m = 0, \pm 1, \pm 2, \cdots$).*

PROOF. It suffices to prove the lemma for positive exponents. S

contains a and $a1a=a^2$ and with a^m, also $aa^ma=a^{m+2}$. Hence the lemma follows by mathematical induction.

THEOREM 2. *If an S-module S contains a, b and ab it contains all the elements of $F(a, b)$.*

PROOF. Without loss of generality, suppose that $a, b\neq 0$. S contains $a^{-1}\cdot ab\cdot a^{-1}=ba^{-1}$, hence ab^{-1} and $b\cdot ab^{-1}\cdot b=ba$. The suppositions are therefore symmetric for left and right and for a and b. Herefrom it follows that $a^\delta b^\epsilon$ and $b^\delta a^\epsilon$ belong to S for ϵ, $\delta=\pm 1$. If $a^rb^s\in S$ then $a^{\pm r}b^{\pm s}\in S$ and $b^{\pm s}a^{\pm r}\in S$. To show that all the terms a^rb^s belong to S, we may therefore restrict ourselves to positive values of r and s. Suppose $ab^m\in S$ for $0\leqq m\leqq n$. This formula holds for $n=1$. Moreover S contains $b\cdot ab^{n-1}\cdot b=bab^n$. As S is supposed to contain b and ab^n, it must also contain ab^{n+1}. Hence it follows by mathematical induction that $ab^m\in S$ for every positive m and therefore $b^ma\in S$. We can now substitute b^m for a and a for b and obtain by the same conclusion that $b^ma^r\in S$ for every positive r and finally we see that for all the integral values of r and s, the elements a^rb^s and b^sa^r belong to S. Let now R be the ring generated by a, b, a^{-1}, b^{-1}. The elements of R can all be represented as sums of terms

$$\pm\, a^{r_1}b^{s_1}\cdots a^{r_n}b^{s_n}, \tag{3}$$

where the exponents take the values $0, \pm 1, \pm 2, \cdots$. To prove that $R\subseteq S$, it suffices to show that all the elements (3) belong to S. The statement has been proved for $n=1$. Now $(b^{-u}a^{-v}\cdot a^{r_1}b^{s_1}\cdots a^{r_n}b^{s_n}\cdot b^{-u}a^{-v})^{-1}=a^vb^{u-s_n}a^{-r_n}\cdots b^{-s_1}a^{v-r_1}b^u$. When $u, v, r_1, \cdots, r_n$, $s_1, \cdots, s_n$ run independently over all the integral numbers, then the same holds for the $2n+2$ exponents on the right-hand side. Thus one obtains by mathematical induction that R is contained in S. For the last steps of the proof one needs the following lemmas:

LEMMA 2. *If an S-module S contains a ring R, then S contains also a ring in which the elements of R and their inverse elements occur.*

PROOF OF LEMMA 2. The ring generated by the elements $\alpha_i, \alpha_k, \cdots$ of R and their inverse elements consists of sums of terms of the type

$$\alpha_1\overset{-1}{\alpha_2}\cdots\alpha_{2n-1}\overset{-1}{\alpha_{2n}}; \tag{4}$$

the element 1 can be used as an α as well as an α^{-1}. To show that this ring is contained in S, it suffices to show that every element of type (4) belongs to S. As α_i, α_k and $\alpha_i\alpha_k$ belong to S, the same holds for $\alpha_i\alpha_k^{-1}$; hence the statement is true for $n=1$. To prove it for an arbitrary n by mathematical induction, we observe that every product of α's is an α and that the corresponding holds for the inverse elements. Thus

$$(\alpha_1\alpha_3\cdot\alpha_4^{-1})(\alpha_4\cdot\alpha_3^{-1}\alpha_2^{-1}\alpha_1^{-1})(\alpha_1\alpha_3\cdot\alpha_4^{-1}) = \alpha_1\alpha_2^{-1}\alpha_3\alpha_4^{-1} \in S.$$

Moreover if the statement holds for any particular $n>1$, it follows that

$$\begin{aligned}(\alpha_1\alpha_{2n+1}\alpha_{2n+2}^{-1})(\alpha_{2n+2}\alpha_{2n+1}^{-1}\alpha_2^{-1}\cdot\alpha_3\alpha_4^{-1}\cdots\alpha_{2n-1}\alpha_{2n}^{-1}\alpha_1^{-1})(\alpha_1\alpha_{2n+1}\alpha_{2n+2}^{-1})\\ = \alpha_1\alpha_2^{-1}\cdot\alpha_3\alpha_4^{-1}\cdots\alpha_{2n-1}\alpha_{2n}^{-1}\cdot\alpha_{2n+1}\alpha_{2n+2}^{-1} \in S.\end{aligned}$$

Hence we have Lemma 2.

LEMMA 3. *If an S-module S contains a ring R, it contains also a skewfield* $F\supseteq R$.

PROOF OF LEMMA 3. From Lemma 2 follows the existence of a ring R' such that $R\subseteq R'\subseteq S$ and R' contains also the elements which are inverse to those of R. If R' contains the inverse elements of all its elements, then it is a skewfield; at any rate it is a subring of a subring R'' of S which contains those inverse elements. By continuing this procedure, one obtains an ascending chain of subrings $R, R', R'', \cdots$ in which each ring contains the preceding rings and their inverse elements. The join of these rings is a skewfield F. Hence we have the lemma.

As, under the suppositions of Theorem 2, S has a subring R which contains a and b, the module S has also a sub-skewfield F which contains R and therefore a and b. Hence $S\supseteq F\supseteq F(a, b)$.

COROLLARY. *When a J-module J contains* a, b, c *and* d, *where* $ab^{-1}cd^{-1}=1$, *then J contains* $dF(d^{-1}c, d^{-1}a)$.

PROOF. $d^{-1}J$ is an S-module which contains $d^{-1}c$, $d^{-1}a$ and $d^{-1}b = d^{-1}cd^{-1}a$.

3. **Finite S- and J-modules.** Let $0, a_1, a_2 \cdots$ be the elements of an S-module S in Σ.

Every "word" of the type

$$a_1a_2\cdots a_n \tag{5}$$

belongs to the ring A generated by S. As every $-a_i$ is also an a, each element of A can be represented as a sum of words (5). Furthermore the sum of the two equal terms can be contracted into a single one, say $2a_1\cdot a_2\cdots a_n$, since $2a_1$ is also an element a. Thus we can suppose that the terms in a sum which represent an element of A are all different. In general such a term affords different representations as a word (5). For the shortest representation, the following lemma holds.

LEMMA 4. *In the shortest representation* (5) *of a product of nonzero elements of S all the letters a_i are different.*

PROOF. Suppose that in (5) the same letter a occurs several times, say $aa_1 \cdots a_m a$ is a portion of a product of type (5). We replace every a_{2k} by $aa^{-1}a_{2k}a^{-1}a$. Now $aa_{2k-1}a = a'_{2k-1}$ and $a^{-1}a_{2k}a^{-1} = a'_{2k}$ are also elements of S. Hence the product under consideration is reduced to $a'_1 \cdots a'_m$ when m is odd and to $a'_1 \cdots a'_m a^2$ when m is even. As $a^2 \in S$, the length of the product has been reduced by the operation. Thus in a shortest representation, no repetition of elements can occur.

It may be mentioned that in a J-module which does not contain 1, no square of any element $a \neq 0$ of J is contained, since $a^2 \in S$ implies $a \cdot a^{-2} \cdot a = 1 \in S$. The lemma therefore does not hold for J-modules. However one can show by the same method that when the module formed by the a's is selfinverse, in the shortest representation (5) no letter occurs more than twice.

THEOREM 3. *Every finite S-module S is a Galois field.*

PROOF. If n is the number of elements of S, then it follows from Lemma 4 that there exist only $m \leqq n^n$ different products of such elements. The ring R generated by S consists of sum of different products and therefore R has not more than 2^m elements. In a finite ring, every element a generates a finite multiplicative cyclic group; hence R contains a^{-1}. R is therefore a skewfield and as R is finite, it is a Galois field.[2] S is therefore an S-module in a Galois field. S-modules in (commutative) fields are known[3] to be subfields, except in the case of characteristic 2. It remains to prove the theorem for the case when S is an S-module in GF_{2^r}. It has been proved[4] that the elements of GF_{2^r} which multiplied with the elements of S give elements of S form a field $M(S)$ and that $a \in S$ implies $a^2 \in M(S)$. As $1 \in S$, we have $M(S) \subseteq S$. In a Galois field of order 2^r, $a = a^{2^r}$. Therefore $a^2 \in M(S)$ implies $a \in M(S)$. Hence $S = M(S)$. This finishes the proof.

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[2] J. H. Maclagan Wedderburn, Trans. Amer. Math. Soc. vol. 6 (1905), p. 349; see also E. Witt, Abh. Hamburgischen Univ. Math. Sem. vol. 8 (1931) p. 413.

[3] Loc. cit. footnote 1, Proposition 4.

[4] Loc. cit. footnote 1, Proposition 2.

COROLLARY. *Every finite J-module is of the form aG where G is a Galois field.*

If in particular the finite J-module is self-inverse, then $a^2 \in G$. In the case of Galois fields of characteristic 2, this relation implies $a \in G$ and therefore $J = G$.

4. **Additional remarks.** Let S be an S-module in Σ and $a \in S$. By $\mu(a)$ denote the set of those elements $x \in S$ for which $ax \in S$. From Theorem 2 it follows that $ax^{-1} \in S$; moreover $\mu(a)$ is a module containing 1. Hence $\mu(a)$ is an S-module. In the same way, one proves that $\mu(a) = \mu(a^{-1})$ and that $\mu(a)$ is also the set of the elements $x \in S$ for which $xa \in S$ holds. The meet of all the modules $\mu(a)$ is a skewfield $M(S)$. Two modules $aM(S)$ and $bM(S)$ are either identical or they have only the element 0 in common; these modules are J-modules. Let $c \in M(S)$ and $c \neq 0$, then $\mu(a) = \mu(ac)$. Furthermore denote the modules $xM(S)$ by $M_1, M_2, \cdots$. For every particular $c \neq 0$ of $M(S)$ the mapping $M_i \rightarrow cM_i$ generates a permutation of the modules M_i which are subdivided into systems of transitivity. $M(S)$ forms a system of transitivity by itself. If M_1 and M_2 belong to the same system of transitivity, then $M_1 = cM_2$, where $c \neq 0$, $c \in M(S)$. As $M(S) = M(S)c^{-1}$, $M_1 = cM_2c^{-1}$. Thus the modules belonging to the same system of transitivity are conjugate, the transforming element belonging to $M(S)$, and conversely. To every M_i there corresponds a subskewfield of $M(S)$ consisting of these elements y for which $yM_i = M_i$ or $yM_i = 0$. The meet of all these skewfields is a skewfield which contains the prime field of Σ.

CALCUTTA, INDIA

ON A CONJECTURE OF CARMICHAEL

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Carmichael [1][2] conjectured that for no integer n can the equation $\phi(x)=n$ (ϕ being Euler's totient) have exactly one solution. To support the conjecture, he showed that each n for which there is a unique solution must satisfy a restriction which implies $n>10^{37}$. In this note we prove the validity of restrictions considerably stronger than those of Carmichael, and raise the lower bound on n to 10^{400}.

We shall denote by X the set of all integers x for which $\phi(y)=\phi(x)$ implies $y=x$. (If the conjecture is correct, X is empty, and the theorems stated are vacuously satisfied.)

(1) THEOREM. *Suppose that* $x=\prod_A p_i^{a_i}$ *is in* X, *where the* p_i*'s are distinct primes and* A *is the range of the index* i. *Let* $m=\prod_B p_i^{a_i-1}(p_i-1) \cdot \prod_C p_i^{c_i}$ *where* B *and* C *are disjoint subsets of* A (*one of them may be empty*) *and* $c_i \leqq a_i-1$ *for* i *in* C. *Then if* p *is prime and* $p-1=m$, *we have* $p \mid x$.

For if $p \nmid x$, we have $\phi(p\cdot\prod_{A-B-C} p_i^{a_i}\cdot\prod_C p_i^{a_i-c_i})=\phi(x)$, contrary to the definition of X.

(1.1) COROLLARY. *Suppose, under the hypotheses of* (1), *that* B *has the following property: if* q *is prime and* $q \mid (p_j-1)$ *for some* j *in* B, *then* $q \mid x$. *We must then have* $p^2 \mid x$.

For under this condition we have $p-1=\prod_D p_i^{d_i}$, D being a subset of A. So if $p \mid x$ but $p^2 \nmid x$, then $\phi(\prod_{A-D} p_i^{a_i}\cdot\prod_D p_i^{a_i+d_i}/p)=\phi(x)$, contrary to the definition of X.

(1.2) COROLLARY. *If, in the hypotheses of* (1), B *is empty, we have* $p^2 \mid x$.

(1.3) COROLLARY. $4 \mid x$. *If* f *is a Fermat prime such that* $f \mid x$, *then* $f^2 \mid x$.

(1.2) and (1.3) are Carmichael's original conditions. From (1.1) and (1.3) it follows that x is divisible by 3^2, 7^2, 43^2, 3^3 or 13^2, $\cdots$. (By extending this list Carmichael showed both x and $\phi(x)$ to be greater than 10^{37}.)

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[1] I should like to thank Professor C. G. Jaeger of Pomona College for arousing my interest in Euler's ϕ-function.

[2] Numbers in brackets refer to the references at the end of the paper.

(2) THEOREM. *Suppose* $u=2^e s\prod_A f_i^{a_i}$ *is in* X, *where the* f_i's *are distinct Fermat primes other than* 3, *and* s *is odd and divisible by no Fermat prime other than* 3. *Suppose that* $C\subset A$, $2\leqq c\leqq a$, $v=2^e s\prod_C f_i^{a_i}$. *Then* v *is in* X *if and only if either* $C=A$, *or for each index* i *in* $A-C$, $2^e|(f_i-1)$.

Clearly if v is in X, one of the stated conditions is valid. Now suppose that y is an even integer such that $\phi(y)=\phi(v)$, and let D denote the subset of $A-C$ consisting of all indices i in $A-C$ for which $f_i|y$. Let $d_i=a_i-1$ for i in D, $d_i=a_i$ for i in $(A-C)-D$, and $e=a-c+\sum_D k_i$ (where $f_i=2^{k_i}+1$). Let $w=2^e y\prod_{A-C} f_i^{d_i}$. Then $\phi(w)=\phi(u)$. If one of the stated conditions holds, $y\neq v$ implies that either y is divisible by some prime q such that $q\nmid u$, or u is divisible by a non-Fermat prime p such that $p\nmid y$. In either case $w\neq u$, contradicting the hypothesis that u is in X. Hence we have $y=v$ and v is in X.

(2.1) COROLLARY. *If* u *is in* X, *there is an element* v *of* X *such that* (i) $v|u$; (ii) $8\nmid v$, *and* v *is divisible by no Fermat prime other than* 3.

An element of X which satisfies condition (ii) of (2.1) will be called a reduced element of X, and the collection of all reduced elements will be denoted by $R(X)$. Clearly the structure of X is fairly well determined by that of $R(X)$, so henceforth we shall confine our attention to $R(X)$. If an element of X has no divisor in X, we shall call it irreducible, and shall denote the collection of all such elements by $I(X)$.

(3) THEOREM. *Let* $x=\prod_B p_i^{b_i}$ *be in* $R(X)$, *where the* p_i's *are distinct primes. For an integer* m, *let* $B(m)$ *denote the subset of* B *consisting of all indices* i *for which* $m|(p_i-1)$. *Let* $w=\prod_{B-B(m)} p_i^{b_i}$. *Then either* w *is in* $R(X)$ *or* $m|\phi(w)$.

Suppose $w\notin R(X)$. Then $w\notin X$, and there is an integer $y\neq w$ such that $\phi(y)=\phi(w)$. And if $(y,\ x/w)=1$, $\phi(yx/w)=\phi(x)$, contradicting the fact that $x\in X$. So for some i in $B(m)$, we have $p_i|y$, whence $(p_i-1)|\phi(w)$. But $m|(p_i-1)$, so $m|\phi(w)$.

(3.1) COROLLARY. *If, in* (3), $x\in I(X)$, m *is prime, and* $B(m)$ *is not empty, then* $m^2|x$.

Since $B(m)$ is not empty, $w<x$. But then, since $x\in I(X)$, $w\notin X$. Hence $m|\phi(w)$, whence from the definition of w it is clear that $m^2|x$.

(3.2) COROLLARY. If $x\in I(X)$, p *is prime and* $p|x$, *then* $p^2|x$.

This follows from (1.1).

(3.3) COROLLARY. *If $x \in I(X)$, $\phi(x)$ is divisible by no Fermat prime other than* 3.

This is an immediate consequence of (3.1). (3.1), (3.2), and (3.3) together eliminate 11, 23, 31, 41, 47, · · · as possible divisors of x or $\phi(x)$ if $x \in I(X)$.

It would be of interest to know the relationship of $R(X)$ to $I(X)$, and of $I(X)$ to its smallest member.

We now apply (1.1) to show the following theorem.

(4) THEOREM. *If x is in X, then both x and $\phi(x)$ are greater than* 10^{400}.

We know $3^2 | x$, and shall consider three possibilities: (I) $3^3 \nmid x$; (II) $3^3 | x$ but $3^4 \nmid x$; (III) $3^4 | x$. Suppose that $x = \prod p_i^{a_i}$, the p_i's being distinct primes, and let $m = \prod p_i^{a_i - 1}$. In each case 3, 7, and 43 are divisors of m.

In case (I) any prime of the form $6k+1$ or $12k+1$, where $k | m$ and $(6, k) = 1$, must be a divisor of m. Applying (1.1), we find the following prime divisors: 13, 79, 157, 547, 1093, 3319, 3613, 6163, 6637, 6709, 36979, 39829, 40507, 42667, 45949, 46957, 74419, 81013, 85333, 91813, 170509, 258847, 282253, 303493, 518083, 529933, 596779, 1041853, 1053157, 1573207, 1834639, 1854763, 1954869, 3623803, 3641917, 3669277, 3856147, 3944389, 6318943, 6772039, 6806893, 7161379, 7207243, 7372093.

In cases (II) and (III), m is divisible by each prime of the form $6k+1$ or $18k+1$, where k and m are as above. So in these cases m is divisible by 19, 127, 2287, 4903, 5419, 13723, 82339, 98299, 101347, 304039, 617761, 688087, 1676827, 3736087, 4130323, 4324363, 4693267.

In case (II) we also have as divisors primes of the form $36k+1$, and in case (III) of the form $54k+1$. This fact gives rise to additional divisors as follows:

Case (II): 37, 223, 1549, 4219, 4663, 4789, 9547, 10837, 25309, 27883, 29527, 176509, 196597, 197359, 200467, 399643, 494029, 544123, 545947, 1059517, 1063159, 1088467, 1184149, 1198927, 1203019, 1235419, 1564309, 2397853, 2407141, 3265399, 3702367, 7082029, 7221439, 7274053, 7619263, 8262367, 8387839, 8647927, 9139519.

Case (III): 379, 6823, 15919, 40939, 43207, 123499, 130483, 143263, 202627, 264763, 302443, 368443, 741043, 859699, 1857859, 2018383, 2053423, 2333467, 4446759, 5030479, 5480287.

The stated divisors were found with the aid of [2]. It is easy to find still more divisors by this method. Those obtained imply the following lower bounds for x and $\phi(x)$: (I) 10^{458}; (II) $10^{5\text{[illegible]}}$; (III) 10^{400}.

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ON THE DARBOUX TANGENTS

V. G. GROVE

1. Introduction. In a recent paper [1][1] Abramescu gave a metrical characterization of the cubic curve obtained by equating to zero the terms of the expansion of a surface S at an ordinary point O_1, up to and including the terms of the third order. This cubic curve is rational and its inflexions lie on the three tangents of Darboux through O_1. In this paper we give a projective characterization of such a curve, and hence a new derivation of the tangents of Darboux. By using the method employed in this characterization to the curve of intersection of the tangent plane of the surface at O_1 with S, a simple characterization of the second edge of Green is found. Another application exhibits the correspondence of Moutard. Finally a new interpretation of the reciprocal of the projective normal is given in terms of the conditions of apolarity of a cubic form to a quartic form. The canonical tangent appears in a similar fashion.

Let S be referred to its asymptotic curves, and let the coordinates (x^1, x^2, x^3, x^4) of the generic point O_1 of S be normalized so that they satisfy the system [2] of differential equations

$$\begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + p x, \\ x_{vv} &= \gamma x_u + \theta_v x_v + q x, \qquad \theta = \log R. \end{aligned} \tag{1.1}$$

The line l_1 joining O_1 to O_4, whose coordinates are x^i_{uv}, is the R-conjugate line, and the line l_2 determined by O_2, O_3, whose respective coordinates are x^i_u, x^i_v, is the R-harmonic line.

If we define the local coordinates (x_1, x_2, x_3, x_4) with respect to

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$O_1O_2O_3O_4$ of a point X by the expression

$$X^i = x_1x^i + x_2x_u^i + x_3x_v^i + x_4x_{uv}^i,$$

and local nonhomogeneous coordinates (x, y, z) by $x=x_2/x_1$, $y=x_3/x_1$, $z=x_4/x_1$, the power series expansion [4] of S at O_1 is

$$z = xy - \frac{1}{3}(\beta x^3 + \gamma y^3) + \frac{1}{12}F_4(x, y) + \cdots, \tag{1.2}$$

wherein

$$\begin{aligned} F_4(x, y) = {} & (2\beta\theta_u - \beta_u)x^4 - 4(\beta\theta_v + \beta_v)x^3y - 6\theta_{uv}x^2y^2 \\ & - 4(\gamma\theta_u + \gamma_u)xy^3 + (2\gamma\theta_v - \gamma_v)y^4. \end{aligned} \tag{1.3}$$

2. **Characteristic points of a plane curve.** Let the triangle of reference $O_1O_2O_3$ to which a plane curve C is referred be covariant to the curve or to a surface to which C bears some geometrical relation. Let the homogeneous coordinates of a point with respect to this triangle be (x_1, x_2, x_3), the nonhomogeneous coordinates being defined by the expressions $x=x_2/x_1$, $y=x_3/x_1$. The line $y=0$ being chosen as the tangent to C at O_1, the power series expansion [4] of C at O_1 is

$$y = a_2x^2 + a_3x^3 + a_4x^4 \cdots. \tag{2.1}$$

Consider at $O_3(0, 0, 1)$ the involution whose double lines are O_1O_3, O_2O_3. Corresponding lines of this involution intersect C in points $P_1(x, y)$, $P_2(-x, y')$, $y'=a_2x^2-a_3x^3+a_4x^4- \cdots$. The line P_1P_2 intersects the tangent to C at O_1 in a point whose limit T as P_1 approaches O_1 along C has coordinates

$$x_1 = a_3, \qquad x_2 = -a_2, \qquad x_3 = 0. \tag{2.2}$$

We shall call the point T with coordinates (2.2) *the characteristic point* of the second order of C at O_1 *relative to* $O_1O_2O_3$.

Let $O_2'(\rho, 1, 0)$ be an arbitrary point on the tangent to C at O_1, but distinct from O_1. The transformation from the triangle $O_1O_2O_3$ to $O_1O_2'O_3$ is

$$x = \frac{Ax'}{1+\rho Ax'}, \qquad y = \frac{By'}{1+\rho Ax'}. \tag{2.3}$$

Under the transformation (2.3), the equation of C may be written in the form

$$y' = a_2'x'^2 + a_3'x'^3 + \cdots,$$

wherein

$$a_2' = A^2 a_2/B, \qquad a_3' = A^3(a_3 - \rho a_2)/B.$$

Hence *the characteristic point of C relative to $O_1O_2O_3'$ has coordinates*

$$(2.4) \qquad x_1 = (a_3 - 2\rho a_2), \qquad x_2 = -a_2, \qquad x_3 = 0$$

referred to $O_1O_2O_3$.

More generally let the equation of C have the form

$$y = a_k x^k + a_{k+1} x^{k+1} + \cdots, \qquad k \geqq 2.$$

Consider through O_3 two lines forming with O_1O_3, O_2O_3 the constant cross ratio l, l being one of the kth roots of unity, but $l \neq 1$. These lines intersect C in two points P_1, P_2 determining a line which intersects the tangent to C at O_1 in a point whose limit as P_1 approaches O_1 has coordinates

$$(2.5) \qquad x_1 = a_{k+1}, \qquad x_2 = -a_k, \qquad x_3 = 0.$$

We shall call the point T whose coordinates are (2.5) *the characteristic point of the kth order of C relative to $O_1O_2O_3$.*

3. **The characteristic curve of** S. Let us consider the section C_π of the surface S by a plane π through the R-conjugate line l_1. Let π intersect the R-harmonic line l_2 in O_π. The local coordinates of O_π are of the form $(0, \lambda, \mu, 0)$, and the local coordinates of any point Q_1 on O_1O_π are $(1, \lambda\xi, \mu\xi, 0)$. The equation of C_π referred to $O_1O_\pi O_4$ in nonhomogeneous coordinates (ξ, z) is

$$(3.1) \qquad z = \lambda\mu\xi^2 - \frac{1}{3}(\beta\lambda^3 + \gamma\mu^3)\xi^3 + \frac{1}{12}F_4(\lambda, \mu)\xi^4 + \cdots.$$

From (2.2) the characteristic point T_π of C_π relative to $O_1O_4O_\pi$ has coordinates

$$(3.2) \qquad \xi = 3\lambda\mu/(\beta\lambda^3 + \gamma\mu^3), \qquad z = 0,$$

referred to $O_1O_\pi O_4$, and coordinates

$$(3.3) \qquad x = 3\lambda^2\mu/(\beta\lambda^3 + \gamma\mu^3), \quad y = 3\lambda\mu^2(\beta\lambda^3 + \gamma\mu^3), \quad z = 0$$

referred to $O_1O_2O_3O_4$. *The locus of T_π* as π rotates about l_1 *is the covariant rational cubic curve Γ_3 whose equation is*

$$(3.4) \qquad 3xy - (\beta x^3 + \gamma y^3) = 0, \qquad z = 0.$$

We shall call this cubic *the characteristic curve of S relative to l_1, l_2. The nodal tangents of Γ_3 are of course the asymptotic tangents of S at O_1, and the inflexions lie on the tangents of Darboux. The R-harmonic line* *is the flex-ray of* Γ_3.

From (3.3) it follows that *the only sections of S through the R-conjugate line whose characteristic points relative to $O_1O_4O_\pi$ lie on the R-harmonic line are those through the tangents of Darboux.*

Another characterization of the cubic Γ_3 may be found in the following manner. The osculating conic of the section C_π has the equation [4]

$$\begin{aligned}(3.5)\qquad &\lambda^3\mu^2(z - \lambda\mu\xi^2) + \frac{1}{3}\lambda^2\mu^2(\beta\lambda^3 + \gamma\mu^3)\xi z\\ &\qquad + \left[\frac{1}{9}(\beta\lambda^3 + \gamma\mu^3)^2 - \frac{1}{12}F_4(\lambda, \mu)\right]z^2 = 0.\end{aligned}$$

The pole of R-conjugate line with respect to this conic is the point T_π' with coordinates

$$\xi = -3\lambda\mu/(\beta\lambda^3 + \gamma\mu^3), \qquad z = 0.$$

The harmonic conjugate of T_π' with respect to O_1O_π is the point T_π defined by (3.2). Incidentally the locus of T_π' is the cubic Γ_3',

$$3xy + \beta x^3 + \gamma y^3 = 0.$$

The tangents of Darboux are thus again exhibited by means of Γ_3'.

Finally we may readily show that *the polar line of the conic* (3.5) *intersects O_4O_π in a point whose locus as π varies is a rational curve of order seven which intersects the R-harmonic line at its intersections with the tangents of Darboux.*

4. The edges of Green.

The expansions [4] of the two branches of the curve of intersection of S at O_1 with its tangent plane are

$$\begin{aligned}(4.1)\qquad &y = \frac{1}{3}\beta x^3 - \frac{1}{12}(2\beta\theta_u - \beta_u)^3 + \cdots, \ z = 0;\\ &x = \frac{1}{3}\gamma y^3 - \frac{1}{12}(2\gamma\theta_v - \gamma_v)^3 + \cdots, \ z = 0.\end{aligned}$$

The characteristic point T_u of the first of (4.1) relative to $O_1O_3O_2$ has coordinates

$$(4.2)\qquad x_1 = \frac{1}{4}\left(2\theta_u - \frac{\beta_u}{\beta}\right), \qquad x_2 = 1, \qquad x_3 = x_4 = 0,$$

and the characteristic point T_v of the second relative to $O_1O_2O_3$ has coordinates

$$(4.3)\qquad x_1 = \frac{1}{4}\left(2\theta_v - \frac{\gamma_v}{\gamma}\right), \qquad x_2 = 0, \qquad x_3 = 1, \qquad x_4 = 0.$$

The line joining the harmonic conjugates of T_u and T_v with respect to O_1O_2 and O_1O_3 respectively is Green's edge of the second kind.

This edge of Green may be characterized in another way. The section of S by the plane through the R-conjugate line and the tangent to the asymptotic curve $v=$const. has the equation

$$(4.4)\qquad z = -\frac{1}{3}\beta x^3 + \frac{1}{12}(2\beta\theta_u - \beta_u)x^4 + \cdots .$$

The characteristic point of the third order of the curve (4.4) *relative to $O_1O_4O_2$, is found from* (2.5) *to have coordinates given by* (4.2); by interchanging the roles of the asymptotic tangents the point (4.3) is characterized. The second edge of Green is therefore given another characterization.

Consider on the tangent to the section (3.1) C_r of S the point $O_r'(\rho, 2\lambda, 2\mu, 0)$. From (2.4) we find readily that the characteristic point T of C_r relative to $O_1O_4O_r'$ has coordinates

$$(4.5)\qquad x_1 = \rho\lambda\mu + \frac{1}{3}(\beta\lambda^3 + \gamma\mu^3), \qquad x_2 = \lambda^2\mu, \qquad x_3 = \lambda\mu^2, \qquad x_4 = 0.$$

The point P_r on the tangent to C_r at O_1 which with O_1 separates O_r' and O_r harmonically has coordinates $(\rho, \lambda, \mu, 0)$. *Equations* (4.4) *therefore represent a cubic transformation of P_r into the characteristic point of C_r relative to $O_1O_4O_r'$.* The polar plane of the point (4.5) with respect to any quadric of Darboux,

$$x_2x_3 - x_1x_4 + k_4x_4^2 = 0,$$

has coordinates

$$(4.6)\qquad \xi_1 = 0, \quad \xi_2 = \lambda\mu^2, \quad \xi_3 = \lambda^2\mu, \quad \xi_4 = -\rho\lambda\mu - \frac{1}{3}(\beta\lambda^3 + \gamma\mu^3).$$

The correspondence (4.6) *between P_r and the polar plane of the characteristic point of C_r relative to $O_1O_4O_r'$ is the correspondence of Moutard* ($k = -1/3$). We have previously [3] given a different derivation of this correspondence.

5. **The projective normal.** The surface S' whose equation is

$$(5.1)\qquad z = xy - \frac{1}{3}(\beta x^3 + \gamma y^3)$$

has a unode at O_4, the plane $O_2O_3O_4$ as uniplane, and has contact of the third order with S at O_1; hence S' is completely determined. The projection on their common tangent plane at O_1 of the curve of intersection of S and S' has a quadruple point at O_1, the quadruple tangents being given by

$$F_4(x, y) = 0 \tag{5.2}$$

where $F_4(x, y)$ is defined by (1.3). The lines (5.2) intersect the R-harmonic line in four points F_i, and the Segre tangents intersect this line in three points S_i. It is easy to verify that *the points S_i are apolar to F_i if and only if the R-harmonic line is the reciprocal of the projective normal*. The projective normal is therefore geometrically determined by reciprocation with respect to the quadrics of Darboux.

Finally let the lines l_1, l_2 be the projective normal and its reciprocal; then it readily follows that the polar of the form $\beta x^3+\gamma y^3$ with respect to $F_4(x, y)$ is

$$\phi x - \psi y \tag{5.3}$$

wherein $\phi=\partial \log (\beta\gamma^2)/\partial u$, $\psi=\partial \log (\beta^2\gamma)/\partial v$. The form (5.3) equated to zero is seen to be *the equation of the canonical tangent*.

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MICHIGAN STATE COLLEGE

ON A LINKAGE THEOREM BY L. CESARI

SAMUEL EILENBERG

In connection with his work on Lebesgue area of surfaces, Lamberto Cesari (*Su di un problema di analysis situs dello spazio ordinario*, R. Istituto Lombardo di Scienze e Lettere, Classe di Scienze Matematiche e Naturali, Rendiconti (3) vol. 6 (1942) pp. 267–291) has stated and proved a linkage theorem (see below), which reveals a rather interesting property of Euclidean 3-space. In view of the simplicity of the theorem and the laboriousness of Cesari's proof, the following concise proof may be of some interest.

In Euclidean three space E^3 consider the set M consisting of the three axes X, Y, Z. Let δ be a positive number and let N be the set consisting of the four lines

$$X_1: \ (y = 0, z = -\delta), \qquad Y_1: \ (x = 0, z = \delta),$$
$$Z_1: \ (x = \delta, y = \delta), \qquad Z_1': \ (x = -\delta, y = -\delta).$$

THEOREM OF CESARI. *Any closed path in E^3-M that has a distance greater than δ from M and is contractible in E^3-N also is contractible in E^3-M.*

Let $\mathcal{A}$ be the fundamental group of E^3-M and $\mathcal{B}$ the fundamental group of E^3-N. We shall assume that the same point of E^3 is used as base point in the definition of both $\mathcal{A}$ and $\mathcal{B}$ and that this point has a distance greater than δ from M.

Given any element $a \in \mathcal{A}$ select a closed path in E^3-M in the class a with distance greater than δ from M. Such a path lies also in E^3-N and determines an element $\phi(a)$ of $\mathcal{B}$. It is easy to see that ϕ is single-valued and yields a homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$.

Cesari's theorem can now be reformulated as follows:

THEOREM. *ϕ maps $\mathcal{A}$ isomorphically into a subgroup of $\mathcal{B}$.*

To prove the theorem we draw projections of M and N and establish generators and relations for $\mathcal{A}$ and $\mathcal{B}$. In terms of these generators the homomorphism ϕ is given an explicit form. The problem thus translates into a problem on free groups which is solved algebraically.

In looking at the diagrams one should consider the eye as the base point of the fundamental group. To each line of the diagram corresponds then an element of the group represented by an arrow which corresponds to the path leading rectilinearly from the eye to the be-

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ginning of the arrow, following the arrow and returning rectilinearly from the end of the arrow to the eye. Each crossing or ramification gives a relation. See K. Reidemeister, *Knotentheorie*, Berlin, 1932, chap. 3.

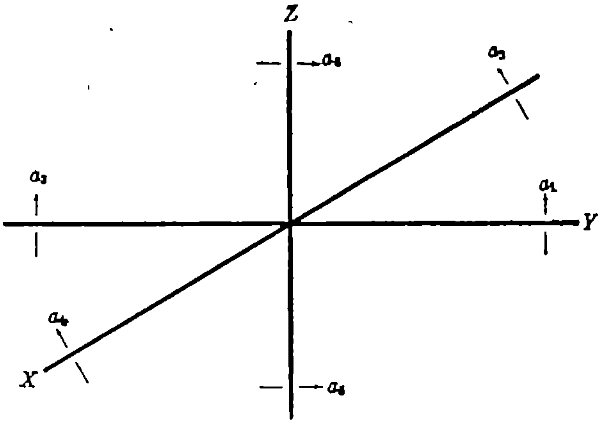

FIG. 1

From Fig. 1 we deduce that $\mathcal{A}$ is generated by six generators $a_1, \cdots, a_6$ with the single relation $a_5a_1a_2=a_4a_3a_6$ derived from the ramification point. Thus $a_6=a_3^{-1}a_4^{-1}a_5a_1a_2$ and $\mathcal{A}$ is the free group generated by $a_1, \cdots, a_5$.

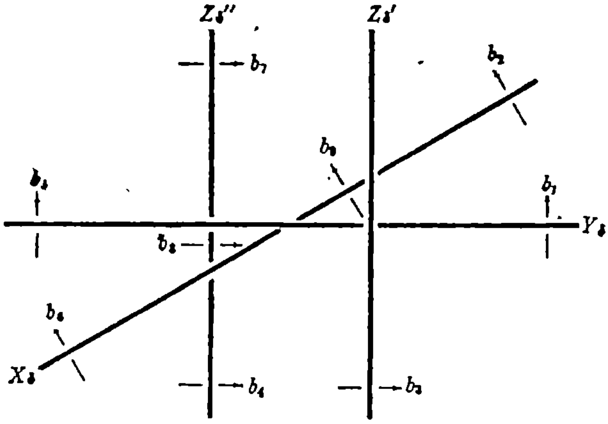

FIG. 2

From Fig. 2 we deduce that $\mathcal{B}$ is generated by the elements $b_1, \cdots, b_9$ and the five relations derived from the five crossings

$$b_5b_3b_1^{-1}b_3^{-1} = 1, \qquad b_9b_3b_2^{-1}b_3^{-1} = 1, \qquad b_6b_5b_9^{-1}b_5^{-1} = 1,$$
$$b_8b_6^{-1}b_4^{-1}b_6 = 1, \qquad b_7b_5^{-1}b_8^{-1}b_5 = 1,$$

which are equivalent with the following:

$$\begin{aligned}
b_5 &= b_3b_1b_3^{-1},\\
b_6 &= b_3b_1b_2b_1^{-1}b_3^{-1},\\
b_7 &= b_3b_2^{-1}b_1^{-1}b_3^{-1}b_4b_3b_1b_2b_3^{-1},\\
b_8 &= b_3b_1b_2^{-1}b_1^{-1}b_3^{-1}b_4b_3b_1b_2b_1^{-1}b_3^{-1},\\
b_9 &= b_3b_2b_3^{-1}.
\end{aligned}$$

Thus $\mathcal{B}$ is the free group generated by $b_1, \cdots, b_4$.

The values of the homomorphism $\phi:\mathcal{A}\rightarrow\mathcal{B}$ on the generators $a_1, \cdots, a_5$ are

$$\text{(I)} \qquad \begin{aligned}
\phi(a_1) &= b_1,\\
\phi(a_2) &= b_2,\\
\phi(a_3) &= b_5 = b_3b_1b_3^{-1},\\
\phi(a_4) &= b_6 = b_3b_1b_2b_1^{-1}b_3^{-1},\\
\phi(a_5) &= b_4b_3.
\end{aligned}$$

We also have $\phi(a_6)=b_7b_3$ but this is a consequence of (I) since $a_6=a_3^{-1}a_4^{-1}a_5a_1a_2$.

We introduce new bases in $\mathcal{A}$ and $\mathcal{B}$ as follows

$$\begin{aligned}
A_1 &= a_1, & B_1 &= b_1,\\
A_2 &= a_2, & B_2 &= b_2,\\
A_3 &= a_3, & B_3 &= b_3,\\
A_4 &= a_3^{-1}a_4a_3, & B_4 &= b_4b_3,\\
A_5 &= a_5.
\end{aligned}$$

It follows from a theorem of F. Levi (Math. Zeit. vol. 37 (1933) p. 95) that $A_1, \cdots, A_5$ are free generators for $\mathcal{A}$ and that $B_1, \cdots, B_4$ are free generators for $\mathcal{B}$. In terms of the new generators we have

$$\text{(II)} \qquad \begin{aligned}
\phi(A_1) &= B_1,\\
\phi(A_2) &= B_2,\\
\phi(A_3) &= B_3B_1B_3^{-1},\\
\phi(A_4) &= B_3B_2B_3^{-1},\\
\phi(A_5) &= B_4.
\end{aligned}$$

Since A_5 and B_4 occur only in the last equation we can suppress these two generators and consider the group $\mathcal{A}'$ generated by $A_1, \cdots, A_4$ and the group $\mathcal{B}'$ generated by B_1, B_2, B_3.

Every element $A \in \mathcal{A}'$ can uniquely be written as

$$A = \cdots W_i(A_1, A_2)V_i(A_3, A_4)W_{i+1}(A_1, A_2)V_{i+1}(A_3, A_4) \cdots$$

where W_i is a word in A_1, A_2 in normal form (that is one in which no element stands next to its inverse) and V_i is similarly a word in A_3, A_4. The expression for A may begin and end with either a W or a V. In view of relations (II) we have

$$\phi W_i(A_1, A_2) = W_i(B_1, B_2),$$
$$\phi V_i(A_3, A_4) = B_3V_i(B_1, B_2)B_3^{-1}.$$

Thus

$$\phi(A) = \cdots W_i(B_1, B_2)B_3V_i(B_1, B_2)B_3^{-1} \cdot W_{i+1}(B_1, B_2)B_3V_{i+1}(B_1, B_2)B_3^{-1} \cdots$$

and the expression on the right is in normal form. Thus $\phi(A)=1$ implies $W_i=1$ and $V_i=1$ and therefore $A=1$. This shows that ϕ is an isomorphism as desired.

It may be worth noting that the analogous theorem is false if the line Zl' is removed. Indeed the removal of this line has the effect of setting $b_4=1$ in the group $\mathcal{B}$. Then $B_3=B_4$ and $W=A_3A_5A_1^{-1}A_5^{-1}$ is an element in $\mathcal{A}$ with $\phi(W)=1$. On the other hand the theorem remains valid if both lines X and X_3 are removed. The effect of this removal is to set $a_2=a_4=1$ or equivalently $A_2=A_4=1$ in the group $\mathcal{A}$, and $B_2=b_2=1$ in the group $\mathcal{B}$. The group $\mathcal{A}$ is then freely generated by A_1, A_3, A_5 while $\mathcal{B}$ is similarly generated by B_1, B_3, B_4. The homomorphism ϕ is defined by

$$\phi(A_1) = B_1,$$
$$\phi(A_3) = B_3B_1B_3^{-1},$$
$$\phi(A_5) = B_4.$$

An argument completely analogous to the preceding one shows that ϕ is an isomorphism of $\mathcal{A}$ into a subgroup of $\mathcal{B}$.

INDIANA UNIVERSITY

ERRATA, VOLUME 53

A. S. Galbraith and J. W. Green, *A note on the mean value of the Poisson kernel.*

p. 316, display (5). For "$m!2^m$" read "$m!2^{m+p}$."

W. H. Mills, *A prime-representing function.*

p. 604, displays (3), (4), and (5). For "$3-n$" read "3^{-n}." For "$3-n-1$" read "3^{-n-1}."

S. Chandrasekhar, *The transfer of radiation in stellar atmospheres.*

p. 679, line 2 from the bottom. For "$\sigma(\mu, \varphi; \mu_0, \varphi_0)\mu$" read "$\sigma(\mu, \varphi; \mu_0, \varphi_0)/\mu$."

T. R. Hollcroft, *The April Meeting in New York.*

p. 723, line 10. For "53-5-121" read "53-3-121."

H. J. Hamilton, *Mertens' theorem and sequence transformations.*

p. 786, line 15. Read "$\neq$" for "$=$."

H. E. Salzer, *An "empirical theorem" which is true for the first* 618 *cases, but fails in the* 619*th*. Abstract 53-9-308.

p. 908, line 4 of the abstract. For "618" read "6186."

p. 908, line 5 of the abstract. For "2006" read "20006."

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